STUDY ON INFERENCES AND APPLICATIONS OF ODD GENERALIZED EXPONENTIAL-RAYLEIGH DISTRIBUTION

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JULY, 2017

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B.Sc STATISTICS (ABU) 2014

P15SCMT8067

A DISSERTATION SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES, AHMADU BELLO UNIVERSITY, ZARIA-NIGERIA.

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF A MASTER DEGREE IN STATISTICS

DEPARTMENT OF STATISTICS

FACULTY OF PHYSICAL SCIENCES

AHMADU BELLO UNIVERSITY, ZARIA

NIGERIA

JULY, 2017

DECLARATION

I declare that the work in this dissertation entitled "Study on Inferences and Applications of Odd Generalized Exponential-Rayleigh Distribution " has been carried out by me in the Department of Statistics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other institution.

Nura UMAR

Date

CERTIFICATION

This dissertation titled "STUDY ON INFERENCES AND APPLICATIONS OF ODD GENERALIZED EXPONENTIAL-RAYLEIGH DISTRIBUTION" by Nura UMAR meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This dissertation is dedicated to Allah (S.W.T) for all he has done for me and also to my late Mother Malama Saudatu Abdul'azeez.

ACKNOWLEDGEMENTS

My first and foremost gratitude goes to Almighty Allah (SWT), unto him I have been saying "ALHAMDULILLAH!". I wish to express my sincere appreciation to my Supervisor, Dr. Abubakar Yahaya, for working tirelessly, showing concern, interest, high commitment, encouragement, careful reading of my work and prompt response, words are not enough to thank you for what you did to me. May Allah (SWT) continue to make best come your way always-Amin.

I would equally express my sincere gratitude to my second supervisor, Dr. S. I. S. Doguwa, for his time, patience, valuable suggestions and meaningful contributions to the success of this research, thank you Sir.

My special appreciation goes to the Head of Department of Statistics, Dr. H.G. Dikko and the entire staff of the department for their positive contributions towards this successful journey.

I would also like to thank all my fellow postgraduate Statistics students for their friendship and support including those in the Departments of Mathematics and Computer Science, you guys have also been good to me from the start. Your very many names are inscribed written in my heart because you all are so dear to me. Thank you all.

I am very much grateful to families of Alhaji Muhammad Zubairu Katsina, Flight Lieutenant A. M. Goma, Suleiman Salisu, ibrahim Ishaq Danbatta and to my cousins Ahmad Dangida and Musa Uzairu to mention a few. Also, to my uncle and siblings Alh. Lawal Isah, Muntari, Murtala, Maryam, Labiba, Mahabuba, Khuzaifa, Al'amin

My sincere appreciation goes to my Father Malam Umar Usman and my step-mother (Maman Nana) for their supports and prayers.

My heartiest appreciation also goes to my mentor(s) in person(s) of Prof. I. I. Funtua, Dr. Aliyu Usman, Dr. A. Yahaya. All their advices and supports I appreciate.

Lastly, my sincere appreciation goes to my dear wife Maryam Lawal for the sacrifice and prayers you made throughout this programme. Thank you so much.

ABSTRACT

The Rayleigh distribution has wide range of applications in Applied Sciences; some of which are related to sea waves, harbor, coastal-engineering and studies on wind-wave heights among others. In this work, we propose a new lifetime model entitled Odd Generalized Exponential-Rayleigh (OGE-R) Distribution and present some of its statistical properties comprising moments, moment generating function, quantile function, median, reliability analysis and order statistics. The plot for the pdf of the distribution showed that OGE-R distribution is positively skewed, which makes it a good candidate for fitting dataset that are positively skewed. Also, the plot of the hazard funtion indicates that the proposed model can adequately fit dataset with J-shape. A method of maximum likelihood was used to estimate the parameters of the new distribution. The applicability of the new distribution using real dataset was also illustrated and its performance compared with other distributions. The result of the comparison shows that the distribution is competitive in modeling positively skewed datasets because it has the smallest values of the AIC=18.1925, AICC=18.5454 and BIC=17.7644 than the other three distributions considered.

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CHAPTER ONE

INTRODUCTION

1.1 Background of the study

The Rayleigh distribution was named after Lord Rayleigh (1842-1919) a British physicist as well as mathematician also known as John William Strutt. In 1895 he discovered the inert gas Argon (Ar), the research that earned him the 1904 Nobel Prize in Physics Venkatesh and Manikandan (2016).

The Rayleigh distribution has wide range of applications in the field of applied sciences, especially in modeling the lifetime of an object or service time. Battjes (1969) stated some areas where the distribution can also be applied, these includes sea waves, harbor, coastal and ocean engineering, heights and periods of wind waves. Despite its applicability, the distribution suffer's the same problem other classical distributions suffered from, which is lack of flexibility due to the fact that it has only one parameter. For instance, Tahir and Cordeiro (2016) stated that, the well-known classical/baseline distributions such as exponential, Rayleigh, Weibull and gamma are limited in their characteristics and are unable to show wide flexibility. Because of this and several other problems; several researchers have worked and some are still trying to overcome these challenges by generalizing some of these classical distributions to come up with compound distributions.

According to Eugene *et al.* (2002) generalization of distributions started in the year 1925; Also, Ahuja and Nash (1967) introduced the generalized Gompertz-Verhulst family of distributions to study growth curve mortality. Gupta *et al.* (1998) added one parameter to the cumulative distribution function of the baseline distribution to define the exponentiated-G class of distributions and several others follows.

Also Gupta and Kundu (1999) pioneered the study of two-parameter generalization, in which they studied two-parameter Generalized Exponential (GE) distribution also called Exponentiated Exponential (EE) distribution, after which several other

authors worked on GE distribution due to its attractive features, among which are Gupta and Kundu (1999), Kundu *et al.* (2005), Nadarajah (2006) and so on.

1.2 Statement of the problem

The importance of the assumption of probability distribution for which a random variable follows cannot be overemphasized in the statistical analysis of such variable for meaningful conclusion. Because of the fact that most of the lifetime data are unique, the need to have appropriate distribution that lifetime data follows is of great importance to improve the validity of the statistical analysis. In recent time, several generalization of probability distributions have been introduced for modeling lifetime data. However, not all datasets can adequately be modeled by the classical distributions due to the uniqueness of the data.

Hence, developing new distributions that would adequately address this problem is of paramount importance, with this in mind, we intend to propose a new distribution called the Odd Generalized Exponential-Rayleigh (OGE-R) distribution.

1.3 Generalized Exponential Distribution and Rayleigh Distribution

The Generalized Exponential (GE) distribution is a continuous probability distribution belonging to the Exponentiated Family (EF) of distributions. Tahir and Nadarajah (2015) said "The origin of this family can be traced back to the early part of the nineteenth century during which Gompertz (1825) and Verhulst (1838, 1845, 1847) used the cumulative distribution function (CDF) $G(t) = (1 - \rho e^{-\lambda t})^{\alpha}$ for $t > \lambda^{-1} log\rho$ where, α, λ and ρ are greater than zero". The generalized Gompertz-Verhulst family of distributions was introduced by Ahuja and Nash (1967) to study growth curve

mortality. The Gompertz-Verhulst's CDF happened to be the first member of the EF of distributions. The GE distribution is a particular case of this family for $\rho = 1$. The CDF and the pdf of the GE distribution is given by equation (1.3.1) and (1.3.2) respectively.

$$G(x) = (1 - e^{-\lambda x})^{\alpha} \tag{1.3.1}$$

$$g(x) = \alpha\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1} \tag{1.3.2}$$

where $x > 0$, and $\alpha, \lambda > 0$

The Rayleigh distribution naturally arises in case when the wind speed data is analysed into two-orthogonal dimensional vector components, where the magnitude of components is independent and normally distributed with equal variances. Also, this distribution arises in the case of random complex numbers whose real and imaginary components are independently and identically normally distributed random variables; other areas where this distribution can be applied include health, agriculture and biology, among others (Gomes *et al.*, 2014).

