A STUDY OF EIGEN FUZZY SETS OF FUZZY RELATIONS

BY

IBRAHIM FOLORUNSHO MUKAILA

DEPARTMENT OF MATHEMATICS

AHMADU BELLO UNIVERSITY, ZARIA

NIGERIA

MARCH, 2017

A STUDY OF EIGEN FUZZY SETS OF FUZZY RELATIONS

BY

Ibrahim Folorunsho MUKAILA
MSC/SCI/21414/2012-2013

A DISSERTATION SUBMITTED TO THE POSTGRADUATE SCHOOL, AHMADU BELLO UNIVERSITY, ZARIA

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF MASTERS OF SCIENCE DEGREE IN MATHEMATICS

DEPARTMENT OF MATHEMATICS,
FACULTY OF PHYSICAL SCIENCE,
AHMADU BELLO UNIVERSITY, ZARIA

NIGERIA

MARCH, 2017

DECLARATION

I declare that the work in this dissertation entitled **A STUDY OF EIGEN FUZZY SETS OF FUZZY RELATIONS** has been performed by me in the Department of Mathematics under the supervision of Dr. A. M. Ibrahim and Prof D. Singh. The information derived from literature has been duly acknowledged in the text and a list of reference provided. No part of this dissertation was previously presented for another degree at any University or Institution.

Ibrahim Folorunsho MUKAILA	____________________	_______________
Name of Student	Signature	Date

CERTIFICATION

The dissertation titled **A STUDY OF EIGEN FUZZY SETS OF FUZZY RELATIONS** by MUKAILA, Ibrahim Folorunsho (MSC/SCI/21414/2012-2013), meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

Dr. A. M. Ibrahim	____________	______________
Chairman, Supervisory Committee	**Signature**	**Date**
Prof. D Singh	___________	______________
Member, Supervisory Committee	**Signature**	**Date**
Prof. A. A. Tijjani	___________	______________
Head of Department	**Signature**	**Date**
________________	___________	______________
External Examiner	**Signature**	**Date**
Prof. S. Z. Abubakar	___________	______________
Dean, Postgraduate School	**Signature**	**Date**

DEDICATION

This dissertation is dedicated to God Almighty for seeing me through this rigorous program.

ACKNOWLEDGMENT

My profound gratitude goes to Almighty ALLAH (S.W.T) for giving me the opportunity to undertake this rigorous program in good health, making this research dissertation a reality, and for many good things happening in my life (Alhamdulillah). My sincere gratitude goes to my supervisors Dr. A. M. Ibrahim and Prof. D. Singh for their effort to make this work a successful one, their suggestions and strictness has really brought out the best in me. My appreciation goes to my Head of Department, Prof. A. A. Tijjani, all Lecturers and entire staff of the Department of Mathematics, Ahmadu Bello University, Zaria.

My special thanks goes to my family, most especially, my caring sister Dr. M. T. Salawudeen for being with me each step of the way, and to my wife Mrs. K. A. Oyekola, also to my friends and entire PG (2012-2013) students of Mathematics Department, Ahmadu Bello University, Zaria. I want to say thank you all.

Finally, my sincere appreciation goes to all the staff of Division of Agricultural Colleges, Ahmadu Bello University, Zaria, most especially, the immediate past Director Prof. M. Y. Fatihu, the present Director, the Provost and entire staff of College of Agriculture, Kabba, Kogi State for their support. I thank you all.

ABSTRACT

In this dissertation, a concept called subreflexive fuzzy relation as a relaxation of the reflexive property of fuzzy relation has been introduced and investigated. Various compositions of fuzzy relations such as max-min, min-max, max-average and max-product were presented. Eigen Fuzzy Sets of fuzzy relation based on the max-min and min-max compositions have been studied, and algorithms for the determination of Greatest Eigen Fuzzy Set (GEFS) based on max-min composition have also been studied and a modified algorithm was presented, and these algorithms have been extended to Least Eigen Fuzzy Set (LEFS) based on min-max composition. Finally, Eigen Fuzzy Sets based on max-product and max-average compositions have also been investigated. It was observed that an Eigen Fuzzy Set do not exist in the max-product composition except if the relation is reflexive, and also, an Eigen Fuzzy Set may or may not exist in the max-average composition, but if the relation is subreflexive such that $\mu_R(x_i, x'_j)_{i=j} \geq \mu_R(x_i, x'_j)_{i \neq j}, \forall (x_i, x'_j) \in R$, *then a constant Eigen Fuzzy Set is guaranteed, and in particular, if the relation is reflexive.*

TABLE OF CONTENTS

CHAPTER ONE

GENERAL INTRODUCTION

1.0 Background of the Study

Most of the existing mathematical tools for formal modeling, reasoning and computing are crisp, deterministic and precise in nature, (Zimmermann, 1996). But, in real life situation, problems do not always involve crisp data. Consequently, we cannot successfully use the traditional classical methods because of various type of uncertainties presented in these problems. To deal with these uncertainties, fuzzy set theory can be considered as one of the mathematical tools.

The notion of a fuzzy set stem from the observation made by Zadeh, (1965) that "more often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership". This observation emphasizes the gap existing between mental representations of reality and usual mathematical representations thereof, which are based on binary logic, precise numbers, differential equations and the like. Classes of objects referred to in (Zadeh, 1965) exist only through mental representations through natural language terms such as high temperature, young man, big size, etc. Classical logic is too rigid to account for such categories where it appears that membership is a gradual notion rather than an all-or-nothing matter. (Dubois *et al*., 2000).

The theory of fuzzy set developed in (Zadeh, 1965), a theory of graded concept, is a generalization of classical set theory where an element in a set is allowed to have varying degree of membership in the real interval [0, 1] representing the degree to which each element belongs to the set, the value zero represents a complete non-membership, the value one (1) represents a complete or absolute membership in the set and the values in between signifies partial membership of the element. Since its introduction, the theory has gain serious recognition due to its contributions and applicability in real world.

Fuzzy relation was also introduced by Zadeh (1965) as a generalization of crisp relation to one that allows partial membership in which two objects can be related to one another to a given degree in the real interval [0, 1]. Starting in early seventies, fuzzy relations have been defined, investigated, and applied in many different ways and have applications in fields such as Artificial Intelligence, Psychology, Medicine, Economics, and Sociology (Beg and Ashraf, 2009).

Eigen Fuzzy Sets associated with the fuzzy relation is a fuzzy set of all invariant elements in the fuzzy relation obtained by the composition of a fuzzy set and the fuzzy relation. This was introduced by Sanchez (1978) and was found to have several applications such as in the area of image retrieval and reconstruction.

1.1 Statement of the Problem

A lot of work in fuzzy set theory and fuzzy logic has progressed since the inception of these fields in 1965. One of the basic notions of mathematics, a crisp relation was fuzzified into its fuzzy counterpart by Zadeh (1965). This paved way for researchers to extend various aspects and properties of crisp relation to fuzzy relation and also developed new concepts. Consequently, Eigen Fuzzy Sets was introduced in (Sanchez, 1978) based on composition of fuzzy relations. This is an important area of fuzzy relation with many applications in the field of science, engineering, medicine and even behavioral science. However, only the max-min composition was used in (Sanchez, 1978) called Greatest Eigen Fuzzy Set (GEFS) associated with a fuzzy relation. So it becomes essential to study various Eigen Fuzzy Set based on other compositions of fuzzy relation.

1.2 Aim and Objectives

The aim of this dissertation is to investigate Eigen Fuzzy Sets associated with fuzzy relations, specifically, the objectives are to:

1. explore fuzzy set and fuzzy relation with their various operations, properties and compositions;
2. study Eigen Fuzzy Set based on max-min composition and the algorithms for the determination of the Greatest Eigen Fuzzy Set (GEFS) and presents a modified algorithm;
3. extend the algorithms for the determination of the Greatest Eigen Fuzzy Set (GEFS) to Least Eigen Fuzzy Set (LEFS) based on the min-max composition;
4. investigate Eigen Fuzzy Sets based on max-product and max-average compositions.

1.3 Methodology

In making up this research dissertation, we began by reviewing various literatures on fuzzy set, fuzzy relation and their properties, study various compositions of fuzzy relations, Eigen Fuzzy Sets associated with fuzzy relations based on the max-min composition and the algorithms for the determination of the Greatest Eigen Fuzzy Set (GEFS). We were able to provide extensions to some of the existing properties and propositions, and provide algorithms as a dual type for the min-max composition for the Least Eigen Fuzzy Set (LEFS). We also investigate the existence of Eigen Fuzzy Sets based on max-product and max-average compositions.

1.4 Organization of the Dissertation

This Dissertation consists of five chapters. Chapter one contains the general introduction and basic definitions on fuzzy set and operations of fuzzy set. In chapter two, review of related literature will be presented. In chapter three, fuzzy relation, its properties, compositions and Eigen fuzzy set associated with fuzzy relation with the algorithms for the determination of the Greatest Eigen Fuzzy Set (GEFS) will be presented. In chapter four a modified algorithm for the determination of GEFS and the algorithms for the determination of Least Eigen Fuzzy Sets (LEFS) will be presented. Finally, in chapter five, the summary, conclusion and recommendation will be presented.

1.5 Basic Definitions

In this section, we present some basic definitions and concepts which are needed for understanding of ideas in this dissertation. These definitions and concepts are mostly taken from (Zadeh, 1965), (Zimmermann, 1996), (Dubois *et al*, 2000), and (Klir and Yuan, 1995).

Definition: 1.5.1 (Fuzzy Set)

Let X be a classical set, a fuzzy set $A \subseteq X$ is characterized by a membership function μ_A which associates to each element $x \in X$, a real number $\mu_A(x)$ in the unit interval $[0, 1]$.

The value $\mu_A(x)$ represents the degree to which x belongs to A, $\mu_A(x)$ can be seen as a mapping that takes each element $x \in X$ to the unit interval $[0,1]$, i.e., $\mu_A : X \mapsto [0,1]$, where the closer the value of $\mu_A(x)$ to unity, the higher the degree of membership of x in A, and the value zero represents a complete non-membership of x in A. Therefore, a fuzzy set A is a set of ordered pairs $\{(x, \mu_A(x)) | x \in X\}$, where $x \in X$ and $\mu_A(x)$ is the membership function of $x \in A$.

Example: 1.5.1

Let X be a set of natural numbers, and A be the set of numbers close to 10, then, this can be written as;

$A = \{(5,0.4), (7,0.6), (8,0.7), (9,0.9), (10,1), (11,0.9), (12, 0.8), (15, 0.5)\}$

Remark 1.5.1

This is a particular representation of A, other models can be given for this particular fuzzy set A, as no general model exist for fuzzy sets. Another model can be defined by membership function given below;

$$\mu_A(x) = \begin{cases} \frac{1}{(11-x)}, if\ x \leq 10 \\ \frac{1}{(x-9)}, if\ x \geq 10 \end{cases}$$

Definition 1.5.2 (Empty Fuzzy Set)

A fuzzy set is said to be empty if its membership degree is identically zero in X, that is;

$$A = \emptyset, \qquad \text{if } \mu_A(x) = 0\ \forall x \in X$$

Definition 1.5.3 (Fuzzy Subset)

Given two fuzzy sets A and B, A is said to be a subset of B, written as; $A \subseteq B$ if

$$\mu_A(x) \leq \mu_B(x), \forall x \in X.$$

A is a proper subset of B, that is; $A \subset B$, If in addition to $\mu_A(x) \leq \mu_B(x)$, there exist some $x \in X$, such that $\mu_A(x) = 0$ and $\mu_B(x) \neq 0$.

Definition 1.5.4 (Equality)

Two fuzzy sets A and B are said to be equal written as $A = B$, if;

$$\mu_A(x) = \mu_B(x)\ \forall x \in X.$$

Definition 1.5.5 (Support)

The support of a fuzzy set A, denoted as $S(A)$ is a crisp set of all element $x \in X$, having membership degree $\mu_A(x)$ greater than zero.

Symbolically,

$$S(A) = \{x \in X | \mu_A(x) > 0\}$$

Considering example 1.5.1, we have; $S(A) = \{5, 7, 8, 9, 10, 11, 12, 15\}$

Definition 1.5.6 (Core)

The core $C(A)$ of a fuzzy set A is the crisp set that contains all elements of X whose membership degree is one. Symbolically,

$$C(A) = \{x \in X | \mu_A(x) = 1\}$$

Definition 1.5.7 (Height)

The height $h(A)$ of a fuzzy set A, is the greatest membership degree of the elements in that set. That is;

$$h(A) = \max_{x \in X} \left(\mu_A(x) \right)$$

Definition 1.5.8 (Normality)

A fuzzy set A is normal if its Core is nonempty. That is, we can always find at least a point $x \in X$ such that $\mu_A(x) = 1$.

Definition 1.5.9 (Fuzzy Singleton)

A fuzzy set whose support is a single point in X with $\mu_A(x) > 0$ is called a Fuzzy Singleton.

Definition 1.5.10 (Crossover Point)

A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$.

1.6 Operations on Fuzzy Set

Let X be a classical set and let $A, B, C, \ldots$, be fuzzy subsets of X with the membership functions $\mu_A(x)$, $\mu_B(x)$,$\mu_C(x), \ldots$, respectively for each $x \in X$, we have the following definitions.

Definition 1.6.1 (Union)

The union of two fuzzy sets A and B written as $A \cup B$ is a fuzzy set say C defined by the membership function; $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}, \forall x \in X$.

Definition 1.6.2 (Intersection)

The intersection of two fuzzy sets A and B written as $A \cap B$ is a fuzzy set say C defined by the membership function

$$\mu_C(x) = \mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \forall x \in X$$

Remark 1.6.1

The union of any finite number of fuzzy sets is the smallest fuzzy set containing them. That is; if,

$$A_t = \bigcup_{i=1}^{n} A_i \text{ and for each } i, \qquad A_i \subseteq A_q, \text{then } A_t \subseteq A_q$$

In particular, if A and B are fuzzy sets such that $C = A \cup B$, and D is any other fuzzy set containing both A and B, then D contains C also. Here since $C = A \cup B$, then it follows that

$$\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\} \forall x \in X.$$

Since A and B are both subsets of D, it implies $\mu_A(x) \leq \mu_D(x)$, and $\mu_A(x) \leq \mu_D(x)$ for each $x \in X$. Hence, $\max\{\mu_A(x), \mu_B(x)\} \leq \mu_D(x) \forall x \in X$.

Therefore, $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\} \leq \mu_D(x)\ \forall x \in X$.

It then follows that $C \subseteq D$, and by mathematical induction, it can be generalized to finite number fuzzy sets.

In the same manner, it can be shown that the intersection of a finite number of fuzzy sets $A_1, A_2, \dots, A_n$ is the largest fuzzy set that they contained.

Definition 1.6.3 (Complement)

The complement of a fuzzy set A denoted by A' is a fuzzy set defined by the membership function $\mu_{A'}(x) = 1 - \mu_A(x), \forall x \in X$.

Example 1.6.1

Let $X = \{x \in \mathbb{N} | 5 \leq x \leq 14\}$, A and B be fuzzy sets given as

$$A = \{(5,0.4), (7,0.6), (8,0.7), (9,0.9), (10,1), (11, 0.9), (12, 0.8)\}$$

and

$$B = \{(5, 0.3), (6, 0.4), (7, 0.6), (8, 0.8), (10, 1), (12, 0.8), (14, 0.6)\}$$

Then we have the following fuzzy sets for Union, intersection and complement

$$A \cup B = \begin{Bmatrix} (5, 0.4), (6, 0.4), (7, 0.6), (8, 0.8), (9, 0.9), (10, 1), \\ (11, .09), (12, 0.8), (14, 0.6) \end{Bmatrix}.$$

$$A \cap B = \{(5, 0.3), (7, 0.6), (8, 0.7), (10, 1), (12, 0.8)\}.$$

$$A' = \{(5,0.6), (6, 1), (7,0.4), (8,0.3), (9,0.1), (11,0.1), (12, 0.2), (13, 1), (14, 1)\}.$$

$$B' = \{(5, 0.7), (6, 0.6), (7, 0.3), (8, 0.2), (9, 1), (11, 1), (12, 0.2), (13,1), (14, 0.4)\}.$$

Remark 1.6.2

These operators give the usual union, intersection and complement of classical set when the membership function is restricted to {0, 1}, that is; the definitions of operators on fuzzy sets include those on ordinary (classical) sets as special cases. Other definitions of these and further set operators have also been proposed. (see (Yager, 1991; and Zimmermann 1996)).

