MOTION IN THE GENERALIZED RESTRICTED THREE-BODY PROBLEM

BY

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DECLARATION

I declare that the work in this thesis entitled *Motion in the Generalized Restricted Three-Body Problem* has been performed by me, under the supervisions of Dr. J. Singh and Prof. B.K. Jha. The information derived from the literature has been duly acknowledged in the text and a list of references. No part of this Thesis was previously presented for another degree at any University.

……………………. ………………………..

Joel John Taura Date

CERTIFICATION

This thesis titled *Motion in the Generalized Restricted Three-body Problem* meets the regulations governing the award of the degree of masters of Science of Ahmadu Bello University Zaria, and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

Through all my endeavors, both past and present, my parents have encouraged, supported, and guided me. My father gave me the desire to set goals worth achieving for myself and education with which to achieve those goals. I am indeed grateful to him for all that he has done for me; sadly, he just passed away before I could compensate him materially or otherwise. This thesis is dedicated to him.

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ABSTRACT

This thesis investigates motion in the generalized restricted three-body problem. It is generalized in the sense that both the primaries are radiating oblate bodies, together with the effect of gravitational potential from a belt. It derives the equations of motion, locates the positions of the equilibrium points and examines their linear stability. It has been found that in addition to the usual five equilibrium points, there appear two new collinear points L_{n1}, L_{n2} due to the potential from the belt, and in the presence of all these perturbations, the equilibrium points L_1, L_3, L_4, L_5 come nearer to the primaries; while L_2, L_{n2} move towards the bigger primary and L_{n1} moves away from it. The collinear equilibrium points remain unstable, while the triangular points are stable in $0 < \mu < \mu_c$ and unstable in $\mu_c \leq \mu \leq \frac{1}{2}$, where μ_c is the critical mass ratio influenced by the oblateness and radiation of the primaries and potential from the belt. This model can be applied in the study of binary systems, especially motion near oblate, radiating binary stars.

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CHAPTER 1

1.0 INTRODUCTION

The restricted three-body problem is a famous model of classical mechanics. It describes the motion of an infinitesimal mass moving under the gravitational effects of the two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The approximate circular motion of the planets around the sun and the small masses of asteroids and the satellites of planets compared to the planet's masses, originally suggested the formulation of the restricted problem.

In certain stellar dynamics problems it is altogether inadequate to consider solely the gravitational interaction force. For example, when a star acts upon a particle in a cloud of gas and dust, the dominant factor is by no means gravity, but the repulsive force of the radiation pressure (Poynting, 1903). Since a large fraction of all stars belong to binary systems Allen (1973), the particle motion in the field of a double star offers special interest. There are disks of dust with various masses in the extra solar planetary systems, which are regarded as young analogues of the kuiper belt in our solar system. The gravitational potential due to these belts also have great influence on the infinitesimal body (Jiang and Yeh, 2004a).

The participating bodies in the classical restricted three-body problem are strictly spherical in shape, but in actual situations we find that, some of the natural and artificial bodies moving in the space are not point masses or spherical rather they are oblate bodies. For instance, Saturn, Jupiter, Regulus (star) and Peanut binary stars are sufficiently oblate. Singh and Ishwar (1999), pointed out that, lack of sphericity, or oblateness of the primaries affects the motion of an infinitesimal body. The motions of artificial Earth satellites are examples.

In the classical problem, the effects of the gravitational attraction of the infinitesimal body and other perturbations have been ignored. Thus, the classical restricted three- body problem is inadequate to explain the motion of the infinitesimal body in the presence of any perturbing forces such as radiation pressure, oblateness of a body and gravitational potential from a belt.

Hence, it becomes imperative to modify the classical restricted three-body problem by including some of these perturbing forces.

1.1 STATEMENT OF THE PROBLEM

Consider an infinitesimal mass (e.g. dust particle) moving in the orbital plane of oblate binary stars. Then, the problem is to study its motion in the generalized restricted three-body problem. The problem is generalized in the sense that both the primaries are radiating oblate bodies, together with the effect of gravitational potential from the belt. Thus we are to study the combined effect of radiation and oblateness of the primaries and gravitational potential from a belt on the stability of equilibrium points in restricted three-body problem.

1.2 JUSTIFICATION/SIGNIFICANCE OF THE STUDY

The human species stands on the edge of a new frontier, the transition from a planet-bound to a space-faring civilization. Just as the transition from hunter-gatherer to farmer necessitated new approaches to solve new problems, so the expansion into the space, in terms of dynamics of artificial satellite , requires the formulation of new models that include the effects of some of the perturbing forces on the satellite. Motion in the generalized restricted three-body problem is one of such models that considered the effects of radiation and oblateness of the primaries and gravitational potential from a belt on the satellite. Thus, this model will be very helpful in the study of binary stars, especially dynamics near oblate binary stars. We choose the primaries as oblate spheroid: peanut-stars, the two stars appear to be nearly identical, each 15 to 20 times the mass of our sun (Jenks, 2008).

1.3 OBJECTIVES OF THE STUDY

The objectives of the study include:

- To derive equations of motion of an infinitesimal body under the influence of radiating oblate primaries and gravitational potential from a belt in the restricted three-body problem.
- To examine the effects of these perturbations on the locations of the equilibrium points;
- To investigate the effects of these perturbations on the linear stability of the equilibrium points in restricted three-body problem.

1.4 THEORETICAL FRAME WORK

The outline of the theoretical bases on which the problem is built, are given here:

1.4.1 Circular Restricted Three-body Problem

The three-body problem involves the motion of three celestial bodies under their mutual gravitational attraction. It is an old problem and logically follows from the two-body problem which was solved by Newton in his *Principia* in 1687. Newton also considered the three-body problem in connection with the motion of the Moon under the influences of the Sun and the Earth, the consequences of which included a headache. Unlike the two body problem, there is no closed form analytical solution for the differential equations governing the motion in the three-body problem. However, it is still possible, although not easy, to gain insight into the qualitative nature of the solutions in this system.

This task is more tractable if several simplifying assumptions are introduced. In reducing the general three-body equations, the first assumption is that the mass of one of the bodies is infinitesimal, that is, it does not affect the motion of the other two bodies. Thus, the two massive bodies, or primaries, move in Keplerian orbits about their common center of mass. This reduced model is called the restricted three-body problem, and was formalized by Euler in the late 18th century (Szebehely, 1967). The problem is further simplified by constraining the primaries to move in circular orbits about their center of mass and are kept fixed on the x- axis, the (x, y), plane is the plane of motion of the primaries, and the z-axis is orthogonal to the (x, y) plane. These coordinates are sometimes called synodical. The resulting simplified model is usually labeled the circular restricted three-body problem (CR3BP). Although a less complex dynamical model than the general problem (in terms of the number of equations and the number of dependent variables), analysis in the circular problem offers further understanding of the motion in a regime that is of increasing interest to space science.

In the restricted circular three body problem, the units are usually chosen in such a way that the properties of the system depend on a single parameter.

- The sum of the masses of the primaries is taken as the unit of mass. The mass of the smaller primary is denoted by μ, whence the mass of the bigger primary $1-\mu$, where $\mu = \dfrac{m_2}{m_1 + m_2}$ is the mass parameter and m_1, m_2 are the masses of the bigger and smaller primaries, respectively.

- The distance between the primaries is the unit of distance. The distances of the smaller primary and bigger primary from the centre of mass are then $1 - \mu$ and μ, respectively.
- The unit of time is chosen so that the mean motion of the primaries is $n = 1$. From these it follows that the gravitational constant is unity. The only remaining parameter is μ.

The equations of motion of the infinitesimal mass m with the coordinates (x, y, z) in the circular restricted three body problem, relative to a frame that rotates with the primaries Figure 1.1 are:

$$\ddot{x} - 2\dot{y} = \Omega_x,$$
$$\ddot{y} + 2\dot{x} = \Omega_y,$$
$$\ddot{z} = \Omega_z$$

where $\Omega(x, y, z)$ is

$$\Omega = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

and $r_1^2 = (x+\mu)^2 + y^2 + z^2$, $r_2^2 = (x+\mu-1)^2 + y^2 + z^2$, where r_1, r_2 are the distances of the infinitesimal mass m from the bigger primary m_1 and smaller primary m_2, respectively.

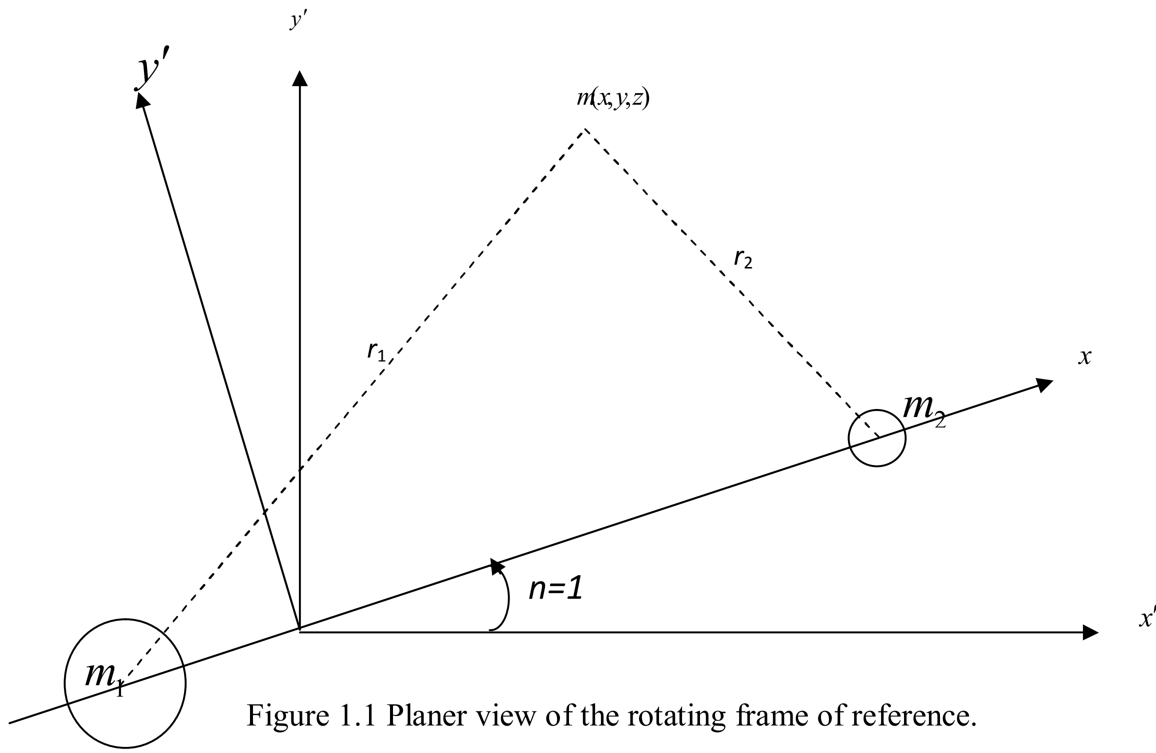

Figure 1.1 Planer view of the rotating frame of reference.

Unfortunately, even with the simplifying assumptions, no general closed form solution to the CR3BP is available. However, particular solutions can be determined. Of notable interest are the equilibrium solutions, first identified by Lagrange in 1772 (Szebehely,1967), which represent locations where the infinitesimal particle remains fixed, relative to a frame of reference that rotates with the primaries.

If the restricted problem is formulated in terms of a coordinate frame that rotates with the primaries, it is possible to identify five equilibrium solutions, also known as libration or Lagrange points. Three of these points $L_1, L_2,\ L_3$ are collinear Figure 1.2, and lie along the line joining the primaries and are unstable equilibrium points. The other two points $L_4,\ L_5$ form equilateral triangles with the primaries, in the plane of primary motion and are conditionally stable equilibrium points. L_4 and L_5 are located at

$$\left(\frac{1}{2}-\mu,\pm\frac{\sqrt{3}}{2}\right)$$

The focus of most investigations into the circular restricted three-body problem is the motion of the infinitesimal particle in the vicinity of the equilibrium points.

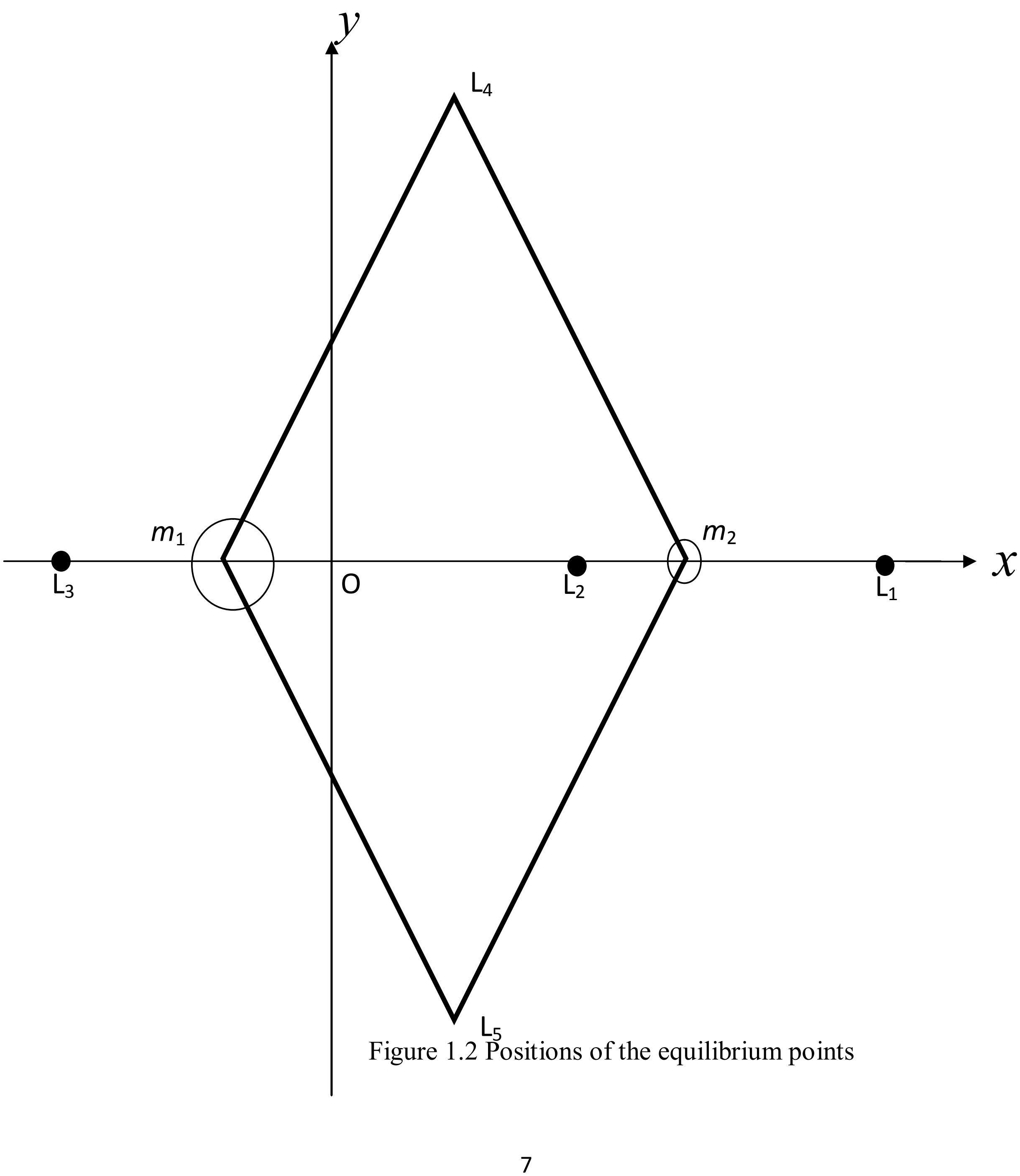

Figure 1.2 Positions of the equilibrium points

1.4.2 Radiation

For our investigation we adopt Radzievskii's simplifying theory that has been used by many investigators (Bhatnagar and Chawla, 1979; Kumar and Choudhry 1988, 1989; Ragos and Zafiropoulos 1995; Papadakis 2006; Kalvouridis and Hadjifotinou, 2008; etc.) and is based on some assumptions that can be summarized as follows:

• The radiation emitted from the primaries (stars) influences the motion of the small body but does not affect the motion of the others.

The mass and the temperature of the radiating stars are almost constant.