A random variable X is said to have a Rayleigh distribution if the CDF and the pdf are respectively given as:

$$G(x;\sigma) = 1 - e^{-\frac{x^2}{2\sigma^2}} \tag{1.3.3}$$

$$g(x;\sigma) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \tag{1.3.4}$$

where $x > 0$, σ is scale parameter and $\sigma > 0$.

1.4 Aim and objectives

The aim of this research is to develop an Odd Generalized Exponential-Rayleigh distribution, study some of its properties and evaluate its performance using real datasets. The stated aim is expected to be achieved through the following objectives. By

i.) defining and expressing the pdf of the proposed distribution.

ii.) deriving some statistical properties of the proposed distribution comprising the moments, moment generating function, Quantile, Median, reliability analysis and the distribution of order statistics.

iii.) estimating the parameters of the proposed distribution using the method of Maximum Likelihood.

iv.) evaluating the performance of the proposed distribution compared to other generalizations of Rayleigh distribution.

1.5 Scope of the study

The study is focused only on extending the Rayleigh distribution and deriving mathematical expressions for some selected properties of the proposed distribution such as moment, moment generating function, quantile, median, density functions for the minimum and maximum order statistics and estimating the model parameters by using only the method of Maximum Likelihood.

1.6 Significance of the study

The study of the proposed distribution, its properties and the parameter estimation will increase the flexibility of the Rayleigh distribution and easily model various datasets that cannot be properly fitted by the existing generalization of Rayleigh distributions. In addition, it will makes the kurtosis more flexible (compared to the baseline distribution) and possibly to construct heavy-tailed distributions for modeling real data. This study will compare the proposed distribution to some existing generalizations of the Rayleigh distribution to identify the distribution that will provide better fit using a real datasets.

1.7 Motivation

Tahir *et al.*(2015) highlighted some special distributions that can be obtained using their generator, these include Weibull, Frechet and normal distributions and gave the CDF and the pdf of these special distributions which can be useful in applied survival analysis, their study also estimated the parameters and compared the fitness of these models using real dataset.

Johnson *et al.* (1994) stated that the use of four-parameter distributions should be sufficient for most practical purposes. According to them, "at least three parameters are needed but they doubted any noticeable improvement arising from including a fifth or sixth parameter".

The Rayleigh distribution is a popular lifetime distribution and it is one of the most important distributions for problems in the field of applied sciences and reliability engineering. However, there has been little or no study on the Odd generalized exponential-Rayleigh distribution since Tahir *et al.* (2015) proposed their generator. Hence, we intend to study the odd generalized exponential-Rayleigh distribution, some of its properties and its application to real dataset.

1.8 Definition of Terms

1.8.1 Continuous Random Variable

A nondiscrete random variable X is said to be absolutely continuous, or simply continuous, if its distribution function may be represented as

$$F(X) = P(X \leq x) = \int_{-\infty}^{x} f(u)du \tag{1.8.1}$$

for every real number x and evaluated from $-\infty$ to a real number x. while the probability density function (pdf) is defined as;

$$f(x) = \frac{dF(x)}{dx} \tag{1.8.2}$$

and it has the following properties:

1. $f(x) \geq 0, \forall x$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P(a < X < b) = \int_{a}^{b} f(x)dx$

1.8.2 Moments

This is defined as a statistical technique used to study some of the most important features and characteristics of a random variable such as mean (central tendency measure), variance (dispersion measure), skewness (Sk) and kurtosis (Ku). Also moments of a probability distributions is a collection of descriptive constants that can be used for measuring its properties. Mathematically, it can be define as follows: let X be a continuous random variable, then the r^{th} moment of X about the origin is given in equation (1.8.3)

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} X^r f(x)dx \tag{1.8.3}$$

where $f(x)$ is any pdf.

And, the r_{th} central moment of X , say μ_r also known as moment about the mean is obtain using equation (1.8.4)

$$\mu_r = E[X - \mu'_1]^n \tag{1.8.4}$$

where μ_1' is the 1^{st} moment about the origin while the variance (σ^2) is the 2^{nd} central moment or moment about the mean.

Quantile Function

Quantile function is essentially the inverse of the cumulative distribution function. It is use for calculating the median, measure of location and for simulation of random numbers. Mathematically, it's defined as

$$F(x_q) = q \tag{1.8.5}$$

where $F(x_q)$ is the cumulative distribution function and $0 < q < 1$

Skewness

Skewness is a measure of the degree of asymmetry or lack of symmetry of a distribution. The coefficient of skewness is the standardized third central moment of X or moment about the mean and can be obtained using the expression in equation (1.8.5);

$$SK = E\left[\frac{(x - \mu_1')}{\sigma}\right]^3 = \frac{\mu_3}{\sigma^3} \tag{1.8.6}$$

where μ_3 is the third moment about mean and σ^3 is standardized third deviation.

Kurtosis

Kurtosis is defined as a measure of the degree of peakedness or flatness of a density near the center. It can also be define as the standardized fourth population moment

about the mean, and is given by;

$$KU = E\left[\frac{(x-\mu_1')}{\sigma}\right]^4 = \frac{\mu_4}{\sigma^4} \tag{1.8.7}$$

where μ_3 is the fourth moment about mean and σ^3 is standardized fourth deviation.

1.8.3 Moment Generating Function

The Moment Generating Function provides an alternative way to analytical result rather than working directly with the pdf. The MGF of a random variable X can be obtain using equation (1.8.7)

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{1.8.8}$$

In other words, the MGF generates the moments of X by differentiation, that is to say, for any real number say j, the j^{th} derivative of $M_x(t)$ evaluated at $t = 0$ is the j^{th} moment μ_j' of X.

1.8.4 Reliability Analysis

Reliability (Survival) analysis is the process of modeling time-to-event dataset or it is a statistical technique used to describe and quantify time-to-event data. This analysis was originally introduced for the purpose of evaluating the treatment efficacy of fatal condition like cancer. It's applicability expands to numerous areas of human endeavor including engineering, medicine, sciences, industry and so on.

Survival Function

This is simply define as the probability that a system will survive beyond a specified time. Mathematically it's defined as

$$S(x) = P(X > x) = \int_x^{\infty} f(u)du = 1 - F(x) \tag{1.8.9}$$

where $F(x)$ is the CDF of any distribution.

Harzard Function

This is define as the probability per unit time that a case which has survived to the beginning of the respective interval will fail in that interval. Mathematically it's defined as

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \tag{1.8.10}$$

where $f(x)$ is pdf of any distribution.

1.8.5 Order Statistics

Order statistics are widely used in many areas of statistical theory and practice, for instance, detection of outlier in statistical quality control processes.

Suppose $X_1, X_2, \cdots, X_n$ is a random sample from a distribution with pdf $f(x)$ and let $X_{1:n}, X_{2:n}, \cdots, X_{i:n}$ denote the corresponding order statistic obtained from this sample. The pdf $f_{i:n}(x)$ of the i^{th} order statistic can be express as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} [1 - F(x)]^{n-i} \tag{1.8.11}$$

1.8.6 Maximum Likelihood Method

From a statistical perspective, the method of maximum likelihood estimation is with some exceptions,considered to be the most robust of the parameter estimation techniques. Suppose $X_1, X_2, \cdots, X_n$ is a random sample from a population with pdf $f(x;\theta)$, where θ is an unknown parameter to be estimated. The likelihood function, L(θ), is defined to be the joint density of the random variables $X_1, X_2, \cdots, X_n$. That is,

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) \tag{1.8.12}$$

The sample statistic that maximizes the likelihood function $L(\theta)$ is known as maximum likelihood estimator of θ and is denoted as $\hat{\theta}$.

CHAPTER TWO

LITERATURE REVIEW

In modern world of Statistics, several researchers have worked seriously and some are still working on to see that they come up with generators aimed at hybridizing some well-known classical/baseline distributions with others to provide them with better fit than when they are alone due to the additional flexibility.

