STUDY ON TRANSMUTED POWER LOMAX DISTRIBUTION

BY

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DEPARTMENT OF STATISTICS
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DEPARTMENT OF STATISTICS,
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MARCH, 2018

DECLARATION

I declare that the work in this dissertation titled "Study on Transmuted Power Lomax distribution" has been carried out by me in the Department of Statistics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other institution.

MOLTOK TONGDYEN TAJAN   Date

CERTIFICATION

This dissertation entitled “STUDY ON TRANSMUTED POWER LOMAX DISTRIBUTION” by MOLTOK TongdyenTajan meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

______________	____________	__________
Dr. H. G Dikko Chairman, Supervisory committee	Signature	Date
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Dr. H. G. Dikko Head of Department	Signature	Date
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Prof. S.Z. Abubakar Dean, School of Postgraduate Studies	Signature	Date

DEDICATION

This research is dedicated to God Almighty my creator whom through his mercy gave me the ability to accomplish this task. Also to my lovely husband, kids and parents, I love you all.

ACKNOWLEDGEMENT

To God be the honor and glory. My indebt gratitude goes to the Alpha and Omega of my life, for in his time, all things have been made beautiful. I am forever grateful Lord.

To the love of my life,my husband who encouraged, motivated and gave me all the needed support throughout my period of study, words cannot say how grateful I am,you are simply the best,I couldn't have wished for a better life partner. May heaven reward you greatly.And to my parents Mr. and Mrs. Amos Jwat,your prayers and encouragement added colour and made the pursuit easy, God bless you.

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ABSTRACT

Adding parameters to an existing and well established distribution helps in obtaining new compound distributions with more flexibility.In this study, a new distribution called theTransmuted Power-Lomax Distribution(TPLD) as an extension of the popular Lomax distribution in its power transformation-form using the Quadratic rank transmutation map is proposed.Using the transmutation map, we defined the Probability Density Function (*PDF*) and Cumulative Distribution Function (*CDF*) of the Transmuted Power Lomax Distribution. Some properties of the new distribution such as moments, moment generating function,characteristic function,quantile function, survival function, hazard function and order statistics were also studied. The distribution's parameters werealsoestimatedusing the maximum likelihoodmethod of estimation. The performance of the proposed probability distribution was checked in comparison with some other generalizations of Lomax distribution using two real life dataset. The results obtainedindicated that *TPLD*performsbetterthan the other distributions comprisingPower Lomax, and the Lomax distribution. The TPLD is recommended for modeling positively skewed dataset with higher peaks and tails. It is also recommended for analyzing age dependent variables.

TABLE OF CONTENTS

Contents

LIST OF FIGURES

LIST OF TABLES

CHAPTER ONE

INTRODUCTION

1.1 Background to the Study

The Lomax or Pareto II distribution is well-known for its use in analyzing and modeling skewed datasets (Lomax,1954). It was specifically introduced to model business failure data, however it has since gained wide application in a variety of contexts. Hassan and Al-Ghamdi (2009) used it for reliability modeling and life testing. The distribution has been used for modeling different data which has beenstudied by so many authors, Harris (1968) used Lomax distribution for income and wealth data.Atkinson and Harrison (1978) used it for modelling business failure data, while Corbelini*et al*., (2007) used it to model firm size and queuing problems. It has also found application in the biological sciences and even for modeling the distribution of the sizes of computer files on servers (Holland *et al.*,2006). Bryson (1974), suggested the use of this distribution as an alternative to the exponential distribution when the data is heavy-tailed.

Researchers over the years have been able to introduce several extensions of the Lomax distribution by way of using a particular class of generator that suites their purpose to introduce a new generalization of the Lomaxfamily,with the aim of obtaining distributions with better performance. Some examples are highlighted in the literature review. Lomax distribution could be used as the basis for generalizations. Al-Awadi and Ghitany (2001) used it as a generator (mixing distribution for the poisson parameter) to derive a discrete Poisson-Lomax distribution.

Adding parameters to an existing and well established distribution helps in obtaining new compound distributions with more flexibility.Recently, a three-parameter continuous distribution namely, Power Lomax (POLO) distribution was proposed by Rady*et al*., (2016).A power parameter was added to the two parameters of the Lomax distribution which further increased the flexibility and usefulness of the Lomax family.

Shaw and Buckley (2007) pioneered a powerful technique that could be used to induce a new parameter to a baseline distribution.This interesting method gave rise to a flexible family of distributions known as the Transmuted Family of distributions also called Transmuted Extended distributions.This new family of distributions has received increased attention in recent times as a lot of research has been attributed to developing transmuted models.The proposed model under study also belongs to this family.

1.2 Statement of the Problem

The effectiveness of a probability distribution in modelling and statistical analysis depends on how well it is able to capture the sensitive part of a given dataset. A major challenge of probability theory is that some of the existing and well known probability distributions are not efficient enough in modelling real datasets due to the nature of these datasets that makes them not to properly fit into some of these classical probability distributions thereby reducing their effectiveness in statistical analysis.This therefore creates room for the introduction of compound probability distributions which could address that limitation.

This study intends to propose a Transmuted Power Lomax Distribution which will provide a compound distribution with well established characteristics for a life time model,offer more distributional flexibility and could also compete with other distributions in terms of performance as reported in literature.

1.3 Aim and Objectives of the Study

The aim of this work is to propose and study a Transmuted Power-Lomax Distribution (TPLD).Thestated aim will be achieved through the following objectives, to:

i. derive the pdf and cdf of the proposed model,

ii. derive some statistical properties of the TPLD such as the moments, the moment generating function, characteristics function, the survival function, hazard function, quantile function and distribution of order statistics,

iii. to obtain non linear system of equations(partial derivative of the log likelihood function with respect to each parameter) which could be used to estimate the model's parameters,

iv. evaluate the performance of the new model using two real life datasets.

1.4 Scope and Limitation of Study

This research basically focused on extending a Power Lomax distribution,derived some selected properties of the proposed distribution such as moment,moment generating function,quantilefunction,survival function and the hazard function.It also derived density functions for the order statistics and estimate the model parameters using the method of maximum likelihood estimation,and finally accessed the performance of the proposed distribution compared to other extensions of Lomax distribution.

This research could not study some characteristics of a probability distribution which includes the limiting behavior of the distribution and expressions for R'enyi entropyamong others.

1.5 Significance of the Study

This dissertationproposes a new extension of the Lomax distribution called transmuted Power Lomax distribution, studies some of its properties and evaluates its performance.A major importance of this study is the fact that it will increase the robustness,effectiveness and applicability of the classical Lomax distribution. This research will use two different real datasets to illustrate the usefulness of the proposed model.

1.6 Motivation

Our motivation stems from the Quadratic Rank Transmutation Map which was introduced by Shaw and Buckley (2007) which can conveniently transmute any baseline continuous distribution.It was realised from their findings that Quadratic transmutation map is a powerful technique that could be used to introduce a parameter into an existing distribution with the aim of increasing its robustness.

More so,Merovci et al., (2014) clearly stated the reason for transmuting a standard distribution is because the transmuted form provides greater flexibility in modelling real life datasets.In view of that,we were motivated to propose and study the TPLD.

CHAPTER TWO

LITERATURE REVIEW

2.1 Extensions of Lomax Distribution

Researchers over the years have tried to improve the applicability of Lomax distribution by introducing new generalizations of the Lomax family.These among others includes theMarshall–Olkin extended-Lomax by Ghitany*et al*., (2007),Exponentiated Lomax by Abdul-Moniem (2012), Gamma-Lomax by Cordeiro*et al*., (2013),Al-Zahrania and Sagorb (2014) introduced Poisson-Lomax distribution, El-Bassiouny*et al.,*(2015) introduced Exponential Lomax distribution, Tahir*et al.,* (2015) introduced the four parameters Weibull Lomax distribution and the Extended Poisson-Lomax distribution was introduced by Al-Zahrani (2015).

Al-Zahrania and Sagorb (2014) proposed a distribution referred to as the Poisson Lomax distribution. A mathematical treatment of the proposed distribution including explicit formulas for the density and hazard functions, moments, order statistics, mean and median deviations were provided. The estimation of the parameters was done by maximum likelihood. Also, the asymptotic variance-covariance matrix of the estimates was obtained. Finally, a real data set was analyzed to show the potential of the proposed Poisson Lomax distribution. The result indicates that the Poisson Lomax distribution may be used for a wider range of statistical applications. The authors mentioned some of possible directions which are still open for further works on the same distribution.

El-Bassiouny*et al*. (2015) introduced a generalization of Lomax distribution called Exponential Lomax distribution. Some statistical properties of this distribution were derived and discussed,the maximum likelihood estimators of the parameters were derived. A real dataset was analyzed using the new distribution and its performance was compared with that of the

Exponentiated Lomax, Marshall-Olkin extended-Lomax, beta-Lomax, Kumaraswamy-Lomax, McDonald-Lomax and Gamma-Lomax. Based on the comparisons between all these models, they concluded that the newly introduced model performed better.