• The particle is assumed spherical and homogeneous with constant density. Its surface has a uniform reflection coefficient and its size must exceed the wavelength of the incident light.

• Light has no fluctuations, propagates in straight lines and screening or shadowing effects produced by other bodies of the system are negligible.

• The force of light pressure is inversely proportional to the square of the distance between the particle and the illuminating body.

Then the effect of radiation pressure of a source on a small particle, is expressed by means of a reduction factor $q = 1 - b$. Parameter b, that is often mentioned as the radiation coefficient, is the ratio of force F_r which is caused by radiation to force F_g which results from gravitation, that is $b = \frac{F_r}{F_g}$. It depends on the physical properties of the radiating bodies, as well as those of the particle. Corben and Stehle (1977) proposed the expression

$$b = \frac{3L}{16\pi cGM\rho s}$$

where coefficient b is directly proportional to luminosity L of the radiating star and inversely proportional to its mass M, to the radius of the particle s, and to its uniform density ρ. G is the gravitational constant and c is the speed of light. Provided that the mass and the luminosity of the star are known, b's can be evaluated for any particle of a given radius and density. Schuerman (1980) proposed a similar formula

$$b=\frac{\int L_v \sigma_v dv}{4\pi cGMm}$$

where m is the mass of the particle, σ_v is its radiation cross section and L_v is the luminosity in the frequency range v to $v+d_v$ of a star of mass M. The radiation pressure cross section may be written as AQ_v, where A is the geometric cross section and Q_v is a dimensionless "efficiency factor" which is a function of particle size, shape and optical properties.

Therefore, the above formula can be written as.

$$b=\frac{LAQ_v}{4\pi cGMm}$$

For a star of mass M and luminosity L, b is roughly determined by the ratio A/m for particle sizes larger than or equal to the wavelength of the incident radiation.

In our case, if we symbolize the radiation factors of the bigger and smaller primaries by q_1, q_2 respectively, then $q_i = 1 - p_i$ where, $p_i = \frac{F_{ri}}{F_{g_i}}$ $i = 1,2$. Since in most cases the gravitational force exceeds radiation, we shall consider that $q_i > 0$ and consequently $0 < p_i << 1$. The particular problem is characterized by two parameters which are the two radiation coefficients p_1 and p_2.

1.4.3 Oblateness Coefficients.

In our problem both the primaries are oblate, we denote $A_i, i = 1,2$ for the oblateness coefficients of bigger m_1 and smaller m_2 primaries respectively, such that $0 < A_i << 1$ and

$$A_1 = \frac{AE_1^2 - AP_1^2}{5R^2}, \qquad A_2 = \frac{AE_2^2 - AP_2^2}{5R^2}$$

there $AE_1$and AE_2 are equatorial radii and AP_1, AP_2 are the polar radii of m_1 and m_2 respectively and R the distance between the primaries.

The mean motion n of the primaries as in Singh and Ishwar (1999) is given by:

$$n^2 = 1 + \frac{3(A_1 + A_2)}{2}.$$

1.4.4 Potential of the Belt

We adopt the potential profile proposed by Miyamoto and Nagai (1975), according to the profile, the potential of the belt is given by

$$V(r,z) = -\frac{M_b}{\sqrt{r^2 + \left(a + \sqrt{z^2 + b^2}\right)^2}}$$

M_b is the total mass of the belt, $r^2 = x^2 + y^2$, a and b are parameters which determine the density profile of the belt. The parameter a controls the flatness of the profile and can be called "flatness parameter". The parameter b controls the size of the core of the density profile and can be called "core parameter". When $a = b = 0$, the potential equals to the one by a point mass.

Restricting ourselves to the $xy-plane$, the potential becomes:

$$V\left(r,0\right)=-\frac{M_b}{\left(r^2+T^2\right)^{1/2}} \qquad T=a+b$$

The mean motion n under the influence of the belt is given by

$$n^2=1-2f_b\left(r\right)$$

where $f_b\left(r\right)$ is the gravitational potential from the belt and $f_b\left(r\right)=-\frac{\partial V}{\partial r}$ (Jiang and Yeh , 2006). Thus $n^2=1+\frac{2M_b r_c}{(r_c^2+T^2)^{3/2}}$, where r_c is the radial distance of the infinitesimal mass in the classical restricted three-body problem.

1.4.5 Stability of Equilibrium Points of a System of Differential Equations

One of the techniques to determine stability of an equilibrium x^* of a system of differential equations, is to analyze the linearized system around that equilibrium. Then, the solutions to linearized differential equations with constant coefficients can be expressed as linear combination of the product of a polynomial and an exponential function. Therefore solving a linear differential equation with constant coefficients can be transformed to solving a linear algebraic problem, more precisely, an eigenvalue-eigenvector problem. Stability or instability then may follow from the eigenvalues of the matrix of the linearized system. Lyapunov proved that an equilibrium point x^* is stable if the real part of every eigenvalue is negative or the eigenvalues are purely imaginary and distinct.

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CHAPTER 2

LITERATURE REVIEW

The three-body problem involves the motion of three celestial bodies under their mutual gravitational attraction. It is an old problem and logically follows from the two-body problem which was solved by Newton in his *Principia* in 1687. Newton also considered the three-body problem in connection with the motion of the Moon under the influences of the Sun and the Earth, the consequences of which included a headache.

After Newton, Euler (1772) studied the lunar theory using the restricted problem of three bodies; he found that no general closed form solution exists. However, at about the same time, the first special solutions L_4 and L_5 of the restricted three-body problem were discovered by J. L Lagrange and later, the collinear points L_1, L_2 and L_3 by L Euler.

Poynting (1903) has pointed out that particles, such as small meteors or cosmic dust are comparably affected by gravitation and light radiation force (photogravitational) as they approach luminous celestial bodies.

Radzievskii (1950) formulated the photogravitational restricted three-body problem. This arises from classical problem when one of the interacting masses is an intense emitter of radiation. He discussed it for three specific bodies: the sun, a planet and a dust particle. He studied the equilibrium points of the photogravitational problem and found that their locations depend on the radiation pressure factor.

Szebehely (1967) studied the stability of the equilibrium points of the restricted three body-problem. He established that, in linear sense the collinear points L_1, L_2 and L_3 are unstable for any value of the mass ratio μ, and the triangular points L_4 and L_5 are stable for $0 < \mu < \mu_0 = 0.03852$..

Bhatnagar and Chawla (1979) investigated the stability of motion around triangular equilibrium points in the photogravitational restricted three body problem. They found that the range of stability decreases due to the radiation pressure.

Sharma (1982) studied the linear stability of the triangular equilibrium points of the restricted three-body problem when the more massive primary is a source of radiation and is an oblate spheroid as well. He found that the eccentricity of the conditional retrograde elliptic periodic orbits around the triangular points at the critical mass μ_c increases with an increase in the oblateness coefficient and the radiation force and becomes unity when μ_c =0. Simmons et al. (1985) obtained a complete solution of the restricted three-body problem. They discussed the existence and linear stability of the equilibrium points for all values of radiation pressures of both luminous bodies and all values of mass ratios.

An investigation of the positions of libration points, when the more massive primary is a source of radiation and the smaller one is an oblate spheroid, was carried out by Sharma (1987). He showed that the triangular points are linearly stable for the mass parameter $0 < \mu < \mu_{Crit}$ and the critical mass value μ_{Crit} decreases with the increase in oblateness and radiation force.

Markellos *et al.* (1996) computed the non-linear stability zones of the triangular equilibrium points in the case of oblate bigger primary in the restricted three body problem. It was found that oblateness reduces the area of the non-linear stability zones of the triangular equilibrium points.

The effect of oblateness and radiation pressure forces of the primaries on the locations and the linear stability of the triangular points in the restricted three-body problem was analyzed by Singh and Ishwar (1999). They considered both primaries as sources of radiation as well as oblate spheroids, and observed that these points are stable for $0 < \mu < \mu_{CO}$ and unstable for $\mu_{CO} < \mu < \frac{1}{2}$, where μ_{CO} is the critical value of the mass parameter and depends on the radiating and oblateness coefficients. Sharma et al. (2001a) examined the existence and stability of libration points in the restricted three body problem when the bigger primary is a triaxial rigid body and source of radiation with one of the axes as the axis of symmetry and its equatorial plane coinciding with the plane of motion. They found five libration points, two triangular and three collinear. The collinear points are unstable, while the triangular points are stable for the mass parameter $0 < \mu < \mu_{Crit}$ and that they have long or short periodic elliptical orbits in the same range of the mass parameter. In the same problem, when the primaries are triaxial rigid bodies and source of radiations, Sharma et al. (2001b) arrived at similar conclusions.

The stability of equilibrium points under the influence of small perturbations in the Coriolis and centrifugal forces, together with the effects of oblateness and radiation of both primaries was investigated by AbdulRaheem and Singh (2006). It was found that the Coriolis force has a stabilizing tendency, while the centrifugal force, radiation, and oblateness of the primaries have destabilizing effects; the presence of any one or more of the latter makes weak the stabilizing ability of the former. The overall effect is that the range of stability of the triangular points decreases. Singh (2009) studied the nonlinear stability of the triangular libration point L_4, when both of the primaries are oblate spheroids as well as sources of radiation. He found that L_4 is stable for all mass ratios in the range of linear stability except for three mass ratios depending upon oblateness coefficients and mass reduction factors.

Many extra solar planetary systems have been discovered, and most of them exhibit interesting features. Some of the discovered extrasolar planetary systems are claimed to have disks of dust, and these disks are regarded as the young analogues of the Kuiper belt (Aumann, et al.1984). For instance, Greaves *et al.* (1998) found a dust ring around a nearby star, e Eridani. Trilling *et al.* (2007) detected debris disks in many main-sequence stellar binary systems using the *Spitzer Space Telescope.* Out of an observed 69 A3-F8 main sequence binary star systems, nearly 60% showed dust disks surrounding binary stars. Jiang and Yeh (2003) studied the effect of disks on planetary orbits and concluded that the planets might prefer to stay near the inner part rather than the outer part of the disk. They in their later paper (2004*a*) modified the restricted three-body problem by considering the effects of additional gravitational forces from the disk on the infinitesimal mass. The influence from the disk makes the structure of the dynamical system quite different, such that new equilibrium points exist under certain condition. In their further paper (2004*b*), they focused on the chaotic orbits of disk-star-planet systems for the modified restricted three-body problem. They found that chaotic boundary does not depend much on the disk mass for type I initial conditions, but can change a lot for different disk masses for type II initial conditions. They also found that the influence from the disk can change the locations of equilibrium points and the orbital behaviors for both types of initial conditions.

Jiang & Yeh (2006) and Yeh & Jiang (2006) examined the conditions for the existence of equilibrium points in the Chermnykh-like problems for different values of mass parameter μ. They have included the potential of the belt, and found three collinear points, two triangular points, and two other new equilibrium points.

Kushvah (2008) investigated the linear stability of equilibrium points in the generalized photogravitational Chermnykh's problem**.** The bigger primary was considered as a source of radiation and small primary as an oblate spheroid. He studied analytically and numerically the effect of radiation pressure, oblateness, and gravitational potential from the belt. The stability of the equilibrium points was also examined using Lypunov's method, and it was concluded that the collinear equilibrium points are linearly unstable while the triangular points are conditionally stable. In his recent paper (2011), he outlined in the same problem, a design of the trajectory and analysis of the stability of collinear point L_2 in the Sun- Earth system. It was found that, the point L_2 was asymptotically stable up to a specific value of time t corresponding to each set of values of parameters and initial conditions.

CHAPTER 3

EQUATIONS OF MOTION

3.0 INTRODUCTION

In this chapter, we shall derive the equations of motion of the infinitesimal body in the gravitational field of radiating oblate primaries (peanut binary stars) together with the influence of gravitational potential from a belt in the restricted three-body problem. We begin with mathematical formulations of the problem, derivation of the equations of motion of the infinitesimal body, and then state the Jacobian integral.

3.1 MATHEMATICAL FORMULATIONS OF THE PROBLEM

Let us consider an infinitesimal body (dust particle) of mass m moving under the gravitational influence of the bigger and smaller primaries of masses m_1 and m_2, respectively. Let us take a coordinate system *oxyz* with origin at the centre of mass of the primaries and the x-axis is the line joining the primaries; while y-axis is perpendicular to it, the z-axis is perpendicular to the orbital plane of the primaries. The distances between m and the primaries m_1 and m_2 are $r_1$and r_2 respectively, and the distance between the primaries is R. The coordinates of m_1, m_2 and m are $(x_1, 0)$, $(x_2, 0)$ and (x, y) respectively as shown in figure (3.1).

Let the barycentric coordinates system (*0xyz*) be rotating about z-axis with constant angular velocity n. Then, the kinetic energy of the infinitesimal mass m is given by

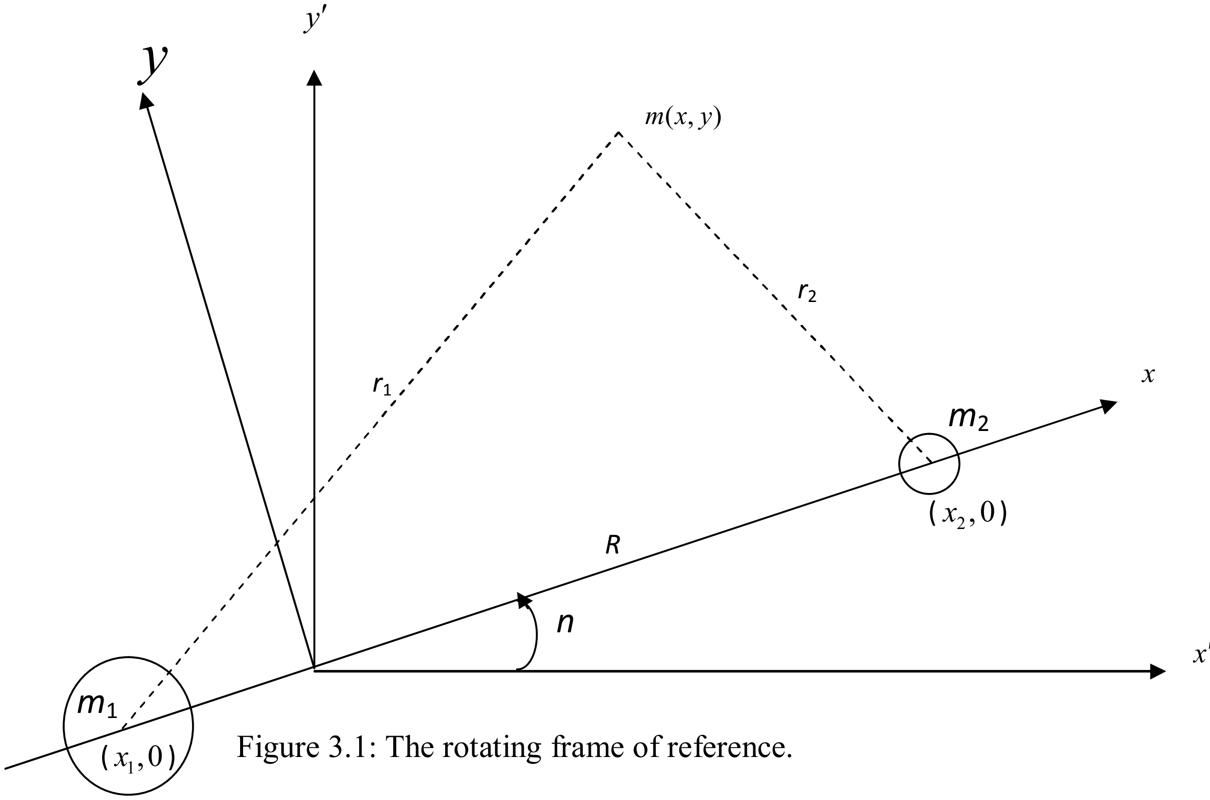

Figure 3.1: The rotating frame of reference.

$$K_e = \frac{1}{2} m \left\{ n^2 \left(x^2 + y^2 \right) + 2n \left(x\dot{y} - \dot{x}y \right) + \left(\dot{x}^2 + \dot{y}^2 \right) \right\} \tag{3.1}$$

$$= T_0 + T_1 + T_2$$

where

$$T_0 = \frac{1}{2} m n^2 \left(x^2 + y^2 \right)$$

$$T_1 = mn \left(x\dot{y} - \dot{x}y \right)$$

$$T_2 = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right)$$

and its potential energy due to the primaries and the belt is given by

$$V = -Gm\left\{ m_1\left(\frac{q_1}{r_1} + \frac{A_1 q_1}{2r_1^3} \right) + m_2\left(\frac{q_2}{r_2} + \frac{A_2 q_2}{2r_2^3} \right) + \frac{M_b}{\left(r^2 + T^2\right)^{1/2}} \right\} \tag{3.2}$$

where,

$$r_1^2 = \left(x - x_1\right)^2 + y^2, \quad r_2^2 = \left(x - x_2\right)^2 + y^2 \tag{3.3}$$

G is the gravitational constant; dots denote differentiations with respect to time t; q_1, A_1 and q_2, A_2 are the radiation and oblatness factors of the bigger and smaller primaries, respectively.

$$\frac{M_b}{\left(r^2 + T^2\right)^{1/2}} \tag{3.4}$$

is the potential due to the belt (Miyamoto and Nagai ,1975) , where

M_b is the total mass of the belt, r is the radial distance of the infinitesimal mass and is given by $r^2 = x^2 + y^2$, $T = a + b$, a and b are parameters which determine the density profile of the belt. The parameter a controls the flatness of the profile and can be called *flatness parameter*. The parameter b controls the size of the core of the density profile and can be called *core parameter*. When $a = b = 0$, the potential equals to the one by a point mass.