1.7 Alpha-Cuts in Fuzzy Set

Definition 1.7.1 (Alpha-Cut)

Given any fuzzy set A, the alpha-cut of A, denoted as A_α, $\alpha \in (0, 1]$, is a crisp set defined as;

$$A_\alpha = \{x | x \in X, \mu_A(x) \geq \alpha\},$$

and strong alpha-cut denoted as $A_{\underline{\alpha}}$ is defined as;

$$A_{\underline{\alpha}} = \{x | x \in X, \mu_A(x) > \alpha\}.$$

Example 1.7.1

$A = \{(5,0.4), (7,0.6), (8,0.7), (9,0.9), (10,1), (11,0.9), (12, 0.8)\}$

$A_{0.4} = \{5, 7, 8, 9, 10, 11, 12\}$, $A_{0.6} = \{7, 8, 9, 10, 11, 12\}$, $A_{0.7} = \{8, 9, 10, 11, 12\}$

$A_{0.8} = \{9, 10, 11, 12\}$, $A_{0.9} = \{9, 10, 11\}$, $A_1 = \{10\}$,

and the strong level set for $\alpha = 0.6$, is given as $A_{\underline{0.6}} = \{8, 9, 10, 11, 12\}$.

Remark 1.7.1

Similar to the concept of level set (alpha cuts) is a fuzzy subset whose membership functions are greater or equal to alpha for each alpha in the interval (0, 1].This concept has first been introduced by (Radecki, 1976) which he called level fuzzy set.

Definition 1.7.2 (Level Fuzzy Subset)

Let A be a fuzzy set over a set X, and let $\alpha \in (0, 1]$ be any point, then the level fuzzy subsets of A is defined as;

$$\mathcal{A}_\alpha = \{(x, \mu_A(x)) | x \in A_\alpha\}$$

In the same way, we can define the strong level set of the fuzzy set A as;

$$\mathcal{A}_{\underline{\alpha}} = \{(x, \mu_A(x)) | x \in A_{\underline{\alpha}}\}.$$

Example 1.7.2

$$A = \{(5,0.4), (7,0.6), (8,0.7), (9,0.9), (10,1), (11,0.9), (12, 0.8)\}$$

Then $\mathcal{A}_{\alpha=0.5} = \{(7,0.6), (8,0.7), (9,0.9), (10,1), (11,0.9), (12, 0.8)\}$

And $\mathcal{A}_{\alpha=0.8} = \{(9,0.9), (10,1), (11,0.9), (12, 0.8)\}$

Also $\mathcal{A}_{\underline{\alpha}=0.8} = \{(9,0.9), (10,1), (11,0.9)\}$

Definition 1.7.3 (Inverse $\boldsymbol{\alpha}$-Cut)

Let X be a classical set, $A \in F(X)$, and $\alpha \in (0,1]$, the inverse $\boldsymbol{\alpha}$-Cut of A, denoted as $A_{\alpha^{-1}}$ is a crisp set defined as;

$$A_{\alpha^{-1}} = \{x \in X | \mu_A(x) < \alpha\}$$

The weak Inverse α-Cut is defined as;

$$A_{\underline{\alpha}^{-1}} = \{x \in X | \mu_A(x) \leq \alpha\}$$

Definition 1.7.4 (Inverse Level Fuzzy Subset)

The inverse level fuzzy subset of A denoted as $\mathcal{A}_{\alpha^{-1}}$ is defined as

$$\mathcal{A}_{\alpha^{-1}} = \{x, \mu_A(x)) | x \in A_{\alpha^{-1}}\}$$

The weak inverse level fuzzy subset $\mathcal{A}_{\underline{\alpha}^{-1}}$ is defined as;

$$\mathcal{A}_{\underline{\alpha}^{-1}} = \{x, \mu_A(x)) | x \in A_{\underline{\alpha}^{-1}}\}$$

CHAPTER TWO

LITERATURE REVIEW

The development of fuzzy set theory originated from the first publications by (Zadeh, 1965), with the intention to generalize the classical notion of a set to accommodate fuzziness in the sense that it is contained in human language, that is, in human judgment, evaluation, and decisions (Zimmermann, 1996)

In his paper (Zadeh, 1965), states, ‘The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables’.

‘Imprecision’ here is meant in the sense of vagueness rather than the lack of knowledge about the value of a parameter. Fuzzy set theory provides a strict mathematical framework in which vague conceptual phenomena can be precisely and rigorously studied. It can also be considered as a modeling language, well suited for situations in which fuzzy relations, criteria, and phenomena exist.

The acceptance of this theory grew slowly in the 1960s and 1970s of the last century. In the second half of the 1970s, however, the first successful practical applications in the control of technological processes via fuzzy rule based systems, called fuzzy control (heating systems, cement factories, etc.), boosted the interest in this area considerably. Successful applications, particularly in Japan, in washing machines, video cameras, cranes, subway trains, and so on triggered further interest and research in the 1980s so that in 1984 already approximately 4000 publications existed and in 2000 more than 30,000. (Zimmermann, 2010)

Roughly speaking, fuzzy set theory during the last decades has developed along two lines:

i. As a formal theory that, when maturing, became more sophisticated and specified and was enlarged by original ideas and concepts as well as by 'embracing' classical mathematical areas, such as algebra, relation and graph theory, topology, and so on by generalizing or 'fuzzifying' them. This development is still ongoing.

ii. As an application-oriented 'fuzzy technology', that is, as a tool for modeling, problem solving, and data mining that has been proven superior to existing methods in many cases and as attractive 'add-on' to classical approaches in other cases.

Regrettably, this adaption has not yet progressed to a satisfactory level, leaving an abundance of challenges for new researchers.

The notion of α-cuts was introduced in (Zadeh, 1965) with the aim of establishing a bridge between fuzzy set theory and classical set theory. A fuzzy set can thus be viewed as a family of nested ordinary subsets, namely the fuzzy sets of elements whose membership is greater than given thresholds. Various properties of alpha-cut were studied and developed by authors such as (Negoita & Ralescu, 1975; Klir & Yuan, 1995).

Sun and Han (2006) introduced the concept of inverse alpha-cut to improve the usage of alpha-cut in solving real life problems, which they showed can also be considered as a bridge between fuzzy set and crisp set. They demonstrated that, like alpha-cut, every fuzzy set can uniquely be represented by the family of its inverse alpha-cuts. (Alkali, 2015).

Fuzzy set theory has progressed into different disciplines of pure mathematics. Among various branches of pure mathematics, relation is one of the first few areas in which the notion of fuzzy set was applied. The concept of relation is one of the most fundamental concepts in pure and applied sciences. Science has been described as the discovery of relations between objects, states and events. Fuzzy relations generalize the concept of relations in the same manner as fuzzy sets generalize the fundamental idea of sets. (Hussain, 2010).

The notion of fuzzy relation was first introduced alongside fuzzy set by Zadeh, (1965 and1971), he defined the notion of fuzzy equivalence (similarity) relation, and gave the concept of fuzzy ordering and provide some basic properties of fuzzy relation. The concept of fuzzy equivalence relation was introduced by generalizing the notion of reflexivity, symmetric and transitivity, thereby facilitating the derivation of known results in various areas in relation and the discovery of new ones.

Several aspects of fuzzy relations have since then been developed by authors such as; (Kaufmann, 1975; Rosenfield, 1975; Zimmermann H. J., 1996; Ovchinnikov, 2000; Kwang, 2005; and Daren et al., 2007).

Beg and Ashraf (2008 and 2009) Presented fuzzy equivalence relation and provide a self contained survey of fuzzy binary relations and some of their applications; such as Fuzzy relational calculus; Indistinguishability relation; fuzzy order; fuzzy equivalence relation; fuzzy inclusion relation; preference structure; continuous fuzzy relation; linear fuzzy relation.

Hussain (2010) also presented an overview of comparison between classical and fuzzy relations, described compositions of fuzzy relations and used these compositions as a model for predicting score in cricket.

Fuzzy relational calculus as an application of fuzzy relation has also been studied and presented by researchers such as; (Sanchez, 1976; 1984; and Di Nola, 1991), the basic aim was to obtain the possible solutions of the fuzzy relation equation $R \circ S = T$, arising from the composition of fuzzy relations where either R or S is unknown. This has also generated a lot of interest by researchers and has since then been under investigation.

Sanchez (1978) introduced the notion of Eigen Fuzzy Sets of fuzzy relation to describe the concept of invariants fuzzy sets associated with a given fuzzy relation using composition of fuzzy relation. He used max-min composition to determine the Greatest Eigen Fuzzy Set (GEFS) associated with fuzzy relation and developed three major algorithms for the determination of the GEFS in his paper titled Eigen Fuzzy Sets and Fuzzy Relations.

Goetschel and Voxman, (1985) extended some results of (Sanchez, 1978) to the context of fuzzy numbers.

Di Martino *et al*., (2004) introduced the concept of Least Eigen Fuzzy Set (LEFS) based on the min-max composition and showed that both GEFS and LEFS are useful in image information retrieval by calculating a similarity measure between two fuzzy relations using their GEFS and LEFS respectively.

Nobuhara *et al*., (2006) formulated and solved a problem of image reconstruction using Eigen fuzzy sets. Treating images as fuzzy relations, they propose two algorithms of generating Eigen fuzzy sets that are used in the reconstruction process using the convex combination of Eigen fuzzy sets of max–min and min–max compositions, and Eigen fuzzy Sets generated by a permutation matrix.

Di Martino & Sessa (2007) present a Genetic Algorithm (GA) based on Eigen Fuzzy Set of Fuzzy Relation for Image Reconstruction in which GEFS and LEFS were used to calculate the fitness value in a GA.

Saleem (2010) reviewed Eigen fuzzy set of fuzzy relation in his M.Sc. thesis titled 'Eigen Fuzzy Sets of Fuzzy Relation with Applications' Blekinge institute of technology. He introduced a concept of least Eigen fuzzy set based on max-min composition. However, this concept is not the least Eigen fuzzy set based on the max-min composition.

CHAPTER THREE

FUZZY RELATION, ITS PROPERTIES AND COMPOSITION

3.1 Introduction

In this chapter, we study and present the basic notions of fuzzy relation, its properties and various compositions and it's alpha-cut. We also present the concept of Eigen Fuzzy Set based on max-min composition and highlight the concept of Greatest Eigen Fuzzy Sets of Fuzzy Relation.

Fuzzy relations are fuzzy subsets of the Cartesian product $X \times Y$and are extension of ordinary relations. They have been studied by several numbers of authors, in particular by (Zadeh, 1971; Kaufmann, 1975; and Rosenfield, 1975). Applications of fuzzy relations are widespread across various fields of knowledge.

Definition 3.1.1 (Fuzzy Relation)

Let X and Y be two known classical sets (universe), and let $(x, y) \in X \times Y$ be the ordered pairs of the Cartesian product of X and Y for every element $x \in X$ and $y \in Y$, then a relation R, which maps every $(x, y) \in X \times Y$ to the unit interval [0, 1], that is; $R \subseteq X \times Y$, defined by

$$R = \{((x, y),\ \mu_R(x, y)) | (x, y) \in X \times Y\}$$

is a fuzzy relation in $X \times Y$, where $\mu_R(x, y): X \times Y \to [0,1]$.

This fuzzy relation can also be represented in the following way

$$R(x, y) = \left\{ \frac{\mu_R(x, y)}{(x, y)} \middle| (x, y) \in X \times Y \right\}$$

When the universes X and Y are both finite, the fuzzy relation R can be represented in an $m \times n$ matrix form, where $\mu_R(x, y)$ are the entries of the matrix $\forall(x, y) \in R$. This matrix is written as;

$$\begin{array}{c} \\ x_1 \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array} \\ \begin{bmatrix} \mu_R(x_1, y_1) & \cdots & \mu_R(x_1, y_n) \\ \vdots & \cdots & \vdots \\ \mu_R(x_m, y_1) & \cdots & \mu_R(x_m, y_n) \end{bmatrix} \end{array}$$

(Hussain, 2010).

Example 3.1.1

Let $X = \{1,2,3\}, Y = \{2,3,4\}$, and define R such that;

$$\mu_R(x, y) = \begin{cases} x/y, & if\ x < y \\ 1, & if\ x = y \\ y/x, & if\ x > y \end{cases}$$

then we have

$$R = \left\{\frac{0.5}{(1,2)}, \frac{0.33}{(1,3)}, \frac{0.25}{(1,4)}, \frac{1}{(2,2)}, \frac{0.66}{(2,3)}, \frac{0.5}{(2,4)}, \frac{0.66}{(3,2)}, \frac{1}{(3,3)}, \frac{0.75}{(3,4)}\right\}$$

this can be written in matrix form as;

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.5 & 0.33 & 0.25 \\ 1 & 0.66 & 0.5 \\ 0.66 & 1 & 0.75 \end{bmatrix} \end{array}$$

Definition: 3.1.2 (Fuzzy Relation of Fuzzy Sets)

Let X and Y be universes of discourse, let $\tilde{A}$ and $\tilde{B}$ be fuzzy subsets of X and Y respectively, i.e. $\tilde{A} \subseteq X$ and $\tilde{B} \subseteq Y$, then a fuzzy relation R can be defined on elements of $\tilde{A} \times \tilde{B}$ such that

$$\mu_R(x,y) \le \mu_{\tilde{A}}(x), \forall x \in \tilde{A}$$

Also

$$\mu_R(x,y) \le \mu_{\tilde{B}}(y), \forall y \in \tilde{B}$$

Therefore, by this definition, we have

$$\mu_R(x,y) \le \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}, \forall (x,y) \in \tilde{A} \times \tilde{B}.$$

Definition 3.1.3 (Domain and Range)

When a fuzzy relation R is defined on crisp sets A and B, then the Domain and Range of this relation are given by the membership functions as;

$$\mu_{Dom(R)}(x) = \max_{y \in B} \mu_R(x,y)$$

and

$$\mu_{Ran(R)}(y) = \max_{x \in A} \mu_R(x,y).$$

Here, the sets A and B becomes the support for $Dom(R)$ and $Ran(R)$ respectively, where $Dom(R) \subseteq A$ and $Ran(R) \subseteq B$. (Zimmermann, 1996)

Considering Example 3.1.1, the domain and range are given as follows;

$$\mu_{Dom(R)}(1) = \max_{2,3,4 \in Y}\{\mu_R(1,2), \mu_R(1,3), \mu_R(1,4)\} = \max\{0.5, 0.33, 0.25\} = 0.5$$

$$\mu_{Dom(R)}(2) = \max_{2,3,4 \in Y}\{\mu_R(2,2), \mu_R(2,3), \mu_R(2,4)\} = \max\{1, 0.67, 0.5\} = 1$$

$$\mu_{Dom(R)}(3) = \max_{2,3,4 \in Y}\{\mu_R(3,2), \mu_R(3,3), \mu_R(3,4)\} = \max\{0.67, 1, 0.75\} = 1$$

Therefore, $Dom(R) = \{(1, 0.5), (2, 1), (3,1)\}$ and also,

$$\mu_{Ran(R)}(2) = \max_{1,2,3, \in X}\{\mu_R(1,2), \mu_R(2,2), \mu_R(3,2)\} = \max\{0.5, 1, 0.67\} = 1$$

$$\mu_{Ran(R)}(3) = \max_{1,2,3, \in X}\{\mu_R(1,3), \mu_R(2,3), \mu_R(3,3)\} = \max\{0.33, 0.67, 1\} = 1$$

$$\mu_{Ran(R)}(4) = \max_{1,2,3, \in X}\{\mu_R(1,4), \mu_R(2,4), \mu_R(3,4)\} = \max\{0.25, 0.5, 0.75\} = 0.75$$

Hence $Ran(R) = \{(2,1), (3,1), (4,0.75)\}$.