Tahir*et al*., (2015) proposed a four-parameter Weibull-Lomax (WL) distribution. They studied some structural properties of the WL distribution including an expansion for the density function and explicit expressions for the ordinary and incomplete moments, mean residual life, mean waiting time, moments generating function and quantile function. The explicit expressions for R´enyi entropy, q entropy and order statistics were also derived.The maximum likelihood method was employed for estimating the model parameters.They also obtained the observed information matrix.Theyfitted the WL model to two real life data sets to show the usefulness of the proposed distribution.The new model provided consistently a better fit than the other models namely: the McDonald-Lomax, Kumaraswamy-Lomax, Gamma-Lomax, Beta-Lomax, Exponentiated-Lomax and Lomax distributions. They posited that the proposed model will attract wider application in areas such as engineering, survival and lifetime data, hydrology, economics (income inequality) and others.

N.M Kilany (2016) proposed and studied the Weighted Lomax distribution.The density function and its behavior, moments, hazard and survival functions, mean residual life and reversed failure rate,extreme values distribution and order statistics were derived and studied.Parameters were estimated using maximum likelihood method and the observed information matrix was derived.Finally,an application of the model to a real dataset was presented and compared with some other well-known distributions.

Rady*et al*., (2016) introduced a three parameter Power Lomax Distribution. The new distribution could exhibit a much more flexible model for life time data especially bladder cancer data than its predecessor Lomax distributions, presenting decreasing, inverted bath tub hazard rate function. Some statistical and reliability properties were derived and studied. Simulation schemes were formulated and provided less bias and mean square error as sample size increases for MLEs of power Lomax distribution parameters. Point Estimation through the method of moments and MLE methods were done. Moreover, the Fisher information matrix for interval estimation was studied for power Lomax distribution. A real data on bladder cancer was used to illustrate and compare the potential of power Lomax distribution with other competing distributions and the results showed that it could offer a better fit than a set of extensions of Lomax distribution.

2.2 The Transmuted Family of Distributions

Recent studies have shown increasing interest of researchers in transmuted distributions. Significant number of work has been attributed towards developing a new transmuted model and subsequently demonstrating its usefulness as regards flexibility in modelling various types of real life data.

Shaw and Buckley (2007) pioneered the family of transmuted distributions using Quadratic Rank Transmutation Map (QRTM) to generate a flexible family of distributions.Theydiscussed the concept of distributional Alchemy which is defined by transmutation maps. They also gave a general definition for Rank Transmutation maps as the functional composition of the *cdf* of one distribution (Base) with the inverse *cdf* (quantile function) of another. The general characteristic of the Transmuted Family was studied by Bourguignon *et al*., (2016). Below are some examples of the transmuted family distributions.

Ashour and Eltehiwy (2013a) introduced a generalization of Exponentiated Lomax distribution called the transmuted Exponentiated Lomax distribution. The distribution was generated using the quadratic rank transmutation map and taking the Exponentiated Lomax distribution as the base distribution. Some mathematical properties along with estimation issues were addressed. The hazard rate function and reliability behavior of the transmuted Exponentiated Lomax distribution showed that the subject distribution can be used to model reliability data. The authors expect that this study will serve as a reference and help to advance future research in the subject area.

Ashour and Eltehiwy (2013b) introduced another generalization of Lomax distribution called the Transmuted Lomax distribution. The distribution is generated by using the quadratic rank transmutation map and taking the Lomax distribution as the base distribution. Some mathematical properties along with estimation issues were addressed. The hazard rate function and reliability behavior of the transmuted Lomax distribution shows that the subject distribution could be used to model reliability data.

Merovci*et al.*, (2014) developed a new distribution called Transmuted Pareto distribution which extends the Pareto distribution in the analysis of data of real support.A clear reason for transmuting a standard distribution lies in the fact that, the transmuted form provides greater flexibility in modelling real life datasets. They derived some properties of the new distribution which includes expansions for the expectation, variance, moments and moment generating function.Parameters were estimated using maximum likelihood approach.The likelihood ratio statistics was used to compare the model with its baseline form.The model was then applied to real data to demonstrate its usefulness,the result obtained showed that the proposed model provides a better fit than the Pareto distribution.

Oguntude and Adejumo (2015), introduced a two-parameter probability model called Transmuted Inverse Exponential (TIE) distribution using the Quadratic Rank Transmutation

Map. Properties of the proposed model were systematically studied (such as moments, moment generating function, quantile function, reliability function and hazard function). Parameters were estimated using the method of maximum likelihood. The hazard function of the model has an inverted bathtub shape. They proposed the usefulness of the TIE distribution in modelling breast and bladder cancer data sets.

Owoloko*et al*., (2015) investigated the performance rating of the Transmuted Exponential (TE) distribution with regard to some other generalized models.The TE distribution appeared to be better than the Beta Exponential distribution, Generalised Exponential distribution in terms of flexibility when applied to real life data.

Khan *et al*., (2016) proposed a generalization of the Kumaraswamy distribution referred to as the Transmuted Kumaraswamy (TK) distribution. The new transmuted distribution was developed using the quadratic rank transmutation map studied by Shaw and Buckley (2007). A comprehensive account of the mathematical properties of the new distribution was provided. Explicit expressions were derived for the moments, moment generating function, entropy, mean deviation, Bonferroni Lorenz curves, and moments for order statistics were derived. The TK distribution parameters were estimated by using the method of maximum likelihood. Monte Carlo simulation was performed in order to investigate the performance of the MLEs. A flood and HIV/ AIDS data were used to illustrate the usefulness of the proposed model.

These literatures basically focused on introducing new probability distributions that could accommodate certain characteristics of real datasets(skewness),derived some of their characteristics and estimate their parameters mostly via method of maximum likelihood. Real dataset would be used to illustrate the usefulness of the new model and recommendations made based on findings.Performance of some of these new models were evaluated and compared with other models using some of the popular and mostly used information criterion.

With good understanding of these literatures reviewed,we propose a Transmuted Power-Lomax distribution which seeks to increase the flexibility of the Power Lomax distribution using the method proposed by Shaw and Buckley.Its application would be demonstrated using two real datasets and the performance compared with that of Lomax,Power-Lomax and the four parameter Weibull Lomax distributions.

CHAPTER THREE

METHODOLOGY

3.1 Definition of the Lomax Distribution

A random variable X is said to follow a Lomax distribution with parameter α *and* Θ if its probability density function (*pdf*) is given as

$$f(x) = \frac{\alpha}{\theta}\left[1+\left(\frac{x}{\theta}\right)\right]^{-(\alpha+1)} \quad (3.1)$$

and the corresponding cumulative distribution function (*cdf*) is given as

$$F(x) = 1-\left[1+\left(\frac{x}{\theta}\right)\right]^{-\alpha} \quad (3.2)$$

for $x > 0, \alpha > 0, \theta > 0$ .whereα *and* θ are the shape and scale parameters respectively

3.2 Definition of the Power Lomax (POLO) Distribution

The POLO distribution was defined by considering the power transformation $X = T^{\frac{1}{\beta}}$, where the random variable T is said to have followed the Lomax distribution withparametersα, Θ. The distribution of $X$is referred to as POLO distribution. The *pdf* of POLO distribution is given as

$$g(x) = \alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}, x > 0, \alpha, \beta, \theta > 0 \quad (3.3)$$

The corresponding cumulative distribution function (*cdf*) of Power Lomax distribution is given as

$$G(x) = 1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}, x > 0, \alpha, \beta, \theta > 0 \quad (3.4)$$

Whereα, Θ and β are the shape, scale and power parameter respectively.

3.3 Definition of the Transmuted Power Lomax Distribution (*TPLD*)

The *cdf* and *pdf* of the Transmuted Power Lomax distribution are obtained using the technique proposed by Shaw and Buckley (2007). A random variable *X* is said to have a transmuted distribution function if its *pdf* and *cdf* are respectively given as

$$f(x)=g(x)\left[(1+\lambda)-2\lambda G(x)\right] \tag{3.5}$$

and

$$F(x)=(1+\lambda)G(x)-\lambda[G(x)]^2 \tag{3.6}$$

$G(x)$ is the *cdf* of any continuous distribution, $f(x)$ and $g(x)$ are the associated *pdf* of $F(x)$ and $G(x)$ respectively. λ is the added transmuted parameter.In order to obtain the *pdf* of the TPLD, we would substitute equations (3.3) and (3.4) into (3.5) and simplify. Hence the proposed *pdf* is given as

$$f(x)=\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}\left[(1+\lambda)-2\lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)\right] \tag{3.7}$$

Also,by substituting (3.4) into (3.6) we obtain the *cdf* of the TPLD.And that is given as

$$F(x)=(1+\lambda)\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)-\lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)^{2} \tag{3.8}$$

Where $x>0,\alpha>0,\theta>0,\beta>0,-1\le\lambda\le 1$ α *and* Θ are the shape and scale parameters respectively,β is the power parameter and λ is called the transmuted parameter.