The generator we use in this research has its root from Generalized Exponential (GE) distribution also known as Exponentiated Exponential (EE) distribution which is one of the members of Exponentiated Family (EF) of distributions. We reviewed some literatures related to GE family of distributions as follows: Gupta and Kundu (1999) pioneered the study of two-parameter generalization of the GE distrbution. Several authors studied the distribution due to its attractive nature, some of them includes: Cho *et al.* (2009) introduced the exponentiated extreme value distribution and studied some of the theoretical properties of the distribution, they derived the MLE's of the scale parameter and shape parameter, also the mean suared error of the parameters were obtained for comparison using censored data.

Cordeiro *et al.* (2013c) proposed the beta exponentiated Weibull distribution which extends works of (Lee *et al.*, 2007) and (Barreto-Souza *et al.*, 2010) it was demonstrated in the work that the density of the new distribution can be expressed as a linear combination of Weibull densities. An expression for the moments and moment generating function were presented in closed-form, explicit expressions are derived for the mean deviations, Bonferroni and Lorenz curves, reliability and entropies . Using the Akaike information criterion (AIC) in application section it was observed that the proposed distribution perform better than the other two distribution compared.

Sarhan *et al.* (2013) studied exponentiated generalized linear exponential distribution, The distribution is important since it contains as special sub-models some widely well known distributions such as the exponentiated Weibull distribution and generalized linear exponential distribution among many others. Some statistical properties of

the new distribution were studied and it was observed that the new distributin provides more flexibility to other distributions compared in the work.

According to Tahir *et al.* (2015), the GE distribution is applicable to lifetime data with monotonic hazard rate function (hrf) but ineffective if the hrf is upside-down, J or reversed-J shapes, because of the above mentioned limitations, they proposed that the random variable X in the CDF of the GE distribution be replaced by the odd ratio of the CDF of the other distribution, and came up with a new family of distributions called Odd Generalized Exponential (OGE) Family of distributions that will be used in generalizing some classical distributions.

The generator introduced by Tahir *et al.* (2015) has received wide attention in the following researches:

El-Damcese *et al.* (2015) in the study of odd generalized exponential Gompertz (OGE-G) distribution adopted the generator of the OGE family of distributions proposed by Tahir *et al.* (2015), they obtained some of the mathematical properties of the distribution and also some structural properties of the new distribution were studied. The method of maximum likelihood was used for estimating the model parameters and they also derived Fisher's information matrix while at thesame time using numerical dataset to compare the performance of their proposed distribution with some existing distributions and it was observed that the OGE-G is the best distribution for fitting data sets among exponential, generalized exponential, Gompertz, generalized Gompertz and beta Gompertz.

El-Damcese *et al.* (2016) used the same generator by Tahir *et al.* (2015) to come up with a new distribution called odd generalized exponential linear failure rate (OGE-FLR) distribution, they obtained some statistical properties of the proposed distribution such as moment, quantile, median and mode, the method of Maximum Likelihood Estimation was used to estimate the parameters of the proposed model, and they subjected the proposed model using data set to compare its performance with existing distributions, based on the results obtained they were able to conclude that model provide better fit to the data rather than other distributions which we

compared with because it has the smallest value of AIC and three other statistics used.

Abdelall (2016) came up with a proposed model called odd generalized exponential modified-Weibull (OGE-MW) distribution using the generator by Tahir *et al.* (2015), he studied some mathematical properties of the new distribution such as moment, order statistics and method of Maximum Likelihood Estimation was employed to estimate the parameters of the new distribution, here also he compared the new model with the existing ones to ascertain the effectiveness of the new distribution and based on the result obtained it was observed that the (OGE-MW) distribution provides better fit to the dataset used than the other distributions it was compared with. These include exponential, generalized exponential, linear failure rate and Weibull distributions.

The Odd Generalized Exponential Log-Logistic Distribution (OGELLD) was proposed by Rosaiah *et al.* (2016) using the OGE generator by Tahir *et al.*(2016), in the article some mathematical and structural properties of the proposed distribution were studied, the parameters of the distribution were derived using method of maximum likelihood. They also compared the performance of the new model with other distributions and found that based on the values of Cramer-von Mises and Anderson Darling statistics that the new model fits the dataset best than the rest. Finally, they also estimated the two sided confidence intervals for the unknown parameters.

Rayleigh distribution as one of the classical distribution with one parameter may likely suffer from the aforementioned problems (lack of flexibility among others), because of these, several researchers tried to hybridize the Rayleigh distribution that will help do away with these problems, some of these studies include among others: Merovci (2013) studied transmuted Rayleigh distribution using quadratic rank transmutation map (QRTM) developed by Shaw and Buckley (2007), in the study, a comprehensive description of some mathematical properties of the new distribution were derived, these are moments, moment generating function (MGF).

Also method of maximum likelihood was employed to estimate the parameters (σ and λ) of the new distribution, $100(1-\alpha)$ confidence interval for the two parameters were obtained aswell as reliability analysis and order statistics . Using a dataset on Nicotine, the study concludes that the new distribution is statistically superior to Rayleigh distribution.

Merovci (2014), proposed transmuted generalized Rayleigh distribution to extend the generalized Rayleigh distribution using the QRTM techniques. The study derived the moments and obtained the mean, variance, skewness and kurtosis. Also quantile and mgf of the new distribution were derived. The parameters of the model were estimated using the method of maximum likelihood. An application of the transmuted generalized Rayleigh distribution to real dataset using the loglikelihood ratio statistic show that the new distribution can be used quite effectively to provide better fit than the generalized Rayleigh distribution.

In the study the kumaraswamy generalized Rayleigh distribution (Gomes *et al.*, 2014) used the generator proposed by Cordeiro and de-Castro (2011) to express the new distribution as a mixture of Generalized Rayleigh density functions. The study derived explicit expressions for moments, quantile and mgf, mean deviations, reliability and Renyi entropy and order statistics. The method of maximum likelihood was used to estimate parameters of the proposed distribution. They compared the fitness of four distributions using their Akaike Information Criteria (AIC) and concluded that the new distribution has better fit to the dataset used over exponentiated generalized Rayleigh, generalized Rayleigh and Rayleigh distributions.

Merovci and Elbatal (2015) defined and studied three parameter Weibull-Rayleigh distribution using the generator proposed by Marcelo *et al.* (2014) called the Weibull-G Family of probability distributions. In the study it was observed that because of the additional flexibility, the new distribution has better fit than the other three distributions it was compared with comprising beta-Weibull, Exponentiated-Weibull and Weibull distributions. Also they derived expressions for the moments and mgf. The parameters of the new distribution were estimated using the method

of maximum likelihood and least squares estimation, also the information matrix was derived.

In the study Transmuted Exponentiated Inverse Rayleigh (TEIR) Distribution Haq (2016), used the QRTM to extend the Exponentiated Inverse Rayleigh Distribution. Some of the properties of the new model were studied among which are moments, incomplete moments, renyi entropy, random number generator and quantile function. Furthermore, expressions of hazard and survival function were derived. method of maximum likelihood was used to obtain the parameters of the new model and it was concluded that the TEIR distribution is competitive distribution compared with other distributions. In fact, using numerical dataset it was observed that based on the values of some test statistics, TEIR distribution happened to be the best among three model considered.

Other studies on Rayleigh distribution include: Kundu and Raqab (2005) studied "Generalized Rayleigh Distribution: Different Methods of Estimations" in the study they compared only different methods of estimation and not generalizing the distribution, also they compared the performance of the different estimators using monte carlo simulations, the distribution they used is the generalized Rayleigh distribution also known as two-parameter Burr Type X distribution.

In the research "proposed Methods for Estimating Parameters of the Generalized Rayleigh Distribution in the Presence of One Outlier" by Hassan *et al.* (2012) studied three method for estimating the parameters of generalized Rayleigh distribution (Least Squares, Moments, MLE), and derived the moment generating function, which is used to obtain the first and second Moment of the dstribution, and then used as an alternative method for estimating Parameters.