Definition 3.1.4 (Inverse Fuzzy Relation)

Let R be a fuzzy relation in $X \times Y$, then the inverse fuzzy relation of R deonted as R^{-1} or simply as $\bar{R}$ is the fuzzy relation in $Y \times X$ defined by the membership function

$$\mu_{\bar{R}}(y, x) = \mu_R(x, y), \forall (x, y) \in R.$$

If R is represented by a matrix M_R where M_R is an $m \times n$ matrix, then the matrix of R^{-1} denoted as $M_{R^{-1}}$ is an $n \times m$ matrix which is a transpose of the matrix M_R of R.

Using the same Example 3.1.1, we have the following;

$$\mu_{\bar{R}}(2,1) = \mu_R(1,2) = 0.5, \mu_{\bar{R}}(2,2) = \mu_R(2,2) = 1, \mu_{\bar{R}}(2,3) = \mu_R(3,2) = 0.67$$

$$\mu_{\bar{R}}(3,1) = \mu_R(1,3) = 0.33, \mu_{\bar{R}}(3,2) = \mu_R(2,3) = 0.67, \mu_{\bar{R}}(3,3) = \mu_R(3,3) = 1$$

$$\mu_{\bar{R}}(4,1) = \mu_R(1,4) = 0.25, \mu_{\bar{R}}(4,2) = \mu_R(2,4) = 0.5, \mu_{\bar{R}}(4,3) = \mu_R(3,4) = 0.75$$

Then

$$R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0.4 & 0.6 & 0.6 & 0.5 \\ 0.4 & 1 & 0.7 & 0.5 \\ 0 & 0 & 0.6 & 0.5 \end{bmatrix} \end{matrix}$$

Here, we can see that $M_{R^{-1}}$ is the transpose of M_R since the entries in each column of M_R are the ones in each row of $M_{R^{-1}}$, and vice versa.

3.1.1 Operations on fuzzy relations

Since fuzzy relations are subsets of the Cartesian products, therefore set theoretic operations can be defined for them in analogy to fuzzy set operations.

Definition 3.1.5

Let R and S be two fuzzy relations from X to Y, that is; $R, S \subseteq X \times Y$, the Union and Intersection of R and S are defined by the following membership functions respectively as;

$$\mu_{R\cup S}(x,y) = \max\{\mu_R(x,y), \mu_S(x,y)\}$$

$$\mu_{R\cap S}(x,y) = \min\{\mu_R(x,y), \mu_S(x,y)\}$$

Definition 3.1.6

Let R be a fuzzy relation on X, that is; $R \subseteq X \times X$, then the complement of R denoted by $\bar{R}$ or R' is also a fuzzy set on X defined by the membership function

$$\mu_{\bar{R}}(x,y) = 1 - \mu_R(x,y), \forall (x,y) \in R.$$

Example 3.1.2

Let us consider two relations

$$R = \begin{matrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0.5 & 0.3 & 0.7 & 0.9 \\ x_3 & 0 & 0.4 & 1 & 0.3 \\ x_4 & 0.7 & 0.5 & 0 & 0.6 \end{matrix}$$

$$S = \begin{matrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.5 & 0.4 & 0.6 & 0.2 \\ x_2 & 1 & 0.2 & 0.5 & 0.7 \\ x_3 & 0.6 & 0.7 & 0.3 & 0.6 \\ x_4 & 0.8 & 1 & 0.4 & 0.3 \end{matrix}$$

Then we have the following;

$$R \cup S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.4 & 0.6 & 0.7 \\ 1 & 0.3 & 0.7 & 0.9 \\ 0.6 & 0.7 & 1 & 0.6 \\ 0.8 & 1 & 0.4 & 0.6 \end{bmatrix} \end{array}.$$

Also,

$$R \cap S = \begin{array}{c} \\ x_1 \\ x_2 \\ x^3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} & & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 & 0.1 & 0.2 \\ 0.5 & 0.2 & 0.5 & 0.7 \\ 0 & 0.4 & 0.3 & 0.3 \\ 0.7 & 0.5 & 0 & 0.3 \end{bmatrix} \end{array}$$

$$\bar{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.2 & 0.9 & 0.8 & 0.3 \\ 0.5 & 0.7 & 0.3 & 0.1 \\ 1 & 0.6 & 0 & 0.7 \\ 0.3 & 0.5 & 1 & 0.4 \end{bmatrix} \end{array}.$$

3.1.2 Composition of fuzzy relations

Given any two fuzzy relations R and S defined on sets A, B, C, where R is a relation from A to B and S from B to C, i.e. $R \subseteq A \times B$ and $S \subseteq B \times C$. The composition of both R and S written as $R \circ S = SR$ is expressed by the fuzzy relation from A to C, i.e., $(R \circ S \subseteq A \times C)$ defined by the membership function

$$\mu_{R \circ S}(x, z) = \max_{y \in B}\{min[\mu_R(x, y), \mu_S(y, z)]\}$$

$$\forall (x, y) \in A \times B, \forall (y, z) \in B \times C.$$

This composition of relation is called the max-min composition of fuzzy relation. (Zimmermann, 1996)

Example 3.1.3

Let

$$R = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 1 & 0.5 & 0.7 \\ x_2 & 0.3 & 0.4 & 1 \\ x_3 & 0.2 & 0.3 & 0 \end{matrix}$$

and

$$S = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.4 & 0.8 & 0.7 \\ y_2 & 1 & 0.3 & 0.2 \\ y_3 & 0 & 0.7 & 0.6 \end{matrix}$$

Now,

$$\mu_{R\circ S}(x_1, z_1) = \max_{y\in X}\left\{\begin{matrix} \min\big(\mu_R(x_1, y_1), \mu_S(y_1, z_1)\big) \\ , \min\big(\mu_R(x_1, y_2), \mu_S(y_2, z_1)\big), \min\big(\mu_R(x_1, y_3), \mu_S(y_3, z_1)\big) \end{matrix}\right\}$$

$$= \max\{\min(1, 0.4), \min(0.5, 1), \min(0.7, 0)\} = \max\{0.4, 0.5, 0\} = 0.5$$

$$\mu_{R\circ S}(x_1, z_2) = \max_{y\in X}\left\{\begin{matrix} \min\big(\mu_R(x_1, y_1), \mu_S(y_1, z_2)\big), \\ \min\big(\mu_R(x_1, y_2), \mu_S(y_2, z_2)\big), \min\big(\mu_R(x_1, y_3), \mu_S(y_3, z_2)\big) \end{matrix}\right\}$$

$$= \max\{\min(1, 0.8), \min(0.5, 0.3), \min(0.7, 0.7)\} = \max\{0.8, 0.3, 0.7\} = 0.8$$

$$\mu_{R\circ S}(x_1, z_3) = \max_{y\in X}\left\{\begin{matrix} \min\big(\mu_R(x_1, y_1), \mu_S(y_1, z_3)\big), \\ \min\big(\mu_R(x_1, y_2), \mu_S(y_2, z_3)\big), \min\big(\mu_R(x_1, y_3), \mu_S(y_3, z_3)\big) \end{matrix}\right\}$$

$$= \max\{\min(1, 0.7), \min(0.5, 0.2), \min(0.7, 0.6)\} = \max\{0.7, 0.2, 0.6\} = 0.7$$

Continuing in the same manner, we obtain the following;

$$\mu_{R\circ S}(x_2, z_1) = 0.4, \mu_{R\circ S}(x_2, z_2) = 0.7, \mu_{R\circ S}(x_2, z_3) = 0.6$$

$$\mu_{R\circ S}(x_3, z_1) = 0.3, \mu_{R\circ S}(x_2, z_2) = 0.3, \mu_{R\circ S}(x_3, z_3) = 0.2$$

then

$$R \circ S = \begin{bmatrix} 1 & 0.5 & 0.7 \\ 0.3 & 0.4 & 1 \\ 0.2 & 0.3 & 0 \end{bmatrix} \circ \begin{bmatrix} 0.4 & 0.8 & 0.7 \\ 1 & 0.3 & 0.2 \\ 0 & 0.7 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.8 & 0.7 \\ 0.4 & 0.7 & 0.6 \\ 0.3 & 0.3 & 0.2 \end{bmatrix}$$

Hence

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.8 & 0.7 \\ 0.4 & 0.7 & 0.6 \\ 0.3 & 0.3 & 0.2 \end{bmatrix} \end{array}$$

3.1.3 Other composition of fuzzy relations

As max-min composition operator is defined on fuzzy relations, other composition operators can also be defined on fuzzy relations, this operators are max-product, max-average, and min-max. In general, it is convenient to represent max-min, max-product and max-average operators by max-($\star$), where the operator '$\star$' stands for any of the min, product, average and other operators that can be defined in this manner.

Definition 3.1.7 (Rosenfeld 1975):

Let R_1 and R_2 be fuzzy relations on $A \times B$ and $B \times C$ respectively, the max-($\star$) composition of R_1 and R_2 is then defined as

$$R_1 \star R_2(x,z) = \left\{(x,z), \max_{y \in B}\{\mu_{R_1}(x,y) \star \mu_{R_2}(y,z)\} \middle| x \in A, y \in B, z \in C\right\}$$

If '$\star$' is replaced by product or average, then the above formula becomes

$$R_1 \odot R_2(x,z) = \left\{(x,z), \max_{y \in B}\{\mu_{R_1}(x,y) \cdot \mu_{R_2}(y,z)\} \middle| x \in A, y \in B, z \in C\right\}$$

$$R_1 \oplus R_2(x,z) = \left\{(x,z), \max_{y \in B}\left\{\text{av}\left(\mu_{R_1}(x,y)\mu_{R_2}(y,z)\right)\right\} \middle| x \in A, y \in B, z \in C\right\},$$

respectively.

Given fuzzy relations R_1 and R_2 as above, the min-max composition of R_1 and R_2 is defined as;

$$R_1 \diamond R_2(x,z) = \left\{(\mathrm{x},\mathrm{z}), \min_{y \in B}\left\{\max\left(\mu_{R_1}(x,y)\mu_{R_2}(y,z)\right)\right\} \middle| x \in A, y \in B, z \in C\right\}.$$

Example 3.1.4

Using R and S in example 3.1.3, we have the following for max-product, max-average and min-max respectively.

$$R_1 \odot R_2 = \begin{matrix} & z_1 & z_2 & z_3 \\ x_1 & 0.5 & 0.8 & 0.7 \\ x_2 & 0.4 & 0.7 & 0.6 \\ x_3 & 0.3 & 0.16 & 0.14 \end{matrix}$$

$$R_1 \oplus R_2 = \begin{matrix} & z_1 & z_2 & z_3 \\ x_1 & 0.75 & 0.9 & 0.85 \\ x_2 & 0.7 & 0.85 & 0.8 \\ x_3 & 0.65 & 0.5 & 0.45 \end{matrix}$$

$$R_1 \diamond R_2 = \begin{matrix} & z_1 & z_2 & z_3 \\ x_1 & 0.7 & 0.5 & 0.5 \\ x_2 & 0.4 & 0.4 & 0.4 \\ x_3 & 0 & 0.3 & 0.3 \end{matrix}$$

Remark 3.1.1

The membership values in max-product operation is always less than and at most equal to the membership values in max-min operation while those in max-average operation on the other hand are greater than or equal to those of the max-min operation. That is; $\forall x, \forall y, \forall z,$ such that $(x, y) \in R_1$ and $(y, z) \in R_2$ then,

$$\max_{y \in B}\{prod[\mu_R(x,y), \mu_S(y,z)]\} \leq \max_{y \in B}\{min[\mu_R(x,y), \mu_S(y,z)]\}$$

$$\leq \max_{y \in B}\{av[\mu_R(x,y), \mu_S(y,z)]\}$$

Here, there is no relationship between max-min composition and min-max composition except that max-min composition of the complement of two fuzzy relations is equivalent to the complement of their min-max composition. i.e., $\bar{R}_1 \circ \bar{R}_2 = \overline{R \diamond R}$.

3.1.4 Properties of fuzzy relation

(Zimmermann, 1996); Given any fuzzy relation $R \subseteq A \times A$, the following properties are defined

1. **Reflexive**: R is said to be a reflexive relation if for every $x \in A, \mu_R(x,x) = 1$. If there exist any $x \in A$ such that $\mu_R(x,x) \neq 1$, then R is called irreflexive or non-reflexive, and if for all $x \in X, \mu_R(x,x) \neq 1,$ then R is called antireflexive.
2. **Symmetric:** A relation $R \subseteq A \times A$ is said to be symmetric if $\forall(x,y) \in A \times A$

   $\mu_R(x,y) = \mu = \mu_R(y,x)$. In other words, $\mu_R(x,y) = \mu_R(y,x) > 0$

   If $\forall(x,y) \in A \times A$, $x \neq y$, $\mu_R(x,y) \neq \mu_R(y,x)$ or $\mu_R(x,y) = \mu_R(y,x) = 0$, then R is called antisymmetric and if there exist $\forall(x,y) \in A \times A$ such that $\mu_R(x,y) \neq \mu_R(y,x)$, then we say that R is not symmetric or asymmetric and Perfectly antisymmetric if $\forall(x,y) \in A \times A$, and $x \neq y$, we have $\mu_R(x,y) > 0 \Longrightarrow \mu_R(y,x) = 0$.
3. **Transitive:** A relation R is called is a transitive relation if it satisfies the following;

   $\forall(x,y),(y,z),(x,z) \in A \times A, \mu_R(x,z) \geq \max_y\{\min\,(\mu_R(x,y), \mu_R(y,z))\}$

   If we use the symbol $'\vee'$ and $'\wedge'$ for max and min operators respectively, the above formula becomes

$$\mu_R(x,z) \geq \bigvee_y\{\mu_R(x,y) \wedge \mu_R(y,z)\}$$

Here, the above formula is identical to the definition of composition of fuzzy relations and therefore, we can conclude that a fuzzy relation R is transitive if $R \supseteq R \circ R = R^2$ or $R^2 \subseteq R$.

Remark 3.1.2

When a fuzzy relation R is represented by a matrix M_R, to investigate whether or not the relation is transitive, we check all the element (entries) in the matrix M_R of R and that of its composition M_{R^2} such that $M_R \geq M_{R^2}$,

i.e., if $\mu_R(x,x) \geq \mu_{R^2}(x,x)$ and $\mu_R(x,y) \geq \mu_{R^2}(x,y)$, $\forall (x,x) \in A \times A$, then R is transitive.

Example 3.1.5

$$R_1 = \begin{bmatrix} 1 & 0.7 & 0.8 & 0.1 \\ 0 & 1 & 0.2 & 0.9 \\ 0.9 & 0.5 & 1 & 0.4 \\ 0.4 & 1 & 0.3 & 1 \end{bmatrix}$$

The above relation R_1 is reflexive.

$$R_2 = \begin{bmatrix} 1 & 0.7 & 0.8 & 0.1 \\ 0.7 & 1 & 0.5 & 1 \\ 0.8 & 0.5 & 1 & 0.4 \\ 0.1 & 1 & 0.4 & 1 \end{bmatrix}$$

R_2 is both reflexive and symmetric

$$R_3 = \begin{bmatrix} 0.3 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0.8 & 0.4 & 0.7 & 0.8 \\ 0.5 & 0.4 & 0.2 & 1 \end{bmatrix}$$

R_3 is only transitive.

3.1.5 Fuzzy equivalence relation

A fuzzy relation R is called equivalence relation (similarity relation) if it satisfies the following three properties of fuzzy relation;

1. Reflexivity: $\forall x \in A, \mu_R(x,x) = 1$

2. Symmetricity: $\forall (x,y) \in A \times A$, $\mu_R(x,y) = \mu_R(y,x) = \mu$

3. Transitivity:

$$\forall (x,y),(y,z),(x,z) \in A \times A, \mu_R(x,z) \geq \max_y \{\min(\mu_R(x,y), \mu_R(y,z))\}$$

Example 3.1.6

Consider the fuzzy relation R represented by following matrix below;

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0.8 & 0.7 & 1 \\ 0.8 & 1 & 0.7 & 0.8 \\ 0.7 & 0.7 & 1 & 0.7 \\ 1 & 0.8 & 0.7 & 1 \end{bmatrix} \end{array}$$

This relation is a similarity relation since it satisfies the three properties of fuzzy equivalence relation. It partitioned the set A into subsets $A_1, A_2, \ldots, A_n$. This is represented by a partition tree below.