3.4 Model validity check

To check the validity of the proposed model,we use the equation below:

$$\int_0^{\infty} f(x)dx=1 \tag{3.9}$$

Proof

$$\int_0^\infty f(x)dx = \int_0^\infty \alpha\beta\theta^\alpha x^{\beta-1}(\theta+x^\beta)^{-(\alpha+1)}[(1+\lambda)-2\lambda(1-\theta^\alpha(x^\beta+\theta)^{-\alpha})]dx \quad (3.10)$$

Let $u = x^\beta + \theta$ (3.11)

$$\frac{du}{dx} = \beta x^{\beta-1} \quad (3.12)$$

$$dx = \frac{du}{\beta x^{\beta-1}} \quad (3.13)$$

Upper limit

$x = \infty, u = \infty$

$x = 0, u = \theta$

$$\int_0^\infty f(x)dx = \int_\theta^\infty x\beta\theta^\alpha x^{\beta-1}u^{-(\alpha+1)}[(1+\lambda)-2\lambda(1-\theta^\alpha u^{-\alpha})]\frac{du}{\beta x^{\beta-1}} \quad (3.14)$$

$$\int_0^\infty f(x)dx = \int_\theta^\infty \alpha\theta^\alpha u^{-(\alpha+1)}[(1+\lambda)-2\lambda(1-\theta^\alpha u^{-\alpha})]du \quad (3.15)$$

$$\int_0^\infty f(x)dx = \alpha\theta^\alpha\int_\theta^\infty u^{-(\alpha+1)}[(1+\lambda)-2\lambda(1-\theta^\alpha u^{-\alpha})]du \quad (3.16)$$

$$\int_0^\infty f(x)dx = \alpha\theta^\alpha\int_\theta^\infty (1+\lambda)u^{-(\alpha+1)} - 2\lambda u^{-(\alpha+1)}(1-\theta^\alpha u^{-\alpha})du \quad (3.17)$$

$$\int_0^\infty f(x)dx = \alpha\theta^\alpha(1+\lambda)\int_\theta^\infty u^{-(\alpha+1)}du - 2\alpha\lambda\theta^\alpha\int_\theta^\infty u^{-(\alpha+1)}du + 2\alpha\lambda\theta^{2\alpha}\int_\theta^\infty u^{-(2\alpha+1)}du \quad (3.18)$$

$$\int_0^\infty f(x)dx = \alpha\theta^\alpha(1+\lambda)\left|\frac{u^{-\alpha}}{-\alpha}\right|_\theta^\infty - 2\alpha\lambda\theta^\alpha\left|\frac{u^{-\alpha}}{-\alpha}\right|_\theta^\infty + 2\alpha\lambda\theta^{2\alpha}\left|\frac{u^{-\alpha}}{-2\alpha}\right|_\theta^\infty \quad (3.19)$$

$$\int_0^\infty f(x)dx = \theta^\alpha (1+\lambda)\left| -\frac{1}{u^\alpha} \right|_\theta^\infty - 2\lambda\theta^\alpha \left| -\frac{1}{u^\alpha} \right|_\theta^\infty + \lambda\theta^{2\alpha} \left| -\frac{1}{u^{2\alpha}} \right|_\theta^\infty \tag{3.20}$$

$$\int_0^\infty f(x)dx = \theta^\alpha (1+\lambda)\left(0+\frac{1}{\theta^\alpha} \right) - 2\lambda\theta^\alpha \left(0+\frac{1}{\theta^\alpha} \right) + \lambda\theta^{2\alpha} \left(0+\frac{1}{\theta^{2\alpha}} \right) \tag{3.21}$$

$$\int_0^\infty f(x)dx = \theta^\alpha (1+\lambda)\left(\frac{1}{\theta^\alpha} \right) - 2\lambda\theta^\alpha \left(\frac{1}{\theta^\alpha} \right) + \lambda\theta^{2\alpha} \left(\frac{1}{\theta^{2\alpha}} \right) \tag{3.22}$$

$$\int_0^\infty f(x)dx = 1+\lambda-2\lambda+\lambda = 1 \tag{3.23}$$

The*pdf* and *cdf*of the *TPLD* using some parameter values are displayed in **fig. 3.1** and **3.2** as follows:

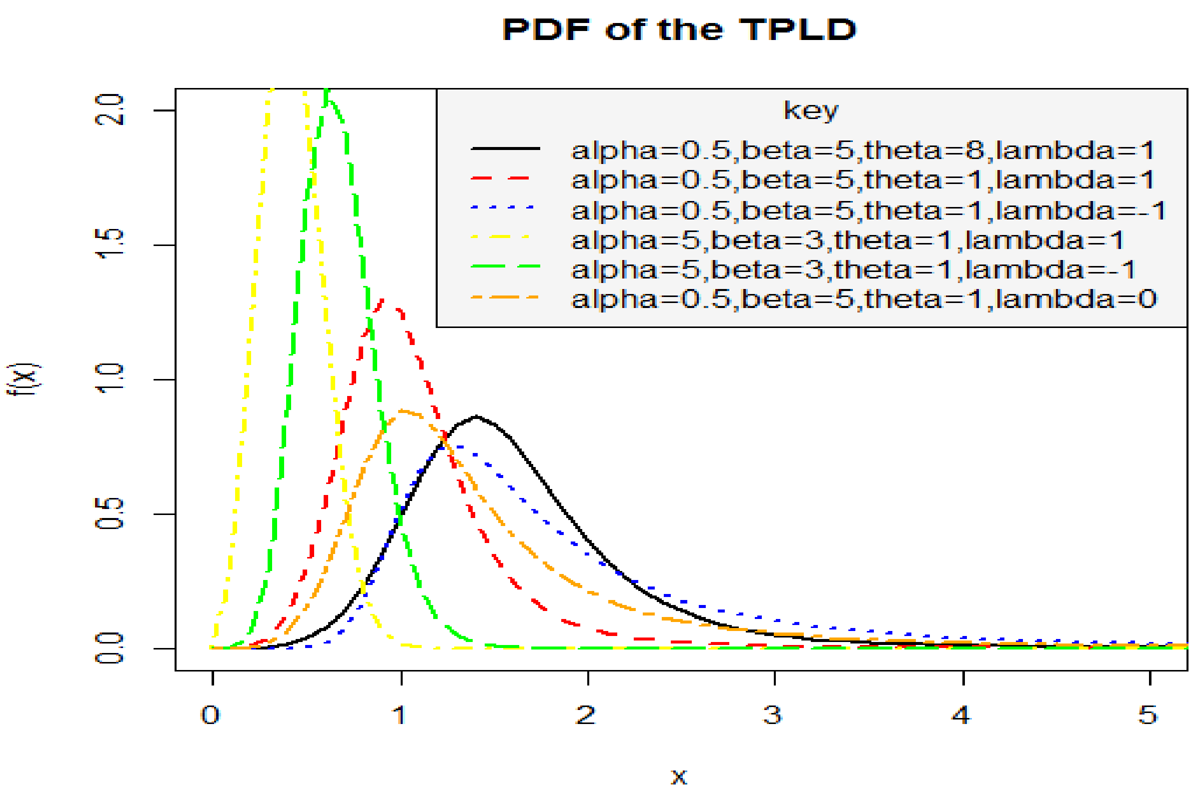

Fig. 3.1: **The graph of *pdf*of the *TPLD* at different parameter values**

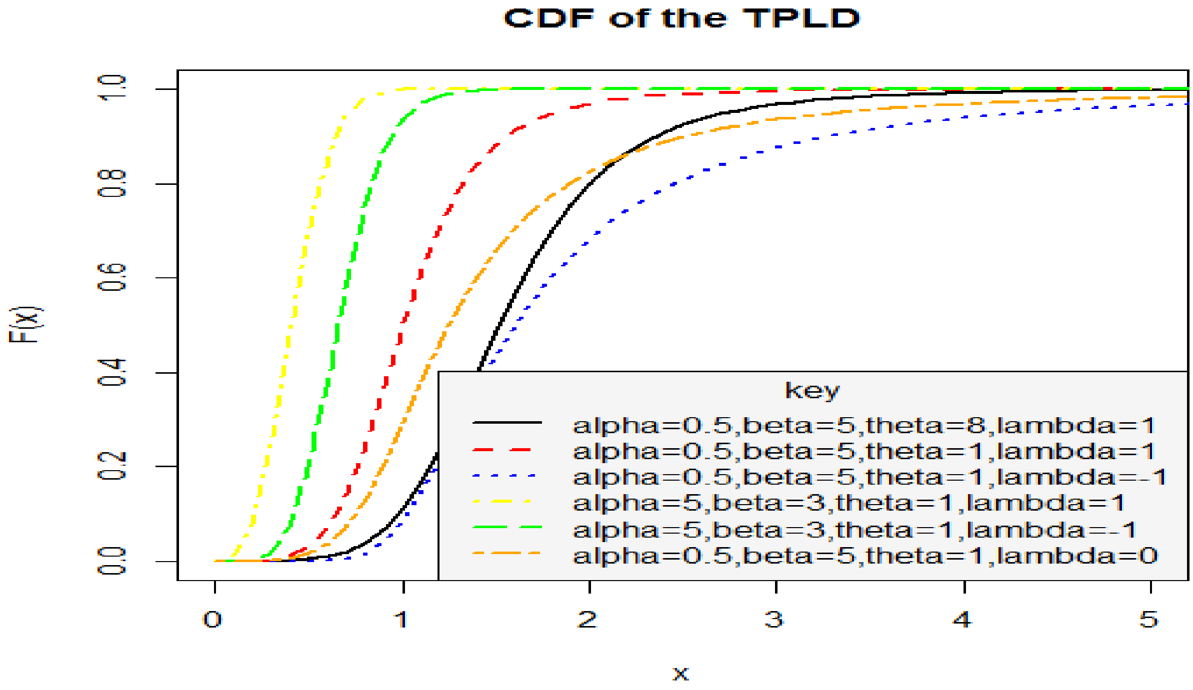

Fig. 3.2: **The graph of *cdf*of the *TPLD*atsome parameter values**

The plot for the *pdf* reveals that the *TPLD* is positively skewed and therefore will be a good model for positively skewed data sets.