Now, let P_x and P_y be the momenta, conjugate to x and y, respectively defined as

$$P_x = \frac{\partial K_e}{\partial x} = m\left(\dot{x} - ny\right)$$
$$P_y = \frac{\partial K_e}{\partial y} = m\left(\dot{y} + nx\right) \tag{3.5}$$

Which imply

$$\dot{x} = \frac{P_x}{m} + ny , \qquad \dot{y} = \frac{P_y}{m} - nx \tag{3.6}$$

The Hamiltonian H is given by

$$H = T_2 - T_0 + V$$

$$H = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}mn^2\left(x^2 + y^2\right) + V \tag{3.7}$$

Using equation (3.6) in (3.7) we have

$$H = \frac{1}{2}m\left(\left(\frac{P_x}{m} + ny\right)^2 + \left(\frac{P_y}{m} - nx\right)^2\right) - \frac{1}{2}mn^2\left(x^2 + y^2\right) + V$$

$$= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + n\left(p_x y - p_y x\right) + V \tag{3.8}$$

Now Hamilton's canonical equations of motion are given by

$$\dot{x} = \frac{\partial H}{\partial p_x}, \qquad \dot{y} = \frac{\partial H}{\partial p_y}$$

and

$$\dot{p}_x = -\frac{\partial H}{\partial x}, \quad \dot{p}_y = -\frac{\partial H}{\partial y}$$

$$\therefore\ \dot{p}_x = -\left(-np_y + \frac{\partial V}{\partial x}\right) = np_y - \frac{\partial V}{\partial x},$$

$$\dot{p}_x = -\left(np_x + \frac{\partial V}{\partial y}\right) = -np_x - \frac{\partial V}{\partial y} \tag{3.9}$$

Thus, equations (3.6) and (3.9) give four first order ordinary differential equations which govern the motion of the infinitesimal mass.

Next, differentiating (3.5) with respect to time t, we have

$$\begin{aligned} \dot{p}_x &= m(\ddot{x} - n\dot{y}) \\ \dot{p}_y &= m(\ddot{y} + n\dot{x}) \end{aligned} \tag{3.10}$$

Equating equations (3.10) and (3.9), we obtain

$$\begin{aligned} m(\ddot{x} - n\dot{y}) &= np_y - \frac{\partial V}{\partial x} \\ m(\ddot{y} + n\dot{x}) &= -np_x - \frac{\partial V}{\partial y} \end{aligned} \tag{3.11}$$

In view of equations (3.5), equations (3.11) become

$$m(\ddot{x}-n\dot{y})=nm(\dot{y}+nx)-\frac{\partial V}{\partial x}$$
$$m(\ddot{y}+n\dot{x})=-nm(\dot{x}-ny)-\frac{\partial V}{\partial y} \tag{3.12}$$

Dividing through by m, we get

$$\ddot{x}-n\dot{y}=n(\dot{y}+nx)-\frac{\partial V}{m\partial x}$$
$$\ddot{y}+n\dot{x}=-n(\dot{x}-ny)-\frac{\partial V}{m\partial y}$$

or

$$\begin{aligned}\ddot{x}-2n\dot{y}&=n^2x-\frac{\partial V}{m\partial x}\\ \ddot{y}+2n\dot{x}&=n^2y-\frac{\partial V}{m\partial y}\end{aligned} \tag{3.13}$$

These are the equations that govern the motion of the infinitesimal mass.

We define a modified potential function as

$$U=V-T_0$$

By the use of (3.1) it gives

$$U=V-\frac{1}{2}mn^2\left(x^2+y^2\right)$$

Then

$$\frac{\partial U}{\partial x}=\frac{\partial V}{\partial x}-mn^2x$$
$$\frac{\partial U}{\partial y}=\frac{\partial V}{\partial y}-mn^2y$$

On multiplying these by $-\frac{1}{m}$, we get

$$-\frac{\partial U}{m\partial x}=-\frac{1}{m}\frac{\partial V}{\partial x}+n^2x$$
$$-\frac{\partial U}{m\partial y}=-\frac{1}{m}\frac{\partial V}{\partial y}+n^2y \tag{3.14}$$

From equations (3.13) and (3.14), we find

$$\ddot{x}-2n\dot{y}=-\frac{1}{m}\frac{\partial U}{\partial x}$$
$$\ddot{y}+2n\dot{x}=-\frac{1}{m}\frac{\partial U}{\partial y} \tag{3.15}$$

where $U=V-\frac{1}{2}mn^2\left(x^2+y^2\right)$

Now in view of equation (3.2), we have

$$\frac{\partial U}{\partial x}=\frac{\partial V}{\partial x}-\frac{\partial}{\partial x}\left(\frac{1}{2}mn^2\left(x^2+y^2\right)\right) \tag{3.16}$$

$$=-Gm\left\{m_1\left(q_1\frac{\partial}{\partial x}(\frac{1}{r_1})+\frac{A_1q_1}{2}\frac{\partial}{\partial x}(\frac{1}{r_1^3})\right)+m_2\left(\frac{q_2\partial}{\partial x}\left(\frac{1}{r_2}\right)+\frac{A_2q_2}{2}\frac{\partial}{\partial x}\left(\frac{1}{r_2^3}\right)\right)+\frac{\partial}{\partial x}\left(\frac{M_b}{\left(r^2+T^2\right)^{1/2}}\right)\right\}-mn^2x$$

From equation (3.3) i.e.

$$r_1^2=\left(x-x_1\right)^2+y^2,\qquad r_2^2=\left(x-x_2\right)^2+y^2$$

we have

$$2r_1\frac{\partial r_1}{\partial x}=2\left(x-x_1\right) \qquad\qquad 2r_2\frac{\partial r_2}{\partial x}=2\left(x-x_2\right)$$

$$\Rightarrow\frac{\partial r_1}{\partial x}=\frac{\left(x-x_1\right)}{r_1} \qquad\qquad \Rightarrow\frac{\partial r_2}{\partial x}=\frac{\left(x-x_2\right)}{r_2}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right)=-\frac{1}{r_1^2}\frac{\partial r_1}{\partial x} \qquad\qquad \frac{\partial}{\partial x}\left(\frac{1}{r_2}\right)=-\frac{1}{r_2^2}\frac{\partial r_2}{\partial x}$$

$$\Rightarrow\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right)=-\frac{\left(x-x_1\right)}{r_1^3} \qquad\qquad \Rightarrow\frac{\partial}{\partial x}\left(\frac{1}{r_2}\right)=-\frac{\left(x-x_2\right)}{r_2^3}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_1^3}\right)=-\frac{3}{r_1^4}\frac{\partial r_1}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{1}{r_1^3}\right)=-\frac{3\left(x-x_1\right)}{r_1^5}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_2^3}\right)=-\frac{3}{r_2^4}\frac{\partial r_2}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{1}{r_2^3}\right)=-\frac{3\left(x-x_2\right)}{r_2^5}$$

In view of equation (3.4), we have

$$\frac{\partial}{\partial x}\left(\frac{M_b}{\left(r^2+T^2\right)^{1/2}}\right)=-\frac{M_b x}{\left(r^2+T^2\right)^{3/2}}$$

$$\frac{\partial}{\partial y}\left(\frac{M_b}{\left(r^2+T^2\right)^{1/2}}\right)=-\frac{M_b y}{\left(r^2+T^2\right)^{3/2}}$$

Substituting these equations in equation (3.16), we get:

$$\frac{\partial U}{\partial x}=-Gm\left\{m_1\left(-\frac{q_1\left(x-x_1\right)}{r_1^3}-\frac{A_1q_1}{2}\frac{\left(x-x_1\right)}{r_1^5}\right)+m_2\left(-\frac{q_2\left(x-x_2\right)}{r_2^3}-\frac{A_2q_2\left(x-x_2\right)}{2r_2^5}\right)-\frac{M_b x}{\left(r^2+T^2\right)^{3/2}}\right\}-mn^2x$$

$$=Gm\left\{m_1\left(\frac{q_1\left(x-x_1\right)}{r_1^3}+\frac{3A_1q_1}{2}\frac{\left(x-x_1\right)}{r_1^5}\right)+m_2\left(\frac{q_2\left(x-x_2\right)}{r_2^3}+\frac{3A_2q_2\left(x-x_2\right)}{2r_2^5}\right)+\frac{M_b x}{\left(r^2+T^2\right)^{3/2}}\right\}-mn^2x \tag{3.17}$$

Similarly,

$$\frac{\partial U}{\partial y}=Gm\left\{m_1\left(\frac{q_1 y}{r_1^3}+\frac{3A_1q_1 y}{2r_1^5}\right)+m_2\left(\frac{q_2 y}{r_2^3}+\frac{3A_2q_2 y}{2r_2^5}\right)+\frac{M_b y}{\left(r^2+T^2\right)^{3/2}}\right\}-mn^2y \tag{3.18}$$

Using equations (3.17) and (3.18) in equation (3.15), we obtain

$$\ddot{x}-2n\dot{y}=-G\left\{m_1\left(\frac{q_1\left(x-x_1\right)}{r_1^3}+\frac{3A_1q_1}{2}\frac{\left(x-x_1\right)}{r_1^5}\right)+m_2\left(\frac{q_2\left(x-x_2\right)}{r_2^3}+\frac{3A_2q_2\left(x-x_2\right)}{2r_2^5}\right)+\frac{M_b x}{\left(r^2+T^2\right)^{3/2}}\right\}+n^2x$$

$$\ddot{y}+2n\dot{x}=-G\left\{m_1\left(\frac{q_1 y}{r_1^3}+\frac{3A_1q_1 y}{2r_1^5}\right)+m_2\left(\frac{q_2 y}{r_2^3}+\frac{3A_2q_2 y}{2r_2^5}\right)+\frac{M_b y}{\left(r^2+T^2\right)^{3/2}}\right\}+n^2y \tag{3.19}$$

3.2 NON-DIMENSIONAL UNITS OF MEASUREMENT

To simplify and generalize the equations of (3.19), it is useful to non-dimensionalize the variables in such a way that the properties of the system depend on a single parameter:

- The sum of the masses of the primaries is taken as the unit of mass. For this, the mass of the smaller primary is denoted by μ, whence the mass of the bigger primary $1-\mu$, where μ is the ratio of the mass of the smaller primary m_2 to the total mass of the primaries and $0 < \mu \leq \frac{1}{2}$.
- For the unit of length, the distance between the primaries is taken as one.
- The unit of time is so chosen that the gravitational constant G is unity i.e., $G=1$.

Now since the origin is at the barycentre of m_1 and m_2, we have

$$m_1 x_1 + m_2 x_2 = 0$$

That is

$$\begin{aligned}
&(1-\mu)x_1 + \mu x_2 = 0 \\
&x_1 - \mu x_1 + \mu x_2 = 0 \\
&x_1 + \mu\left(-x_1 + x_2\right) = 0 \\
&x_1 + \mu\left(1\right) = 0 \\
&x_1 = -\mu
\end{aligned}$$

and

$$x_2 = 1 - \mu$$

Hence, the coordinates of the bigger primary *and* smaller one are $\left(-\mu, 0\right)$ and $\left(1-\mu, 0\right)$.

3.3 EQUATIONS OF MOTION IN THE NON-DIMENSIONAL UNITS

Using $m_1 = 1-\mu, m_2 = \mu, \; x_1 = -\mu, \; x_2 = 1-\mu$ and G=1 in equation (3.19), the equations become:

$$\ddot{x} - 2n\dot{y} = n^2 x - \left\{ (1-\mu)\left(\frac{q_1(x+\mu)}{r_1^3} + \frac{3A_1 q_1 (x+\mu)}{2r_1^5} \right) + \mu \left(\frac{q_2(x+\mu-1)}{r_2^3} + \frac{3A_2 q_2 (x+\mu-1)}{2r_2^5} \right) + \frac{M_b x}{\left(r^2+T^2\right)^{3/2}} \right\}$$

$$\ddot{y} + 2n\dot{x} = n^2 y - \left\{ (1-\mu)\left(\frac{q_1 y}{r_1^3} + \frac{3A_1 q_1 y}{2r_1^5} \right) + \mu \left(\frac{q_2 y}{r_2^3} + \frac{3A_2 q_2 y}{2r_2^5} \right) + \frac{M_b y}{\left(r^2+T^2\right)^{3/2}} \right\}$$

or

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x \\ \ddot{y} + 2n\dot{x} &= \Omega_y \end{aligned} \qquad (3.20)$$

where

$$\Omega = \frac{n^2(x^2+y^2)}{2} + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)A_1 q_1}{2r_1^3} + \frac{\mu A_2 q_2}{2r_2^3} + \frac{M_b}{(r^2+T^2)^{1/2}}$$

$$\Omega_x = n^2 x - \frac{(1-\mu)(x+\mu)q_1}{r_1^3} - \frac{\mu(x+\mu-1)q_2}{r_2^3} - \frac{3(1-\mu)(x+\mu)A_1 q_1}{2r_1^5} - \frac{3\mu(x+\mu-1)A_2 q_2}{2r_2^5} - \frac{M_b x}{\left(r^2+T^2\right)^{3/2}}$$

$$\Omega_y = n^2 y - \frac{(1-\mu)q_1 y}{r_1^3} - \frac{\mu q_2 y}{r_2^3} - \frac{3}{2}\frac{(1-\mu)A_1 q_1 y}{r_1^5} - \frac{3}{2}\frac{\mu A_2 q_2 y}{r_2^5} - \frac{M_b y}{(r^2+T^2)^{3/2}}$$

with

$$r_1^2 = (x+\mu)^2 + y^2, \qquad r_2^2 = (x+\mu-1)^2 + y^2 \qquad (3.21)$$

and the mean motion n for the system is given by

$$n^2 = 1 + \frac{3}{2}(A_1 + A_2) + \frac{2M_b r_c}{(r_c^2+T^2)^{3/2}}, \qquad (3.22)$$

where r_c is the radial distance of the infinitesimal mass in the classical restricted three - body problem.

3.4. THE JACOBIAN INTEGRAL

Multiplying the first and second equations of (3.20) by $\dot{x}$ and $\dot{y}$ respectively, we get

$$\dot{x}\ddot{x} - 2n\dot{x}\dot{y} = \dot{x}\Omega_x$$
$$\dot{y}\ddot{y} + 2n\dot{y}\dot{x} = \dot{y}\Omega_y$$

Adding them together, we get

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = \dot{x}\Omega_x + \dot{y}\Omega_y$$
$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{d}{dt}\Omega\left(x, y\right)$$
$$\Rightarrow \frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2\right) = 2\frac{d}{dt}\Omega\left(x, y\right) \qquad (3.23)$$

Integrating equation (3.23) gives

$$\dot{x}^2 + \dot{y}^2 = 2\Omega\left(x, y\right) - C \qquad (3.24)$$

where C is the constant of integration. Since we know that $\dot{x}^2 + \dot{y}^2 = v^2$, the square of the velocity in the system is

$$v^2 = 2\Omega\left(x, y\right) - C$$

and from that we can write C as

$$2\Omega\left(x, y\right) - v^2 = C$$

This means that the quantity $2\Omega\left(x, y\right) - v^2$ is a constant of the problem and therefore, it is called the Jacobian constant. Equation (3.24) is known as the Jacobian integral.