Tahir *et al.* (2015) states that their generator can extend several classical distributions, in the paper, they studied some special OGE distributions, these includes; the OGE-Weibull distribution, the OGE-Frechet distribution and OGE-Normal distribution. They also showed that the OGE density function can be expressed as a mixture of exponentiated-G density functions, this mixture representation is important to

derive several structural properties of this family in full generality; the structural properties include Moments and incomplete moments, generating function, mean deviations, enthropies and order statistics. They derived general mathematical properties of the OGE family of distributions.

Based on the understanding of many concepts developed by several researchers, this research seeks to increase the flexibility of the Rayleigh distribution using the Odd generalized exponential family of distributions by Tahir *et al.* (2015). Hence, this study proposed Odd Generalized Exponential-Rayleigh (OGE-R) distribution.

CHAPTER THREE

METHODOLOGY

3.1 Definition of the Proposed Distribution

Given any continuous distribution with CDF $G(x;\xi)$, pdf $g(x;\xi)$ and survival function $(1-G(x;\xi))$, Tahir *et al.* (2015) proposed the Odd Generalized Exponential (OGE) family of distributions that provides greater flexibility in modeling of real datasets and effectively used to analyse right-skewed datasets. The CDF of the OGE family of distributions according to Tahir *et al.* (2015) is defined as

$$F(x) = F(x;\alpha,\lambda,\xi) = \left[1 - e^{-\lambda\frac{G(x;\xi)}{1-G(x;\xi)}}\right]^{\alpha} \tag{3.1.1}$$

where $\alpha > 0$ and $\lambda > 0$ are shape and rate (inverse scale) parameters respectively. By replacing the random variable x in the CDF of the GE distribution $(1-e^{-\lambda x})^{\alpha}$ with the odd ratio $\frac{G(x;\xi)}{1-G(x;\xi)}$. Hence, equation (3.1.1) was obtained which is the CDF of the OGE family of distributions proposed by Tahir *et al.* (2015). Also, the pdf of the OGE family of distributions can be obtained from equation (3.1.1) by taking the derivative of the CDF with respect to x as follows:
Recall that equation (1.8.2) defined $f(x)$ as

$$f(x) = f(x;\alpha,\lambda,\xi) = \frac{dF(x;\alpha,\lambda,\xi)}{dx} \tag{3.1.2}$$

where

$$F(x;\alpha,\lambda,\xi) = \left[1 - e^{-\lambda\frac{G(x;\xi)}{1-G(x;\xi)}}\right]^{\alpha} \tag{3.1.3}$$

Now, let

$$y = \left[1 - e^{-\lambda\frac{G(x;\xi)}{1-G(x;\xi)}}\right]^{\alpha}$$

$t = \frac{G(x;\xi)}{1-G(x;\xi)}$, $n = \lambda t$ and $u = (1 - e^{-n})$

such that

$$y = (1 - e^{-n})^{\alpha} = u^{\alpha}$$

$\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dn} * \frac{dn}{dt} * \frac{dt}{dx}$ (chain rule or function of a function rule)

then,

$\frac{dy}{du} = \alpha u^{\alpha-1}$, $\frac{du}{dn} = e^{-n}$, $\frac{dn}{dt} = \lambda$ and $\frac{dt}{dx} = \frac{g(x,\xi)}{(1-G(x;\xi))^2}$

hence,

$$\frac{dy}{dx} = \frac{\lambda\alpha g(x;\xi)}{(1-G(x;\xi))^2} e^{-\lambda\frac{G(x;\xi)}{1-G(x;\xi)}} [1 - e^{-\lambda\frac{G(x;\xi)}{1-G(x;\xi)}}]^{\alpha-1} \tag{3.1.4}$$

Now,

$$f(x) = f(x;\alpha,\lambda,\xi) = \frac{dF(x;\alpha,\lambda,\xi)}{dx}$$

becomes

$$f(x) = f(x;\alpha,\lambda,\xi) = \frac{\lambda\alpha g(x;\xi)}{(1-G(x;\xi))^2} e^{-\lambda\frac{G(x;\xi)}{1-G(x;\xi)}} [1 - e^{-\lambda\frac{G(x;\xi)}{1-G(x;\xi)}}]^{\alpha-1}$$

Hence, the corresponding pdf of the OGE family of distributions is given by

$$f(x) = f(x;\alpha,\lambda,\xi) = \frac{\lambda\alpha g(x;\xi)}{(1-G(x;\xi))^2} e^{-\lambda\frac{G(x;\xi)}{1-G(x;\xi)}} [1 - e^{-\lambda\frac{G(x;\xi)}{1-G(x;\xi)}}]^{\alpha-1} \tag{3.1.5}$$

where $G(x;\xi)$, $g(x;\xi)$ and $1 - G(x;\xi)$ are the CDF, pdf and survival function of any continuous distribution respectively also $\alpha > 0$ and $\lambda > 0$ are shape and rate parameters respectively.

Equation (3.1.5) is the pdf of any OGE family of distributions and it will be most tractable when the CDF $G(x;\xi)$ and pdf $g(x;\xi)$ have explicit expressions. The main justifications for using the OGE family are; to make the kurtosis more flexible (compared to the baseline distribution) and also to make it possible to construct heavy-tailed distributions that are not long-tailed for modeling real datasets.

Henceforth, we define a random variable X with density function (3.1.5) to be denoted by $X \sim OGE(\alpha,\lambda,\xi)$.

The CDF and pdf of the Rayleigh distribution with scale parameter $\sigma > 0$ and $x > 0$ are given by

$$G(x;\sigma) = 1 - e^{-\frac{x^2}{2\sigma^2}} \tag{3.1.6}$$

and

$$g(x;\sigma) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \tag{3.1.7}$$

respectively.

Now, inserting the CDF and pdf of Rayleigh distribution in (3.1.1) and (3.1.5) we obtained the CDF and pdf of OGE-R distribution as

$$F(x) = F(x;\alpha,\lambda,\sigma) = (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha} \tag{3.1.8}$$

and

$$f(x) = f(x;\alpha,\lambda,\sigma) = \frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1} \tag{3.1.9}$$

respectively. where $x > 0$, $\alpha > 0$ is the shape parameter, $\lambda > 0$ is the rate (inverse scale) parameter and $\sigma > 0$ is the scale parameter.

3.2 Validity Check of the new distribution

The new distribution is valid if and only if:

$$\int_{-\infty}^{\infty} f(x)dx = 1 \tag{3.2.1}$$

Now,

$$f(x) = f(x;\alpha,\lambda,\sigma) = \frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1} \tag{3.2.2}$$

$$\int_0^{\infty} f(x)dx = \int_0^{\infty} \frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1} dx \tag{3.2.3}$$

Let

$$m = e^{\frac{x^2}{2\sigma^2}} - 1 \tag{3.2.4}$$

Then,

$$\frac{dm}{dx} = \frac{x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} \Rightarrow dx = \frac{\sigma^2 dm}{x e^{\frac{x^2}{2\sigma^2}}} \tag{3.2.5}$$

as $x \to \infty, m \to \infty$ and as $x \to 0, m \to 0$

Substituting (3.2.5) and (3.2.4) in (3.2.3), we have

$$\int_0^\infty f(x)dx = \alpha\lambda \int_0^\infty e^{-\lambda m}(1 - e^{-\lambda m})^{\alpha-1} dm \tag{3.2.6}$$

Next, we let

$$n = 1 - e^{-\lambda m} \tag{3.2.7}$$

$$\frac{dn}{dm} = \lambda e^{-\lambda m} \Rightarrow dm = \frac{dn}{\lambda e^{-\lambda m}} \tag{3.2.8}$$

as $m \to \infty, n \to 1$ and as $m \to 0, n \to 0$

Substituting (3.2.8) and (3.2.7) in (3.2.6), we have

$$\int_0^\infty f(x)dx = \alpha \int_0^1 n^{\alpha-1} dn \tag{3.2.9}$$

$$\alpha(\frac{n^\alpha}{\alpha}) = n^\alpha|_0^1 = 1 - 0 = 1 \tag{3.2.10}$$

Hence, (3.2.10) shows that the distribution in (3.2.10) is a valid pdf.