Some Properties of the Transmuted Power Lomax Distribution

3.5 Moments, Mean and Variance

Let X denote a continuous random variable that follows the TPLD, the n^{th} moment of X is given by;

$$\mu_n' = E\left(X^n\right) = \int_0^\infty x^n f(x)dx \tag{3.24}$$

Where *f(x)* is as given in equation (3.7)

but

$$f(x)=\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}\left[1+\lambda-2\lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)\right]$$

Expansion and simplification of the *pdf* gives:

$$f(x)=\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}[1+\lambda-2\lambda+2\lambda\theta^{\alpha}\left(\theta+x^{\beta}\right)^{-\alpha}] \quad (3.25)$$

$$f(x)=\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}[1-\lambda+2\lambda\theta^{\alpha}\left(\theta+x^{\beta}\right)^{-\alpha}] \quad (3.26)$$

$$f(x)=\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}-\lambda\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}+2\lambda\alpha\beta\theta^{2\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-2\alpha-1} \quad (3.27)$$

$$f\left(x\right)=\left(1-\lambda\right)\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}+2\lambda\alpha\beta\theta^{2\alpha}x^{\beta-1}\left(x^{\beta}+\theta\right)^{-2\alpha-1} \quad (3.28)$$

Hence,

$$\mu_{n}^{'}=E\left(X^{n}\right)=\int_{0}^{\infty}x^{n}f(x)dx=\int_{0}^{\infty}X^{n}\left\{\left(1-\lambda\right)\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}+2\lambda\alpha\beta\theta^{2\alpha}x^{\beta-1}\left(x^{\beta}+\theta\right)^{-2\alpha-1}\right\}dx \quad (3.29)$$

$$\mu_{n}^{'}=\int_{0}^{\infty}\left(1-\lambda\right)\alpha\beta\theta^{\alpha}x^{n+\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}dx+\int_{0}^{\infty}2\lambda\alpha\beta\theta^{2\alpha}x^{n+\beta-1}\left(x^{\beta}+\theta\right)^{-2\alpha-1}dx \quad (3.30)$$

$$\mu_{n}^{'}=\left(1-\lambda\right)\alpha\beta\theta^{\alpha}\int_{0}^{\infty}x^{n+\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}dx+2\lambda\alpha\beta\theta^{2\alpha}\int_{0}^{\infty}x^{n+\beta-1}\left(x^{\beta}+\theta\right)^{-2\alpha-1}dx \quad (3.31)$$

Using integration by substitution in (3.31) above, Let

$$y=\theta+x^{\beta}$$

Where

$$\frac{dy}{dx}=\beta x^{\beta-1}$$

$$dx = \frac{dy}{\beta x^{\beta-1}}$$

Therefore, substituting for *dx* in equation (3.31) and simplifying, we obtain

$$\mu_n^{'} = (1-\lambda)\alpha\theta^{\alpha}\int_0^{\infty} x^n y^{-(\alpha+1)}dy + 2\lambda\alpha\theta^{2\alpha}\int_0^{\infty} x^n y^{-2\alpha-1}dy \qquad (3.32)$$

Recall from the previous statements that *x* can be express in terms of *y* as follows:

$$y = \theta + x^{\beta}$$

$$\Rightarrow (y-\theta) = x^{\beta}$$

$$\Rightarrow x = (y-\theta)^{\frac{1}{\beta}} = \left\{\theta\left(\frac{y}{\theta}-1\right)\right\}^{\frac{1}{\beta}}$$

$$x = \left\{(-1)\theta\left(1-\frac{y}{\theta}\right)\right\}^{\frac{1}{\beta}}$$

$$\Rightarrow x = (-1)^{\frac{1}{\beta}}\theta^{\frac{1}{\beta}}\left(1-\tfrac{y}{\theta}\right)^{\frac{1}{\beta}} \qquad (3.33)$$

Substituting for x in (3.32), we have:

$$\mu_n^{'} = (1-\lambda)\alpha(-1)^{\frac{n}{\beta}}\theta^{\frac{n}{\beta}+\alpha}\int_0^{\infty}\left(1-\tfrac{y}{\theta}\right)^{\frac{n}{\beta}} y^{-(\alpha+1)}dy + 2\lambda\alpha(-1)^{\frac{n}{\beta}}\theta^{\frac{n}{\beta}+2\alpha}\int_0^{\infty}\left(1-\tfrac{y}{\theta}\right)^{\frac{n}{\beta}} y^{-2\alpha-1}dy \qquad (3.34)$$

$$\mu_n^{'} = (1-\lambda)\alpha(-1)^{\frac{n}{\beta}}\theta^{\frac{n}{\beta}+\alpha}\int_0^{\infty} y^{-(\alpha+1)}\left(1-\tfrac{y}{\theta}\right)^{\frac{n}{\beta}}dy + 2\lambda\alpha(-1)^{\frac{n}{\beta}}\theta^{\frac{n}{\beta}+2\alpha}\int_0^{\infty} y^{-2\alpha-1}\left(1-\tfrac{y}{\theta}\right)^{\frac{n}{\beta}}dy \qquad (3.35)$$

$$\mu_n^{'} = (1-\lambda)\alpha\theta^{\frac{n}{\beta}}\int_0^{\infty}\left(\tfrac{y}{\theta}\right)^{\alpha-\frac{n}{\beta}-1}\left(1-\tfrac{y}{\theta}\right)^{\frac{n}{\beta}+1-1}dy + 2\lambda\alpha\theta^{\frac{n}{\beta}}\int_0^{\infty}\left(\tfrac{y}{\theta}\right)^{2\alpha-\frac{n}{\beta}-1}\left(1-\tfrac{y}{\theta}\right)^{\frac{n}{\beta}+1-1}dy \qquad (3.36)$$

Recall that:

$$B(x,y)=\int_0^{\infty} t^{x-1}(1-t)^{y-1}\,dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (3.37)$$

Hence,

$$\mu_n^{'}=(1-\lambda)\alpha\theta^{\frac{n}{\beta}}\frac{\Gamma\left(\alpha-\frac{n}{\beta}\right)\Gamma\left(\frac{n+\beta}{\beta}\right)}{\Gamma(\alpha+1)}+2\lambda\alpha\theta^{\frac{n}{\beta}}\frac{\Gamma\left(2\alpha-\frac{n}{\beta}\right)\Gamma\left(\frac{n+\beta}{\beta}\right)}{\Gamma(2\alpha+1)} \quad (3.38)$$

The Mean

The mean of the *TPLD* can be obtained from the n^{th} moment of the distribution when $n=1$ as follows:

$$\mu_1^{'}=(1-\lambda)\alpha\theta^{\frac{1}{\beta}}\frac{\Gamma\left(\alpha-\frac{1}{\beta}\right)\Gamma\left(\frac{1+\beta}{\beta}\right)}{\Gamma(\alpha+1)}+2\lambda\alpha\theta^{\frac{1}{\beta}}\frac{\Gamma\left(2\alpha-\frac{1}{\beta}\right)\Gamma\left(\frac{1+\beta}{\beta}\right)}{\Gamma(2\alpha+1)} \quad (3.39)$$

Also the second moment of the *TPLD* is obtained from the n^{th} moment of the distribution when $n=2$ as

$$\mu_2^{'}=(1-\lambda)\alpha\theta^{\frac{2}{\beta}}\frac{\Gamma\left(\alpha-\frac{2}{\beta}\right)\Gamma\left(\frac{2+\beta}{\beta}\right)}{\Gamma(\alpha+1)}+2\lambda\alpha\theta^{\frac{2}{\beta}}\frac{\Gamma\left(2\alpha-\frac{2}{\beta}\right)\Gamma\left(\frac{2+\beta}{\beta}\right)}{\Gamma(2\alpha+1)} \quad (3.40)$$

The Variance

The n^{th} central moment or moment about the mean of *X*, say μ_n, can be obtained as

$$\mu_n=E\left(X-\mu_1^{'}\right)^n=\sum_{i=0}^{n}(-1)^i\binom{n}{i}\mu_1^{'i}\mu_{n-i}^{'} \quad (3.41)$$

The variance of *X* for *TPLD* is obtained from the central moment when $n=2$, that is,

$$Var(X)=E\left(X^2\right)-\left\{E(X)\right\}^2 \quad (3.42)$$

$$Var(X)=\mu_2^{'}-\left\{\mu_1^{'}\right\}^2 \quad (3.43)$$

Where $\mu_1^{'}$ and $\mu_2^{'}$ are the mean and second moment of the *TPLD* all obtainable from equation (3.38)

3.6 Moment Generating Function (mgf)

The moment generating function is an important shape characteristic of a distribution and is always in one function that represents all the moments. In other words, the mgf produces all the moments of a random variable

The mgf of the TPLD is derived below:

$$M_x(t)=E\left(e^{tx}\right)=\int_0^\infty e^{tx}f(x)dx \tag{3.44}$$

$$M_x(t)=E\left(e^{tx}\right)=\int_0^\infty e^{tx}\left((1-\lambda)\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}+2\lambda\alpha\beta\theta^{2\alpha}x^{\beta-1}\left(x^{\beta}+\theta\right)^{-2\alpha-1}\right)dx \tag{3.45}$$