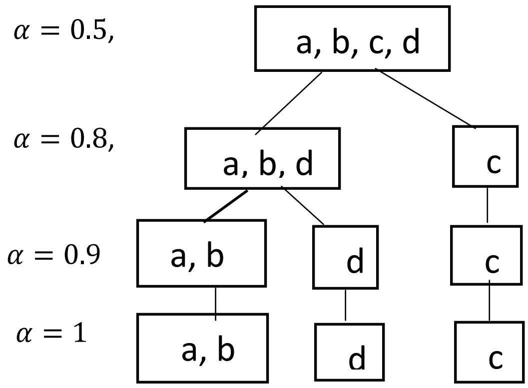

Figure 1. partition tree

3.2 Subreflexive Fuzzy Relation

A fuzzy relation R on a fuzzy set A is called a subreflexive fuzzy relation if for any

$$\alpha \in [0.5, 1], \forall x, y \in A, \mu_R(x,x) = \mu_R(y,y) = \alpha.$$

This definition is a relaxation of the usual reflexive property of a fuzzy relation since $\mu_R(x,x) = \mu_R(y,y) = \alpha$ which collapse to the definition of reflexivity when $\alpha = 1$.

Definition 3.2.1 (Minimal Reflexivity)

A relation $R \subseteq A \times A$, is said to be minimal reflexive if

$\mu_R(x,x) = \mu_{\bar{R}}(x,x), \forall x \in A$, where $\mu_{\bar{R}}(x,x) = 1 - \mu_R(x,x)$

That is, if $\mu_R(x,x)$and $\mu_{\bar{R}}(x,x)$ converge at a point $\mu_R(x,x) = \mu_{\bar{R}}(x,x) = 0.5$, then the relation is said to be minimal reflexive.

Remark 3.1.3

For any R, a fuzzy relation such that $\mu_R(x,x) < \alpha$, where α is a minimal reflexive point, is not reflexive even if $\mu_R(x,x) = \mu_R(y,y), \forall x, y \in A$.

Theorem 3.1

Let R be a fuzzy relation on a set A, i.e., $R \subset A \times A$. If R is minimal reflexive and symmetric, then $\bar{R}$ is also a minimal reflexive and symmetric relation on A.

Proof

Since R is minimal reflexive, then $\mu_R(x,x) = \mu_{\bar{R}}(x,x), \forall x \in A$ and

$\mu_R(x,x) = 0.5, \forall x \in A$, this shows that $\mu_{\bar{R}}(x,x) = \mu_R(x,x), \forall x \in A$, therefore, $\mu_{\bar{R}}(x,x) = 0.5, \forall x \in A$.

Hence $\bar{R}$ is minimal reflexive.

Also, to show that $\bar{R}$ is symmetric. Since R is symmetric, we have;

$$\mu_R(x,y) = \mu_R(y,x), \forall x,y \in A$$

But $\mu_{\bar{R}}(x,y) = 1 - \mu_R(x,y) = 1 - \mu_R(y,x) = \mu_{\bar{R}}(y,x), \forall x,y \in A$

Hence, $\mu_{\bar{R}}(x,y) = \mu_{\bar{R}}(y,x), \forall x,y \in A$.

Therefore, $\bar{R}$ is symmetric and this concludes the proof.

Remark 3.1.4

Subreflexive property of any fuzzy relation is the convex combination of both the minimal and maximal reflexive properties of the relation.

Theorem 3.2

Given any two reflexive fuzzy relations R and S, the max-min composition of these two relations is also reflexive.

Proof

The membership functions of the max-min composition of R and S is given as follows;

$$\mu_{R\circ S}(x_i, z_j) = \max\left\{min\left(\mu_R(x_i,\ y_1), \mu_S(y_1 z_j)\right), \dots, min\left(\mu_R(x_i,\ y_n), \mu_S(y_n z_j)\right)\right\}$$

Since both R and S are both reflexive, then we have the following;

$\mu_R(x_i, y_i) = 1, \forall i = 1,2, \dots, n$, and $\mu_S(y_i, z_i) = 1, \forall i = 1,2, \dots, n$

Therefore,

$$\mu_{R\circ S}(x_i, z_i) = \max\left\{\begin{matrix} min\left(\mu_R(x_i,\ y_1), \mu_S(y_1 z_j)\right), \dots, min(\mu_R(x_i,\ y_i), \mu_S(y_i z_i)), \\ \dots, min(\mu_R(x_i,\ y_n), \mu_S(y_n z_i)) \end{matrix}\right\}$$

$$i = 1,2, \dots, n$$

where each $min(\mu_R(x_i,\ y_k), \mu_S(y_k z_i)) \leq 1, k \neq i$,

and $min(\mu_R(x_i,\ y_i), \mu_S(y_i z_i)) = 1$.

Therefore,

we have $\mu_{R\circ S}(x_i, z_i) = 1$ whenever $\mu_R(x_i, y_i) = 1$ and $\mu_S(y_i, z_i) = 1, i = 1,2, \dots n$,

and hence $R \circ S$ is reflexive whenever R and S are both reflexive.

Theorem 3.3

Given any two subreflexive fuzzy relations R and S, the max-min composition of R and S is not necessarily a subreflexive relation.

Proof

The result follows in particular when there exist any $y_k \in A, 1 \leq k \leq n$, such that

$$\min(\mu_R(x_i,\ y_k), \mu_S(y_k z_i)) > \min(\mu_R(x_i,\ y_i), \mu_S(y_i z_i)).$$

Example 3.1.7

Consider the following two relations;

$$R = \begin{bmatrix} 0.7 & 0.5 & 0 & 0.4 \\ 0.2 & 0.7 & 0.3 & 0 \\ 0.8 & 0.3 & 0.7 & 0.6 \\ 0.5 & 1 & 0.2 & 0.7 \end{bmatrix}, \text{and } S = \begin{bmatrix} 0.7 & 0.5 & 0.9 & 0.5 \\ 1 & 0.7 & 0.8 & 0.1 \\ 0.7 & 0.6 & 0.7 & 0 \\ 0.4 & 0.5 & 0.2 & 0.7 \end{bmatrix}$$

where both R and S are subreflexive at $\alpha = 0.7$, but the max-min composition of R and S yields;

$$R \circ S = \begin{bmatrix} 0.7 & 0.5 & 0.7 & 0.5 \\ 0.7 & 0.7 & 0.7 & 0.2 \\ 0.7 & 0.6 & 0.8 & 0.6 \\ 0.4 & 0.5 & 0.8 & 0.9 \end{bmatrix}$$

which is not subreflexive.

Example 3.1.8

Consider the fuzzy equivalence relation

$$R = \begin{bmatrix} 1 & 0.2 & 1 & 0.6 \\ 0.2 & 1 & 0.2 & 0.2 \\ 1 & 0.2 & 1 & 0.6 \\ 0.6 & 0.2 & 0.6 & 1 \end{bmatrix}$$

This relation is reflexive, symmetric and transitive. If the reflexivity is relaxed to a level $\alpha = 0.8$ from the above equivalence relation, we get a subreflexive fuzzy relation below

$$R_2 = \begin{bmatrix} 0.8 & 0.2 & 1 & 0.6 \\ 0.2 & 0.8 & 0.2 & 0.2 \\ 1 & 0.2 & 0.8 & 0.6 \\ 0.6 & 0.2 & 0.6 & 0.8 \end{bmatrix}$$

This relation is subreflexive with level 0.8 and symmetric. To investigate the transitivity of this relation R_2, we know that for any fuzzy relation R to be transitive, $R \circ R \subseteq R$, so,

$$R_2 \circ R_2 = \begin{bmatrix} 0.8 & 0.2 & 1 & 0.6 \\ 0.2 & 0.8 & 0.2 & 0.2 \\ 1 & 0.2 & 0.8 & 0.6 \\ 0.6 & 0.2 & 0.6 & 0.8 \end{bmatrix} \circ \begin{bmatrix} 0.8 & 0.2 & 1 & 0.6 \\ 0.2 & 0.8 & 0.2 & 0.2 \\ 1 & 0.2 & 0.8 & 0.6 \\ 0.6 & 0.2 & 0.6 & 0.8 \end{bmatrix} = \begin{bmatrix} 1 & 0.2 & 0.8 & 0.6 \\ 0.2 & 0.8 & 0.2 & 0.2 \\ 0.8 & 0.2 & 1 & 0.6 \\ 0.6 & 0.2 & 0.6 & 0.8 \end{bmatrix}$$

Here

$$R_2 \circ R_2 = \begin{bmatrix} 1 & 0.2 & 0.8 & 0.6 \\ 0.2 & 0.8 & 0.2 & 0.2 \\ 0.8 & 0.2 & 1 & 0.6 \\ 0.6 & 0.2 & 0.6 & 0.8 \end{bmatrix} \nsubseteq R_2$$

With the result above, it was observed that if there exist any $x, y \in A$ such that

$$\min\big(\mu_R(x,y), \mu_R(y,x)\big) > \mu_R(x,x), \forall x \in A,$$

transitivity cannot hold in the subreflexive fuzzy relation.

Theorem 3.4

Let R_2be any subreflexive fuzzy relation $R_2 \subseteq A \times A$, if there exist any $x, y \in A$ Such that $\min\big(\mu_R(x,y), \mu_R(y,x)\big) > \mu_R(x,x), \forall x \in A$, then the relation is not transitive.

Proof

For transitivity to hold in any fuzzy relation R, $R \circ R \subseteq R$ must be satisfied. i.e. every $\mu_{R^2} \leq \mu_R$.

Let $R_2 = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix}$, where $\mu_{11} = \mu_{22} = \mu_{33}$,

Now, by the max-min composition, we have

$$r_1 \circ c_1 \leq \mu_{11}, \qquad r_1 \circ c_2 \leq \mu_{12}, \qquad r_1 \circ c_3 \leq \mu_{13}$$

$$r_2 \circ c_1 \leq \mu_{21}, \qquad r_2 \circ c_2 \leq \mu_{22}, \qquad r_{2\circ}c_3 \leq \mu_{23}$$

$$r_3 \circ c_1 \leq \mu_{31}, \qquad r_3 \circ c_2 \leq \mu_{32}, \qquad r_3 \circ c_3 \leq \mu_{33}$$

where $r_1 \circ c_1 = [\mu_{11} \quad \mu_{12} \quad \mu_{13}] \circ \begin{bmatrix} \mu_{11} \\ \mu_{21} \\ \mu_{31} \end{bmatrix}$,

$$\max\{\min(\mu_{11}, \mu_{11}), \min(\mu_{12}, \mu_{21}), \min(\mu_{13}, \mu_{31})$$

$$= \max\{\mu_{11}, \min(\mu_{12}, \mu_{21}), \min(\mu_{13}, \mu_{31})\}$$

Now, if $\max\{\mu_{11}, \min(\mu_{12}, \mu_{21}), \min(\mu_{13}, \mu_{31})\} = \min(\mu_{12}, \mu_{21})$

We have $\min(\mu_{12}, \mu_{21}) > \mu_{11}$, hence transitivity fails.

Also, if $\max\{\mu_{11}, \min(\mu_{12}, \mu_{21}), \min(\mu_{13}, \mu_{31})\} = \min(\mu_{13}, \mu_{31})$, transitivity fails also.

But transitivity will only hold if $\max\{\mu_{11}, \min(\mu_{12}, \mu_{21}), \min(\mu_{13}, \mu_{31})\} = \mu_{11} \leq \mu_{11}$.

Similarly, for $r_i \circ c_j \leq \mu_{ij}, 1 \leq i, j \leq 3$.

Therefore, transitivity fails and hence, the result follows.

Corollary 3.1

If R is a subreflexive and symmetric fuzzy relation and there exist $(x, y) \in R$ such that $\mu_R(x, y) > \mu_R(x, x)$, then R is not transitive.

Proof

The proof of this corollary is a consequence of the above result since if R is symmetric then

$$\min(\mu_R(x, y), \mu_R(y, x)) = \mu_R(x, y) = \mu_R(y, x)$$

Hence $\mu_R(x, y) = \min(\mu_R(x, y), \mu_R(y, x)) > \mu_R(x, x)$,

$$\Longrightarrow \mu_R(x, y) > \mu_R(x, x).$$

Theorem 3.5

Let R be a fuzzy relation, if R is antireflexive and symmetric; then R is not transitive under max-min composition.

Proof

Suppose on the contrary that there exist a fuzzy relation R which is both antireflexive, symmetric and transitive, then by definition $\forall(x,x) \in R,\ \mu_R(x,x) = 0$,

Now, assume $(x,y) \in R, (y,x) \in R$, since R is transitive, then

$$\mu_R(x,x) \geq \max_{y_i}\{\min\left(\ \mu_R(x,y_i),\ \mu_R(y_i,x)\right)\}, i = 1,2,\dots,n$$

,

for $i = 1$, we have $\mu_R(x,x) \geq \min\{\ \mu_R(x,y),\ \mu_R(y,x)\}$, and since R is symmetric, then

$$\min\{\ \mu_R(x,y),\ \mu_R(y,x)\} = \ \mu_R(x,y) = \ \mu_R(y,x) > 0$$

Hence $\mu_R(x,x) > 0$.

But this is a contradiction to the hypothesis that $\mu_R(x,x) = 0$.

Therefore, we have the proof.

3.2.1 Alpha-cuts in fuzzy relation

Definition 3.8 (α-cut relation)

Given any fuzzy relation R, the α-cut relation of R is a crisp relation defined by the membership function $\mu_{R_\alpha}(x,y) = \begin{cases}1, \text{if } \mu_R(x,y) \geq \alpha \\ 0, \text{if } \mu_R(x,y) < \alpha\end{cases} \qquad \alpha \in (0,1]$

Written as $R_\alpha = \{(x,y)|\mu_{R_\alpha}(x,y) \geq \alpha, \forall x \in A, \forall y \in A$

Example 3.1.9

Let $R = \begin{bmatrix} 1 & 0.7 & 0 & 0.8 & 0.3 \\ 0.2 & 0 & 0.3 & 0.5 & 0.1 \\ 0.5 & 0.4 & 0.9 & 0 & 0.7 \end{bmatrix}$, then for $\alpha \in (0,1] R_{0.5} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$ and

$$R_{0.7} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Remark 3.1.5

1. For any fuzzy relation R, if the matrix of R, M_R is a square matrix then R_0 is an equivalence crisp relation.
2. If R is a reflexive fuzzy relation then $\{R_\alpha | \alpha \in (0,1]\}$ is also reflexive.
3. If R is symmetric, so also is $\{R_\alpha | \alpha \in (0,1]\}$.
4. If R is transitive, then $\{R_\alpha | \alpha \in (0,1]\}$ is also transitive.

In general, if R is a similarity relation then $\{R_\alpha | \alpha \in (0,1]\}$ is an equivalence relation.

Definition: 3.9 (Fuzzy Level Subrelation)

Let R be a fuzzy relation on a fuzzy set A, then a fuzzy level subrelationS_α of R is a fuzzy relation defined by the membership function

$$\mu_{S_\alpha} = \begin{cases} \mu_R(x,x), if\ \mu_R(x,x) \geq \alpha \\ 0, \quad otherwise \end{cases} \quad \alpha \in (0,1]$$

Example 3.1.10

Let

$$R = \begin{bmatrix} 1 & 0.7 & 0.5 & 0.3 \\ 0.2 & 0.1 & 0.4 & 1 \\ 0.8 & 0.6 & 0.2 & 0 \\ 0.5 & 0.9 & 0.1 & 0.4 \end{bmatrix}$$

Then

$$S_{0.5} = \begin{bmatrix} 1 & 0.7 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \\ 0.8 & 0.6 & 0 & 0 \\ 0.5 & 0.9 & 0 & 0 \end{bmatrix}$$

$$S_{0.7} = \begin{bmatrix} 1 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \end{bmatrix}$$

Theorem 3.6

If R is a reflexive, symmetric and transitive relation, then every level subrelations of R are also reflexive, symmetric and transitive.