The plots of the CDF and pdf of the new distribution for some selected parameter values were shown in figures 3.2.1 and 3.2.2 respectively. The CDF plot shows that the distribution has an increasing upward trend that approaches value one (1) as required for any CDF plot. Also, the pdf plot shows that the distribution is skewed to the right (positively skewed) this indicate that the new distribution will adequately model datasets that are positively skewed.

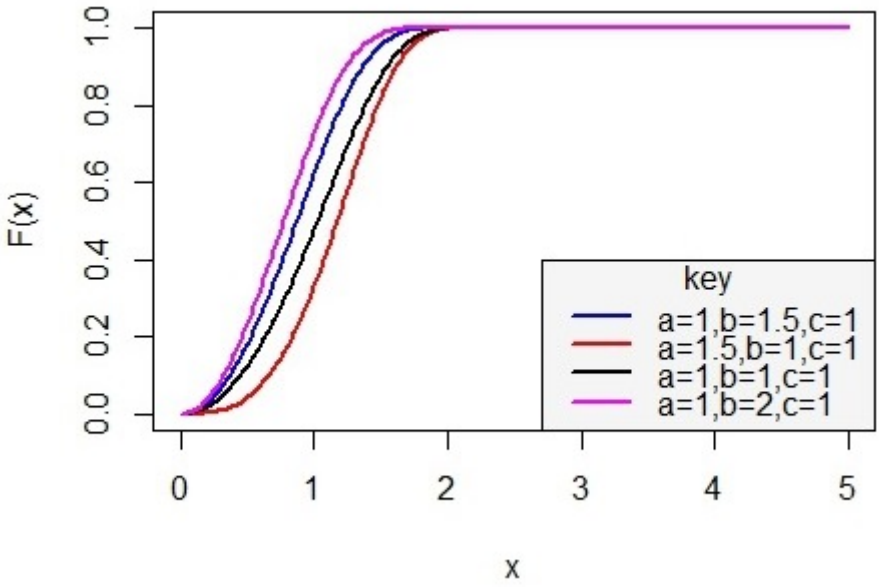


Figure 3.2.1: CDF of OGE-R Distribution

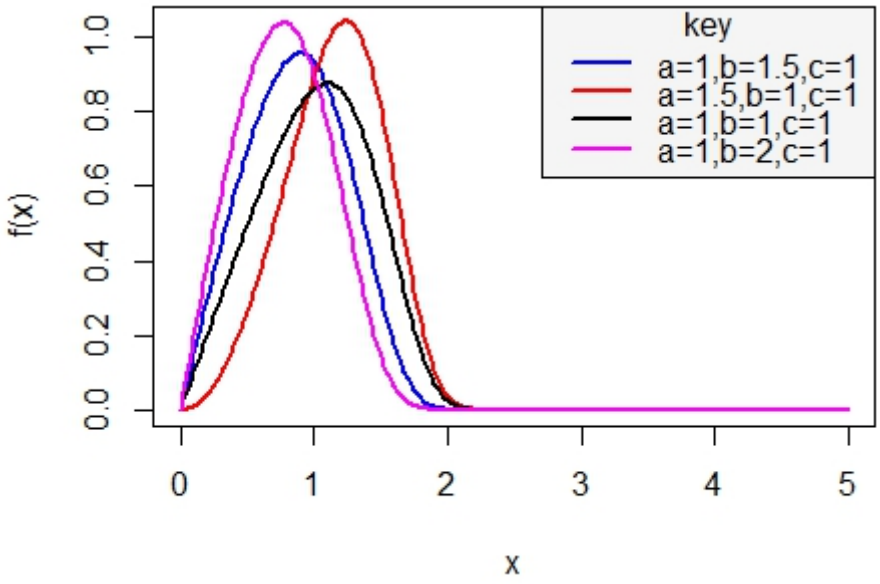


Figure 3.2.2: pdf of OGE-R Distribution

3.3 Statistical Properties

3.3.1 Moments

Tahir *et al.* (2015) stated that some of the most important features and characteristics of a distribution can be studied through moments, among which are: mean, variance, skewness and kurtosis.

Mathematically, it can be defined as let X be a continuous random variable from OGE-R distribution, then the r^{th} moment of X about the origin is given in (3.3.1)

$$\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r f(x) dx \tag{3.3.1}$$

where $f(x)$ is the OGE-R distribution.

Substituting (3.1.9) into (3.3.1), we get

$$\mu_r' = E(x^r) = \int_0^{\infty} x^r \frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1} dx \tag{3.3.2}$$

using the binomial series expansion for $(1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1}$ we obtain

$$(1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1} = \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i e^{-\lambda i(e^{\frac{x^2}{2\sigma^2}}-1)} \tag{3.3.3}$$

Substituting (3.3.3) into (3.3.2), we get

$$\mu_r' = \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \frac{\alpha\lambda}{\sigma^2} \int_0^{\infty} x^{r+1} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} e^{-\lambda i(e^{\frac{x^2}{2\sigma^2}}-1)} dx \tag{3.3.4}$$

Using series expansion of $e^{-\lambda(i+1)(e^{\frac{x^2}{2\sigma^2}}-1)}$, we get

$$e^{-\lambda(i+1)(e^{\frac{x^2}{2\sigma^2}}-1)} = \sum_{j=0}^{\infty} \frac{(-1)^j \lambda^j (1+i)^j (e^{\frac{x^2}{2\sigma^2}} - 1)^j}{j!} \tag{3.3.5}$$

Substituting (3.3.5) in (3.3.4), we get

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\alpha-1}{i}(-1)^{i+j}(i+1)^j\frac{\alpha\lambda^{j+1}}{\sigma^2}\int_0^{\infty}x^{r+1}e^{\frac{x^2}{2\sigma^2}}(e^{\frac{x^2}{2\sigma^2}}-1)^j dx \tag{3.3.6}$$

Using binomial series expansion on $(e^{\frac{x^2}{2\sigma^2}}-1)^j$, we get

$$(e^{\frac{x^2}{2\sigma^2}}-1)^{\infty} = \sum_{k=0}^{\infty}\binom{j}{k}(-1)^k e^{\frac{(j-k)x^2}{2\sigma^2}} \tag{3.3.7}$$

Substituting (3.3.7) in (3.3.6), we get

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j\frac{\alpha\lambda^{j+1}}{\sigma^2 j!}\int_0^{\infty}x^{r+1}e^{\frac{x^2}{2\sigma^2}}e^{\frac{(j-k)x^2}{2\sigma^2}}dx \tag{3.3.8}$$

(3.3.8) can be written as

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j\frac{\alpha\lambda^{j+1}}{\sigma^2 j!}\int_0^{\infty}x^{r+1}e^{\frac{(j-k+1)x^2}{2\sigma^2}}dx \tag{3.3.9}$$

Now (3.3.9) can be rewritten as

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j\frac{\alpha\lambda^{j+1}}{\sigma^2 j!}\int_0^{\infty}x^{r+1}e^{\frac{-(k-j-1)x^2}{2\sigma^2}}dx \tag{3.3.10}$$