According to Maclaurin's series expansion

$$e^{tx}=\sum_{n=0}^{\infty}\frac{t^n x^n}{n!} \tag{3.46}$$

$$M_x(t)=E\left(e^{tx}\right)=\sum_{n=0}^{\infty}\frac{t^n}{n!}E(x^n) \tag{3.47}$$

$$M_x(t)=E\left(e^{tx}\right)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\mu_n^{'} \tag{3.48}$$

where

$$\mu_n^{'}=(1-\lambda)\alpha\theta^{\frac{n}{\beta}}\frac{\Gamma\left(\alpha-\frac{n}{\beta}\right)\Gamma\left(\frac{n+\beta}{\beta}\right)}{\Gamma(\alpha+1)}+2\lambda\alpha\theta^{\frac{n}{\beta}}\frac{\Gamma\left(2\alpha-\frac{n}{\beta}\right)\Gamma\left(\frac{n+\beta}{\beta}\right)}{\Gamma(2\alpha+1)} \tag{3.49}$$

is as defined in equation (3.38) previously

3.7 QUANTILE FUNCTION

The quantile function Q of a distribution is the inverse of its cumulative distribution function F. In probability and statistics, the quantile specifies for a given probability in the probability distribution of a random variable,the value at which the probability of the given random variable is less than or equal to the given probability.It could also be used to find the median, skewness, kurtosis of a distribution, as well as to simulate random numbers of a distribution.

Let $Q(q) = F^{-1}(q)$ be the quantile function of $F(x)$ for $0<q<1$, solving for $F(x)=q$, the quantile function of x gives:

$$x = Q(q) = F^{-1}(q) \tag{3.50}$$

Quantile function of the TPLD is derived below:

$$q = F(x) \tag{3.51}$$

Rewriting equation (3.51) gives:

$$(1+\lambda)\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)-\lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)^{2}+q=0 \tag{3.52}$$

Going by

λ

Simplifying the equation above will give a negative and positive root.We would therefore futher our derivation with the negative root so as to avoid having a negative quantile function which is unacceptable.

Let

$$a=\lambda, b=-(1+\lambda), c=q, m=1-\theta^{\alpha}(x^{\beta}+\theta)^{-\alpha}$$

$$m = \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda} \tag{3.53}$$

Substituting *m*above into (3.7.4) gives:

$$1 - \theta^{\alpha}(x^{\beta} + \theta)^{-\alpha} = \left[\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda} \right] \tag{3.54}$$

$$\theta^{\alpha}(x^{\beta} + \theta)^{-\alpha} = 1 - \left[\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda} \right] \tag{3.55}$$

$$(x^{\beta} + \theta)^{-\alpha} = \frac{1 - \left[\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda} \right]}{\theta^{\alpha}} \tag{3.56}$$

Let

$$a = \frac{1 - \left[\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda} \right]}{\theta^{\alpha}} \tag{3.57}$$

It implies that,

$$x^{\beta} + \theta = a^{\frac{1}{\alpha}} \tag{3.58}$$

$$x^{\beta} = a^{\frac{1}{\alpha}} - \theta \tag{3.59}$$

Therefore,

$$x_q = \left(a^{\frac{1}{\alpha}} - \theta \right)^{\frac{1}{\beta}} \tag{3.60}$$

Is the derived Quantile function of the TPLD.

The median being quantile of order $\frac{1}{2}$ is derived when q=0.5

To derive the median

$$x_{0.5} = \left(a^{\frac{1}{\alpha}} - \theta \right)^{\frac{1}{\beta}}$$

and

$$a = \frac{1 - \left[(1+\lambda) - \frac{\sqrt{(1+\lambda)^2 - 4\lambda(0.5)}}{2\lambda} \right]}{\theta^{\alpha}}$$

3.8 Characteristics Function

This is useful and has some properties which gives it a genuine role in mathematical statistics. It is used for generating moments, characterization of distributions and in analysis of linear combination of independent random variables.

The characteristics function of a random variable X is given by;

$$\varphi_x(t) = E\left(e^{itx}\right) = E\left[\cos(tx) + i\sin(tx)\right] = E\left[\cos(tx)\right] + E\left[i\sin(tx)\right] \quad (3.61)$$

Recall from power series expansion that

$$\cos(tx) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} x^{2n} \quad (3.62)$$

$$E\left[\cos(tx)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \mu'_{2n} \quad (3.63)$$

$$E\left(x^{2n}\right)=\mu'_{2n} \tag{3.64}$$

Since

And also that

$$\sin(tx)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}x^{2n+1}$$

$$E\left[\sin(tx)\right]=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\mu'_{2n+1}$$

Simple algebra and power series expansion proves that

$$\phi_x(t)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}\mu'_{2n}+i\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\mu'_{2n+1}$$

Where μ'_{2n} and $\mu'_{2n+1}$are the moments of X for $n=2n$ and $n=2n+1$ respectively and can be obtained from μ'_n as

$$\mu'_{2n}=(1-\lambda)\alpha\theta^{\frac{2n}{\beta}}\frac{\Gamma\left(\alpha-\frac{2n}{\beta}\right)\Gamma\left(\frac{2n+\beta}{\beta}\right)}{\Gamma(\alpha+1)}+2\lambda\alpha\theta^{\frac{2n}{\beta}}\frac{\Gamma\left(2\alpha-\frac{2n}{\beta}\right)\Gamma\left(\frac{2n+\beta}{\beta}\right)}{\Gamma(2\alpha+1)} \tag{3.65}$$

and

$$\mu'_{2n+1}=(1-\lambda)\alpha\theta^{\frac{2n+1}{\beta}}\frac{\Gamma\left(\alpha-\frac{2n+1}{\beta}\right)\Gamma\left(\frac{2n+\beta+1}{\beta}\right)}{\Gamma(\alpha+1)}+2\lambda\alpha\theta^{\frac{2n+1}{\beta}}\frac{\Gamma\left(2\alpha-\frac{2n+1}{\beta}\right)\Gamma\left(\frac{2n+\beta+1}{\beta}\right)}{\Gamma(2\alpha+1)} \tag{3.66}$$

respectively.

3.9 Reliability Analysis of the TPLD

This is a statistical technique that is used to evaluate the treatment efficiency of fatal condition like cancer,time of machine breakdown etc.

3.9.1 Survival Function

Survival function is the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \tag{3.67}$$

Where $F(x)$ is *cdf* of the TPLD

$$F(x) = (1+\lambda)\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right) - \lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)^{2}$$

$$S(x) = 1 - \left\{(1+\lambda)\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right) - \lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)^{2}\right\} \tag{3.68}$$

$$S(x) = 1 - \left\{1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha} + \lambda - \lambda\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha} - \lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)^{2}\right\} \tag{3.69}$$

$$S(x) = \theta^{\alpha}(x^{\beta}+\theta)^{-\alpha} - \lambda + \lambda\theta^{\alpha}(x^{\beta}+\theta)^{-\alpha} + \lambda(1-\theta^{\alpha}(x^{\beta}+\theta)^{-\alpha})^{2} \tag{3.70}$$

$$S(x) = (1+\lambda)\left(\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right) + \lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)^{2} - \lambda \tag{3.71}$$

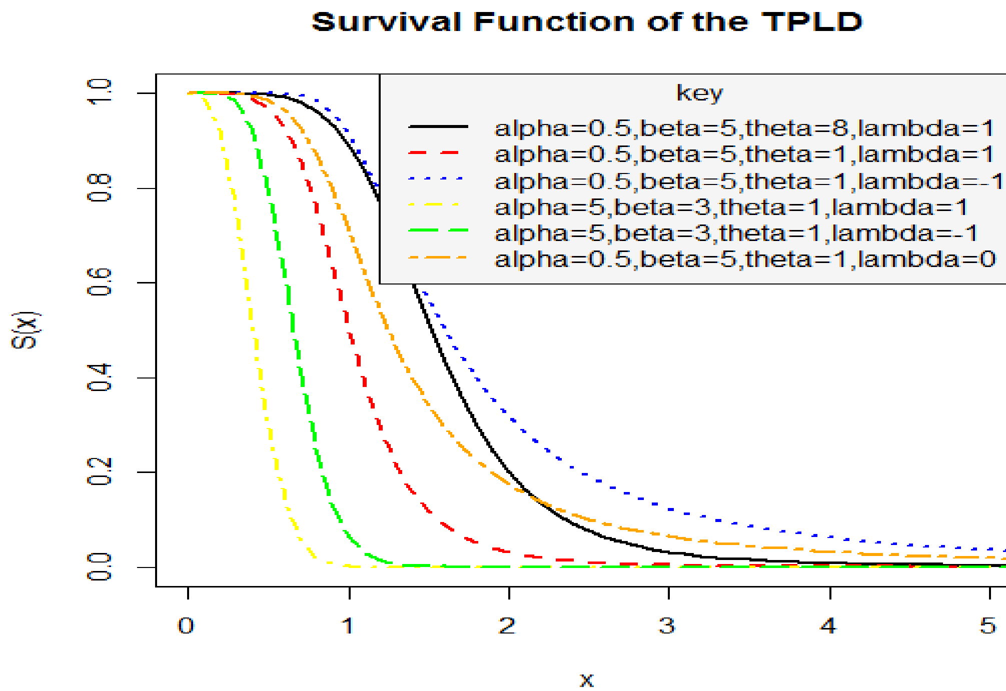

Fig. 3.3: Survival function of the *TPLD* at different parameter values.