3.5 DISCUSSION

We have obtained the equations of motion of the generalized restricted three-body problem. Equation (3.20) gives the equations and they are different from those obtained by Kushvah (2008) due to the presence of radiation of the smaller primary and oblateness of the bigger primary. However, if these are ignored (i.e. $q_2 = 1, A_1 = 0$) ,the equations will tally with those obtained by him. If the potential from the belt is neglected (i.e. $M_b = 0$), the equations of (3.20) are analogous to those of Singh and Ishwar (1999) , and Abdulraheem and Singh (2006) in the absence of perturbations in the coriolis and centrifugal forces. If the primaries are spherical and the potential from the belt is absent (i.e. $A_1 = A_2 = M_b = 0$), the equations of (3.20) fully coincide with those of Zheng and Yu (1993). In the presence of the oblateness of the bigger primary only (i.e. $q_1 = q_2 = 1, A_2 = M_b = 0$), the equations of (3.20) agree with those of Kalantonis *et al.* (2006). In the presence of the potential from the belt only (i.e. $q_1 = q_2 = 1, A_1 = A_2 = 0$), the equations of (3.20) are in agreement with those of Jiang and Yeh (2006) and Yeh and Jiang (2006). In the presence of radiation of the bigger primary only (i.e. $q_2 = 1, A_1 = A_2 = M_b = 0$), the equations of (3.20) tally with those of Bhatnagar and Chawla (1979). If the primaries are spherical and non-radiating and in the absence of the potential from the belt (i.e. $q_1 = q_2 = 1, A_1 = A_2 = M_b = 0$), the equations of (3.20) reduce to those of classical case of Szebehely (1967).

3.6 CONCLUSION

We have obtained the equations of motion (3.20) and Jacobian integral (3.24) for the infinitesimal body in the rotating coordinate system. We observed that the equations of motion (3.20) are affected by the radiation and oblateness factors and the potential from the belt.

CHAPTER 4

LOCATIONS OF EQUILIBRIUM POINTS

4.0 INTRODUCTION

The equilibrium points are places where the infinitesimal body can sit “motionless" relative to the primaries that are orbiting each other. Hence at the equilibrium positions, all forces acting on the infinitesimal mass in the rotating frame of reference vanish, that is $\ddot{x} = \ddot{y} = \dot{x} = \dot{y} = 0.$

Thus, we have from equations (3.20) :

$$\Omega_x = 0, \quad \Omega_y = 0$$

or

$$n^2 x - \frac{(1-\mu)(x+\mu)q_1}{r_1^3} - \frac{\mu(x+\mu-1)q_2}{r_2^3} - \frac{3(1-\mu)(x+\mu)A_1 q_1}{2r_1^5} - \frac{3\mu(x+\mu-1)A_2 q_2}{2r_2^5} - \frac{M_b x}{\left(r^2+T^2\right)^{3/2}} = 0$$

$$n^2 y - \frac{(1-\mu)q_1 y}{r_1^3} - \frac{\mu q_2 y}{r_2^3} - \frac{3}{2}\frac{(1-\mu)A_1 q_1 y}{r_1^5} - \frac{3}{2}\frac{\mu A_2 q_2 y}{r_2^5} - \frac{M_b y}{(r^2+T^2)^{3/2}} = 0$$

Re-writing these equations, we have

$$x\left(n^2 - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu q_2}{r_2^3} - \frac{3}{2}\frac{(1-\mu)A_1 q_1}{r_1^5} - \frac{3}{2}\frac{\mu A_2 q_2}{r_2^5} - \frac{M_b}{(r^2+T^2)^{3/2}}\right)$$

$$-\frac{(1-\mu)\mu q_1}{r_1^3} - \frac{\mu(\mu-1)q_2}{r_2^3} - \frac{3(1-\mu)\mu A_1 q_1}{2r_1^5} - \frac{3\mu(\mu-1)A_2 q_2}{2r_2^5} = 0 \tag{4.1}$$

$$y\left(n^2 - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu q_2}{r_2^3} - \frac{3}{2}\frac{(1-\mu)A_1 q_1}{r_1^5} - \frac{3}{2}\frac{\mu A_2 q_2}{r_2^5} - \frac{M_b}{(r^2+T^2)^{3/2}}\right) = 0 \tag{4.2}$$

Here two cases arise: $y \neq 0$ gives triangular points and $y = 0$ collinear points.

4.1 LOCATIONS OF THE TRIANGULAR POINTS

The triangular points are the solutions of equations (4.1) and (4.2) when $y \neq 0$.

Now from (4.2) we have

$$n^2 - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu q_2}{r_2^3} - \frac{3}{2}\frac{(1-\mu)A_1 q_1}{r_1^5} - \frac{3}{2}\frac{\mu A_2 q_2}{r_2^5} - \frac{M_b}{(r^2+T^2)^{3/2}} = 0 \tag{4.3}$$

Using (4.3) in (4.1) gives

$$-\frac{(1-\mu)\mu q_1}{r_1^3} - \frac{\mu(\mu-1)q_2}{r_2^3} - \frac{3(1-\mu)\mu A_1 q_1}{2r_1^5} - \frac{3\mu(\mu-1)A_2 q_2}{2r_2^5} = 0$$

$$-(1-\mu)\mu\left(\frac{q_1}{r_1^3} - \frac{q_2}{r_2^3} + \frac{3A_1 q_1}{2r_1^5} - \frac{3A_2 q_2}{2r_2^5}\right) = 0$$

$$\Rightarrow \frac{q_1}{r_1^3} - \frac{q_2}{r_2^3} + \frac{3A_1 q_1}{2r_1^5} - \frac{3A_2 q_2}{2r_2^5} = 0 \tag{4.4}$$

From (4.3) we have

$$n^2 - \frac{q_1}{r_1^3} + \frac{\mu q_1}{r_1^3} - \frac{\mu q_2}{r_2^3} - \frac{3A_1 q_1}{2r_1^5} + \frac{3\mu A_1 q_1}{2r_1^5} - \frac{3}{2}\frac{\mu A_2 q_2}{r_2^5} - \frac{M_b}{(r^2+T^2)^{3/2}} = 0$$

or

$$n^2 - \frac{q_1}{r_1^3} - \frac{3A_1 q_1}{2r_1^5} + \mu\left(\frac{q_1}{r_1^3} - \frac{q_2}{r_2^3} + \frac{3A_1 q_1}{2r_1^5} - \frac{3A_2 q_2}{2r_2^5}\right) - \frac{M_b}{(r^2+T^2)^{3/2}} = 0 \tag{4.5}$$

Using (4.4) in (4.5) we get

$$n^2 - \frac{q_1}{r_1^3} - \frac{3A_1 q_1}{2r_1^5} - \frac{M_b}{(r^2+T^2)^{3/2}} = 0 \tag{4.6}$$

Combining (4.4) and (4.6) we have

$$n^2 - \frac{q_2}{r_2^3} - \frac{3A_2 q_2}{2r_2^5} - \frac{M_b}{(r^2+T^2)^{3/2}} = 0 \tag{4.7}$$

When the effects of radiation and oblateness of the primaries and the potential from the belt are neglected, equations (4.6) and (4.7) are reduced to classical case with the solutions $r_1=r_2=1$. Thus we can assume the solutions of the equations (4.6) and (4.7) to be

$$r_1 = 1+\varepsilon_1 \qquad r_2 = 1+\varepsilon_2 \tag{4.8}$$

where ε_1 , ε_2 are very small and are due to the combined effects of the radiation, oblateness and potential from the belt.

The exact coordinates of the triangular points are found by solving equations of (3.21) for x and y, i.e.

$$r_1^2 = (x+\mu)^2 + y^2, \quad r_2^2 = (x+\mu-1)^2 + y^2$$

$$(1+\varepsilon_1)^2 = (x+\mu)^2 + y^2 \tag{4.9}$$

$$(1+\varepsilon_2)^2 = (x+\mu-1)^2 + y^2 \tag{4.10}$$

Subtracting equation (4.10) from (4.9), we obtain

$$(1+\varepsilon_1)^2 - (1+\varepsilon_2)^2 = (x+\mu)^2 - (x+\mu-1)^2$$

Considering only linear terms in ε_i*'s*, we get

$$x = \frac{1}{2} - \mu + \varepsilon_1 - \varepsilon_2 \tag{4.11}$$

Using equation (4.11) in equation (4.9), we have

$$\begin{aligned} y^2 &= (1+\varepsilon_1)^2 - \left(\frac{1}{2} + \varepsilon_1 - \varepsilon_2\right)^2 \\ &= 1 + 2\varepsilon_1 - \left(\frac{1}{4} + \varepsilon_1 - \varepsilon_2\right) \\ &= \frac{3}{4} + \varepsilon_1 + \varepsilon_2 \\ &= \frac{3}{4}\left(1 + \frac{4}{3}(\varepsilon_1 + \varepsilon_2)\right) \\ &= \frac{3}{4}\left(1 + \frac{2}{3}(\varepsilon_1 + \varepsilon_2)\right)^2 \end{aligned}$$

$$\therefore y = \pm\sqrt{\frac{3}{4}\left(1+\frac{2}{3}(\varepsilon_1+\varepsilon_2)\right)^2}$$

$$y = \pm\sqrt{3}\left(\frac{1}{2}+\frac{1}{3}(\varepsilon_1+\varepsilon_2)\right) \tag{4.12}$$

Also $r^2 = x^2 + y^2$

$$r^2 = \left(\frac{1}{2}-\mu+\varepsilon_1-\varepsilon_2\right)^2+\frac{3}{4}+\varepsilon_1+\varepsilon_2$$

This simplifies to

$$r^2 = r_c^{\,2} + 2(1-\mu)\varepsilon_1 + 2\mu\varepsilon_1 \tag{4.13}$$

where $r_c^{\,2} = 1-\mu+\mu^2$

Substituting equations (4.8), (4.11), (4.12) and (4.13) in equation (4.6), using $q_i = 1-p_i$, $i = 1,2$ where $p_i = 1-q_i \ll 1$ and restricting ourselves to the linear terms in $\varepsilon_i, A_i, p_i, M_b$ we obtain

$$\varepsilon_1 = -\frac{p_1}{3}-\frac{A_2}{2}-\frac{M_b(2r_c-1)}{3(r_c^2+T^2)^{3/2}} \tag{4.14}$$

Similarly, from equation (4.7) we get

$$\varepsilon_2 = -\frac{p_2}{3}-\frac{A_1}{2}-\frac{M_b(2r_c-1)}{3(r_c^2+T^2)^{3/2}} \tag{4.15}$$

So, from (4.8) we have

$$r_1 = 1-\left(\frac{p_1}{3}+\frac{A_2}{2}+\frac{M_b(2r_c-1)}{3(r_c^2+T^2)^{3/2}}\right)$$

$$r_2 = 1-\left(\frac{p_2}{3}+\frac{A_1}{2}+\frac{M_b(2r_c-1)}{3(r_c^2+T^2)^{3/2}}\right) \tag{4.16}$$

Using (4.14) and (4.15) in equations (4.11) and (4.12), we obtain the triangular equilibrium points L_4 and L_5 as:

$$x = \frac{1}{2} - \mu - \left(\frac{p_1 - p_2}{3} - \frac{A_1 - A_2}{2} \right)$$

$$y = \pm\sqrt{3}\left(\frac{1}{2} - \frac{p_1 + p_2}{9} + \frac{A_1 + A_2}{6} + \frac{2M_b(2r_c - 1)}{9(r_c^2 + T^2)^{3/2}} \right) \tag{4.17}$$

In the absence of potential from the belt, these are same as those of Singh and Ishwar (1999) , Abdulraheem and Singh(2006) when perturbations are absent, and Kushvah(2008) while radiation of the smaller primary and obleteness of the bigger primary are neglected.

4.1.1 Numerical Investigation of Triangular Points

For numerical investigations, we take the masses of the bigger and smaller primaries (peanuts) as $20M_{\odot}$ and $16M_{\odot}$ respectively. So, the mass parameter μ for this system is

$$\mu = \frac{16}{16 + 20} = 0.444444$$

For simplicity we set T=0.01.

In order to show the effects of various parameters on the position of triangular points $L_{4(5)}$ we have used equation (4.17) in the following cases:

1 Absence of radiation, oblateness and potential from the belt (classical case).

2 Oblateness of the bigger primary only.

3 Oblateness of the smaller primary only

4 Radiation of the bigger primary only

5 Radiation of the smaller primary only,

6 Potential from the belt only.

7 Radiation and Oblateness of the primaries and potential from the belt.

Table 4.1: Coordinates of triangular points $L_{4,5}$ (μ=0.444444 and T=0.01)

Case	q_1	q_2	A_1	A_2	M_b	$L_{4,5}$ (x, y)
1	1	1	0	0	0	0.0555560, ± 0.8660254
2	1	1	0.03	0	0	0.0705560, ± 0.8746857
3	1	1	0	0.02	0	0.0455560, ± 0.8717989
4	0.75	1	0	0	0	-0.0277773, ± 0.6495190
5	1	0.85	0	0	0	0.1055560, ± 0.7361216
6	1	1	0	0	0.01	0.0555560, ± 0.8703569
7	0.75	0.85	0.03	0.02	0.01	0.0272227, ± 0.5383805

We see that the positions of the triangular points are affected by the oblateness and radiation of the primaries and the potential from the belt such that they do not form equilateral triangles with the primaries. We also observe that the triangular points come nearer to the primaries in the presence of all these perturbations (Table (4.1) case 7).

4.2 LOCATIONS OF COLLINEAR POINTS

The collinear points are the solutions of the equations (4.1) and (4.2) when y = 0. That is, the collinear points lie on the line joining the primaries. We denote expression on the left hand side of equation (4.1) by $F(x, y)$ i.e.

$$F(x,y) = n^2 x - \frac{(1-\mu)q_1(x+\mu)}{r_1^3} - \frac{\mu q_2(x+\mu-1)}{r_2^3} - \frac{3}{2}\frac{(1-\mu)A_1 q_1(x+\mu)}{r_1^5} - \frac{3}{2}\frac{\mu A_2 q_2(x+\mu-1)}{r_2^5} - \frac{M_b x}{(r^2+T^2)^{3/2}}$$

Then $F(x, 0) = f(x) = P(x) + Q(x)$ *(4.18)* where,

$$P(x) = n^2 x - \frac{(1-\mu)q_1(x+\mu)}{|x+\mu|^3} - \frac{\mu q_2(x+\mu-1)}{|x+\mu-1|^3} - \frac{3}{2}\frac{(1-\mu)A_1 q_1(x+\mu)}{|x+\mu|^5} - \frac{3}{2}\frac{\mu A_2 q_2(x+\mu-1)}{|x+\mu-1|^5}$$

$$Q(x) = -\frac{M_b x}{(x^2+T^2)^{3/2}}$$

To investigate the position of collinear equilibrium points, we divide the orbital plane *Oxy* into three parts with respect to the primaries: $x<-\mu$, $-\mu < x < 1-\mu$ and $1-\mu < x$. The function *P(x)* is defined as follows:

$$P(x) = \begin{cases} n^2 x + \dfrac{(1-\mu)q_1}{(x+\mu)^2} + \dfrac{\mu q_2}{(x+\mu-1)^2} + \dfrac{3(1-\mu)A_1 q_1}{2(x+\mu)^4} + \dfrac{3\mu A_2 q_2}{2(x+\mu-1)^4}, if\ x < -\mu \\ n^2 x - \dfrac{(1-\mu)q_1}{(x+\mu)^2} + \dfrac{\mu q_2}{(x+\mu-1)^2} - \dfrac{3(1-\mu)A_1 q_1}{2(x+\mu)^4} + \dfrac{3\mu A_2 q_2}{2(x+\mu-1)^4}, if\ -\mu < x < 1-\mu \\ n^2 x - \dfrac{(1-\mu)q_1}{(x+\mu)^2} - \dfrac{\mu q_2}{(x+\mu-1)^2} - \dfrac{3(1-\mu)A_1 q_1}{2(x+\mu)^4} - \dfrac{3\mu A_2 q_2}{2(x+\mu-1)^4}, if\ 1-\mu < x \end{cases} \quad (4.19)$$

First case:

$x < -\mu$ i.e. $x \in (-\infty, -\mu)$ then,

$$P(x) = n^2 x + \frac{(1-\mu)q_1}{(x+\mu)^2} + \frac{\mu q_2}{(x+\mu-1)^2} + \frac{3(1-\mu)A_1 q_1}{2(x+\mu)^4} + \frac{3\mu A_2 q_2}{2(x+\mu-1)^4}$$

and

$$P'(x) = n^2 - \frac{2(1-\mu)q_1}{(x+\mu)^3} - \frac{2\mu q_2}{(x+\mu-1)^3} - \frac{6(1-\mu)A_1 q_1}{(x+\mu)^5} - \frac{6\mu A_2 q_2}{(x+\mu-1)^5}$$

or

$$p'(x) = n^2 + \frac{2(1-\mu)q_1}{|x+\mu|^3} + \frac{2\mu q_2}{|x+\mu-1|^3} + \frac{6(1-\mu)A_1 q_1}{|x+\mu|^5} + \frac{6\mu A_2 q_2}{|x+\mu-1|^5}$$

$$\Rightarrow P'(x) > 0$$

Since $P'(x) > 0$ for $x \in (-\infty, -\mu)$, $P(x)$ is a monotonically increasing function, because $\lim_{x\to-\infty} P(x) = -\infty$, $\lim_{x\to-\mu^-} P(x) = \infty$.