Proof

Since R is reflexive then

$$\mu_R(x, x) = \mu_R(y, y) = 1, \forall x, y \in X$$

Now, for any level subset S_α such that $\alpha \in (0, 1]$, we have

$$\mu_S(x, x) = \mu_S(y, y) \geq \alpha, \forall x, y \in X, \alpha \in (0, 1]$$

Therefore, S_α is reflexive.

For symmetric, assume R is symmetric then

$$\mu_R(x, y) = \mu_R(y, x), \forall x, y \in X$$

Let S_α be any level subset of R, if $(x, y) \in S_\alpha$, then $\mu_S(x, y) \geq \alpha$,

But

$$\mu_R(x, y) = \mu_R(y, x), \forall (x, y), (y, x) \in R$$

and $S_\alpha \subseteq R$, hence

$$\mu_S(x, y) = \mu_S(y, x) \geq \alpha, \forall (x, y), (y, x) \in S_\alpha$$

Therefore every level subset of R is symmetric.

Finally, suppose R is transitive and let S_α be any level subset of R, and let

$(x, y), (y, x) \in S_\alpha$, then

$$\mu_S(x, y) \geq \alpha, \mu_S(y, z) \geq \alpha$$

We want to show that $\mu_S(x, z) \geq \alpha$

But by transitivity of R, we have;

$$\mu_R(x, z) \geq \max_{y_i}\{min(\mu_R(x, y_i), \mu_R(y_i, z))\}$$

If $i = 1$, then we have $\mu_R(x, z) \geq \min\{\mu_R(x, y), \mu_R(y, z)\}$

Since $S_\alpha \subseteq R$, then $\mu_S(x, z) \geq \min\{\mu_S(x, y), \mu_S(y, z)\} \geq \alpha$

Hence, S is transitive. This concludes the proof.

Remark: 3.1.6

The level fuzzy subrelations of a fuzzy relation preserve the three properties of the fuzzy relation.

Definition: 3.9 (Inverse Alpha Cut)

Let R be a fuzzy relation, the inverse $\boldsymbol{\alpha}$**-cut** of R denoted as $R_{\alpha^{-1}}$is a crisp relation defined by the membership function

$$\mu_{R_{\alpha^{-1}}}(x, y) = \begin{cases} 1, \text{if } \mu_R(x, y) < \alpha \\ 0, \text{if } \mu_R(x, y) \geq \alpha \end{cases} \quad \alpha \in (0, 1]$$

Remark 3.1.7

The only property of fuzzy relation that is been retained by inverse $\boldsymbol{\alpha}$**-cut** of any fuzzy relation R is the symmetric property.

Theorem 3.7

Let R be a similarity fuzzy relation on X, then $R_{\alpha^{-1}}$ is not an equivalence relation.

Proof

Since R is reflexive, then $\mu_R(x,x) = 1, \forall x \in X$, now for $\alpha \in (0,1]$, $\mu_{R_{\alpha^{-1}}}(x,x) \geq \alpha$, therefore, $\mu_{R_{\alpha^{-1}}}(x,y) = 0$, hence, $R_{\alpha^{-1}}$ is not reflexive.

Also, since R is symmetric, we have $\mu_R(x,y) = \mu_R(y,x)$, then for some $\alpha \in (0,1]$, $\mu_R(x,y) = \mu_R(y,x) \geq \alpha$. this implies $\mu_{R_{\alpha^{-1}}}(x,y) = \mu_{R_{\alpha^{-1}}}(y,x) = 1$.

Hence $R_{\alpha^{-1}}$ is symmetric.

Now, for transitivity, $\mu_R(y,x) < \alpha$ and $\mu_R(y,z) < \alpha$, then,

$$\max_{y \in X}\big(\mu_R(x,y), \mu_R(y,z)\big) < \alpha$$

but since R is transitive, we have

$$\mu_R(x,z) \geq \max_{y \in X}\big(\mu_R(x,y), \mu_R(y,z)\big)$$

therefore, there exist $\alpha \in (0,1]$ such that $\mu_R(x,z) > \alpha$ where

$$\max_{y \in X}\big(\mu_R(x,y), \mu_R(y,z)\big) < \alpha$$

this shows that the relation $R_{\alpha^{-1}}$ is not transitive.

Hence the relation is not an equivalence relation.

Definition 3.10 (Inverse Level Subrelation)

Let R be a fuzzy relation on a set X, an inverse level subrelation $S_{\alpha^{-1}}$ of R is a fuzzy relation defined by the membership function

$$\mu_{S_{\alpha^{-1}}} = \begin{cases} \mu_R(x,y), if\ \mu_R(x,y) < \alpha \\ 0, \qquad\qquad otherwise \end{cases} \qquad \alpha \in (0,1]$$

Remark 3.1.8

Inverse level subrelation of a similarity fuzzy relation R is not a similarity fuzzy relation.

Example 3.1.11

Let R be a similarity relation defined as follows;

$$R = \begin{bmatrix} 1 & 0.7 & 0.5 & 0.8 \\ 0.7 & 1 & 0.5 & 0.7 \\ 0.5 & 0.5 & 1 & 0.5 \\ 0.8 & 0.7 & 0.5 & 1 \end{bmatrix}$$

Then,

$$S_{0.8} = \begin{bmatrix} 0 & 0.7 & 0.5 & 0 \\ 0.7 & 0 & 0.5 & 0.7 \\ 0.5 & 0.5 & 0 & 0.5 \\ 0 & 0.7 & 0.5 & 0 \end{bmatrix}$$

This fuzzy level subrelation is antireflexive, not transitive but symmetric.

3.3 Eigen Fuzzy Sets Associated with Fuzzy Relation

Definition 3.11 (Sanchez, 1981)

Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be two fuzzy relations, from the max-min composition of R and S we obtain a fuzzy relation $T = R \circ S \subseteq X \times Z$.

In the same way, if $A \subseteq F(X)$ and $R \subseteq X \times Y$, then using the same max-min composition on A and R we have a fuzzy set $B \subseteq F(Y)$.

That is; for every $y \in Y$

$$\mu_{(A\circ R)}(y) = \max_{x\in X}\big(\min\big(\mu_A(x), \mu_R(x,y)\big)\big), \forall x \in X$$

If '∨' and '∧' represents 'max' and 'min' operators respectively, then the above becomes

$$\forall y \in Y, \forall (x,y) \in R, \mu_{(A\circ R)}(y) = \bigvee_{x\in X}\big(\mu_A(x) \wedge \mu_R(x,y)\big), \forall x \in X$$

Example 3.1.12

Let $X = \{x, y, z\}, R = \begin{bmatrix} 0.4 & 0.6 & 0.8 \\ 0.7 & 0.1 & 1 \\ 0.5 & 0.9 & 0.6 \end{bmatrix}$, and $A = [0.7, 0.2, 0.5]$, then we have

$B = A \circ R$ by max-min composition, i.e.;

$$B = [0.7, 0.2, 0.5] \circ \begin{bmatrix} 0.4 & 0.6 & 0.8 \\ 0.7 & 0.1 & 1 \\ 0.5 & 0.9 & 0.6 \end{bmatrix} = [0.5, 0.6, 0.7]$$

3.4 Eigen Fuzzy Set (Max-min Composition)

If R is a fuzzy relation on X, that is $R \subseteq X \times X$ and a fuzzy set $A \subseteq F(X)$, by max-min composition of R and A gives a fuzzy set $B \subseteq F(X)$, if $B = A$, then A becomes an eigen fuzzy set associated with R. That is; if $\mu_{(A\circ R)}(x) = \mu_A(x), \forall x \in X$, then A is an eigen fuzzy set associated with R. (Sanchez, 1978)

Example 3.1.13

Let

$$R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}$$

then $A = [0.7, 0.7, 0.7, 0.7]$, and $B = [0.9 \quad 0.9 \quad 0.7 \quad 0.9]$ are Eigen Fuzzy Sets associated with R because, $A \circ R = A$ and $B \circ R = B$.

3.4.1 Greatest eigen fuzzy set (GEFS)

Let R be a fuzzy relation on X, and $\mathcal{F}(A)$ be the set of all fuzzy subset A of $F(X)$ such that $A \circ R = A$ using Max-Min composition, i.e, $\mathcal{F}(A) = \{A \subseteq X | A \circ R = A\}$, then there exist $A \in \mathcal{F}(A)$ such that for every $A_i \in \mathcal{F}(A), i = 1,2, \dots, |X|$, $A_i \subseteq A$, this is called the Greatest Eigen Fuzzy Set (GEFS) associated with R.

Let $A_1 \in F(X)$ defined by $\forall x' \in X,\ \mu_{A_1}(x') = \vee_{x \in X} \mu_R(x, x')$, such that $(x, x') \in R$.

If $A_0 \in F(X)$, defined by $\mu_{A_0}(x) = a_0$, where $a_0 = \wedge_{x' \in X} \vee_{x \in X} \mu_R(x, x'), \forall x \in X$, that is; a_0 is the minimum of $\mu_{A_1}(x')$, then we can easily see that $A_0$is an eigen fuzzy set, that is, $A_0 \in \mathcal{F}(A)$. Because;

$$\forall x' \in X,\ \mu_{A_0 \circ R}(x') = \bigvee_{x \in X} \left(\mu_{A_0}(x) \wedge \mu_R(x, x')\right)$$

$$= \bigvee_{x \in X} \left(a_0 \wedge \mu_R(x, x')\right) = a_0 \wedge \left(\bigvee_{x \in X} \mu_R(x, x')\right)$$

$$= a_0 \wedge \mu_{A_1}(x') = \mu_{A_0}(x'), \forall x' \in X.$$

Therefore, $A_0 \in \mathcal{F}(A)$. But A_0 is not the GEFS associated with the fuzzy relation R.

Remark 3.1.9

Note that since $\forall x' \in X,\ \mu_{A_0}(x') = a_0 = \bigwedge_{x' \in X} \mu_{A_1}(x')$, we have $A_0 \subseteq A_1$.

Theorem 3.8 (Sanchez, 1981)

Let A_n be a sequence of fuzzy sets defined as follows;

$$A_1 \circ R = A_2,$$

$$A_2 \circ R = A_1 \circ R^2 = A_3,$$

$$. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .,$$

$$A_n \circ R = A_1 \circ R^n = A_{n+1}, \forall n > 0$$

The sequence A_n is decreasing and bounded by A_0 and A_1, that is;

$$A_0 \subseteq \cdots \subseteq A_{n+1} \subseteq A_n \subseteq \cdots \subseteq A_3 \subseteq A_2 \subseteq A_1, \forall n > 0.$$

Remark 3.1.10 (Sanchez, 1981)

If there exist $k > 0$ and $x' \in X$, such that $\mu_{A_k}(x') = a_0$, then for every integer $n \geq k$, we have $\mu_{A_n}(x') = a_0$. Because since

$$A_0 \subseteq \cdots \subseteq A_{n+1} \subseteq A_n \subseteq \cdots \subseteq A_k \subseteq \cdots \subseteq A_2 \subseteq A_1$$

And also

$$\mu_{A_0}(x') \leq \cdots \leq \mu_{A_{n+1}}(x') \leq \mu_{A_n}(x') \leq \cdots \leq \mu_{A_k}(x')$$

That is; $a_0 \leq \cdots \leq \mu_{A_n}(x') \leq \cdots \leq a_0$, hence $\mu_{A_n}(x') = a_0, \forall n \geq k.$

Remark 3.1.11 (Sanchez, 1981)

If there exist $n > 0$ such that $A_n \in \mathcal{F}(A)$, then A_n is the greatest element in $\mathcal{F}(A)$, that is, A_n is the Greatest Eigen Fuzzy Set associated with R.

Let $n > 0$ be an integer such that $A_n \in \mathcal{F}(A)$ and let A be any element in $\mathcal{F}(A)$, i.e., $A \circ R = A$. We know that for any $k > 0, A \circ R^k = A$.

Now, for all $x' \in X,\ \mu_A(x') = \vee_{x \in X}\big(\mu_A(x) \wedge \mu_R(x, x')\big) \leq \bigvee_{x \in X} \mu_R(x, x') = \mu_{A_1}(x')$.

Hence, $A \subseteq A_1$, this implies $A \circ R^{n-1} \subseteq A_1 \circ R^{n-1}$, then $A \subseteq A_n$, because $A_1 \circ R^{n-1} = A_n$.

Moreover, if $A_{n+1} = A_n \circ R = A_n$ implies $A_{n+k} = A_n \circ R^k = A_n$, for all integers $k > 0$.

Theorem 3.9 (Sanchez, 1981)

For all integers $n > 0$, and $\forall x' \in X$,

$$\bigvee_{x \in X} \mu_{R^n}(x, x') = \mu_{A_1 \circ R^{n-1}}(x') = \mu_{A_n}(x').$$

CHAPTER FOUR

EIGEN FUZZY SETS OF FUZZY RELATION

4.0 Introduction

In this chapter, we study and present the algorithms for determination of Greatest Eigen Fuzzy Set (GEFS) based on max-min composition of fuzzy relations, introduced a modified algorithm, and extend the algorithms to Least Eigen Fuzzy Set (LEFS) based on the min-max composition of fuzzy relations. We also investigate the existence of Eigen Fuzzy Sets based on max-product and max-average composition.

4.1 Determination of Greatest Eigen Fuzzy set (GEFS)

There are basically three fundamental algorithm for determination of the Greatest Eigen Fuzzy Set associated with a given fuzzy relation R. (Sanchez, 1981)

4.1.1 First algorithm for determination of GEFS

From the given relation $R \subseteq X \times X$ with the membership function $\mu_R(x, x')$

1. Find the set $A_1 \subseteq F(X)$ defined by

$$\mu_{A_1}(x') = \max_{x \in X} \mu_R(x, x'), \forall x' \in X$$

2. Calculate A_2 given by the composition

$$A_2 = A_1 \circ R$$

3. Continue this process until we get a convergence of the sequence A_n given below

$$A_n \circ R = A_1 \circ R^n = A_{n+1} = A_n$$

4. Accept A_n as the Greatest Eigen Fuzzy Set (GEFS).

4.1.2 Second algorithm for determination of GEFS

In this method, there is no need to evaluate any composition of R and any set. What is required to find the invariant element from successive reduction of R. At each step of this method, the invariant elements are exactly the ones of the previous method.

Steps

1. Find the greatest element in each column of R as in the first algorithm of the previous method to give A_1.
2. The smallest (minimum) of these elements will be chosen, the column containing that element and the corresponding row number will be deleted to give R_2. Where
$R_2 \subset R$.
3. Form A^* from A with the same number of cells whose elements are blank.
4. Fill the column number of A^* with the chosen element.
5. Repeat steps 1-4 until no more cell left to be filled in A^*. Our filled A^* is the required eigen fuzzy set

Remark 4.1.1

At each step of this method, the maximum between the present value and the previous value is chosen to fill the cell corresponding to the deleted column of R.

4.1.3 Third algorithm for determination of GEFS

This involves the continuous composition of R itself and taking

$$\max_{x \in X} \mu_R(x, x'), \forall x' \in X$$

column wise until the Greatest Eigen Fuzzy Set is obtained.

Steps

1. Produce A_1 from R as in the previous methods.
2. Compose R with itself to get $R^2 = R \circ R$ and produce A_2.
3. Compare $A_1$with A_2, if $A_1 = A_2$, then accept A_1 as the GEFS, else,
4. Compose R^2 with R to produce $R^3 = R^2 \circ R$ and produce A_3.
5. Continue this process until $A_{n+1} = A_n$, then accept A_n as the GEFS.

Example 4.1.1

Suppose we have a fuzzy relation given below

$$R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}$$

First Algorithm

We have $A_1 = [1 \quad 0.9 \quad 0.7 \quad 1]$

And $A_0 = [0.7 \quad 0.7 \quad 0.7 \quad 0.7]$ which is a constant Eigen Fuzzy Set.