Now, we let $y = \frac{(k-j-1)x^2}{2\sigma^2} \Rightarrow x = (\frac{2y\sigma^2}{k-j-1})^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{(k-j-1)x}{\sigma^2} \Rightarrow dx = \frac{\sigma^2 dy}{(k-j-1)x}$$

replacing x in the last expression and simplifying we have

$$dx = \frac{\sigma^2 dy}{(k-j-1)^{\frac{1}{2}}(2y\sigma^2)^{\frac{1}{2}}}$$

Substituting the above equations in (3.3.10), we get

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j\frac{\alpha\lambda^{j+1}}{\sigma^2 j!}\int_0^{\infty}\left(\frac{2y\sigma^2}{k-j-1}\right)^{\frac{r+1}{2}}e^{-y}\frac{\sigma^2 dy}{(k-j-1)^{\frac{1}{2}}(2y\sigma^2)^{\frac{1}{2}}} \tag{3.3.11}$$

After some simplications (3.3.11) becomes

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j\frac{\alpha\lambda^{j+1}\sigma^r 2^{\frac{r}{2}}}{j!(k-j-1)^{\frac{r+2}{2}}}\int_0^{\infty}y^{\frac{r}{2}}e^{-y}dy \tag{3.3.12}$$

By using the gamma function and simplifying (3.2.12) reduces to

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j\frac{\alpha\lambda^{j+1}\sigma^r 2^{\frac{r}{2}}}{j!(k-j-1)^{\frac{r+2}{2}}}(\frac{r}{2})! \tag{3.3.13}$$

3.3.2 Moment Generating Function

The moment generating function of an Odd Generalized Exponential Rayleigh random variable X $(X \sim OGE-R(\alpha,\lambda,\sigma))$ can be obtained as

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty}e^{tx}f(x)dx \tag{3.3.14}$$

Now, by substituting x^r by e^{tx} from (3.3.10) we obtain the mgf as

$$M_x(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j\frac{\alpha\lambda^{j+1}}{\sigma^2 j!}\int_0^{\infty}xe^{tx}e^{\frac{-(k-j-1)x^2}{2\sigma^2}}dx \tag{3.3.15}$$

But using series expansion

$$e^{tx} = \sum_{m=0}^{\infty}\frac{(tx)^m}{m!} \tag{3.3.16}$$

Then, using the result in (3.3.16) and simplifying the integral in (3.3.15) we get

$$M_x(t) = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^{j}\frac{\alpha\lambda^{j+1}t^{m}}{\sigma^{2}j!m!}\int_{0}^{\infty}x^{m+1}e^{\frac{-(k-j-1)x^{2}}{2\sigma^{2}}}dx \tag{3.3.17}$$

Now, we let $y = \frac{(k-j-1)x^2}{2\sigma^2} \Rightarrow x = (\frac{2y\sigma^2}{k-j-1})^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{(k-j-1)x}{\sigma^2} \Rightarrow dx = \frac{\sigma^2 dy}{(k-j-1)x}$$

replacing x in the last expression and simplifying we have

$$dx = \frac{\sigma^2 dy}{(k-j-1)^{\frac{1}{2}}(2y\sigma^2)^{\frac{1}{2}}}$$

Then,

$$M_x(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^{j}\frac{\alpha\lambda^{j+1}t^{m}\sigma^{m}2^{\frac{m}{2}}}{j!m!(k-j-1)^{\frac{m+2}{2}}}\int_{0}^{\infty}y^{\frac{m}{2}}e^{-y}dy \tag{3.3.18}$$

Using the gamma function and simplifying (3.3.18) we get

$$M_x(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^{j}\frac{\alpha\lambda^{j+1}t^{m}\sigma^{m}2^{\frac{m}{2}}}{j!m!(k-j-1)^{\frac{m+2}{2}}}(\frac{m}{2})! \tag{3.3.19}$$

3.3.3 Quantile and Median of OGE-R Distribution

The quantile function is obtain using the relation

$$F(x_q) = q \tag{3.3.20}$$

where $F(x_q)$ is the CDF of OGE-R distribution in (3.1.8) and $0 < q < 1$

Making x_q subject we obtain the quantile function as follows;

$$(1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}} - 1)})^{\alpha} = q \tag{3.3.21}$$

Then,

$$q^{\frac{1}{\alpha}} = 1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}} - 1)} \tag{3.3.22}$$

Simplifying (3.2.22) and making x_q subject, we obtain

$$x_q = \{2\sigma^2 ln(1 - \frac{1}{\lambda} ln(1 - q^{\frac{1}{\alpha}}))\}^{\frac{1}{2}}, 0 < q < 1 \tag{3.3.23}$$

The median of a random variable X that has probability density function $f(x)$ is a number x_m such that

$$\int_{-\infty}^{x_m} f(x)dx = P(X \leq x_m) = 0.5 \tag{3.3.24}$$

Using (3.2.23) we obtain the median of OGE-R distribution by setting $q = 0.5$ as follows;

$$Median = \{2\sigma^2 ln(1 - \frac{1}{\lambda} ln(1 - \frac{1}{2}^{\frac{1}{\alpha}}))\}^{\frac{1}{2}} \tag{3.3.25}$$

3.4 Reliability Analysis of OGE-R Distribution

3.4.1 Survival Function

Survival function is the probability that an item does not fail before some time, mathematically its given by

$$S(x) = 1 - F(x) \tag{3.4.1}$$

The survival function of OGE-R distribution is given by

$$S(x) = 1 - (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}} - 1)})^{\alpha} \tag{3.4.2}$$

plot of the survival function of the OGE-R distribution at different parameter values is shown in figure 3.3.1, from the graph, it shows that the value of the survival function equals one (1) at the origin and decreases as x increases and tends to zero as x gets larger.

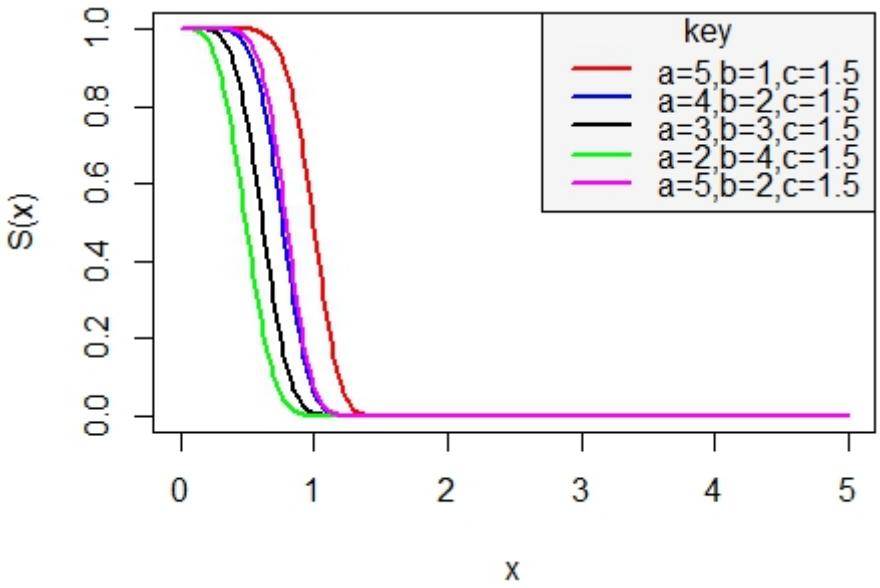


Figure 3.4.1: Survival function of OGE-R Distribution

3.4.2 Harzard Function

Hazard function is also called the failure or risk function and is the probability that a component which has survived to the beginning of the respective interval will fail in that interval of time. The hazard function is defined as;

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \tag{3.4.3}$$

The hazard function of OGE-R distribution is given by

$$h(x) = \frac{\frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} \left(1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)}\right)^{\alpha-1}}{1 - \left(1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)}\right)^{\alpha}} \tag{3.4.4}$$

Figure 3.3.2 provides curves for the hazard rate at some values of the model parameters. From the graph, we observed that the value of the hazard function increases when x increases. It gets higher as the value of x increases.