Also

$$Q(x)=\frac{M_b|x|}{(x^2+T^2)^{3/2}}$$

$Q'(x)=\dfrac{M_b}{(x^2+T^2)^{3/2}}-\dfrac{3M_bx^2}{(x^2+T^2)^{5/2}}<0$ and the $\lim_{x\to-\infty}Q(x)=0$, $\lim_{x\to-\mu^-}Q(x)>0$.

$\Rightarrow Q(x)$ is a monotonically decreasing function.

Now the $\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\{P(x)+Q(x)\}$

$$=\lim_{x\to-\infty}P(x)+\lim_{x\to-\infty}Q(x)$$

$$=-\infty+0$$

$\therefore \lim_{x\to-\infty}f(x)<0$

And $\lim_{x\to-\mu^-}f(x)=\lim_{x\to-\mu^-}\{P(x)+Q(x)\}$

$$=\lim_{x\to-\mu^-}P(x)+\lim_{x\to-\mu^-}Q(x)$$

$$=\infty+positive$$

$\therefore \lim_{x\to-\mu^-}f(x)>0$

$\lim_{x\to-\infty}f(x)<0$ and $\lim_{x\to-\mu^-}f(x)>0$ which imply that there is a unique point $x\in(-\infty,-\mu)$ for which $f(x)=0$. We call this point L_3.

Second case:

$-\mu<x<1-\mu$ i.e. $x\in(-\mu,1-\mu)$, we have from equation (4.19):

$$P(x)=n^2x-\frac{(1-\mu)q_1}{(x+\mu)^2}+\frac{\mu q_2}{(x+\mu-1)^2}-\frac{3(1-\mu)A_1q_1}{2(x+\mu)^4}+\frac{3\mu A_2q_2}{2(x+\mu-1)^4}$$

$$Q(x)=-\frac{M_bx}{(x^2+T^2)^{3/2}}$$

We first consider $x \in (0, 1-\mu)$.

$$P(0) = -\frac{(1-\mu)q_1}{\mu^2} + \frac{\mu q_2}{(\mu-1)^2} - \frac{3(1-\mu)A_1 q_1}{2\mu^4} + \frac{3\mu A_2 q_2}{2(\mu-1)^4}$$

Since $0 < \mu < 0.5$, it is clear that $\frac{(1-\mu)}{\mu^2} > \frac{\mu}{(\mu-1)^2}$ and so $P(0) < 0$. $\lim_{x \to (1-\mu)^-} P(x) = \infty$.

Also

$$Q(0) = 0 \text{ and } Q(1-\mu) < 0.$$

Now, $f(0) = P(0) + Q(0)$

$$= negative + 0$$

$$\therefore f(0) = P(0) + Q(0) < 0$$

and $\lim_{x \to (1-\mu)^-} f(x) = \lim_{x \to (1-\mu)^-} P(x) + Q(1-\mu)$

$$= \infty + negative$$

$$\Rightarrow \lim_{x \to (1-\mu)^-} f(x) > 0$$

$f(0) < 0$ and $\lim_{x \to (1-\mu)^-} f(x) > 0$ show that there is a unique point $x \in (0, 1-\mu)$ such that $f(x) = 0$. We label this point L_2.

Next we examine $x \in (-\mu, 0)$

Let $x \in (-\mu, 0)$ and if $T < \sqrt{2}\mu$ then $-\frac{T}{\sqrt{2}} \in (-\mu, 0)$. So, we consider two sub cases: $x \in \left(-\frac{T}{\sqrt{2}}, 0\right)$ and $x \in \left(-\mu, -\frac{T}{\sqrt{2}}\right)$.

Sub case A: $x \in \left(-\frac{T}{\sqrt{2}}, 0\right)$

Now if $P\left(-\frac{T}{\sqrt{2}}\right)+Q\left(-\frac{T}{\sqrt{2}}\right)>0$, then $f\left(-\frac{T}{\sqrt{2}}\right)=P\left(-\frac{T}{\sqrt{2}}\right)+Q\left(-\frac{T}{\sqrt{2}}\right)>0$ and since $f(0)<0$, it implies that there is a point $x\in\left(-\frac{T}{\sqrt{2}},0\right)$ for which $f(x)=0$. We denote this point by L_{n1}.

Sub case B: $x\in\left(-\mu,-\frac{T}{\sqrt{2}}\right)$

If $P\left(-\frac{T}{\sqrt{2}}\right)+Q\left(-\frac{T}{\sqrt{2}}\right)>0$, then $f\left(-\frac{T}{\sqrt{2}}\right)=P\left(-\frac{T}{\sqrt{2}}\right)+Q\left(-\frac{T}{\sqrt{2}}\right)>0$

Also, $\lim_{x\to-\mu^+} P(x)=-\infty$ and $\lim_{x\to-\mu^+} Q(x)>0$,

$\therefore \lim_{x\to-\mu^+} f(x)=\lim_{x\to-\mu^+}\{P(x)+Q(x)\}<0$

$=\lim_{x\to-\mu^+} P(x)+\lim_{x\to-\mu^+} Q(x)$

$= -\infty$ + Positive number

$\Rightarrow \lim_{x\to-\mu^+} f(x)<0$

$f\left(-\frac{T}{\sqrt{2}}\right)>0$ and $\lim_{x\to-\mu^+} f(x)<0$ imply that there is a unique point $x\in\left(-\mu,-\frac{T}{\sqrt{2}}\right)$ for which $f(x)=0$. We refer this point to L_{n2}

Thus, we have two new equilibrium points L_{n1}, L_{n2} whenever $T < \sqrt{2}\mu$ and $P\left(-\frac{T}{\sqrt{2}}\right)+Q\left(-\frac{T}{\sqrt{2}}\right)>0$ in the interval $(-\mu,0)$. We also notice that there is no equilibrium point in the interval $(-\mu,0)$ if $P\left(-\frac{T}{\sqrt{2}}\right)+Q\left(-\frac{T}{\sqrt{2}}\right)<0$.

Third case:

$x > 1-\mu$ i.e. $x \in (1-\mu,\infty)$, we have

$$P(x) = n^2 x - \frac{(1-\mu)q_1}{(x+\mu)^2} - \frac{\mu q_2}{(x+\mu-1)^2} - \frac{3(1-\mu)A_1 q_1}{2(x+\mu)^4} - \frac{3\mu A_2 q_2}{2(x+\mu-1)^4}$$

$$Q(x) = -\frac{M_b x}{(x^2+T^2)^{3/2}}$$

Now

$$P'(x) = n^2 + \frac{2(1-\mu)q_1}{(x+\mu)^3} + \frac{2\mu q_2}{(x+\mu-1)^3} + \frac{6(1-\mu)A_1 q_1}{(x+\mu)^5} + \frac{6\mu A_2 q_2}{(x+\mu-1)^5}$$

$$\Rightarrow P'(x) > 0$$

$P'(x)>0$ implies that $P(x)$ is a monotonically increasing function for $x \in (1-\mu,\infty)$, since the $\lim_{x\to(1-\mu)^+} P(x) = -\infty$ and $\lim_{x\to\infty} p(x) = \infty$

Also $Q'(x) = -\frac{M_b}{(x^2+T^2)^{3/2}} + \frac{3M_b x^2}{(x^2+T^2)^{5/2}} > 0$, $Q(1-\mu)<0$ and $\lim_{x\to\infty} Q(x) = 0$. So, $Q(x)$ is a monotonically increasing function for $x \in (1-\mu,\infty)$.

Now $\lim_{x\to(1-\mu)^+} f(x) = \lim_{x\to(1-\mu)^+} P(x) + Q(1-\mu)$

$$= -\infty + \text{Negative number}$$

$$\Rightarrow \lim_{x \to (1-\mu)^+} f(x) < 0$$

and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \{P(x) + Q(x)\}$$

$$= \lim_{x \to \infty} P(x) + \lim_{x \to \infty} Q(x)$$

$$= \infty + \text{Zero}$$

$$\Rightarrow \lim_{x \to \infty} f(x) > 0$$

Hence,

$\lim_{x \to (1-\mu)^+} f(x) < 0$ and $\lim_{x \to \infty} f(x) > 0$ show that there is a unique point $x \in (1-\mu, \infty)$ for which $f(x) = 0$. We denote this point by L_1.

4.2.1 Numerical Investigations of Collinear Points

In order to show the effects of various parameters, using equation (4.18) we obtain the coordinates of the collinear equilibrium points (Table 4.2) for different cases:

1 Absence of radiation, oblateness and potential from the belt (classical case).

2 Oblateness of the bigger primary only.

3 Oblateness of the smaller primary only

4 Radiation of the bigger primary only

5 Radiation of the smaller primary only,

6 Potential from the belt only

7 Radiation and oblateness of the primaries and potential from the belt

Table 4.2: x-coordinate of collinear points L_i (i=1,2,3) and L_{n1} L_{n2} when μ=0.44444, T=0.01

Case	q_1	q_2	A_1	A_2	M_b	L_1	L_2	L_3	L_{n1}	L_{n2}
1	1	1	0	0	0	1.21703	0.78507	-1.17858		
2	1	1	0.03	0	0	1.20546	0 .09550	-1.18603		
3	1	1	0	0.02	0	1.22407	0.06482	-1.17024		
4	0.75	1	0	0	0	1.20543	0.04428	-1.10773		
5	1	0.85	0	0	0	1.17986	0.09709	-1.17302		
6	1	1	0	0	0.01	1.21122	0.11881	-1.17265	-0.00014	-0.06017
7	0.75	0.85	0.03	0.02	0.01	1.16587	0.11183	-1.10490	-0.00012	-0.05999

Table 4.2 indicates clearly that the positions of the collinear points are also affected by the oblateness and radiation of the primaries and potential from the belt. It shows that there are two new collinear points L_{n1} and L_{n2} due to the potential from the belt. The overall effect is that the collinear points L_1, L_3 come nearer to the primaries, while L_2 and L_{n2} move towards the bigger primary and L_{n1} moves away from it, in the presence of these perturbations (see case7).

4.3 DISCUSSION

Equation (4.17) gives the coordinates of the triangular equilibrium points. It is evident that these positions are affected by the radiation and oblateness of the primaries and potential from the belt (Table 4.1). If the radiation of the smaller primary and oblateness of the bigger primary are neglected (i.e. $q_2 = 1, A_1 = 0$), the coordinates(4.17) fully agree with those of Kushvah (2008) when only linear terms in small quantities are considered. On ignoring the potential from the belt (i.e. $M_b = 0$), the coordinates(4.17) confirm those of AbdulRaheem and Singh (2006) in the absence perturbations in the Coriolis and centrifugal forces, and coincide with those of Singh and Ishwar (1999). On further ignoring the oblateness of the primaries (i.e. $A_1 = A_2 = M_b = 0$), the coordinates (4.17) agree with those of Zheng and Yu (1993). In the presence of radiation of the bigger primary only (i.e. $q_2 = 1, A_1 = A_2 = M_b = 0$), the equations of (4.17) boil down to those of Bhatnagar and Chawla(1979).

We established in (4.2) that in addition to the usual collinear equilibrium points, there appear two new collinear points L_{n1} L_{n2} due to the potential from the belt. In the presence of the potential from the belt only (i.e. $q_1 = q_2 = 1, A_1 = A_2 = 0$), the findings (4.2) are in concordance with Yeh and Jiang (2006).

4.4 CONCLUSION

We have determined the locations of the triangular and collinear equilibrium points. We have found that:

1 The positions of the equilibrium points are affected by oblateness and radiation of the primaries and potential from the belt (tables 4.1 and 4.2).

2 In addition to the usual collinear equilibrium points, there appear two new collinear points L_{n1} L_{n2} due to the potential from the belt (table 4.2).

CHAPTER 5

STABILITY OF EQUILIBRIUM POINTS

5.0 INTRODUCTION

After determining locations of equilibrium points, it will be interesting to understand the stability properties around these points. In order to arrive at a conclusion about the stability of an equilibrium point, we need to displace the infinitesimal body a little with a small velocity from its equilibrium point under consideration. If the infinitesimal body oscillates considerably around the equilibrium point and returns to the same point as time elapses, the point is said to be *stable*. If however, motion departs considerably from this point and never returns to it, the point is said to be *unstable.*

In order to study the stability of an equilibrium point (x_0, y_0), we derive variational equations of motion corresponding to (3.20). Then, we obtain the characteristic equation and its roots. Finally, we discuss the stability according to the nature of the roots.

5.1 VARIATIONAL EQUATIONS

Let us give small displacement *(α, β)* in the coordinates (x_0, y_0) of the equilibrium point. The coordinates, velocity and acceleration are:

$$x = x_0 + \alpha \qquad y = y_0 + \beta$$

$$\dot{x} = \dot{\alpha} \qquad \dot{y} = \dot{\beta}$$

$$\ddot{x} = \ddot{\alpha} \qquad \ddot{y} = \ddot{\beta} \tag{5.1}$$

respectively.

Now, the function $\Omega(x, y)$ in equation (3.20) becomes

$$\Omega(x, y) = \Omega(x_0 + \alpha, y_0 + \beta) \tag{5.2}$$

Expanding equation (5.2) by Taylor's series and considering only the linear terms in α and β, we have

$$\Omega(x_0+\alpha, y_0+\beta)=\Omega(x_0, y_0)+\alpha\Omega_x(x_0, y_0)+\beta\Omega_y(x_0, y_0)+\ldots$$

Similarly, we obtain

$$\Omega_x(x_0+\alpha, y_0+\beta)=\Omega_x(x_0, y_0)+\alpha\Omega_{xx}(x_0, y_0)+\beta\Omega_{xy}(x_0, y_0)+\ldots$$

$$\Omega_y(x_0+\alpha, y_0+\beta)=\Omega_y(x_0, y_0)+\alpha\Omega_{yx}(x_0, y_0)+\beta\Omega_{yy}(x_0, y_0)+\ldots \tag{5.3}$$

We know that at the equilibrium point (x_0, y_0),

$$\Omega_x(x_0, y_0)=\Omega_y(x_0, y_0)=0$$

So, equations of (5.3) become

$$\Omega_x(x_0+\alpha, y_0+\beta)=\alpha\Omega_{xx}(x_0, y_0)+\beta\Omega_{xy}(x_0, y_0)$$

$$\Omega_y(x_0+\alpha, y_0+\beta)=\alpha\Omega_{yx}(x_0, y_0)+\beta\Omega_{yy}(x_0, y_0) \tag{5.4}$$

Substituting equations (5.1) and (5.4) in the equations of motion (3.20), we obtain the variational equations of motion as

$$\ddot{\alpha}-2n\dot{\beta}=\alpha\Omega_{xx}(x_{0,}y_0)+\beta\Omega_{xy}(x_{0,}y_0)$$
$$\ddot{\beta}+2n\dot{\alpha}=\alpha\Omega_{yx}(x_{0,}y_0)+\beta\Omega_{yy}(x_{0,}y_0)$$

or

$$\ddot{\alpha}-2n\dot{\beta}=\Omega_{xx}^0\alpha+\Omega_{xy}^0\beta$$

$$\ddot{\beta}+2n\dot{\alpha}=\Omega_{yx}^0\alpha+\Omega_{yy}^0\beta \tag{5.5}$$

Here, the second partial derivatives of Ω are denoted by subscripts. The superscript 0 indicates that the derivatives are to be evaluated at the equilibrium point (x_0, y_0).