Now $A_1 \circ R = [1 \quad 0.9 \quad 0.7 \quad 1] \circ \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix} = A_2$

Which by max-min composition, we get the following;

$$\max\{\min(1,0.7), \min(0.9,1), \min(0.7,0.3), \min(1,0.8)\} = \max\{0.7, 0.9, 0.3, 0.8\}$$

$$= 0.9$$

$$\max\{\min(1,0.7), \min(0.9,0.2), \min(0.7,0.5), \min(1,0.9)\} = \max\{0.7, 0.2, 0.5, 0.9\}$$

$$= 0.9$$

$$\max\{\min(1{,}0.6)\,,\min(0.9{,}0.7)\,,\min(0.7{,}0.6)\,,\min(1{,}0)\} = \max\{0.6, 0.7, 0.6, 0\}$$

$$= 0.7$$

$$\max\{\min(1{,}1)\,,\min(0.9{,}0.1)\,,\min(0.7{,}0.4)\,,\min(1{,}0.8)\} = \max\{1, 0.1, 0.4, 0.8\} = 1$$

Therefore, $A_2 = [0.9 \quad 0.9 \quad 0.7 \quad 1]$.

Since $A_2 \neq A_1$, we proceed to compute $A_3 = A_2 \circ R$

$$A_2 \circ R = [0.9 \quad 0.9 \quad 0.7 \quad 1] \circ \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix} = A_3$$

By applying the max-min composition on A_2 and R as above, we obtain

$$A_3 = [0.9 \quad 0.9 \quad 0.7 \quad 0.9], \qquad A_3 \neq A_2$$

We repeat the process again to obtain A_4, where $A_4 = [0.9 \quad 0.9 \quad 0.7 \quad 0.9]$

Since $A_4 = A_3$, we accept A_3 as the (GEFS)

Second Algorithm

$$R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}$$

$$A_1 = [1 \quad 0.9 \quad 0.7 \quad 1], \min(1, 0.9, 0.7, 1) = 0.7$$

Deleting column three and row three from R we obtain $R_2 = \begin{bmatrix} 0.7 & 0.7 & 1 \\ 1 & 0.2 & 0.1 \\ 0.8 & 0.9 & 0.8 \end{bmatrix}$, and

$$A^* = [\qquad 0.7 \qquad], \qquad A_2 = [1 \quad 0.9 \quad 1], min(1, 0.9, 1) = 0.9$$

$$R_3 = \begin{bmatrix} 0.7 & 1 \\ 0.8 & 0.8 \end{bmatrix}, \text{and } A^* = [\qquad 0.7 \quad 0.9]$$

Now from R_3 we obtain $A_3 = [0.8 \quad 1]$, and $\min(1, 0.8) = 0.8$.

Since the new value is less than the previous value, we take the previous value, hence $A^* = [\quad 0.9 \quad 0.7 \quad 0.9]$and $R_4 = [0.7]$, $A_4 = [0.7]$, and A_4 has single element and the value is also less than the previous value, therefore we also use the previous value to fill the required cell. Hence $A^* = [0.9 \quad 0.9 \quad 0.7 \quad 0.9]$ is the required GEFS.

Third Algorithm

$$A_1 = [1 \quad 0.9 \quad 0.7 \quad 1]$$

$$R \circ R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix} =$$

$$R^2 = \begin{bmatrix} 0.8 & 0.9 & 0.7 & 0.8 \\ 0.7 & 0.7 & 0.6 & 1 \\ 0.5 & 0.5 & 0.6 & 0.4 \\ 0.9 & 0.8 & 0.7 & 0.8 \end{bmatrix}$$

$$A_2 = [0.9 \quad 0.9 \quad 0.7 \quad 1]$$

$$R^2 \circ R = \begin{bmatrix} 0.8 & 0.9 & 0.7 & 0.8 \\ 0.7 & 0.7 & 0.6 & 1 \\ 0.5 & 0.5 & 0.6 & 0.4 \\ 0.9 & 0.8 & 0.7 & 0.8 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix} =$$

$$R^3 = \begin{bmatrix} 0.9 & 0.8 & 0.7 & 0.8 \\ 0.8 & 0.9 & 0.7 & 0.8 \\ 0.5 & 0.5 & 0.6 & 0.5 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix}$$

$$A_3 = [0.9 \quad 0.9 \quad 0.7 \quad 0.9]$$

$$R^3 \circ R = \begin{bmatrix} 0.9 & 0.8 & 0.7 & 0.8 \\ 0.8 & 0.9 & 0.7 & 0.8 \\ 0.5 & 0.5 & 0.6 & 0.5 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix} =$$

$$R^4 = \begin{bmatrix} 0.8 & 0.8 & 0.7 & 0.9 \\ 0.9 & 0.8 & 0.7 & 0.8 \\ 0.5 & 0.5 & 0.6 & 0.5 \\ 0.8 & 0.9 & 0.7 & 0.8 \end{bmatrix}$$

$$A_4 = [0.9 \quad 0.9 \quad 0.7 \quad 0.9]$$

Since $A_4 = A_3$, then A_3 becomes the GEFS.

Remark 4.1.2

In all cases, the GEFS is the same regardless of the method used.

4.2 Modified Algorithm for the Determination of GEFS

This algorithm is a modification of the first algorithm for determining the GEFS associated with a given fuzzy relation.

1. Define A_1 by $\mu_{A_1}(x') = \bigvee_{x \in X} \mu_R(x, x'), \forall x' \in X$ as the greatest element in each column of R, then the greatest eigen fuzzy set associated with R, A_n is given by the following formula

$$\mu_{A_n}(x') = \left[\bigvee_{x \in X} \mu_R(x, x')\right] \wedge \mu_{A_{n-1}}(x_R), \forall x' \in X$$

$\mu_{A_{n-1}}(x_R)$ is the element of A_{n-1} which corresponds with the row number where $\left[\bigvee_{x \in X} \mu_R(x, x')\right]$ occur.

If $\left[\bigvee_{x \in X} \mu_R(x, x')\right] \wedge \mu_{A_{n-1}}(x_R) \leq \left[\bigvee_{x \in X} \mu_R(x, x')_n\right] \wedge \mu_{A_{n-1}}(x_R)$, then

$$\mu_{A_n}(x') = \left[\bigvee_{x \in X} \mu_R(x, x')_n\right] \wedge \mu_{A_{n-1}}(x_R)$$

$\bigvee_{x \in X} \mu_R(x, x')_n$ is the next maximum in the column of (x, x').

2. Compare A_n and A_{n+1}, if $A_n = A_{n+1}$ accept A_n as the Eigen Fuzzy Set.

3. Repeat the above procedure until the Eigen Fuzzy Set is obtained.

Example 4.2.1

By using the same example as in Example 3.3, we have;

$$R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}$$

$$A_1 = [1 \quad 0.9 \quad 0.7 \quad 1]$$

Now, $\mu_{A_2}(x_1) = \left[\bigvee_{x\in X} \mu_R(x, x_1)\right] \wedge \mu_{A_1}(x_1) = \{1 \wedge 0.9\} = 0.9$

$$\mu_{A_2}(x_2) = \left[\bigvee_{x\in X} \mu_R(x, x_2)\right] \wedge \mu_{A_1}(x_4) = \{0.9 \wedge 1\} = 0.9$$

$$\mu_{A_2}(x_3) = \left[\bigvee_{x\in X} \mu_R(x, x_3)\right] \wedge \mu_{A_1}(x_2) = \{0.7 \wedge 0.9\} = 0.7$$

$$\mu_{A_2}(x_4) = \left[\bigvee_{x\in X} \mu_R(x, x_4)\right] \wedge \mu_{A_1}(x_R) = \{1 \wedge 1\} = 1$$

$$A_2 = [0.9, 0.9, 0.7, 1]$$

Since $A_2 \neq A_1$, we repeat the procedure,

Now, $\mu_{A_3}(x_1) = \left[\bigvee_{x\in X} \mu_R(x, x_1)\right] \wedge \mu_{A_2}(x_1) = \{1 \wedge 0.9\} = 0.9$

$$\mu_{A_3}(x_2) = \left[\bigvee_{x\in X} \mu_R(x, x_2)\right] \wedge \mu_{A_2}(x_4) = \{0.9 \wedge 1\} = 0.9$$

$$\mu_{A_3}(x_3) = \left[\bigvee_{x\in X} \mu_R(x, x_3)\right] \wedge \mu_{A_2}(x_2) = \{0.7 \wedge 0.9\} = 0.7$$

$$\mu_{A_3}(x_4) = \left[\bigvee_{x\in X} \mu_R(x, x_4)\right] \wedge \mu_{A_2}(x_R) = \{1 \wedge 0.9\} = 0.9$$

Therefore, $A_3 = [0.9, 0.9, 0.7, 0.9]$, also $A_2 \neq A_3$ hence we proceed to obtain A_4.

Here, $A_4 = [0.9, 0.9, 0.7, 0.9]$, since $A_4 = A_3$, we accept A_3 as the GEFS.

Theorem 4.1

Let R be a fuzzy relation over X defined by the membership function $\mu_R(x_i, x'_j)$ such that $x_i, x'_j \in X$, $i, j = 1,2, \dots, n$, if there exist any $i, j \leq n$, such that

$$\mu_R(x_i, x'_j)_{i=j} = \bigvee_{x_i \in X} \mu_R(x_i, x'_j)$$

then, for any A_n,

$$\mu_{A_n}(x'_j) = \mu_R(x_i, x'_j)_{i=j}, n > 0$$

Proof

Since A_1 is defined by

$$\mu_{A_1}(x'_j) = \bigvee_{x_i \in X} \mu_R(x_i, x'_j) = \mu_R(x_i, x'_j)_{i=j}$$

Therefore, $\mu_{A_1}(x'_j) = \mu_R(x_i, x'_j)_{i=j}$, we want to show that for any $n > 0$,

$$\mu_{A_n}(x'_j) = \mu_R(x_i, x'_j)_{i=j}$$

Now setting the index $n = 1$, we have $\mu_{A_1}(x_j) = \mu_R(x_i, x'_j)_{i=j}$

Now, by induction, assume for n, $\mu_{A_n}(x'_j) = \mu_R(x_i, x'_j)_{i=j}$

then, putting $n = n + 1$, we have $A_{n+1} = A_n \circ R = A_1 \circ R^n$, then

$$\mu_{A_{n+1}}(x'_j) = \bigvee_{x_i \in X} \{\mu_{A_n}(x_i) \wedge \mu_R(x_i, x'_j)\}, i = 1,2, \dots, j, \dots n$$

$$= \bigvee \left\{ \begin{matrix} \mu_{A_n}(x_1) \wedge \mu_R(x_1, x'_j), \mu_{A_n}(x_2) \wedge \mu_R(x_2, x'_j), \dots, \mu_{A_n}(x_j) \wedge \mu_R(x_j, x'_j), \dots, \\ \mu_{A_n}(x_n) \wedge \mu_R(x_n, x'_j) \end{matrix} \right\}$$

$$= \mu_{A_n}(x_i) \wedge \mu_R(x_i, x'_j)_{i=j}$$

Since, $\mu_{A_n}(x_j) = \mu_R(x_i, x'_j)_{i=j}$, hence $\mu_{A_{n+1}}(x'_j) = \mu_R(x_i, x'_j)_{i=j}$

Therefore, for any $n > 0$, and for any A_n, $\mu_{A_n}(x'_j) = \mu_R(x_i, x'_j)_{i=j}$,

4.3 Eigen Fuzzy Set (Min-Max Composition)

Let R be a fuzzy relation on X, that is $R \subseteq X \times X$ and a fuzzy set $B \subseteq F(X)$, by the min-max composition of R and B gives a fuzzy set $A \subseteq F(X)$, if $B = A$, then B becomes an eigen fuzzy set associated with R. That is; if $\mu_{(B \circ R)}(x) = \mu_B(x), \forall x \in X$, then B is an eigen fuzzy set associated with R with respect to min-max composition. This is defined by (Nobuhara et al., 2004) as a dual type of (Sanchez, 1978).

4.3.1 Least eigen fuzzy set (LEFS)

Let R be a fuzzy relation on X, and $\mathcal{F}(B)$ be the set of all fuzzy subset B of $F(X)$ such that $B \diamond R = B$ using Min-Max composition, i.e, $\mathcal{F}(B) = \{B \subseteq F(X) | B \diamond R = B\}$, then there exist $B \in \mathcal{F}(B)$ such that for every $B_i \in \mathcal{F}(B), i = 1,2, \dots, |X|$, $B \subseteq B_i$, called the Least Eigen Fuzzy Set (LEFS) associated with R.

Let $B_1 \in F(X)$ defined by $\forall x' \in X,\ \mu_{B_1}(x') = \wedge_{x \in X} \mu_R(x, x')$, such that $(x, x') \in R$.

If $B_0 \in F(X)$, is defined by $\mu_{B_0}(x) = b_0$, where $b_0 = \vee_{x' \in X} \wedge_{x \in X} \mu_R(x, x')$, $\forall x \in X$.

We have that $B_0 \in \mathcal{F}(A)$. Since;

$$\mu_{B_0 \diamond R}(x') = \bigwedge_{x \in X} \left(\mu_{B_0}(x) \vee \mu_R(x, x')\right) = b_0, \forall x' \in X$$

Definition 4.3.1

Let B_n be a sequence of fuzzy sets defined as follows;

$$B_1 \diamond R = B_2,$$

$$B_2 \diamond R = B_1 \diamond R^2 = B_3,$$

$$. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .,$$

$$B_n \diamond R = B_1 \diamond R^n = B_{n+1}, \forall n > 0$$

Theorem 4.2

The sequence B_n is an increasing sequence and bounded by B_1 and B_0, that is;

$B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n = B_{n+1} \subseteq B_0$, for any $n > 0$

Proof

Given any fuzzy relation R and fuzzy set B_1, where $B_1 \diamond R = B_2$ is defined as;

$$\mu_{B_1 \diamond R}(x') = \bigwedge_{x \in X} \left(\mu_{A_1}(x') \vee \mu_R(x, x')\right) \geq \bigwedge_{x \in X} \mu_R(x, x') = \mu_{B_1}(x'), \forall x' \in X$$

Hence $B_1 \subseteq B_2$. Therefore, $B_1 \diamond R \subseteq B_2 \diamond R = B_1 \diamond R^2$, and $B_1 \diamond R^2 = B_3$, then $B_1 \subseteq B_3$, and by induction, we have that

$$B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n = B_{n+1}$$

We now show that for any $n > 0$, $B_n \subseteq B_0$.

By definition,

$$\mu_{B_0}(x) = \bigvee_{x' \in X} \mu_{B_1}(x'), \forall x \in X$$

that is; $B_1 \subseteq B_0$, and also, $B_0 \diamond R = B_0$, $B_0 \diamond R^2 = B_0$, …, $B_0 \diamond R^n = B_0$, for nay $n > 0$, we have $B_2 = B_1 \diamond R \subseteq B_0 \diamond R = B_0$.

Therefore, $B_2 \subseteq B_0$. Also since $B_n = B_1 \diamond R^{n-1}$, and $B_1 \subseteq B_0$.

Then $B_n = B_1 \diamond R^{n-1} \subseteq B_0 \diamond R^{n-1}$, this shows that $B_n \subseteq B_0$.

This concludes the proof.