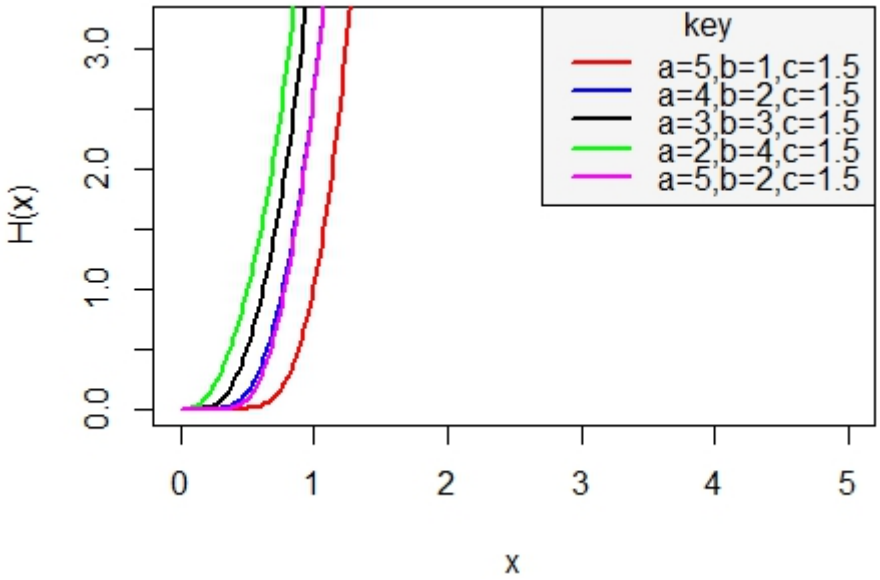


Figure 3.4.2: Hazard function of OGE-R Distribution

3.5 Order Statistics

Order statistics are widely used in many areas of statistical theory and practice, for instance, detection of outlier in statistical quality control processes. In this section, we derive closed form expressions for the pdf of the i^{th} order statistic of the OGE-R distribution.

Suppose $X_1, X_2, \cdots, X_n$ is a random sample from a distribution with pdf $f(x)$ and let $X_{1:n}, X_{2:n}, \cdots, X_{i:n}$ denote the corresponding order statistic obtained from this

sample. The pdf $f_{i:n}(x)$ of the i^{th} order statistic can be express as:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} [1 - F(x)]^{n-i} \tag{3.5.1}$$

where $F(x)$ and $f(x)$ are the CDF and pdf of OGE-R distribution.
Using the binomial expansion on $[1 - F(x)]^{n-i}$ we get

$$[1 - F(x)]^{n-i} = \sum_{r=0}^{n-i} \binom{n-i}{r} (-1)^r F(x)^r \tag{3.5.2}$$

Substituting (3.4.2) in (3.4.1) we get

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) \sum_{r=0}^{n-i} \binom{n-i}{r} (-1)^r F(x)^{r+i-1} \tag{3.5.3}$$

Substituting (3.1.8) and (3.1.9) in (3.4.3) we get

$$f_{i:n}(x) = \sum_{r=0}^{n-i} \frac{n!\,(-1)^r}{(i-1)!(n-i-r)!r!} \frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1+\alpha(r+i-1)} \tag{3.5.4}$$

Simplifying (3.4.3) we get

$$f_{i:n}(x) = \sum_{r=0}^{n-i} \frac{n!\,(-1)^r}{(i-1)!(n-i-r)!r!} \frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha(i+r)-1} \tag{3.5.5}$$

Finally, the pdf of the minimum order statistic x_1 and maximum order statistic x_n of the OGE-R distribution are given by

$$f_{1:n}(x) = \sum_{r=0}^{n-i} \frac{n!\,(-1)^r}{(n-1-r)!r!} \frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha(i+r)-1} \tag{3.5.6}$$

and

$$f_{i:n}(x) = \sum_{r=0}^{n-i} \frac{n\,(-1)^r}{(-r)!r!} \frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha(i+r)-1} \tag{3.5.7}$$

respectively.

3.6 Estimation of Parameters of the OGE-R Distribution

This section is aimed at estimating the parameters of the OGE-R distribution using the method of maximum likelihood estimation. Let $X_1, X_2, \cdots, X_n$ be a sample of size n from the OGE-R distribution with unknown parameter vector $\boldsymbol{\Theta} = (\alpha, \lambda, \sigma)^T$. Then the likelihood function for $\boldsymbol{\Theta}$ is given as:

$$L(\Theta) = L(X_1, X_2, \cdots, X_n/\Theta) = \Pi_{i=1}^{n} f(x_i/\Theta) = \Pi_{i=1}^{n} \{\frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} (1-e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1}\} \tag{3.6.1}$$

The total log-likelihood function for $\boldsymbol{\Theta}$ is

$$lnL(\boldsymbol{\Theta}) = nln\alpha + nln\lambda - 2nln\sigma + \sum_{i=1}^{n} lnx_i + \sum_{i=1}^{n} \frac{x_i^2}{2\sigma^2} + \lambda \sum_{i=1}^{n} (e^{\frac{x_i^2}{2\sigma^2}} - 1) + (\alpha - 1) \sum_{i=1}^{n} ln(1 - e^{-\lambda(e^{\frac{x_i^2}{2\sigma^2}}-1)}) \tag{3.6.2}$$

Hence, differentiating $lnL(\boldsymbol{\Theta})$ partially with respect to each parameters $(\alpha, \lambda, \sigma)$ and setting the results equal to zero gives the estimates of the respective parameters. The partial derivatives of $lnL(\boldsymbol{\Theta})$ with respect to each parameter is given by:

$$\frac{\partial lnL(\boldsymbol{\Theta})}{\partial \alpha} = \frac{n}{\hat{\alpha}} + \sum_{i=1}^{n} ln(1 - e^{-\lambda(e^{\frac{x_i^2}{2\sigma^2}}-1)}) \tag{3.6.3}$$

$$\frac{\partial lnL(\boldsymbol{\Theta})}{\partial \lambda} = \frac{n}{\hat{\lambda}} - \sum_{i=1}^{n} (e^{\frac{x_i^2}{2\sigma^2}} - 1) + (\alpha - 1) \sum_{i=1}^{n} \{\frac{e^{-\hat{\lambda}(e^{\frac{x_i^2}{2\sigma^2}}-1)} (e^{\frac{x_i^2}{2\sigma^2}} - 1)}{(1 - e^{-\hat{\lambda}(e^{\frac{x_i^2}{2\sigma^2}}-1)})}\} \tag{3.6.4}$$

$$\frac{\partial lnL(\boldsymbol{\Theta})}{\partial \sigma} = \frac{-2n}{\hat{\sigma}} - \sum_{i=1}^{n} \frac{x_i^2}{\hat{\sigma}^3} + \frac{\lambda x^2}{\hat{\sigma}^3} e^{\frac{x^2}{2\hat{\sigma}^2}} + (\alpha - 1) \sum_{i=1}^{n} \{\frac{\frac{\lambda x^2}{\hat{\sigma}^3} e^{-\lambda(e^{\frac{x_i^2}{2\hat{\sigma}^2}}-1) + \frac{x^2}{2\hat{\sigma}^2}} (e^{\frac{x_i^2}{2\hat{\sigma}^2}} - 1)}{(1 - e^{-\lambda(e^{\frac{x_i^2}{2\hat{\sigma}^2}}-1)})}\} \tag{3.6.5}$$

Setting these equations (3.5.3) to (3.5.5) equals zero, and solving simultaneously gives the maximum likelihood estimate $\widehat{\boldsymbol{\Theta}}$ of $\boldsymbol{\Theta}$.

These set of equations cannot be solved analytically, hence we result to using

statistical software to solve them numerically.