We now obtain the second partial derivatives of Ω.

From equation (3.20) we have

$$\Omega_x = n^2 x - \frac{(1-\mu)(x+\mu)q_1}{r_1^3} - \frac{\mu(x+\mu-1)q_2}{r_2^3} - \frac{3(1-\mu)(x+\mu)A_1q_1}{2r_1^5} - \frac{3\mu(x+\mu-1)A_2q_2}{2r_2^5} - \frac{M_b x}{\left(r^2+T^2\right)^{3/2}}$$

$$\Omega_y = n^2 y - \frac{(1-\mu)q_1 y}{r_1^3} - \frac{\mu q_2 y}{r_2^3} - \frac{3}{2}\frac{(1-\mu)A_1q_1 y}{r_1^5} - \frac{3}{2}\frac{\mu A_2 q_2 y}{r_2^5} - \frac{M_b y}{(r^2+T^2)^{3/2}}$$

So that

$$\Omega_{xx} = \frac{\partial}{\partial x}\left\{ n^2 x - \frac{(1-\mu)(x+\mu)q_1}{r_1^3} - \frac{\mu(x+\mu-1)q_2}{r_2^3} - \frac{3(1-\mu)(x+\mu)A_1q_1}{2r_1^5} - \frac{3\mu(x+\mu-1)A_2q_2}{2r_2^5} - \frac{M_b x}{\left(r^2+T^2\right)^{3/2}} \right\}$$

$$= n^2 - \frac{(1-\mu)q_1}{r_1^3} - (1-\mu)(x+\mu)q_1\frac{\partial}{\partial x}\left(\frac{1}{r_1^3}\right) - \frac{\mu q_2}{r_2^3} - \mu(x+\mu-1)q_2\frac{\partial}{\partial x}\left(\frac{1}{r_2^3}\right) - \frac{3(1-\mu)A_1q_1}{2r_1^5}$$
$$- \frac{3(1-\mu)(x+\mu)A_1q_1}{2}\frac{\partial}{\partial x}\left(\frac{1}{r_1^5}\right) - \frac{3\mu A_2q_2}{2r_2^5} - \frac{3\mu(x+\mu-1)A_2q_2}{2}\frac{\partial}{\partial x}\left(\frac{1}{r_2^5}\right) - \frac{M_b}{\left(r^2+T^2\right)^{3/2}}$$
$$- M_b x\frac{\partial}{\partial x}\frac{1}{\left(r^2+T^2\right)^{3/2}}$$

(5.6)

From equation (3.21) that is

$$r_1^2 = (x+\mu)^2 + y^2, \qquad r_2^2 = (x+\mu-1)^2 + y^2$$

We have

$$2r_1\frac{\partial r_1}{\partial x} = 2(x+\mu) \qquad\qquad 2r_2\frac{\partial r_2}{\partial x} = 2(x+\mu-1)$$

$$\Rightarrow \frac{\partial r_1}{\partial x} = \frac{(x+\mu)}{r_1} \qquad\qquad \Rightarrow \frac{\partial r_2}{\partial x} = \frac{(x+\mu-1)}{r_2}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_1^3}\right) = -\frac{3}{r_1^4}\frac{\partial r_1}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{1}{r_1^3}\right) = -\frac{3(x+\mu)}{r_1^5}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_2^3}\right) = -\frac{3}{r_2^4}\frac{\partial r_2}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{1}{r_2^3}\right) = -\frac{3(x+\mu-1)}{r_2^5}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right) = -\frac{5}{r_1^6}\frac{\partial r_1}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{1}{r_1}\right) = -\frac{5(x+\mu)}{r_1^7}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_2}\right) = -\frac{5}{r_2^6}\frac{\partial r_2}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{1}{r_2}\right) = -\frac{5(x+\mu-1)}{r_2^7}$$

In view of equation (3.4), we have

$$\frac{\partial}{\partial x}\left(\frac{1}{\left(r^2+T^2\right)^{3/2}}\right) = -\frac{3}{2}\frac{1}{\left(r^2+T^2\right)^{5/2}}\frac{\partial r^2}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{1}{\left(r^2+T^2\right)^{3/2}}\right) = -\frac{3}{2}\frac{1}{\left(r^2+T^2\right)^{5/2}}2x$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{1}{\left(r^2+T^2\right)^{3/2}}\right) = -\frac{3x}{\left(r^2+T^2\right)^{5/2}}$$

and

$$\frac{\partial}{\partial y}\left(\frac{1}{\left(r^2+T^2\right)^{3/2}}\right) = -\frac{3}{2}\frac{1}{\left(r^2+T^2\right)^{5/2}}\frac{\partial r^2}{\partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{\left(r^2+T^2\right)^{3/2}}\right) = -\frac{3}{2}\frac{2y}{\left(r^2+T^2\right)^{5/2}}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{1}{\left(r^2+T^2\right)^{3/2}}\right) = -\frac{3y}{\left(r^2+T^2\right)^{5/2}}$$

Substituting these in equation (5.6) gives

$$\Omega_{xx} = n^2 - \frac{(1-\mu)q_1}{r_1^3} + \frac{3(1-\mu)\left(x+\mu\right)^2 q_1}{r_1^5} - \frac{\mu q_2}{r_2^3} + \frac{3\mu\left(x+\mu-1\right)^2 q_2}{r_2^5} - \frac{3(1-\mu)A_1 q_1}{2r_1^5}$$
$$+\frac{15(1-\mu)\left(x+\mu\right)^2 A_1 q_1}{2r_1^7} - \frac{3\mu A_2 q_2}{2r_2^5} + \frac{15\mu\left(x+\mu-1\right)^2 A_2 q_2}{2r_2^7} - \frac{M_b}{\left(r^2+T^2\right)^{3/2}} + \frac{3M_b x^2}{\left(r^2+T^2\right)^{5/2}} \quad (5.7)$$

In a similar manner, we obtain

$$\Omega_{yy} = n^2 - \frac{(1-\mu)q_1}{r_1^3} + \frac{3(1-\mu)y^2 q_1}{r_1^5} - \frac{\mu q_2}{r_2^3} + \frac{3\mu y^2 q_2}{r_2^5} - \frac{3(1-\mu)A_1 q_1}{2r_1^5}$$
$$+\frac{15(1-\mu)y^2 A_1 q_1}{2r_1^7} - \frac{3\mu A_2 q_2}{2r_2^5} + \frac{15\mu y^2 A_2 q_2}{2r_2^7} - \frac{M_b}{\left(r^2+T^2\right)^{3/2}} + \frac{3M_b y^2}{\left(r^2+T^2\right)^{5/2}} \quad (5.8)$$

and

$$\Omega_{xy} = \Omega_{yx} = \frac{3(1-\mu)\left(x+\mu\right)yq_1}{r_1^5} + \frac{3\mu\left(x+\mu-1\right)yq_2}{r_2^5} + \frac{15(1-\mu)\left(x+\mu\right)yA_1 q_1}{2r_1^7}$$
$$+\frac{15\mu\left(x+\mu-1\right)yA_2 q_2}{2r_2^7} + \frac{3M_b xy}{\left(r^2+T^2\right)^{5/2}} \quad (5.9)$$

5.2 CHARACTERISTIC EQUATION

Consider trial solutions for the variational equations (5.5) to be in the form:

$$\alpha = Ae^{\lambda t} \qquad \beta = Be^{\lambda t}$$

where A, B and λ are parameters to be determined.

Now

$$\dot{\alpha} = A\lambda e^{\lambda t} \qquad \dot{\beta} = B\lambda e^{\lambda t}$$

$$\ddot{\alpha} = A\lambda^2 e^{\lambda t} \qquad \ddot{\beta} = B\lambda^2 e^{\lambda t}$$

Substituting these values in equations of (5.5), we have

$$A\lambda^2 e^{\lambda t} - 2n\lambda B e^{\lambda t} = \Omega^0_{xx} A e^{\lambda t} + \Omega^0_{xy} B e^{\lambda t}$$
$$B\lambda^2 e^{\lambda t} + 2n\lambda A e^{\lambda t} = \Omega^0_{yx} A e^{\lambda t} + \Omega^0_{yy} B e^{\lambda t}$$

Dividing by $e^{\lambda t}$, we have

$$A\lambda^2 - 2n\lambda B = \Omega^0_{xx} A + \Omega^0_{xy} B$$
$$B\lambda^2 + 2n\lambda A = \Omega^0_{yx} A + \Omega^0_{yy} B$$

or

$$A\lambda^2 - 2n\lambda B - \Omega^0_{xx} A - \Omega^0_{xy} B = 0$$
$$B\lambda^2 + 2n\lambda A - \Omega^0_{yx} A - \Omega^0_{yy} B = 0$$

or

$$\left(\lambda^2 - \Omega^0_{xx}\right) A + \left(-2n\lambda - \Omega^0_{xy}\right) B = 0$$
$$\left(2n\lambda - \Omega^0_{yx}\right) A + \left(\lambda^2 - \Omega^0_{yy}\right) B = 0$$

These will have a non –trivial solution for *A* and *B* if

$$\begin{vmatrix} \lambda^2 - \Omega^0_{xx} & -2n\lambda - \Omega^0_{xy} \\ 2n\lambda - \Omega^0_{yx} & \lambda^2 - \Omega^0_{yy} \end{vmatrix} = 0$$

Expanding the determinant yields

$$\left(\lambda^2 - \Omega^0_{xx}\right)\left(\lambda^2 - \Omega^0_{yy}\right) + \left(2n\lambda + \Omega^0_{xy}\right)\left(2n\lambda - \Omega^0_{yx}\right) = 0$$

or

$$\lambda^4 - \Omega^0_{xx}\lambda^2 - \Omega^0_{yy}\lambda^2 + \Omega^0_{xx}\Omega^0_{yy} + 4n^2\lambda^2 - 2n\lambda\Omega^0_{yx} + 2n\lambda\Omega^0_{xy} - \Omega^0_{xy}\Omega^0_{yx} = 0$$

In view of equation (5.9) we have

$$\lambda^4 - \Omega^0_{xx}\lambda^2 - \Omega^0_{yy}\lambda^2 + \Omega^0_{xx}\Omega^0_{yy} + 4n^2\lambda^2 - \Omega^{0\,2}_{xy} = 0$$

or

$$\lambda^4 + (4n^2 - \Omega^0_{xx} - \Omega^0_{yy})\lambda^2 + \Omega^0_{xx}\Omega^0_{yy} - \Omega^{0\,2}_{xy} = 0 \qquad (5.10)$$

Equation (5.10) is called the characteristic equation corresponding to the variational equations (5.5).

5.3 STABILITY OF TRIANGULAR POINTS

In order to determine the stability of the triangular points obtained in (4.17) we:

- evaluate the second partial derivatives equations (5.7), (5.8) and (5.9) at the triangular point;
- substitute the values of the second partial derivatives in the characteristic equation (5.10)
- proceed to find out the eigenvalues of the characteristic equation.

Now, the triangular points from (4.17) are:

$$x = \frac{1}{2} - \mu - \left(\frac{p_1 - p_2}{3} - \frac{A_1 - A_2}{2} \right)$$

$$y = \pm\sqrt{3}\left(\frac{1}{2} - \frac{p_1 + p_2}{9} + \frac{A_1 + A_2}{6} + \frac{2M_b(2r_c - 1)}{9(r_c^2 + T^2)^{3/2}} \right)$$

Using equations (3.22), (4.16) and (4.17) in equations (5.7), (5.8) and (5.9), restricting ourselves to only linear terms in p_1, p_2 $A_1,$ A_2 and M_b ,we obtain respectively :

$$\Omega_{xx}^0 = \frac{3}{4} + a_1 + \mu b_1 + \frac{5M_b(2r_c - 1)}{4(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{1}{4} - \mu + \mu^2)}{(r_c^2 + T^2)^{5/2}}$$

$$\Omega_{yy}^0 = \frac{9}{4} + a_2 + \mu b_2 + \frac{7M_b(2r_c - 1)}{4(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{3}{4})}{(r_c^2 + T^2)^{5/2}}$$

$$\Omega_{xy}^0 = \sqrt{3}\left\{ \frac{3}{4} + a_3 + \mu b_1 + \frac{11M_b(2r_c - 1)}{12(r_c^2 + T^2)^{3/2}} + \mu(b_3 - \tfrac{3}{2} - \frac{11M_b(2r_c - 1)}{6(r_c^2 + T^2)^{3/2}}) + \frac{\frac{3}{2}M_b(\frac{1}{2} - \mu)}{(r_c^2 + T^2)^{5/2}} \right\}$$

where

$$a_1 = -\frac{p_1}{2} + p_2 + \frac{27A_1}{8} + \frac{3A_2}{8} \qquad b_1 = \frac{3p_1}{2} - \frac{3p_2}{2} - 3A_1 + 3A_2 \qquad (5.11)$$

$$a_2 = \frac{p_1}{2} - p_2 + \frac{33A_1}{8} + \frac{33A_2}{8} \qquad b_2 = -\frac{3p_1}{2} + \frac{3p_2}{2}$$

$$a_3 = -\frac{p_1}{6} + \frac{p_2}{3} + \frac{19A_1}{8} + \frac{7A_2}{8} \qquad b_3 = -\frac{p_1}{6} - \frac{p_2}{6} - \frac{13A_1}{4} - \frac{13A_2}{4}$$

Here each of $|a_i|$, $|b_i|$ is very small as $|A_i| \ll 1$, $|p_i| \ll 1 \left(i = 1,2,3\right)$. Putting the values of $\Omega_{xx}^0, \Omega_{yy}^0$, Ω_{xy}^0 found above and the value of n^2 from (3.22) in equation (5.10), the characteristic equation becomes

$$\lambda^4 + b\lambda^2 + c = 0 \tag{5.12}$$

or

$$\Pi^2 + b\Pi + c = 0$$

where,

$$\Pi = \lambda^2$$

$$b = 1 + 6(A_1 + A_2) - (a_1 + a_2) - \mu(b_1 + b_2) + \frac{M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{3M_b r_c^2}{(r_c^2 + T^2)^{5/2}}$$

$$c = \left(-\frac{27}{4} + 9b_3 - \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} - \frac{27M_b}{4(r_c^2 + T^2)^{5/2}} \right)\mu^2$$

$$+\left(\frac{27}{4} + \frac{9b_1}{4} + \frac{3b_2}{4} - \frac{9b_3}{2} + 9a_3 + \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right)\mu \ + \frac{9a_1}{4} + \frac{3a_2}{4} - \frac{9a_3}{2}$$

its roots are:

$$\Pi = \frac{-b \pm \sqrt{\Delta}}{2} \tag{5.13}$$

where $\Delta = b^2\text{-}4c$ is the discriminant

$$\Delta = \left(27 - 36b_3 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right)\mu^2$$

$$\text{-}\left(27 + 11b_1 + 5b_2 - 18b_3 + 36a_3 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{108M_b}{(r_c^2 + T^2)^{5/2}} \right)\mu$$

$$+1+12(A_1+A_2)-11a_1-5a_2+18a_3+\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_br_c^2}{(r_c^2+T^2)^{5/2}} \tag{5.14}$$

Thus, the roots

$$\lambda_{1,2}=\pm\sqrt{\Pi_1}, \qquad \lambda_{3,4}=\pm\sqrt{\Pi_2}, \tag{5.15}$$

are functions of the mass parameter μ, and their nature depend on the discriminant Δ.