The graph in fig.3**.**3 shows that the value of the survival function equals one (1) at initial time or early age and it decreases as*X* increases and equals zero (0) as *X* becomes larger.

3.9.2 Hazard Function

Hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} \qquad (3.72)$$

Taking *f(x)* and *F(x)* to be the *pdf* and *cdf* of the TPLD and substituting in (3.72) above gives;

$$f(x) = \alpha\beta\theta^{\alpha} x^{\beta-1}\left(\theta + x^{\beta}\right)^{-(\alpha+1)}\left[1+\lambda-2\lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)\right]$$

$$=\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}\left[1-\lambda+2\lambda\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right]$$

and

$$F(x)=(1+\lambda)\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)-\lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)^{2}$$

respectively.

Substituting for *f(x)* and *F(x)* and simplifying gives

$$h(x)=\frac{\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}\left[1-\lambda+2\lambda\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right]}{(1+\lambda)\left(\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)+\lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)^{2}-\lambda} \quad (3.73)$$

The following are some possible curves for the hazard rate at various values of the model parameters

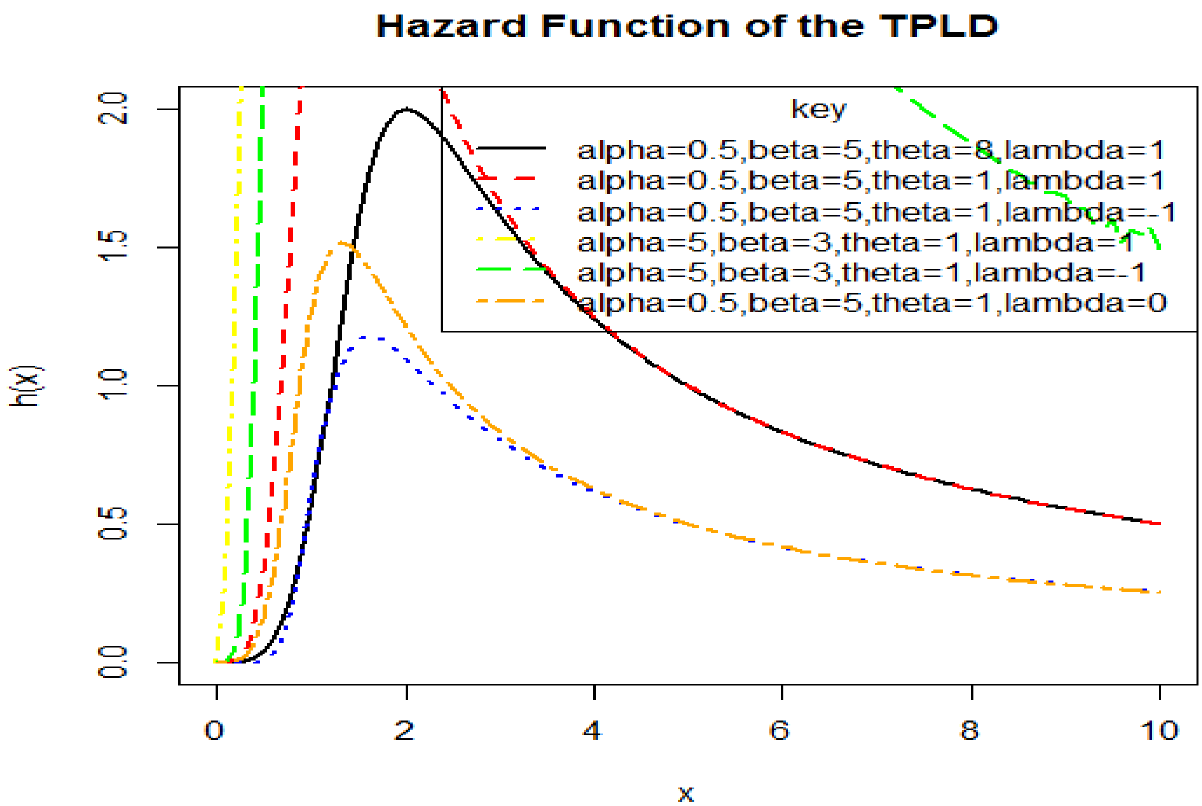

Fig. 3.4: Hazard function of the *TPLD* at different parameter values

We can see from figure 3.4 that the value of the hazard function increases at the beginning and slowly dropsas*X* increases. This means that the *TPLD* may be appropriate for modeling time dependent events, where risk or hazard increases at early stage and then drops with time.

3.10 Order Statistics

Sample values such as the smallest, largest, or middle observation from a random sample provide important information. For example, the highest rainfall, flood or minimum temperature recorded during past years might be useful when planning for future emergencies.Order Statistics could used to determine the distribution of the smallest (minimum) order statistic and the largest(maximum) order statistic of a given distribution. Let $X_{(1)}$ denote the smallest of

$X_1, X_2, ..., X_n$, $X_{(2)}$ denote the second smallest of $X_1, X_2, ..., X_n$, and similarly $X_{(i)}$ denote the i^{th} smallest of $X_1, X_2, ..., X_n$.Then the random variables $X_{(1)}, X_{(2)}, ..., X_{(n)}$, called the order statistics of the sample $X_1, X_2, ..., X_n$, has probability density function of the i^{th} order statistic, $X_{(i)}$, as:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \left[1 - F(x)\right]^{n-i} \tag{3.74}$$

Where $f\left(x\right)$ and $F\left(x\right)$ are the *pdf* and *cdf* of the *TPLD* respectively.

By binomial expansion

$$(1-F(x))^{n-i} = \sum_{k=0}^{n-i} \binom{n-i}{k} (-1)^k \left[f(x)\right]^k \tag{3.75}$$

Putting (3.75) in (3.74) we obtain

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k n!}{(i-1)!(n-i)!} f(x) \left[F(x)\right]^{i+k-1} \tag{3.76}$$

But

$$\binom{n-i}{k} = \frac{(n-i)!}{(n-1-k)!k!} \tag{3.77}$$

Substituting (3.77) into (3.76),

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \frac{(-1)^k n!}{(n-i-k)!(i-1)!k!} f(x) \left[F(x)\right]^{i+k-1} \tag{3.78}$$

Now putting *cdf* and *pdf* of TPLD into (3.78) gives

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \frac{(-1)^k n!}{(n-i-k)!(i-1)!k!} \alpha\beta\theta^{\alpha} x^{\beta-1} (\theta + x^{\beta})^{-(\alpha+1)} \left[1 + \lambda - 2\lambda(1 - \theta^{\alpha}(x^{\beta}+\theta)^{-\alpha})\right] \left[(1+\lambda)(1-\theta^{\alpha}(x^{\beta}+\theta)^{-\alpha})^2\right]^{i+k-1} \tag{3.79}$$

Hence, substituting $i=1$ and $i=n$ in gives us the *pdf* of the smallest (minimum) order statistic and largest (maximum) order statistic as given in (3.80) and (3.81) respectively.

$$f_{1:n}(x)=\sum_{k=0}^{n-i}\frac{(-1)^k n!}{(n-1-k)!k!}\alpha\beta\theta^{\alpha}x^{\beta-1}(\theta+x^{\beta})^{-(\alpha+1)}\left[1+\lambda-2\lambda(1-\theta^{\alpha}(x^{\beta}+\theta)^{-\alpha})\right]\left[(1+\lambda)(1-\theta^{\alpha}(x^{\beta}+\theta)^{-\alpha})^2\right]^k \quad (3.80)$$

$$f_{n:n}(x)=n\left[\alpha\beta\theta^{\alpha}x^{\beta-1}(\theta+x^{\beta})^{-(\alpha+1)}\left[1+\lambda-2\lambda\left(1-\theta^{\alpha}(x^{\beta}+\theta)^{-\alpha}\right)\right]\left[(1+\lambda)\left(1-\theta^{\alpha}(x^{\beta}+\theta)^{-\alpha}\right)\right]-\lambda\left(1-\theta^{\alpha}(x^{\beta}+\theta)^{-\alpha}\right)^2\right]^{n-1}$$
(3.81)

3.11Estimation of Parameters

Parameters of the TPLD will be estimated using the maximum likelihood method. Let $X_1, X_2, ..., X_n$ be a sample of size ‘n’ independently and identically distributed random variables from the *TPLD* with sample values $x_1, x_2, ..., x_n$ having a joint probability density function as $f(x_1, x_2, ..., x_n; \Theta)$ where $\Theta=(\alpha,\beta,\theta,\lambda)^T$ is a vector of unknown parameters. Then the likelihood function $L(\Theta)$ of the random samples is defined

$$L(\Theta; x_1, x_2, ..., x_n)=f(x_1, x_2, ..., x_n; \Theta)$$

$$L(\Theta)=\prod_{i=1}^{n} f(x_i,\Theta) \quad (3.82)$$

Recall that

$$f(x;\alpha,\beta,\theta,\lambda)=\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}\left[1+\lambda-2\lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)\right] \quad (3.83)$$