CHAPTER FOUR

ANALYSIS AND DISCUSSIONS

4.1 Application

This chapter illustrate the usefulness of the OGE-R distribution to real dataset and also to compare the fitness of the distribution with other known distributions including Transmuted Rayleigh distribution, Transmuted Exponentiated Inverse Rayleigh distribution and the Rayleigh distribution. The pdf for these distributions are given as in the following equations:

The Rayleigh (R) Distribution

The pdf of R distribution is as follows:

$$g(x;\sigma) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \tag{4.1.1}$$

where $x > 0$, $\sigma > 0$

The Transmuted Rayleigh (TR) Distribution

The pdf of the TR Distribution is as follows:

$$g(x;\sigma,\lambda) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \left(1 - \lambda + 2\lambda e^{-\frac{x^2}{2\sigma^2}}\right) \tag{4.1.2}$$

where $x > 0$, $\sigma > 0$, $|\lambda| \leq 1$

The Transmuted Exponentiated Inverse Rayleigh (TEIR) Distribution

The pdf of the TEIR Distribution is as follows:

$$g(x;\sigma,\theta,\lambda) = \frac{2\theta\sigma}{x^3} e^{-\frac{\sigma\theta}{x^2}} \left(1 - \lambda + 2\lambda e^{-\frac{\sigma\theta}{x^2}}\right) \tag{4.1.3}$$

where $x > 0$, $\sigma, \theta > 0$, $|\lambda| \leq 1$

Dataset: The real dataset corresponds to the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The dataset consist

of 72 exceedances for the years 1958 – 1984, rounded to one decimal. The dataset was analyzed by Akinsete *et al.* (2008) and Khan (2014), and is given below as:

Table 4.1: The dataset from Akinsete *et al.* (2008) and Khan (2014)

1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0

Table 4.2: Summary of the dataset

Parameters	N	Q1	Median	Q3	Mean	Variance	Skewness	Kurtosis
Value	72	2.125	9.500	20.120	12.200	151.2215	1.504	3.189

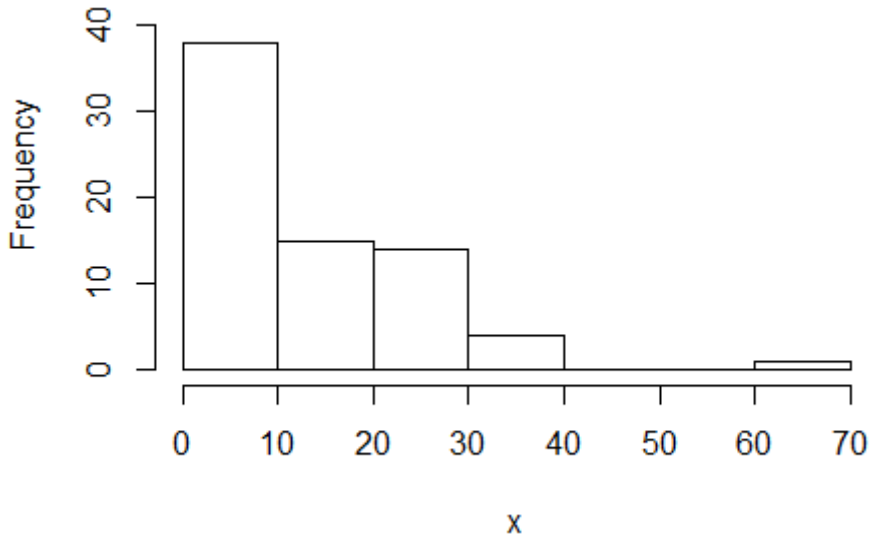


Figure 4.1.1: Histogram of the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada

From the summary and histogram for the dataset, it shows that the dataset is skewed to the right and also from the pdf of OGE-R distribution in Figure 3.1.2, it shows that the distribution is skewed to the right, hence the dataset can be applied on the distribution.

To compare the performance of these distributions, we will adopt some criteria including: Akaike Information Criterion (AIC), Akaike Information Criterion with

Correction (AICC) and Bayesian Information Criterion (BIC) and these statistics are given as:

$$AIC = 2k - 2\ell\ell \tag{4.1.4}$$

$$AICC = AIC + \frac{2k(k+1)}{n-k-1} \tag{4.1.5}$$

and

$$BIC = klog(n) - 2\ell\ell \tag{4.1.6}$$

respectively, where k is the number of parameters in the statistical distribution, n the sample size and $\ell\ell$ is the maximized value of the log-likelihood function under the considered distributions.

Hence, the distribution with the lowest values for these statistics would be considered as the best distribution to fit the dataset.

Table 4.3: MLE's of the parameters

Distribution	MLE's of the parameter(s)
$R(\sigma)$	$\widehat{\sigma}$ = 12.208
$TR(\sigma, \lambda)$	$\widehat{\sigma}$ = 8.4205, $\widehat{\lambda}$ = 0.1066
$TEIR(\sigma, \theta, \lambda)$	$\widehat{\sigma}$ = 2.700, $\widehat{\theta}$ = 3.3109, $\widehat{\lambda}$ = 1.000
$OGE-R(\sigma, \alpha, \lambda)$	$\widehat{\sigma}$ = 3.7426, $\widehat{\alpha}$ = 1.0223, $\widehat{\lambda}$ = 0.0040

Table 4.4: Model Performance Comparisons

Distribution	**Measures**			
	$\ell\ell$	AIC	$AICC$	BIC
$R(\sigma)$	-302.8379	607.6757	608.0286	611.2478
$TR(\sigma, \alpha)$	-348.4632	700.9264	701.2793	702.4984
$TEIR(\sigma, \theta, \lambda)$	-771.9796	1549.9590	1550.3119	1549.5312
$OGE-R(\sigma, \alpha, \lambda)$	-6.0962	18.1925	18.5454	17.7644

Table 4.3 shows MLE of the parameters to each of the four fitted distributions for the dataset, Table 4.4 shows the corresponding values of $\ell\ell$, AIC, AICC and BIC for each of the distributions, the result shows that the OGE-R distribution has better performance and could be used as the best model compared to the other three distributions used for fitting the dataset. Since the dataset is skewed we can likewise say that OGE-R distribution is a competitive distribution for analyzing asymmetric or non-normally distributed data particularly those skewed to the right.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 Summary

This dissertation presented a new three-parameter distribution called the Odd Generalized Exponential-Rayleigh Distribution. Which extend the Rayleigh distribution by adding two parameter thereby adding more flexibility to the distribution. Some statistical properties of the OGE-R distribution including moments, moment generating function, quantile function, reliability analysis and order statistics have been studied appropriately. The parameters of the distribution were obtained using the method of maximum likelihood. We use application on dataset to compare the usefulness of the OGE-R distribution with other known distributions including R, TR and TEIR distributions. The new distribution provides better fits compared to R, TR, TEIR distributions. Hence, OGE-R distribution is the competitive distribution for fitting skewed dataset.

5.2 Conclusion

We propose a new three-parameter model called the OGE-R distribution. Some mathematical and statistical properties of the proposed distribution have been studied. We have derived explicit expressions for its moments, moment generating function, survival function, hazard function, quantile funtion, median and order statistics. Some plots of the distribution shows that it is a skewed distribution. We estimated the model parameters using the method of maximum likelihood estimation. The plots for the survival and hazard functions indicate that the OGE-R distribution would be suitable in modeling time or age-dependent events, where survival and failure rate decreases with time or age.

5.3 Recommendation

Based on the results of this research work, we recommend that the proposed distribution should be used for modeling dataset that is skewed most especially if it is positively skewed. We also recommend that this distribution can be used in modeling time or age dependent events random variables.

5.4 Contribution to Knowledge

We proposed a three parameter lifetime distribution called OGE-R Distribution. We obtained some of its statistical properties, we also estimated the parameters of the new distribution. Finally, we established that the new distribution is a good candidate for fitting positively skewed dataset.

5.5 Areas for Further Research

Based on this research work, Further studies on this work can explore it more, some of which are: One can look at estimation of confidence intervals for the parameters of the proposed distribution, can also obatin the enthropies which is a measure of variation or uncertainty of a random variable X. Also, researchers can estimate the parameters of the new distribution using a non-classical approach for the purpose of theoretical comparison and validation of methodology.

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