SOME MIXED BOUNDARY VALUE PROBLEMS IN CYLINDERICAL COORDINATES

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MIXED BOUNDARY VALUE PROBLEMS AND THEIR REDUCTION TO DUAL INTEGRAL EQUATIONS AND DUAL SERIES RELATIONS

1.1 BOUNDARY VALUE PROBLEMS AND THEIR TYPES

Most problems coming up in mathematical physics require the solution of ordinary or partial differential equations and this forms a classical branch of analysis. There are many important applications of the subject in the physical sciences and engineering.

In the general solution of an ordinary differential equation, one or more arbitrary constants appear. There is only one independent variable, and the arbitrary constants serve to fix the initial values of the solution and its derivatives at a given point. With partial differential equations, the situation is basically different. Several independent variables are present and the general solution involves arbitrary functions of the independent variables rather than arbitrary constants. Also, instead of specifying the values of a solution (and possibly its derivatives) at a point, the values are prescribed along a curve or over a surface.

In a boundary value problem for a function U, we usually require that U satisfies not only the differential equation throughout the given domain of its independent variables but also some conditions of that domain. The equations that represent those conditions may involve values of the derivatives of U as well as U itself at points on the domain. In addition, some conditions on the continuity of U and its derivatives within the domain may be required.

These conditions may appear as "The boundary condition", "The initial condition", or "The jump condition". In case the boundary conditions are prescribed, it may be that either a single relation is given over the whole of the boundary or one relation is given over a part of the boundary while an altogether different type of condition may be prescribed over the other part of the boundary. In the former case, the problem of finding the solution of a given ordinary or partial differential equation is usually called a "boundary value problem" while in the latter case, it is dintinguished by calling it a "mixed boundary value problem".

It is worthwhile mentioning a few types of boundary value problems that are encountered. In our further discussions, we shall deal only with linear boundary value problems. A differential equation in U or a boundary

condition on U is linear, if it is an equation of the first degree in U and its derivatives. Thus, the terms of the equation are either, functions of the independent variables alone including constants or such functions multiplied by either U or one of its derivatives nz:

$$p_o(x)U^{(n)} + p_1(x)U^{(n-1)} + \ldots + p_n(x)\,U = r(x) \quad \ldots\ldots\ldots (1.1.1)$$

where $U^{(n)}$ denotes the n^{th} derivative of U with respect to x is an ordinary linear differential equation of order n in U and its derivatives. Also the equation

$$zU_{xx} + xy^2U_{yy} - e^xU_z = f(x,y,z) \quad \ldots\ldots\ldots\ldots\ldots (1.1.2)$$

where U_x denotes the first partial derivative of U with respect to x, U_{xx} denotes the second partial derivative of U with respect to x and so on, is a second order linear partial differential equation.

A linear ordinary or partial differential equation may also be either homogeneous or non-homogeneous. A linear differential equation is homogeneous if each of its non-zero terms is of the first degree either in the function U or its derivatives. If a linear differential equation also contains one or more free terms, that is, a function involving only independent variables, then it is called non-homogeneous.

If in equation (1.1.1) and (1.1.2) the functions r(x) and f(x,y,z) are put equal to zero separately, the equations become homogeneous, otherwise they are non-homogeneous.

If $u_1(x)$ is a solution of a linear homogeneous differential equation and $u_2(x)$ is another solution of the same equation, then the linear combination of $u_1(x)$ and $u_2(x)$, that is

$$u(x) = c_1u_1(x) + c_2u_2(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1.3)$$

is also a solution of the linear equation where c_1, c_2 are arbitrary constants and the solution (1.1.3) above is called a solution obtained by superposition.

Many differential equations like Legendre equation, Bessel equation, Helmoltz equation, Wave equation, Heat equation, Laplace equation etc are examples of linear homogeneous differential equations. These equations are

of fundamental importance in mathematical description of a wide range of physical phenomena extending from electrostatics and potential theory to Heat conduction and from gravitational forces to sound waves.

Let us consider the second - order linear homogeneous differential equation

$$p_o(x)U^{(2)} + p_1(x)\,U^{(1)} + p_2(x)\,U = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.1.4)$$

The problem of finding a solution of equation (1.1.4) subject to the boundary condition

$$U(a) = U(b) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1.5)$$

where a and b are constants such that $a \leqslant x \leqslant b$ is an example of a linear homogeneous boundary value problem. A boundary value problem is said to be homogeneous if the boundary conditions vanish at the end points, otherwise, it is non-homogeneous.