Now

$$(\Delta)_{\mu=0}=1+12(A_1+A_2)-11a_1-5a_2+18a_3+\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_br_c^2}{(r_c^2+T^2)^{5/2}}$$

$$=1+\left(12(A_1+A_2)-11a_1-5a_2+18a_3+\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_br_c^2}{(r_c^2+T^2)^{5/2}}\right)>0$$

and

$$(\Delta)_{\mu=\frac{1}{2}}=\left(27-36b_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)\frac{1}{4}$$

$$-\left(27+11b_1+5b_2-18b_3+36a_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{108M_b}{(r_c^2+T^2)^{5/2}}\right)\frac{1}{2}$$

$$+1+12(A_1+A_2)-11a_1-5a_2+18a_3+\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_br_c^2}{(r_c^2+T^2)^{5/2}}$$

$$=\frac{27}{4}+\left(-36b_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)\frac{1}{4}$$

$$-\frac{27}{2}-\left(11b_1+5b_2-18b_3+36a_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{108M_b}{(r_c^2+T^2)^{5/2}}\right)1/2$$

$$+1+12(A_1+A_2)-11a_1-5a_2+18a_3+\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_br_c^2}{(r_c^2+T^2)^{5/2}}$$

$$\therefore(\Delta)_{\mu=\frac{1}{2}}=-\frac{23}{4}+\{\left(-36b_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)\frac{1}{4}$$

$$-\left(11b_1+5b_2-18b_3+36a_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{108M_b}{(r_c^2+T^2)^{5/2}}\right)1/2$$

$$+12(A_1 + A_2) - 11a_1 - 5a_2 + 18a_3 + \frac{2M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{6M_b r_c^2}{(r_c^2 + T^2)^{5/2}}\} < 0$$

Also

$$\frac{d\Delta}{d\mu} = \left(27 - 36b_3 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}}\right)2\mu - \left(27 + 11b_1 + 5b_2 - 18b_3 + 36a_3 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{108M_b}{(r_c^2 + T^2)^{5/2}}\right)$$

$$\left(\frac{d\Delta}{d\mu}\right)_{\mu=0} = -27 - \left(11b_1 + 5b_2 - 18b_3 + 36a_3 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{108M_b}{(r_c^2 + T^2)^{5/2}}\right)$$

$$\square\ -27 < 0$$

and

$$\left(\frac{d\Delta}{d\mu}\right)_{\mu=1/2} = \left(27 - 36b_3 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}}\right) - \left(27 + 11b_1 + 5b_2 - 18b_3 + 36a_3 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{108M_b}{(r_c^2 + T^2)^{5/2}}\right)$$

$$\square\ 0$$

$\Rightarrow \dfrac{d\Delta}{d\mu} < 0$ in the interval $\left(0, \dfrac{1}{2}\right)$

Thus, it follows that Δ is a strictly decreasing function of μ in the interval (0, 1/2) and has values of opposite signs at the end points: μ=0 and μ= ½.

The opposite signs of the discriminant Δ at $\mu = 0$ and $\mu = 1/2$ indicate that there is only one value of μ in the open interval (0, 1/2) for which Δ vanishes. This value of μ is called the critical value of the mass parameter and is denoted by μ_c. Hence, we examine the three regions: $0 < \mu < \mu_c, \quad \mu_c < \mu < \dfrac{1}{2} \quad$ *and* $\mu = \mu_c$

Region 1: when $0 < \mu < \mu_c,$ *the value of* Δ *is positive i.e.* $b^2 - 4c > 0 \Rightarrow |b| > \sqrt{\Delta}$

In view of equation (5.12) we see that

$$b = 1 + 6(A_1 + A_2) - (a_1 + a_2) - \mu(b_1 + b_2) + \frac{M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{3M_b r_c^2}{(r_c^2 + T^2)^{5/2}} > 0$$

Thus, from (5.13) we have

$$\Pi = \frac{-b \pm \sqrt{\Delta}}{2} < 0$$

or

$$\Pi_1 = \frac{-b + \sqrt{\Delta}}{2} < 0, \qquad \Pi_2 = \frac{-b - \sqrt{\Delta}}{2} < 0$$

or

$$\Pi_1 = -s_1 \qquad \Pi_2 = -s_2$$

where $s_1 = \left| \frac{-b + \sqrt{\Delta}}{2} \right|$, and $s_2 = \left| \frac{-b - \sqrt{\Delta}}{2} \right|$,

Now, the roots (5.15) becomes

$$\lambda_{1,2} = \pm\sqrt{-s_1}, \qquad \lambda_{3,4} = \pm\sqrt{-s_2}$$

or

$$\lambda_{1,2} = \pm i\sqrt{s_1}, \qquad \lambda_{3,4} = \pm i\sqrt{s_2}$$

where $i = \sqrt{-1}$

Thus, since all the eigenvalues are distinct and pure imaginaries, in view of (1.4.5), the triangular point is stable .

Region2: when $\mu_c < \mu < \frac{1}{2}$ in this region, the discriminant of the characteristic equation is negative *i.e.* $b^2 - 4c < 0$.

Thus, from (5.13) we have

$$\Pi = \frac{-b \pm i\sqrt{|\Delta|}}{2}$$

or

$$\Pi_1 = \frac{-b + i\delta}{2} \qquad \Pi_2 = \frac{-b - i\delta}{2}$$

where,

$$\delta = +\sqrt{|\Delta|}$$

The roots (5.15) of the characteristic equation (5.12) are:

$$\lambda_{1,2} = \pm\sqrt{\frac{-b + i\delta}{2}} \quad , \quad \lambda_{3,4} = \pm\sqrt{\frac{-b - i\delta}{2}}$$

or

$$\lambda_1 = \frac{1}{\sqrt{2}}\sqrt{-b + i\delta} \qquad \lambda_3 = \frac{1}{\sqrt{2}}\sqrt{-b - i\delta}$$

$$\lambda_2 = -\frac{1}{\sqrt{2}}\sqrt{-b + i\delta} \qquad \lambda_4 = -\frac{1}{\sqrt{2}}\sqrt{-b - i\delta}$$

These indicate that the real parts of two of the characteristic roots are positive; hence by (1.4.5), the triangular point is unstable in this region.

Region3: when $\mu = \mu_c$ in this region, the discriminant of the characteristic equation is zero

i.e. *the value of* Δ *is zero* $\Rightarrow \sqrt{\Delta} = 0$

And so, from (5.13) we have

$$\Pi = \frac{-b \pm 0}{2}$$

or

$$\Pi_1 = \Pi_2 = \frac{-b}{2}$$

The roots (5.15) of the characteristic equation (5.12) now become:

$$\lambda_{1,2} = \pm\sqrt{\frac{-b}{2}} \qquad \lambda_{3,4} = \pm\sqrt{\frac{-b}{2}}$$

or

$$\lambda_1 = \lambda_3 = i\sqrt{\frac{b}{2}}$$

$$\lambda_2 = \lambda_4 = -i\sqrt{\frac{b}{2}}$$

Thus we have double roots and so, in view of (1.4.5) the triangular point is unstable when $\mu = \mu_c$.

Hence, the triangular points are stable in $0 < \mu < \mu_c$, and unstable in $\mu_c \le \mu \le \frac{1}{2}$.

5.3.1 The Critical Mass Parameter μ_c

The critical value μ_c of the mass parameter is the value of the mass ratio μ when the discriminant (5.14) is equal to zero. That is, the root of the equation Δ=0 gives the value of the mass parameter.

From equation (5.14) we have

$$\begin{aligned}&\left(27-36b_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)\mu^2\\ &-\left(27+11b_1+5b_2-18b_3+36a_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{108M_b}{(r_c^2+T^2)^{5/2}}\right)\mu\\ &+1+12(A_1+A_2)-11a_1-5a_2+18a_3+\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_br_c^2}{(r_c^2+T^2)^{5/2}}=0\end{aligned} \tag{5.16}$$

The root of (5.16) which gives the value of μ_c is

$$\mu_c = \mu_0 + \mu_p + \mu_a + \mu_b, \tag{5.17}$$

where

$$\mu_0 = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right)$$

$$\mu_p = -2\frac{(p_1 + p_2)}{27\sqrt{69}}$$

$$\mu_a = \frac{1}{9}\left(1 - \frac{13}{\sqrt{69}}\right)A_2 - \frac{1}{9}\left(1 + \frac{13}{\sqrt{69}}\right)A_1$$

$$\mu_b = \frac{(78 - 8r_c)M_b}{27\sqrt{69}\left(r_c^2 + T^2\right)^{3/2}} + \left(\frac{3}{2} - \frac{83 + 12r_c^2}{6\sqrt{69}}\right)\frac{M_b}{\left(r_c^2 + T^2\right)^{5/2}}$$

The value of the critical mass parameter to fifteen decimal places is

μ_c =0.038520896504551-0.285001787790556A_1-0.062779565568333A_2-

0.008917470598946P_1- 0.008917470598946P_2 -0.233068119248071M_b (5.18)

Evidently, μ_c represents the total effect of radiation and oblateness of the primaries and potential from the belt on the critical mass value. However, in the absence of these perturbations, the critical mass value μ_c becomes μ_0, which corresponds to the classical restricted problem (Szebehely, 1967).

The effects of these perturbations are shown in Table (5.1). It indicates that, the perturbations reduce the range of stability. We use the following cases in equation (5.18):

1 Absence of radiation, oblateness and potential from the belt (classical case).

2 Oblateness of the bigger primary only.

3 Oblateness of the smaller primary only.

4 Radiation of the bigger primary only.

5 Radiation of the smaller primary only.

6 Potential from the belt only.

7 Radiation and oblateness of the primaries and potential from the belt .

Table 5.1. The effects of the perturbations on the critical mass value.

Case	q_1	q_2	A_1	A_2	M_b	μ_c
1	1	1	0	0	0	0.038521
2	1	1	0.03	0	0	0.029971
3	1	1	0	0.02	0	0.037265
4	0.75	1	0	0	0	0.036292
5	1	0.85	0	0	0	0.037183
6	1	1	0	0	0.01	0.036190
7	0.75	0.85	0.03	0.02	0.01	0.022818

5.4 STABILITY OF COLLINEAR POINTS

In order to determine the stability of the collinear equilibrium point $(x_0, 0)$ we:

- evaluate the second partial derivatives equations (5.7), (5.8) and (5.9) at the collinear point;

- substitute the values of the second partial derivatives in the characteristic equation (5.10);
- proceed to find out the eigenvalues of the characteristic equation.

Now, for any collinear equilibrium point $(x_0, 0)$, we have from equation (3.21)

$$r_1^2 = (x_0 + \mu)^2, \qquad r_2^2 = (x_0 + \mu - 1)^2$$

$$\Rightarrow r_1 = |x_0 + \mu|, \qquad r_2 = |x_0 + \mu - 1| \tag{5.19}$$

Substituting (5.19) in equations (5.7), (5.8) and (5.9), we obtain respectively

$$\Omega^0{}_{xx} = n^2 + \frac{2(1-\mu)q_1}{|x_0+\mu|^3} + \frac{2\mu q_2}{|x_0+\mu-1|^3} + \frac{6(1-\mu)A_1 q_1}{|x_0+\mu|^5} + \frac{6\mu A_2 q_2}{|x_0+\mu-1|^5} + \left(\frac{3x_0{}^2}{(x_0{}^2+T^2)^{5/2}} - \frac{1}{(x_0{}^2+T^2)^{3/2}}\right)M_b > 0$$

$$\Omega^0{}_{yy} = n^2 - \frac{2(1-\mu)q_1}{|x_0+\mu|^3}\left(1+\frac{3A_1}{2|x_0+\mu|^2}\right) - \frac{\mu q_2}{|x_0+\mu-1|^3}\left(1+\frac{3A_2}{2|x_0+\mu-1|^2}\right) - \frac{M_b}{(x_0{}^2+T^2)^{3/2}} < 0$$

(in the range containing the roots). Which is also obvious from the Figure 5.1 where, $g(x) = \Omega^0{}_{yy}$

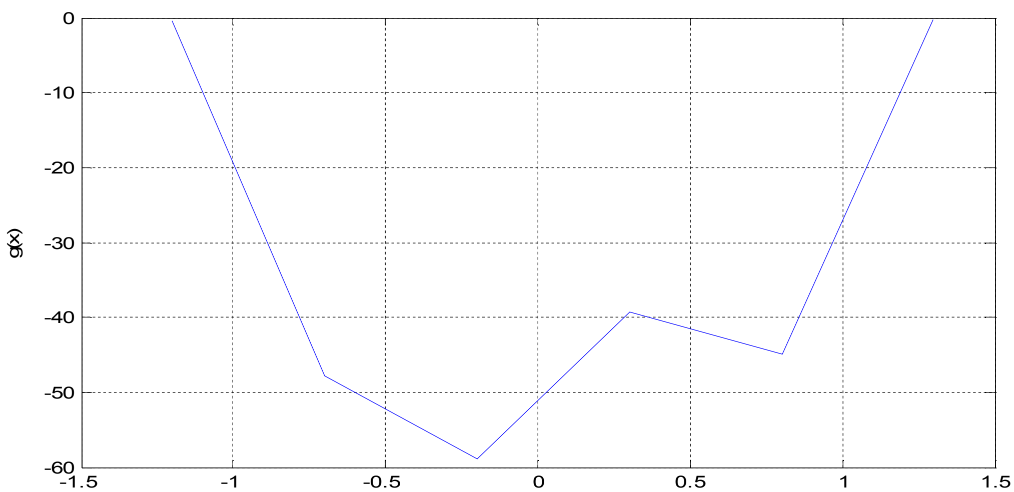

Figure 5.1: Graph of g(x) against x for -1.2<x<1.3

$$\Omega^0{}_{xy} = \Omega^0{}_{yx} = 0$$

The characteristic equation (5.10) now becomes

$$\lambda^4 + b\lambda^2 + c = 0 \tag{5.20}$$

where

$$b = 4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0 \quad , \; c = \Omega_{xx}^0 \Omega_{yy}^0 < 0$$

Its eigenvalues are:

$$\lambda_1^{\;2} = \frac{-b + \sqrt{b^2 - 4c}}{2} \qquad , \quad \lambda_2^{\;2} = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

These two relations will determine the eigenvalues and the properties of the equilibrium point. In fact, the signs of $b^2 - 4c$, $-b + \sqrt{b^2 - 4c}$, and $-b - \sqrt{b^2 - 4c}$ will completely determine the results. However, when one knows the signs of b and c, the details of possible eigenvalues can be directly determined by them. We therefore list all the combinations in Table 5.2, in terms of the signs of $b^2 - 4c$, b and c.

Table 5.2: The Nature of the Eigenvalues

Row	b^2-4c	b	$c.$	$\lambda_1^{\ 2}$	$\lambda_2^{\ 2}$
1	<0			Complex number	Complex number
2	=0	<0		Positive number	Positive number
3	=0	=0		Zero	Zero
4	=0	>0		Negative number	Negative number
5	>0	<0	<0	Positive number	Negative number
6	>0	<0	=0	Positive number	Zero
7	>0	<0	>0	Positive number	Positive number
8	>0	=0	<0	Positive number	Negative number
9	>0	=0	=0	Impossible	Impossible
10	>0	=0	>0	Impossible	Impossible
11	>0	>0	<0	Positive number	Negative number
12	>0	>0	=0	Zero	Negative number
13	>0	>0	>0	Negative number	Negative number

In Row 1, λ_1^2, λ_2^2 are complex conjugate numbers, since $b^2 - 4c < 0$. Thus, λ_1, λ_2 can be expressed as $\lambda_1 = \pm d \pm ie$ and $\lambda_2 = \pm f \pm ig$ where, d, e, f, g are positive real numbers. Since not all the real parts of the eigenvalues are negative, in view of (1.4.5) the equilibrium point is an unstable point.

In Row 2, $b^2 - 4c = 0$ and $b < 0$ we have $\lambda_1^2 = \lambda_2^2 = \frac{|b|}{2}$ i.e. $\lambda_1 = \lambda_2 = \pm\sqrt{\frac{|b|}{2}}$. The equilibrium point is unstable. In Row 3, $b^2 - 4c = 0$ and $b = 0$ implies $\lambda_1^2 = \lambda_2^2 = 0$, all the real part of the eigenvlues are zeroes. The equilibrium point is unstable. In Row 4, $b^2 - 4c = 0$ and $b > 0$, imply $\lambda_1^2 = \lambda_2^2 = -\frac{b}{2}$ i.e. $\lambda_1 = \lambda_2 = \pm ie$. Thus in view of (1.4.5), the equilibrium point is unstable.

In Row 5, $b^2 - 4c > 0$, $b < 0$ and $c < 0$. Since $c < 0$ we have, $b^2 - 4c > b^2$ this implies $\sqrt{b^2 - 4c} > |b|$. Thus $\lambda_1^2 = \frac{|b| + \sqrt{b^2 - 4c}}{2} > 0$, $\lambda_2^2 = \frac{|b| - \sqrt{b^2 - 4c}}{2} < 0$, and so λ_1, λ_2 can be expressed as $\lambda_1 = \pm d$ and $\lambda_2 = \pm ie$. The equilibrium point is also an unstable point.