Hence the likelihood function of (3.82) is givens by

$$L(\Theta)=\prod_{i=1}^{n}\left[\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}\left[1+\lambda-2\lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)\right]\right] \quad (3.84)$$

$$L(\Theta)=\prod_{i=1}^{n}\left[\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}\left[1+\lambda-2\lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)\right]\right] \quad (3.85)$$

Taking the natural logarithm on both sides of 3.85 gives

$$l = n\log\alpha + n\log\beta + n\alpha\log\theta + (\beta - 1)\sum_{i=1}^{n}\log x_i - (\alpha+1)\sum_{i=1}^{n}\log\left(\theta + x_i^{\beta}\right)$$

$$+\sum_{i=1}^{n}\log\left(1+\lambda-2\lambda\left(1-\theta^{\alpha}\left(x_i^{\beta}+\theta\right)^{-\alpha}\right)\right) \qquad (3.86)$$

Where

Log $L(\Theta) = l$

Differentiating 3.86 partially with respect to each parameter,setting the results to zero and solving simultaneously gives the maximum likelihood estimates of the respective parameters.Below are the equations:

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} - n\log\theta - \sum_{i=1}^{n}\log\left(x_i^{\beta}+\theta\right) - 2\lambda\sum_{i=1}^{n}\left\{\frac{\left(\theta^{-1}\left(\theta+x_i^{\beta}\right)\right)^{-\alpha}\log\left(\theta^{-1}\left(\theta+x_i^{\beta}\right)\right)}{1-\lambda+2\lambda\theta^{\alpha}\left(x_i^{\beta}+\theta\right)^{-\alpha}}\right\} = 0 \qquad (3.87)$$

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log x_i - (\alpha+1)\sum_{i=1}^{n}\left\{\frac{x_i^{\beta}\log x}{\theta+x_i^{\beta}}\right\} - 2\alpha\lambda\theta^{\alpha}\sum_{i=1}^{n}\left\{\frac{\left(\theta+x_i^{\beta}\right)^{-\alpha-1}x_i^{\beta}\log x}{1-\lambda+2\lambda\theta^{\alpha}\left(x_i^{\beta}+\theta\right)^{-\alpha}}\right\} = 0 \qquad (3.89)$$

$$\frac{\partial l}{\partial\theta} = \frac{n\alpha}{\theta} - (\alpha+1)\sum_{i=1}^{n}\left\{\frac{1}{\theta+x_i^{\beta}}\right\} + 2\alpha\lambda\theta^{-2}\sum_{i=1}^{n}\left\{\frac{\left(\theta^{-1}\left(\theta+x_i^{\beta}\right)\right)^{-\alpha-1}x_i^{\beta}\log x}{1-\lambda+2\lambda\theta^{\alpha}\left(x_i^{\beta}+\theta\right)^{-\alpha}}\right\} = 0 \qquad (3.90)$$

$$\frac{\partial l}{\partial\lambda} = \sum_{i=1}^{n}\left\{\frac{2\theta^{\alpha}\left(\theta+x_i^{\beta}\right)^{-\alpha}-1}{1-\lambda+2\lambda\theta^{\alpha}\left(x_i^{\beta}+\theta\right)^{-\alpha}}\right\} = 0 \qquad (3.91)$$

However, the above system of equations cannot be solved analytically except numerically with the aid of suitable statistical software R.

CHAPTER FOUR

ANALYSIS AND DISCUSSION

4.1 Introduction

This chapter presents twodatasets,their descriptive statistics, graphs and application to some selected generalizations of the Lomax distribution. We have compared the adequacy of the *TPLD* to those of three generalizations of the Lomax model including the Power Lomax (POLO) distribution, and Lomax distribution (*LD*). The density function of these distributions are given as follows;

The Transmuted Power Lomax Distribution (*TPLD*)

The *pdf* of the *TPLD* distribution is given as;

$$f(x;\alpha,\beta,\theta,\lambda)=\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)}\left[1+\lambda-2\lambda\left(1-\theta^{\alpha}\left(x^{\beta}+\theta\right)^{-\alpha}\right)\right] \quad (4.1)$$

The Power Lomax (POLO)Distribution

The *pdf*of *POLO* is given as;

$$f(x;\alpha,\beta,\theta)=\alpha\beta\theta^{\alpha}x^{\beta-1}\left(\theta+x^{\beta}\right)^{-(\alpha+1)},x>0,\alpha,\beta,\theta>0 \quad (4.2)$$

The Lomax Distribution (*LD*)

The *pdf* of the Lomax distribution is given by

$$f(x;\alpha,\theta)=\frac{\alpha}{\theta}\left[1+\left(\frac{x}{\theta}\right)\right]^{-(\alpha+1)} \quad (4.3)$$

4.2 The Datasets,Summary statistics and Histogram plots

Dataset I: Table 4.1 represents the remission times (in months) of a random sample of 128 bladder cancer patients. It has previously been used by Lee and Wang (2003) and Rady*et al.*, (2016). It is given and summarized as follows:

Table 4.1 Dataset 1

0.080, 0.200, 0.400, 0.500, 0.510, 0.810, 0.900, 1.050, 1.190, 1.260, 1.350, 1.400, 1.460, 1.760, 2.020, 2.020, 2.070, 2.090, 2.230, 2.260, 2.460, 2.540, 2.620, 2.640, 2.690, 2.690, 2.750, 2.830, 2.870, 3.020, 3.250, 3.310, 3.360, 3.360, 3.480, 3.520, 3.570, 3.640, 3.700, 3.820, 3.880, 4.180, 4.230, 4.260, 4.330, 4.340, 4.400, 4.500, 4.510, 4.870, 4.980, 5.060, 5.090, 5.170, 5.320, 5.320, 5.340, 5.410, 5.410, 5.490, 5.620, 5.710, 5.850, 6.250, 6.540, 6.760, 6.930, 6.940, 6.970, 7.090, 7.260, 7.280, 7.320, 7.390, 7.590, 7.620, 7.630, 7.660, 7.870, 7.930, 8.260, 8.370, 8.530, 8.650, 8.660, 9.020, 9.220, 9.470, 9.740, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

Table 4.2: Summary Statistics for dataset I

Parameters	**n**	**Minimum**	Q_1	**Median**	Q_3	**Mean**	**Maximum**	**Variance**	**Skewness**	**Kurtosis**
Values	128	0.0800	3.348	6.395	11.840	9.366	79.05	110.425	3.3257	19.1537

We also provide a histogram plot for the data as shown in **Fig. 4.1**below.

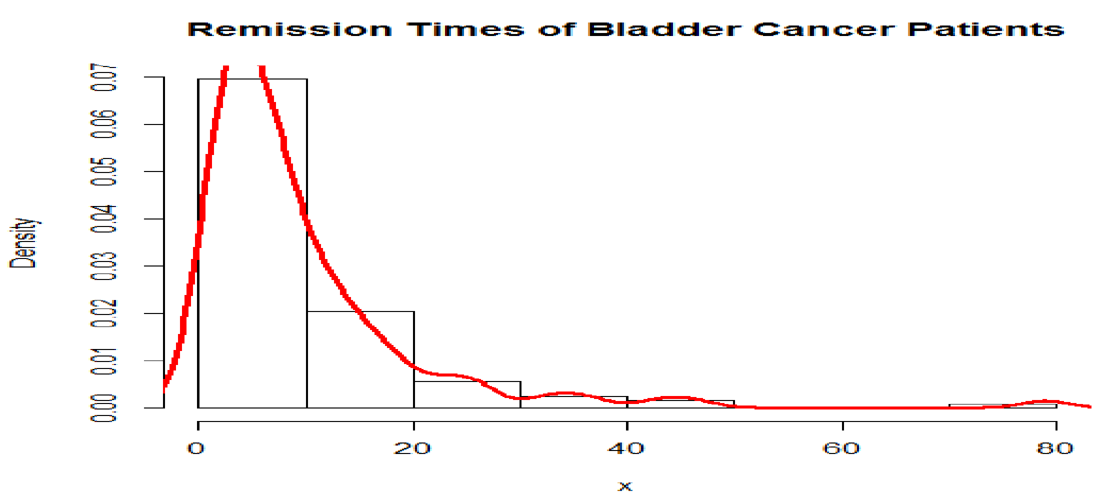

Fig.4.1: A Histogram and density plot on the remission times of bladder cancer patients.

From the descriptive statistics in Table 4.1 as well as the histogram shown above in fig. 4.1 we observed that the data set is positively skewed, therefore suitable for distributions that are skewed to the right.