Theorem 4.3

Let R be a fuzzy relation and B_1 be defined by the membership function

$\mu_{B_1(x')} = \bigwedge_{x \in X} \mu_R(x, x')$, then for any integer $n > 0$, and $\forall x' \in X$, we have

$$\bigwedge_{x \in X} \mu_{R^n}(x, x') = \mu_{B_1 \diamond R^{n-1}}(x') = \mu_{B_n}(x').$$

Proof

For all integers $n > 0$, and $\forall x' \in X$, we have

$$\mu_{B_1 \diamond R^{n-1}}(x') = \bigwedge_{y \in X} \left(\mu_{B_1}(y) \vee \mu_{R^{n-1}}(y, x') \right)$$

Since, $\mu_{B_1}(y) = \bigwedge_{x \in X} \mu_R(x, y)$, we have,

$$\mu_{B_1 \diamond R^{n-1}}(x') = \bigwedge_{y \in X} \left(\left[\bigwedge_{x \in X} \mu_R(x, y) \right] \vee \mu_{R^{n-1}}(y, x') \right)$$

$$= \bigwedge_{y \in X} \bigwedge_{x \in X} \{ \mu_R(x, y) \vee \mu_{R^{n-1}}(y, x') \}$$

$$= \bigwedge_{x \in X} \mu_{R \diamond R^{n-1}}(x, x') = \bigwedge_{x \in X} \mu_{R^n}(x, x')$$

Remark 4.3.1

Given a fuzzy relation R, B_1, a fuzzy set defined by $\mu_{B_1}(y) = \bigwedge_{x \in X} \mu_R(x, y)$, if B_0 is a constant fuzzy set such that $B_0 \subseteq B_1$, then $B_0 \diamond R = B_1$, because;

$\forall x \in X, \mu_{B_0 \diamond R}(x) \wedge_{x \in X} \left(\mu_{B_0}(x) \vee \mu_R(x, x') \right) = \bigwedge_{x \in X} \mu_R(x, x') = \mu_{A_k}(x')$

4.3.2 First algorithm for determination of LEFS

Given a fuzzy relation $R \subseteq X \times X$ with the membership function $\mu_R(x, x')$

1. Find the set $B_1 \subseteq F(X)$ defined by

$$\mu_{B_1}(x') = \min_{x \in X} \mu_R(x, x'), \forall x' \in X$$

2. Calculate A_2 given by the composition

$$B_2 = B_1 \diamond R$$

Continue this process until we get a convergence of the sequence B_n given below

$$B_n = B_{n-1} \diamond R = B_1 \diamond R^n = B_{n+1} = B_n$$

3. Accept B_n as the Least Eigen Fuzzy Set (LEFS).

4.3.3 Second algorithm for determination of LEFS

Again, in this method, there is no need to evaluate any composition of R and any set, we are only required to find the invariant element from successive reduction of R. At each step of this method, the invariant elements are exactly the ones of the previous method. This is in conformity with the second methods of obtaining GEFS using the max-min composition.

Steps

1. Find the least element in each column of R as in the first algorithm of the previous method to give B_1.
2. The greatest (maximum) of these elements will be chosen, the column containing that element and the corresponding row number will be deleted to give R_2, where $R_2 \subset R$.
3. Form B^* from B with the same number of cells whose elements are blank.
4. Fill the column number of B^* with the chosen element.
5. Repeat steps 1-4 until no more cell left to be filled in B^*. The filled B^* is the required Least Eigen Fuzzy Set.

Remark 4.3.2

At each step of this method, the smallest between the present value and the previous value is chosen to fill the corresponding present cell.

4.3.4 Third algorithm for determination of LEFS

This involves the continuous (min-max) composition of R itself and taking

$$\min_{x \in X} \mu_R(x, x'), \forall x' \in X$$

column wise at each composition until the Least Eigen Fuzzy Set is obtained.

Steps

1. Produce B_1 from R as in the previous methods
2. Compose R with itself to get $R^2 = R \diamond R$ and produce B_2.
3. Compare B_1 with B_2, if $B_1 = B_2$, then accept B_1 as the GEFS
4. Compose R^2 with R to produce $R^3 = R^2 \diamond R$ and produce B_3.
5. Continue this process until $B_{n+1} = B_n$, then accept B_n as the LEFS.

4.3.5 Fourth algorithm for determination of LEFS

1. Define B_1 by $\mu_{B_1}(x') = \bigwedge_{x \in X} \mu_R(x, x'), \forall x' \in X$ as the least element in each column of R, then the least eigen fuzzy set associated with R, B_n is given by the formula

$\mu_{B_n}(x') = \left[\bigwedge_{x \in X} \mu_R(x, x')\right] \vee \mu_{B_{n-1}}(x_R), \forall x' \in X$, and $\mu_{B_{n-1}}(x_R)$ is the element of B_{n-1} which corresponds with the row number where $\bigwedge_{x \in X} \mu_R(x, x')$ occur.

If $\left[\bigwedge_{x \in X} \mu_R(x, x')\right] \vee \mu_{B_{n-1}}(x_R) \geq \left[\bigwedge_{x \in X} \mu_R(x, x')_n\right] \vee \mu_{B_{n-1}}(x_R)$,

Then, $\mu_{B_n}(x') = \left[\bigwedge_{x \in X} \mu_R(x, x')_n\right] \vee \mu_{B_{n-1}}(x_R)$, where $\bigwedge_{x \in X} \mu_R(x, x')_n$ is the next minimum in the column of (x, x').

2. Compare B_n and B_{n+1}, if $B_n = B_{n+1}$ accept B_n as the Least Eigen Fuzzy Set.
3. Repeat step 1 and 2 until the Least Eigen Fuzzy Set is obtained.

Example 4.3.1

$$R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}$$

First Algorithm

Here,

$$B_1 = [0.3 \quad 0.2 \quad 0 \quad 0.1]$$

$$B_0 = [0.3 \quad 0.3 \quad 0.3 \quad 0.3]$$

$$B_1 \diamond R = [0.3 \quad 0.2 \quad 0 \quad 0.1] \diamond \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix} = B_2$$

$$B_2 = [0.3 \quad 0.2 \quad 0.1 \quad 0.2]$$

$$B_2 \diamond R = [0.3 \quad 0.2 \quad 0.1 \quad 0.2] \diamond \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix} = B_3$$

$$B_3 = [0.3 \quad 0.2 \quad 0.2 \quad 0.2]$$

Also,

$$B_3 \diamond R = [0.3 \quad 0.2 \quad 0.2 \quad 0.2] \diamond \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix} = B_4$$

$$B_4 = [0.3 \quad 0.2 \quad 0.2 \quad 0.2]$$

Here,

$$B_4 = B_3 \diamond R = B_3 \diamond R$$

Hence, B_3 is accepted as the Least Eigen Fuzzy Set associated with the fuzzy relation R.

Second Algorithm

$$B_1 = [0.3 \quad 0.2 \quad 0 \quad 0.1]$$

$$\max(0.3, 0.2, 0, 0.1) = 0.3$$

Deleting column one and row one from R to get;

$$R_2 = \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.5 & 0.6 & 0.4 \\ 0.9 & 0 & 0.8 \end{bmatrix}$$

$$B^* = [0.3 \qquad\qquad\qquad]$$

$$B_2 = [0.2 \quad 0 \quad 0.1], \max(0.2, 0, 0.1) = 0.2$$

$$B^* = [0.3 \quad 0.2 \qquad\qquad]$$

Now, deleting the column containing 0.2 and the row number from R_2, we get

$$R_3 = \begin{bmatrix} 0.6 & 0.4 \\ 0 & 0.8 \end{bmatrix}$$

Now from R_3 we obtain $B_3 = [0 \quad 0.4]$ and $\max(0, 0.4) = 0.4$.

Since the new value is greater than the previous value, we put the previous in the present cell, hence $B^* = [0.3 \quad 0.2 \qquad 0.2]$, and $R_4 = [0.6], B_4 = [0.6]$, here also, we put the previous value in the new cell.

Therefore, $B^* = [0.3 \quad 0.2 \quad 0.2 \quad 0.2]$ is the required (LEFS).

Third Algorithm

$$R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}, B_1 = [0.3, 0.2, 0, 0.1]$$

Then we have $R \diamond R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix} \diamond \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}$

$$R^2 = \begin{bmatrix} 0.6 & 0.6 & 0.6 & 0.6 \\ 0.7 & 0.2 & 0.1 & 0.2 \\ 0.6 & 0.5 & 0.4 & 0.5 \\ 0.3 & 0.5 & 0.6 & 0.4 \end{bmatrix}, B_2 = [0.3, 0.2, 0.1, 0.2]$$

Since $B_1 \neq B_2$, we compute $R^3 = R \diamond R^2$.

$$R^3 = \begin{bmatrix} 0.6 & 0.6 & 0.6 & 0.6 \\ 0.3 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.5 & 0.5 & 0.4 \\ 0.6 & 0.5 & 0.4 & 0.5 \end{bmatrix}, B_3 = [0.3, 0.2, 0.2, 0.2]$$

Also, $B_3 \neq B_2$, we compute R^4.

$$R^4 = \begin{bmatrix} 0.6 & 0.6 & 0.6 & 0.6 \\ 0.3 & 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.5 & 0.4 \end{bmatrix}, B_4 = [0.3, 0.2, 0.2, 0.2]$$

Since $B_3 = B_4$, we accept B_3 as our LEFS.

Forth Algorithm

$$R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}, B_1 = [0.3, 0.2, 0, 0.1]$$

Now, $\mu_{B_2}(x_1) = \left[\bigwedge_{x \in X} \mu_R(x, x_1)\right] \vee \mu_{B_1}(x_1) = \{0.3 \vee 0\} = 0.3$

$$\mu_{B_2}(x_2) = \left[\bigwedge_{x \in X} \mu_R(x, x_2)\right] \vee \mu_{B_1}(x_2) = \{0.2 \vee 0.2\} = 0.2$$

$$\mu_{B_2}(x_3) = \left[\bigwedge_{x \in X} \mu_R(x, x_3)\right] \vee \mu_{B_1}(x_3) = \{0 \vee 0.1\} = 0.1$$

$$\mu_{B_2}(x_4) = \left[\bigwedge_{x \in X} \mu_R(x, x_4)\right] \vee \mu_{B_1}(x_4) = \{0.1 \vee 0.2\} = 0.2$$

Therefore, $B_2 = [0.3, 0.2, 0.1, 0.2]$, since $B_1 \neq B_2$, we then compute B_3.

$B_3 = [0.3, 0.2, 0.2, 0.2]$, also $B_3 \neq B_2$, we therefore compute B_4, and we have

$B_4 = [0.3, 0.2, 0.2, 0.2]$, since $B_3 = B_4$, we accept B_3 as our LEFS.

Theorem 4.4

Let R be a fuzzy relation over X defined by the membership function $\mu_R(x_i, x'_j)$ for every $x_i, x'_j \in X$, $i, j = 1, 2, \dots, n$, if there exist any $i, j \leq n$, such that

$$\mu_R(x_i, x'_j)_{i=j} = \bigwedge_{x_i \in X} \mu_R(x_i, x'_j)$$

Then, for any B_n,

$$\mu_{B_n}(x'_j) = \mu_R(x_i, x'_j)_{i=j}, n > 0$$

Proof

Since B_1 is defined by

$$\mu_{B_1}(x'_j) = \bigwedge_{x_i, \in X} \mu_R(x_i, x'_j) = \mu_R(x_i, x'_j)_{i=j}$$

Therefore, $\mu_{B_1}(x'_j) = \mu_R(x_i, x'_j)_{i=j}$, to show that for any $n > 0$,

$$\mu_{B_n}(x'_j) = \mu_R(x_i, x'_j)_{i=j}$$

Now, $\mu_{B_1}(x_j) = \mu_R(x_i, x'_j)_{i=j}$, for $n = 1$,

By induction hypothesis, suppose that for $n > 0$, $\mu_{B_n}(x'_j) = \mu_R(x_i, x'_j)_{i=j}$

Putting $n = n + 1$, we have $B_{n+1} = B_n \diamond R = B_1 \diamond R^n$, then

$$\mu_{B_{n+1}}(x'_j) = \bigwedge_{x_i, \in X} \{\mu_{B_n}(x_i) \vee \mu_R(x_i, x'_j)\}, i = 1,2, \dots, j, \dots n$$

$$= \wedge \{\mu_{B_n}(x_1) \vee \mu_R(x_1, x'_j), \mu_{B_n}(x_2) \vee \mu_R(x_2, x'_j), \dots, \mu_{B_n}(x_j) \vee$$

$$\vee \mu_R(x_j, x'_j), \dots, \mu_{B_n}(x_n) \vee \mu_R(x_n, x'_j)\}$$

$$= \mu_{B_n}(x_i) \vee \mu_R(x_i, x'_j)_{i=j}$$

But since $\mu_{B_n}(x_j) = \mu_R(x_i, x'_j)_{i=j}$, hence $\mu_{B_{n+1}}(x'_j) = \mu_R(x_i, x'_j)_{i=j}$

Therefore, for any $n > 0$, and for any B_n, $\mu_{B_n}(x'_j) = \mu_R(x_i, x'_j)_{i=j}$.

4.4 Eigen Fuzzy Set (max-product)

In this section, we investigate the existence of Eigen Fuzzy Set using the max-product composition.

Example 4.4.1

Let R be a fuzzy relation on X and let A_1 be a fuzzy subset of X defined by the membership function $\mu_{A_1}(x') = \bigvee_{x \in X} \mu_R(x, x'), \forall x' \in X$, i.e.,

$R = \begin{bmatrix} 0.7 & 0.6 & 0.9 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.8 & 0.4 \end{bmatrix}$, and $A_1 = [0.7, 0.8, 0.9]$, by the max-product of A_1 and R, we obtain the following; $A_2 = A_1 \odot R$, as follows;

$$A_1 \odot R = [0.7, 0.8, 0.9] \odot \begin{bmatrix} 0.7 & 0.6 & 0.9 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.8 & 0.4 \end{bmatrix}$$

$$A_2 = [0.49, 0.72, 0.63]$$

and

$$A_3 = A_2 \odot R = [0.343, 0.504, 0.441]$$

also, we obtain the following in the same manner;

$A_4 = [0.240, 0.353, 0.309]$, $A_5 = [0.168, 0.247, 0.216]$, $A_6 = [0.118, 0.173, 0.216]$

$A_7 = [0.083, 0.121, 0.106]$, $A_8 = [0.058, 0.085, 0.075]$, $A_9 = [0.041, 0.060, 0.052]$

Remark 4.4.1

The sequence $A_n,\ n > 0$, is decreasing and bounded by A_1 and 0, that is;

$$\{0\} \subseteq \cdots \subseteq A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_2 \subseteq A_1$$

With example 4.4.1 above and several others generated, it was observed that, in the max-product composition, if we define a sequence $A_n,\ n > 0$, by;

$A_{n+1} = A_n \odot R, n \geq 1$, this sequence only converge to the set of zero, and as such, does not produce an Eigen Fuzzy Set except the trivial case. The case where the leading diagonals of the fuzzy relation have membership function identically equal to one, i.e., R is reflexive ($\mu_R(x_i, x'_j)_{i=j} = 1$), then

$\mu_{A_n}(x_j) = \mu_R(x_i, x'_j)_{i=j} = 1$, where $i, j = 1, \dots, |X|$

4.5 Eigen Fuzzy Set (Max-average composition)

We also investigate the existence of Eigen Fuzzy Set using the max-average composition, here, if we define a sequence $A_n,\ n > 0$, by $A_{n+1} = A_n \oplus R, n \geq 1$, using the max-average composition, this sequence neither decrease nor increase and an Eigen Fuzzy Set may or may not exist.

Example 4.5.1

Let R be a fuzzy relation on X and let A_1 be a fuzzy subset of X defined by the membership function $\mu_{A_1}(x') = \bigvee_{x \in X} \mu_R(x, x'), \forall x' \in X$, i.e.,

$$R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}$$

$$A_1 = [1 \quad 0.9 \quad 0.7 \quad 1]$$

Now, using the max-average composition of $A_1$and R, we obtain the following;

$$A_2 = A_1 \oplus R = [0.95 \quad 0.95 \quad 0.8 \quad 1]$$

$$A_3 = [0.975 \quad 0.95 \quad 0.825 \quad 0.975]$$

$$A_4 = [0.975 \quad 0.9375 \quad 0.825 \quad 0.9875]$$

$$A_5 = [0.9688 \quad 0.9438 \quad 0.8188 \quad 0.9575]$$

$$A_6 = [0.9719 \quad 0.9438 \quad 0.8219 \quad 0.9844]$$

$$A_7 = [0.9719 \quad 0.9422 \quad 0.8219 \quad 0.9860]$$

$$A_8 = [0.9711 \quad 0.9430 \quad 0.8211 \quad 0.9860]$$

$$A_9 = [0.9715 \quad 0.9430 \quad 0.8215 \quad 0.9858]$$

This does not produce an Eigen Fuzzy Set.