In Row 6, because $c = 0$, we have $\lambda_1^2 = \frac{-b + |b|}{2}$, $\lambda_2^2 = \frac{-b - |b|}{2}$ and because $b < 0$ we get

$\lambda_1^2 = \frac{|b| + |b|}{2}$, $\lambda_2^2 = \frac{|b| - |b|}{2}$ i.e. $\lambda_1 = \pm d$ and $\lambda_2 = \pm 0$. The equilibrium point is also an unstable point.

In Row 7, due to $c > 0$, we have $b^2 - 4c < b^2 \Rightarrow \sqrt{b^2 - 4c} < |b|$ and because $b < 0$, we get

$\lambda_1^{\ 2}=\dfrac{|b|+\sqrt{b^2-4c}}{2}>0,\qquad \lambda_2^{\ 2}=\dfrac{|b|-\sqrt{b^2-4c}}{2}>0,$ i.e. $\lambda_1=\pm d$ and $\lambda_2=\pm f$, all real numbers. The point is unstable.

Row 8, since $b=0$, $\lambda_1^{\ 2}=\dfrac{\sqrt{-4c}}{2}$ $\lambda_2^{\ 2}=\dfrac{-\sqrt{-4c}}{2}$ and for $c<0$ we have, $\lambda_1^{\ 2}=\sqrt{|c|}$, $\lambda_2^{\ 2}=-\sqrt{|c|}$, thus $\lambda_1=\pm d$ and $\lambda_2=\pm ie$. So, the equilibrium point is an unstable point.

In Rows 9 and 10, $c=0$ and $c>0$ contradicts $b^2-4c>0$ after we used $b=0$. Thus, these two are impossible.

In Row 11, $c<0$, we have $b^2-4c>b^2\Rightarrow\sqrt{b^2-4c}>|b|$ and since $b>0$, we get $\lambda_1^{\ 2}=\dfrac{-b+\sqrt{b^2-4c}}{2}>0,\ \lambda_2^{\ 2}=\dfrac{-b-\sqrt{b^2-4c}}{2}<0$. $\lambda_1=\pm d$ and $\lambda_2=\pm ie$, The equilibrium point is also an unstable point.

In Row 12, $c=0$ and $b>0$ imply $\lambda_1^{\ 2}=0,\ \lambda_2^{\ 2}=-b$. Thus, $\lambda_1=\pm 0$ and $\lambda_2=\pm ie$, the equilibrium point is unstable.

In Row 13, due to $c>0$ and $b>0$, we have $b^2-4c<b^2\Rightarrow\sqrt{b^2-4c}<b$. Furthermore, since $b^2-4c>0$ we get: $\lambda_1^{\ 2}=\dfrac{-b+\sqrt{b^2-4c}}{2}<0,\ \lambda_2^{\ 2}=\dfrac{-b-\sqrt{b^2-4c}}{2}<0$. Both $\lambda_1=\pm ie$ and $\lambda_2=\pm ig$ are pure imaginary. But in this problem c<0, thus Row 13 becomes Row 11.

In general, for the system under consideration, it is impossible that all the four eigenvalues' real parts are negative. Hence, in view of (1.4.5), the collinear equilibrium points are unstable.

Thus, the radiation and oblateness of the primaries and potential from the belt do not affect the stability of the collinear equilibrium points. That is, the collinear equilibrium points remain unstable in presence of these perturbations.

5.5 DISCUSSION

The characteristic equation for the triangular points is given by equation (5.12), it differs from that of Singh and Ishwar (1999) only due to the presence of the potential from the belt. If the effect of the belt is neglected, the characteristic equation (5.12) corresponds to that of AbdulRaheem and Singh (2006) in the absence of perturbations in the Coriolis and centrifugal forces.

Equation (5.18) gives the value of the critical mass parameter μ_c. It shows the combined effect of oblateness of the radiating primaries and potential from the belt on the critical mass value. It is obvious that these perturbations reduce the range of stability (Table 5.1). However, in the absence of these perturbations, the critical mass value μ_c becomes μ_0, which corresponds to the classical restricted problem (Szebehely, 1967). But in the absence of the potential from the belt only, the value of μ_c agrees with Singh and Ishwar (1999), Sharma et al. (2001b) when the primaries are oblate spheroids, Abdulraheem and Singh (2006) when the perturbations in the Coriolis and the centrifugal forces are neglected and Singh (2009) in the case of linear stability. If radiation of the bigger primary and oblatenesss of the smaller primary are considered only, μ_c confirms the result of Sharma (1987). Also, presence of oblateness of the bigger primary only, μ_c tallies with that given by Markellos et al. (1996) in the case of linear stability. If the radiation of the bigger primary is only considered, μ_c confirms the result of Bhatnager and Chawla (1979). If both oblateness and radiation of the bigger primary are considered only, μ_c agrees with Sharma et al. (2001*a*) when the bigger primary is an oblate spheroid instead of triaxial.

Nevertheless, when the radiation of the bigger primary and oblateness of the smaller primary and potential from the belt are considered only, μ_c (= 0.0385208965-0.0627795656A_2-0.0089174706P_1- 0.2330681192M_b) does not fully agree with Kushvah (2008) (μ_c= μ_1 =0.0385208965+0.0375419787A_2−0.0089174706P_1 −0.0579486176Mb). Here, we observe that the coefficients of A_2 and Mb differ from those of Kushvah (2008). We believe that coefficient of A_2 in our μ_c is correct because this agrees with that of Sharma (1987), Ishwar and Elipe (2001), Sharma et al. (2001b) when smaller primary is oblate spheroid, Singh and Ishwar (1999), Abdulraheem and Singh (2006) , Singh (2009) and Singh and Begha (2011) . The coefficient of M_b obtained in our case does not confirm that of Kushvah (2008) since he defined the mean motion, n as $n^2 = 1 + \frac{3}{2}A_2 + \frac{2M_b r_c}{(r_c^2 + T^2)^{3/2}}$, where $r_c^{\,2} = (1-\mu)q_1^{2/3} + \mu^2$ which implies that the mean motion depends on the radiation factor contrary to the usual concept. In our opinion, the definition of mean motion should be independent of radiation factor because the radiation pressure force changes with the distance by the same law as the gravitational attraction force and acts opposite to it. The second reason is that in the computation of μ_c, he has assigned an arbitrary value to r_c which is not proper.

5.6 CONCLUSION

We have found that the collinear equilibrium points remain unstable, while the triangular points are stable in $0 < \mu < \mu_c$ and unstable in $\mu_c \leq \mu \leq \frac{1}{2}$, where μ_c is the critical mass ratio influenced by the oblateness and radiation of the primaries and potential from the belt. We have established, these perturbations reduce the range of stability (Table 5.1).

CHAPTER 6

SUMMARY AND CONCLUSION

6.1 SUMMARY

We have deduced the equations of motion of an infinitesimal body moving in the gravitational field of radiating oblate primaries, together with the influence of gravitational potential from a belt in the restricted three-body problem. The equations are different from those of the classical case due to these perturbations. We have observed that the equations of motion are affected by the radiation and oblateness factors and the potential from the belt.

We have determined analytically and numerically the locations of equilibrium points. We have found that in addition to the usual five (two triangular, three collinear) equilibrium points, there exist two new collinear equilibrium points due to the potential from the belt, which we call L_{n1} and L_{n2}. Numerical investigations reveal that in the presence of the perturbations mentioned, the equilibrium points L_1, L_3, L_4, L_5 come nearer to the primaries; while L_2, L_{n2} move towards the bigger primary and L_{n1} moves away from it. The linear stability of the equilibrium points has also been examined. It has been found that the triangular points are stable for $0 < \mu < \mu_c$, and unstable for $\mu_c \le \mu \le \frac{1}{2}$, where μ_c is the critical mass parameter defined as

μ_c =0.038520896504551-0.285001787790556A_1-0.062779565568333A_2-0.008917470598946P_1- 0.008917470598946P_2 -0.233068119248071M_b

This is obvious that these perturbations reduce the size of stability. We have also seen that the collinear points remain unstable despite the introduction of the various aforementioned perturbations.

6.2 CONCLUSION

The equations that govern the motion of an infinitesimal body in the generalized restricted three-body problem have been obtained and they are found to be affected by the radiation and oblateness factors and potential from the belt. We have examined the locations of the equilibrium points and their linear stability and found that:

- In addition to the usual five equilibrium points, there appear two new collinear points L_{n1} and L_{n2} due to the potential from the belt.
- The positions of the equilibrium points are affected by oblateness and radiation of the primaries and potential from the belt.
- The triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \le \mu \le \frac{1}{2}$, where μ_c is the critical mass parameter influenced by the oblateness and radiation of the primaries and potential from the belt. These perturbations have destabilizing tendency.
- The collinear equilibrium points remain unstable for any mass ratio in the presence of these perturbations.

REFERENCES

Abdul Raheem, A. and Singh, J. (2006). Combined effects of perturbations, radiation, and oblateness on the stability of equilibrium points in the restricted three-body problem. *The Astronomical Journal,* 131: 1880–1885.

Allen, C. W. (1973). *Astrophysical Quantites*. London; Athlone press.

Aumann H.H.,Beichman,C.A.,Gillett, F.C.,de Jong, T., Houck,J.R., Low,F.J., Neugebauer,G.,Walker,R.G. and Wesselius,P.R. (1984). Discovery of a shell around Alpha Lyrae. *Astrophysical Journal*, 278: L23-L27.

Bhatnagar, K.B.and Chawla, J.M. (1979). A study of the Lagrangian points in the photogravitational restricted three-body problem. *Indian J. Pure Appl. Math.* **10**(11): 1443–1451.

Corben, H.C.and Stehle, P. (1977). *Classical Mechanics,* 2nd ed. Dover Publishers, New York.

Euler, L. (1772). Theoria Motuum Lunae, Typis Academiae Imperialis Scientiarum Petropoli. In: *Opera Omnia, Series 2*, ed. L. Courvoisier, 22, Lausanne:Orell Fussli Turici, 1958.

Greaves, J. S., Holland, W. S. , Moriarty-Schieven, G. , Jenness,T., Dent,W. R. F., Zuckerman,B. , Mccarthy, C., Webb, R. A., Butner, H. M., Gear, W. K., And Walker, H. J. (1998). A Dust Ring around ε Eridani: Analog to the young Solar System. *The Astrophysical Journal,* 506:L133 L137.

Ishwar, B.and Elipe, A. (2001). Secular solutions at triangular points in the generalized photogravitational restricted three-body problem. *Astrophysics and Space Science.* 277: 67–81.

Jenks E. (2008). *Peanut-Stars | New Discovery of Double Star Systems: Oblate Spheroid.* Retrieved April 1, 2010, from htt: newdiscory/oblate/

Jiang, I.G. and Yeh, L.C.(2003). Bifurcation for dynamical systems of planet-belt interaction. *Int. J. Bifurcation and Chaos*, 13: 534-539

Jiang, I.G.and Yeh, L.C (2004a). The modified restricted three body problems. *RevMex de A A* , 21:152–155.

Jiang, I.G.and Yeh, L.C (2004b). On the chaotic orbits of disk-star-planet systems. *Astron. J.* ,128:923–932.

Jiang, I.G. and, Yeh, L.C. (2006). On the Chermnykh-like problems: I. The mass parameter $\mu = 0.5$. *Astrophys Space Sci,* 305:341–348.

Kalantonis, V.S., Douskos C.N. and Perdios E.A. (2006). Numerical determination of homoclinic and heteroclinic orbits at collinear equilibria in the restricted three-body problem with oblateness. *Celestial Mechanics and Dynamical Astronomy,* 94: 135–153.

Kalvouridis, T.J.and Hadjifotinou, K. (2008). Bifurcations from planar to three dimensional periodic orbits in the photo-gravitational restricted four-body problem. *Int. J. Bifurc. Chaos,* 18(2): 1–15 .

Kumar, V.and Choudhry, R.K. (1988). On the stability of the triangular libration points for the photo-gravitational circular restricted problem of three bodies under the resonances of the third and the fourth order. *Celest. Mech. Dyn. Astron*., 41: 161–173.

Kumar, V. and Choudhry, R.K. (1989). Linear stability and the resonance cases for the triangular libration points for the doubly photo-gravitational elliptic restricted problem of three bodies. *Celest. Mech. Dyn. Astron*., 46: 59–77.

Kushvah, B.S (2008). Linear stability of equilibrium points in the generalized photogravitational Chermnykh's problem. *Astrophys Space Sci* ,318: 41–50.

Kushvah, B.S (2011**).** Trajectory and stability of Lagrangian point L_2 in the Sun-Earth system. *Astrophys Space Sci* ,332(1): 99-106.

Miyamoto, M.and Nagai, R. (1975). Three-dimensional models for the distribution of mass in galaxies. *Publ. Astron. Soc. Jpn*. **27**, 533–543.

Newton, I. (1687). *Philosophiae Naturalis Principia Mathematica*, London: Royal Society. In: *The Mathematical Principles of Natural Philosophy*, New York: Philosophical Library, 1964.

Papadakis, K.E. (2006). Asymptotic orbits at the triangular equilibria in the photo-gravitational restricted three-body problem. . *Astrophysics and Space Science*, 305: 57–66.

Poynting, J.H. (1903). Radiation in the solar system: its effect on temperature and its pressure on small bodies. Philos. *Trans. R. Soc. Lond***.,** 202: 525–552.

Radzievskii, V. V. (1950). The restricted problem of three-body taking account of light pressure. *Astronomical Journal Zh,* 27(5): 250-256.

Ragos, O. and Zafiropoulos, F.A. (1995). A numerical study of the influence of the Poynting-Robertson effect on the equilibrium points of the photo-gravitational restricted three-body problem. Astron. Astrophys., 300:568–578.

Schuerman, D. (1980). The restricted three-body problem including radiation pressure. *Astrophys.* J., 238: 337–342.

Sharma, R.K. (1982). On linear stability of triangular liberation points of the photo-gravitational restricted three-body problem when the massive primary is an oblate spheroid. *In: Sun and Planetary System, ed*. W. Fricke & G. Teleki (Dordrecht: Reidel), 435

Sharma, R.K. (1987). The linear stability of libration points of the photogravitational restricted three- body problem when the smaller primary is an oblate spheroid. *Astophysics and space Science,* 135: 271-281.

Sharma, R.V.,Taqvi, Z.A. and Bhatnagar, K.B.(2001a). Existence and stability of libration points in the restricted three body problem when the bigger primary is a triaxial rigid body and source of radiation. *Indian J. Pure Appl. Math* ,32(2):255-266

Sharma, R.V.,Taqvi, Z.A. and Bhatnagar, K.B.(2001b). Existence and stability of libration points in the restricted three body problem when the primaries are triaxial rigid bodies and source of radiations. *Indian J. Pure Appl. Math,* 32(7): 981-994.

Simmons, J. F. L., McDonald, J. C. and Brown, J. C. (1985). The three-body problem with radiation pressure.*Celest.Mech*., 35:145-187.

Singh, J. and Ishwar, B. (1999). Stability of triangular points in the generalized photogravitational restricted three body problem. *Bull. Astron. Soc. India,* 27: 415-424.

Singh, J.(2009). Combined effects of oblateness and radiation on the nonlinear stability of L_4 in the restricted three-body problem. *The Astronomical Journal*, 137:3286–3292.

Singh J· and Begha J.M (2011). Stability of equilibrium points in the generalized perturbed restricted three-body problem. Astrophys Space Sci .,331: 511-519.

Szebehely, V. (1967). *Theory of Orbits: The Restricted Problem of Three Bodies*. Academic Press, San Diego.

Trilling D. E., Stansberry J. A., Stapelfeldt K. R., Rieke G. H., Su K. Y. L., Gray R. O., Corbally C. J., Bryden G., Chen C.H., Boden A., and Beichman C. A.(2007). Debris disks in main-sequence binary systems. *The Astrophysical Journal*, 658: 1289-1311.

Yeh, L.C.and Jiang, I.G (2006). On the Chermnykh-like problems: II. The equilibrium points. *Astrophys. Space Sci.,* 306: 189–200.

Zheng and Yu (1993). Photogravitational restricted three-body problem and coplanar libration point. *Chinese Phys.Letter*, 10(1): 61-64.