Dataset II

Table 4.3 represents the actual taxes dataset.It consist of the monthly actual taxes revenue in Egypt from January 2006 to November 2010.The data (in 1000 million Egyptian pounds) is provided below.Nassar and Nada (2011) and Mead (2014) used it in their study. It is given as follows:

Table 4.3 Dataset II

5.9 20.4 14.9 16.2 17.27.8 6.1 9.2 10.2 9.6 13.3 8.5 21.6 18.5 5.1 6.7 17.0 8.6 9.7 9.2 35.7
15.7 9.7 10.0 4.1 36.0 8.5 8.0 9.2 26.2 21.9 16.0 21.3 35.4 14.3 8.5 10.6 19.1 20.5 7.1 7.7
18.1 16.5 11.9 7.0 8.6 12.5 10.3 11.2 6.1 8.4 11.0 11.6 11.9 5.2 6.8 8.9 7.1 10.8

Table 4.4 Summary Statistics for dataset II

Parameters	**N**	**Minimum**	**Q_1**	**Median**	**Q_3**	**Mean**	**Maximum**	**Variance**	**Skewness**	**Kurtosis**
Value	59	4.10	8.45	10.30	12.97	16.35	36.00	64.8266	8.8430	21.20

We also provide a histogram plot for the data as shown **fig. 4.2**

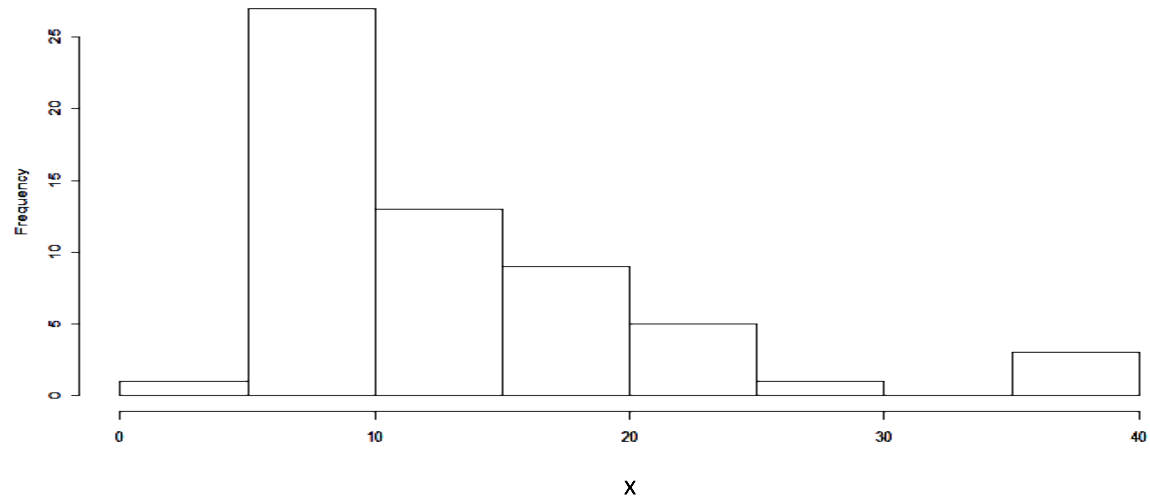

Fig. 4.2: A Histogram plot on actual taxes data set

The descriptive statistics in table 4.2 as well as the histogram above clearly shows that the dataset is positively skewed, therefore suitable for distributions that are skewed to the right.

To assess these models above, we made use of some criteria: the *AIC* (Akaike Information Criterion), *CAIC* (Consistent Akaike Information Criterion), *BIC* (Bayesian Information Criterion) and *HQIC* (HannanQuin information criterion). The formulas for these statistics are given as follows:

$AIC = -2ll + 2k$, $BIC = -2ll + k\log(n)$, $CAIC = -2ll + \frac{2kn}{(n-k-1)}$ and

$HQIC = -2ll + 2k\log\left[\log(n)\right]$

Where ll denotes the log-likelihood function evaluated at the *MLEs*, k is the number of model parameters and n is the sample size.

Decision statement: The model with the lowest values of these statistics would be chosen as the best model to fit the data.The computations for fitting the Transmuted-G (TG) family to real life data in practical problems is performed using the Adequacy Model Script in R statistical software.

Table 4.5: Estimation of models parameters and selection of best model for the bladder cancer data.

Distributions	**Parameter estimates**	**-ll=(-log-likelihood value)**	***AIC***	***CAIC***	***BIC***	***HQIC***	**Ranks of models performance**
TPLD	$\hat{\alpha}$=2.4607 $\hat{\beta}$=1.0209 $\hat{\theta}$=9.2035 $\hat{\lambda}$=-0.9668	411.2126	830.4252	830.7504	841.8334	835.0604	**1**
WLD	$\hat{\boldsymbol{\alpha}}$=0.2293 $\hat{\boldsymbol{\beta}}$=2.9999 $\hat{\boldsymbol{a}}$=7.7805 $\hat{\boldsymbol{b}}$=2.3801	427.3615	861.723	863.0482	874.1311	867.3581	**2**
PLD	$\hat{\alpha}$=0.8682 $\hat{\beta}$=1.4287 $\hat{\theta}$=8.2907	418.2888	842.5776	842.7712	851.1337	846.054	**3**
LD	$\hat{\alpha}$=1.7345 $\hat{\theta}$=9.5593	423.8695	851.739	851.835	857.4431	854.0566	**4**

From **Table 4.5,** the values of the parameter MLEs and the corresponding values of *-ll*, *AIC*, *BIC*, *CAIC* and *HQIC* for each model show that the *TPLD* has better performance compared to the*PLD* and *LD*. This also agrees with the fact that generalizing any continuous distribution provides a compound distribution with a better fit than the classical distribution,also that distributions with higher number of parameters tend to perform better than those with lesser parameters.

Table 4.6: Estimation of models parameters and selection of best model for the actual taxes data.

Distributions	Parameter estimates	-ll=(-log-likelihood value)	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks of models performance
WLD	$\hat{\boldsymbol{\alpha}}$=0.3608 $\hat{\boldsymbol{\beta}}$=2.9639 $\hat{\boldsymbol{a}}$=1.3101 $\hat{\boldsymbol{b}}$=3.1020	190.5505	389.101	389.8418	397.4112	392.345	**1**
TPLD	$\hat{\alpha}$=0.9195 $\hat{\beta}$=1.4738 $\hat{\theta}$=9.7390 $\hat{\lambda}$=-0.9185	205.1158	418.2315	418.9863	426.4733	421.4419	**2**
PLD	$\hat{\alpha}$=0.818 $\hat{\beta}$=1.3150 $\hat{\theta}$=7.7604	225.2949	456.5899	457.0343	462.7712	458.9976	**3**
LD	$\hat{\alpha}$=1.2786 $\hat{\theta}$=9.3176	228.5533	457.0850	457.8397	465.3267	460.2953	**4**

From **Table 4.6,** the values of the parameter MLEs and the corresponding values of *-ll*, *AIC*, *BIC*, *CAIC* and *HQIC* for each model shows that *TPLD* has better performance compared to the*PLD* and *LD*. This has further confirmed the fact that generalizing any continuous distribution provides a compound distribution with a better fit than the classical distribution due to the effect of the added parameter.

Comparing the performance of TPLD with the two different datasets also shows that the four parameters Weibull-Lomax performed slightly better than the TPLD followed by the PLD and LD. This indicates that the Weibull-Lomax could be used in place of TPLD to model datasets from the economic domain when sample size is small. TPLD actually performed better with the bladder cancer data than the actual taxes data, this implies that TPLD will more appropriate in analyzing datasets from the health domain with large sample size.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 Summary

One important and obvious fact about compound distributions is that they perform better than their classical form. In this dissertation, we have developed another compound distribution known as transmuted power Lomax distribution. Some properties of the proposed distribution such as its ordinary moments, moment generating function, characteristics function, reliability analysis, and order statistics were derived. The estimation of parameters has been done using the method of maximum likelihood estimation. The performance of the new distribution is tested by means of some applications to two real life datasets.

5.2 Conclusion

In this dissertation, a new distribution has been proposed. Some mathematical and statistical properties of the proposed distribution have been studied appropriately. The derivations of some expressions for its moments, moment generating function, characteristics function, survival function, hazard function, and ordered statistics has been done appropriately. Some plots of the distribution revealed that its shape is skewed to the right independent on values of the parameters. The model parameters have been estimated with the aid of R statistical software using the goodness.fit command. The implications of the plots for the survival function indicate that the transmuted power Lomax distribution could be used to model age-dependent events or variables whose survival decreases as time grows or where survival rate decreases with time. The results of the application showed that the proposed distribution (Transmuted Power Lomax distribution) performed better than the four parameters Weibull-Lomax, Power-Lomax, and the Lomax distributions with the bladder cancer data than the actual taxes data. The Weibull-Lomax competed quite well with TPLD as it performed slightly better with the actual taxes data. This implies that the performance of TPLD will depend on the nature of dataset used.

5.3 Recommendations

We recommend based on the findings of this dissertation that the proposed distribution should be used to model positively skewed data sets with higher peaks and tails. We also recommend the use of the new model for analyzing age dependent variables.

5.4 Contribution to knowledge

In this study, we have developed a new distribution which is useful for modeling positively skewed data sets with greater peaks and tails. We also provided expressions for some of its properties which are useful for studying the shape characteristics of the distribution as well as the reliability functions which are useful in the field of engineering.

Lastly,performance of the new distribution has been assessed in comparison with some extensions of Lomax distributions and we discovered that it performed better than the Weilbull-Lomax, Power Lomax and Lomax distributions with the bladder cancer data. TPLD will be very useful in analyzing positively skewed dataset from the health domain.

5.5 Areas of Future Research

Subsequent studies on this new distribution can consider estimation of confidence intervals for the parameters of the new distribution. Also, others can estimate the parameters of the new distribution using another classical or Bayesian method.

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