Example 4.5.2

Let R be a fuzzy relation defined below

$$R = \begin{bmatrix} 0.8 & 0.6 & 0.2 & 0.5 \\ 0.7 & 0.5 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.3 & 0.6 \\ 0.5 & 0.3 & 0.1 & 0.5 \end{bmatrix}$$

Thus, we have;

$$A_1 = [0.8 \quad 0.6 \quad 0.3 \quad 0.6]$$

$$A_2 = [0.8 \quad 0.7 \quad 0.5 \quad 0.65]$$

$$A_3 = [0.8 \quad 0.7 \quad 0.5 \quad 0.65]$$

Here, we have $A_3 = A_2$, therefore, A_2 is an Eigen Fuzzy set.

Theorem 4.5

Let R be a subreflexive fuzzy relation such that; $\mu_R(x_i, x'_j)_{i=j} \geq \mu_R(x_i, x'_j)_{i\neq j}$, then by the max-average composition, for any $n > 0$, $\mu_{A_n}(x_j) = \alpha$, $i, j = 1,2, \dots, |X|$. In particular, if R is reflexive, then $\mu_{A_n}(x_j) = 1$.

Proof

We have

$$\mu_{A_1}(x'_j) = \bigvee_{x_i \in X} \mu_R(x_i, x'_j) = \alpha$$

By max-average composition we have for any $n > 0$,

$$\mu_{A_n}(x'_j) = \bigvee_{x_i \in X} \left\{ \frac{\mu_{A_{n-1}}(x_i) + \mu_R(x_i, x'_j)}{2} \right\}$$

But for $n = 2$, we have;

$$\mu_{A_2}(x'_j) = \bigvee_{x_i \in X} \left\{ \frac{\mu_{A_1}(x_i) + \mu_R(x_i, x'_j)}{2} \right\} = \frac{1}{2}\left\{\alpha + \mu_R(x_i, x'_j)_{i=j}\right\}, j = 1,2, \dots, |X|$$

$$= \frac{1}{2}\{\alpha + \alpha\} = \alpha$$

Therefore, $\mu_{A_2}(x'_j) = \alpha, j = 1,2, \dots, |X|$

Now, suppose that for, $\mu_{A_n}(x'_j) = \alpha$

Putting $n = n + 1$, we have $A_{n+1} = A_n \oplus R$, that is;

$$\mu_{A_{n+1}}(x'_j) = \bigvee_{x_i \in X} \frac{1}{2}\left\{\alpha + \mu_R(x_i, x'_j)\right\}, j = 1,2, \dots, |X|$$

$$= \frac{1}{2}\left\{\alpha + \mu_R(x_i, x'_j)_{i=j}\right\}$$

but $\mu_R(x_i, x'_j)_{i=j} = \alpha$, hence $\mu_{A_{n+1}}(x'_j) = \alpha$

Therefore, for any A_n, $\mu_{A_n}(x'_j) = \alpha, j = 1,2, \dots |X|$, $\forall n > 0$

Now, suppose R is reflexive, then, from above, we have;

$$\mu_{A_n}(x'_j) = 1, j = 1,2, \dots |X|, \forall n > 0$$

Remark 4.5.1

In general, in the max-average composition, the sequence is neither decreasing nor increasing, and an Eigen Fuzzy Set may or may not exist. But if the relation is subreflexive such $\mu_R(x_i, x'_j)_{i=j} \geq \mu_R(x_i, x'_j)_{i\neq j}, \forall (x_i, x'_j) \in R$, then a constant Eigen Fuzzy Set is guaranteed, particularly, when the relation is reflexive.

4.6 Construction of Eigen Fuzzy Set by Various Approximation

In this section, we investigate and construct Eigen Fuzzy Set by the approximations of max-product and max-average compositions.

4.6.1 Max-product approximation

Let R be a fuzzy relation on X, the max-product approximation composition is a refinement of max-product composition defined by rounding off to the nearest decimal to produce an Eigen Fuzzy Set.

Example 4.6.1

Let R be a fuzzy relation on X and let A_1 be a fuzzy subset of X defined by the membership function $\mu_{A_1}(x') = \bigvee_{x\in X} \mu_R(x, x'), \forall x' \in X$, i.e.,

$R = \begin{bmatrix} 0.7 & 0.6 & 0.9 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.8 & 0.4 \end{bmatrix}$, and $A_1 = [0.7, 0.8, 0.9]$, by the max-product approximation of A_1 and R, we obtain the following;

$A_2 = [0.5, 0.7, 0.6]$, $A_3 = [0.4, 0.5, 0.5]$, $A_4 = [0.3, 0.4, 0.4]$, $A_5 = [0.2, 0.3, 0.3]$,

$A_6 = [0.1, 0.2, 0.2]$, $A_7 = [0.1, 0.2, 0.1]$, $A_8 = [0.1, 0.2, 0.1]$.

Here, $A_8 = A_7$, hence A_7 is an Eigen Fuzzy Set based on max-product approximation.

4.6.2 Max-average approximation

This composition is also a refinement of the max-average composition in the same analogy with the max-product approximation.

Example 4.6.2

Let R be a fuzzy relation on X and let A_1 be a fuzzy subset of X defined by the membership function $\mu_{A_1}(x') = \bigvee_{x \in X} \mu_R(x, x'), \forall x' \in X$, i.e.,

$$R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}, A_1 = [1, 0.9, 0.7, 1]$$

Then by the max-average approximation we have the following;

$A_2 = [1, 1, 0.8, 1]$, $A_3 = [1, 1, 0.9, 1]$, $A_4 = [1, 1, 0.9, 1]$.

Since $A_4 = A_3$, then A_3 is an Eigen Fuzzy Set based on the max-average approximation.

4.7 Extensions of Max-product and Max-Average Approximation

In this section, we introduce the extensions of max-product, max-average compositions and their approximations as min-product, min-average compositions and also investigate the existence of Eigen Fuzzy Sets on these compositions.

4.7.1 Min-product composition

Min-product composition is an extension of max-product composition where instead of taking maximum of products, we take minimum of the products.

Example 4.7.1

Let R be a fuzzy relation on X and let A_1 be a fuzzy subset of X defined by the membership function $\mu_{A_1}(x') = \bigwedge_{x \in X} \mu_R(x, x'), \forall x' \in X$, i.e.,

$$R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}, A_1 = [0.3, 0.2, 0, 0.1]$$

Then, by min-product composition, we have the following;

$$A_2 = [0, 0, 0, 0]$$

This is an empty Eigen Fuzzy set, hence we conclude that the min-product composition do not produce an Eigen Fuzzy Set.

4.7.2 Min-product approximation

This is also a refinement of the Min-Product Composition.

Example 4.7.2

Let R be a fuzzy relation on X and let A_1 be a fuzzy subset of X defined by the membership function $\mu_{A_1}(x') = \bigwedge_{x \in X} \mu_R(x, x'), \forall x' \in X$, i.e.,

$$R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}, A_1 = [0.3, 0.2, 0, 0.1]$$

Then, by min-product approximation composition, we have the following;

$$A_2 = [0, 0, 0, 0]$$

Here also, the min-product approximation composition produced an empty Eigen Fuzzy Set and hence and Eigen Fuzzy Set may not exist particularly, when there exist an element $x' \in X$ such that $\mu_R(x, x') = 0$ for some $x \in X$.

4.7.3 Min-average approximation

This composition is an extension of max-average composition by approximating the averages and taking their minimum.

Example 4.7.3

Let R be a fuzzy relation on X and let A_1 be a fuzzy subset of X defined by the membership function $\mu_{A_1}(x') = \bigwedge_{x \in X} \mu_R(x, x'), \forall x' \in X$, i.e.,

$$R = \begin{bmatrix} 0.7 & 0.7 & 0.6 & 1 \\ 1 & 0.2 & 0.7 & 0.1 \\ 0.3 & 0.5 & 0.6 & 0.4 \\ 0.8 & 0.9 & 0 & 0.8 \end{bmatrix}, A_1 = [0.3, 0.2, 0, 0.1]$$

By min-average approximation we obtain the following;

$A_2 = [0.2, 0.2, 0, 0.1]$, $A_3 = [0.2, 0.2, 0, 0.1]$.

Since $A_2 = A_3$, then, A_2 is an Eigen Fuzzy Set with respect to min-average approximation composition.

Table 4.1 showing existence of Eigen Fuzzy Sets based on various compositions

Summary of Eigen Fuzzy Sets based on Various Composition			
COMPOSITION	EIGEN FUZZY SET	GEFS	LEFS
Max-Min	Yes	Yes	No
Min-Max	Yes	No	Yes
Max-Product	Special case	No	No
Max-Average	Some cases	No	No
Max-Average(app)	Yes	Yes	No
Max-Product(app)	Yes	Yes	No
Min-Average(app)	Yes	No	Yes
Min-Product	No	No	No

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 Summary

In this dissertation, we presented the basic notions on fuzzy set, fuzzy set operations such as Union, Intersection and Complement, and various alpha-cuts in fuzzy set and their inverse. We presented fuzzy relation, inverse fuzzy relation, operations on fuzzy relations, various compositions of fuzzy relations including max-min, min-max, max-average and max-product compositions, discussed properties of fuzzy relations such as reflexive, symmetric and transitive, we introduced the concept of subreflexive property of fuzzy relation as a relaxation of the usual reflexive property. We gave highlight on an equivalence fuzzy relation (similarity relation) via its representation on a partition tree, and discussed alpha-cuts in fuzzy relation.

We also presented the concept of Eigen Fuzzy Sets associated with fuzzy relations based on max-min composition, min-max composition, the three basic algorithms for the determination of Greatest Eigen Fuzzy Sets (GEFS) based on max-min composition, introduced a modified algorithm for the determination of the GEFS, and extend these algorithms to the context of Least Eigen Fuzzy set (LEFS) based on min-max composition.

Finally, we investigated the existence of Eigen Fuzzy Sets based on the max-product, max-average compositions and their duals as well as their approximations, and we established some results.

5.2 Conclusion

The notion of fuzzy set and fuzzy relation has been a viable tool for the fuzzification of various aspect of classical set.

The concept of Eigen Fuzzy Sets of fuzzy relation introduced by (Sanchez, 1981) based on max-min composition and its algorithms have been extended to min-max composition. This was motivated by the work of (Saleem, 2010), where the concept of Least Eigen Fuzzy Set based on the same max-min composition was introduced. In this work, we refer to the definition of Least Eigen Fuzzy Sets as defined by (Nobuhara, *et al.*, 2004) and extend the algorithms for the determination of GEFS to LEFS. Also, max-product and max-average compositions have been investigated to determine the existence of Eigen Fuzzy Sets based on these compositions.

5.3 Recommendation

Various compositions of fuzzy relation have been presented in this work, however, only the algorithms for the determination of Greatest Eigen Fuzzy Set and Least Eigen Fuzzy Set based max-min and min-max compositions respectively that has been extensively studied and developed. We therefore recommend that the max-product and max-average composition be further investigated due to their potential applications in image retrieval and reconstruction.

REFERENCES

Alkali, A. J. (2015). *Study of Fuzzy Multisets and their Algebra.* (A Doctoral thesis, Ahmadu Bello University, Zaria). Retrieved from Department of Mathematics Library, Ahmadu Bello University, Zaria. Nigeria.

Beg, I., and Ashraf, S. (2008). Fuzzy equivalence relations. *Kuwait Journal Science and Engineering*, *35*(1A): 191-206.

Beg, I., and Ashraf, S. (2009). *Fuzzy Relations.* Germany: Lambert Academic Publisher.

Daren, Y., Qinghua, H., and Congxin, W. (2007). Uncertainty measures for fuzzy relations and their applications. *Applied Soft Computing*, *7*: 1135–1143.

DiMartino, F., and Sessa, S. (2007). *A Genetic Algorithm based on Eigen Fuzzy Sets for Image Reconstruction.* F. Masulli, S. Mitra, and G. Pasi (Eds.): LNAI 4578, Springer, 342-348.

DiMartino, F., Sessa, S., and Nobuhara, H. (2004). Eigen Fuzzy Sets and Image Information Retrieval. *Proceedings of the International Conference on Fuzzy Information Systems*, 3: 1385-1390, Budapest.

DiNola, A., Pedricz, W., Sessa, S., and Sanchez, E., (1991). Fuzzy Relation Equations Theory as a basis of Fuzzy Modelling: An overview. *Fuzzy Sets and Systems*, 40: 415-429.

Dubois, D., Walenty, O., and Prade, H. (2000). Fuzzy Sets: History and Basic Notion. In: D. D. Prade (Ed.), *Fundamentals of Fuzzy sets*, 7: 21-124.

Goetschel, R., and Voxman, W. (1985). Eigen Fuzzy Number Sets. *Fuzzy Sets and Systems, 16:* 75–85.

Hussain, M. (2010). *Fuzzy Relations.* (A Master Thesis, Blekinge Institute of Technology, Sweden). Retrieved from http://www.diva-portal.org/smash/get/diva2:832500/FULLTEX01.

Kaufmann, A. (1975). *Introduction to the Theory of Fuzzy Subsets.* Academic Press.

Klir, G. J., and Yuan, B. (1995). *Fuzzy Sets and Fuzzy Logic: Theory and Applications.* Upper Saddle River, NJ: Prentice Hall.

Kwang, H. L. (2005). *First Course on Fuzy Set Theory and Application.* Germany: Springer Berlin Heidelberg NewYork.

Negoita, C. V., and Ralescu, D. A. (1975). *Applications of Fuzzy Sets to Systems Analysis,*. John Wiley and Sons, New York,

Nobuhara, H., Bede, B., and Hirota, K. (2006). On Various Eigen Fuzzy Sets and Their Application to Image Reconstruction. *Information Sciences*, 176(20): 2988-3010.

Ovchinnikov, S. (2000). An Introduction to Fuzzy Relations. In D. D. (eds.), *Fundamentals of Fuzzy Sets* (pp. 233-259). Boston, Dordrecht, London. Kluwer Academic Publishers.

Radecki, T. (1976). Mathematical model of Information Retrieval System based on the Concept of Fuzzy Thesaurus. *Journal* of *Information Processing and Management*, 12: 313–318.

Rosenfield, A. (1975). Fuzzy graphs. In *Fuzzy sets and their applications*, 77-95. New York, USA: Academic Press.

Saleem, M. N. (2010). *Eigen Fuzzy Sets of Fuzzy Relation with Applications.* (A Master Thesis, Blekinge Institute of Technology, Sweden). Retrieved from www.diva-portal.org/smash/get/diva2:831380/FULLTEX01.

Sanchez, E. (1976). Resolution of Composite Fuzzy Relation Equation. *Information and Control, 30*: 38–48.

Sanchez, E. (1977). Resolution of Eigen Fuzzy Sets Equations. *Fuzzy Set and Systems, 1*: 69-74.

Sanchez, E. (1981). Eigen Fuzzy Sets and fuzzy Relations. *Journal of Mathematical Analysis and Applications, 81*: 399-421.

Sanchez, E. (1984). Solution of Fuzzy Equations with Extended Operations. *Fuzzy Sets and Systems, 12*: 237-248.

Singh, D., Alkali, A. J., and Isah, A. I. (2015). An Extension of the Properties of Inverse α-Cuts to Fuzzy Multisets. *Annals of Fuzzy Mathematics and Informatics, 9*(6): 917-923.

Steimann, F. (1997). Fuzzy Set Theory in Medicine. *Artificial Intelligence in Medicine, 11*: 1-7.

Sun, Z., & Han, J. (2006). , Inverse alpha-Cuts and Interval [a, b)-Cuts. *Proceedings of the International Conference on Innovative Computing, Information and Control* (pp. 441-444). Beijing : IEEE Press.

Yager, R. R. (1991). Connectives and Quantifiers in Fuzzy Sets. *Fuzzy Sets and Systems*, *40*: 39–75.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8: 338-353.

Zadeh, L. A. (1971). Similarity relations and fuzzy orderings. *Information Science*, *3*: 177-200.

Zimmermann, H. J. (1996). *Fuzzy Set Theory and its Apllications* (3rd Ed.). Boston MA: Kluwer academic publishers.

Zimmermann, H. J. (2010). Fuzzy Set Theory. *Advanced Review*, *2*: 317-332.