If instead of (1.1.5) the function U satisfies ~~the boundary condition (1.1.5), it should satisfy~~ the periodic boundary condition

$$\left.\begin{aligned} U(a) &= U(b) \\ U'(a) &= U'(b) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.1.6)$$

then the problem of solving equation (1.1.4) subject to the periodic boundary conditions (1.1.6) is called a periodic boundary value problem.

If $\mathbb{R}$ is a bounded region with a boundary S and if $\bar{\mathbb{R}} = R + S$ (= the union of $\mathbb{R}$ with its boundary S); that is, the closure of $\mathbb{R}$, the problem of finding the solution of a partial differential equation in U when the function U is prescribed as a function of position over S is known as the Dirichlet problem. That is, if

$$\left.\begin{aligned} L\,U &= f \text{ in } \mathbb{R} \\ \text{and} \quad U &= g \text{ over } S \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.1.7)$$

where L is a second-order linear differential operator.

A problem of somewhat different type is to determine a solution of a partial differential equation in U which has a continuous second partial derivatives in $\mathbb{R}$ and satisfies the condition

$$\left.\begin{aligned} LU &= f \text{ in } \mathbb{R} \\ \text{and} \quad \frac{\partial U}{\partial n} &= g \text{ over } S \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.1.8)$$

where n is the outer normal to the surface S and $\frac{\partial U}{\partial n}$ denotes the normal or directional derivative of U in the direction of the outer normal to S. Such a

problem is called the Neumann problem.

If in the place of equation (1.1.7) of (1.1.8) we are given that

$$\begin{aligned} & LU = f \text{ in } R \\ \text{and} \quad & aU + b\frac{\partial U}{\partial n} = g \text{ over } S \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1.9)$$

where g,a and b are assumed to be continous over S and a and b do not vanish simultaneously; such a condition is called a mixed boundary condition. The problem of finding a solution of a partial differential equation in U where U itself should satisfy the condition (1.1.9) is called a mixed boundary value problem.

1.2 Laplace's Equation and the Properties of Harmonic Functions

The special functions of mathematical physics arise in the solution of partial differential equations governing the behaviour of certain physical entities. Probably the most frequently occuring of this type in mathematical physics is Laplace's equation

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \ldots\ldots\ldots\ldots (1.2.1)$$

Many of the problems in potential theory which is the study of solutions of Laplace's equations and their properties, are concerned with finding the solutions for which boundary conditions are prescribed on boundaries which have either cylinderical or spherical symmetry. Therefore, it is worthwhile mentioning the form of Laplace's equation (1.2.1) in polar coordinates.

If ρ,ϕ,z are cylinderical polar coordinates such that

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad \text{and} \quad z = z \qquad \ldots\ldots\ldots\ldots\ldots (1.2.2)$$

the Laplace's equation becomes

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} = 0 \qquad \ldots\ldots\ldots\ldots\ldots (1.2.3)$$

If ρ,ϕ and θ are spherical polar coordinates such that

$$x = \rho\sin\theta\cos\phi, \quad y = \rho\sin\theta\sin\phi, \text{ and } z=\rho\cos\theta \qquad \ldots\ldots\ldots\ldots (1.2.4)$$

the Laplace's equation becomes

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 U}{\partial \phi^2} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial U}{\partial \theta} = 0 \qquad \ldots\ldots\ldots\ldots (1.2.5)$$

A solution of a linear second-order partial differential equation in a region is a function with continuous second derivatives which satisfies the differential equation in that region.

The classical problems associated with Laplace's equation is to find a solution, that is, a function U in the interior of the region $\mathbb{R}$ when the function U itself is prescribed as a function of position on the boundary S of $\mathbb{R}$. A function U is said to be harmonic at a point P in three dimensional space if there exists a sphere Q with centre P such that U has continuous second derivatives and satisfies Laplace's equation at each point inside Q. And a function U is harmonic in a region $\mathbb{R}$ if it is harmonic at each point of $\mathbb{R}$. Corresponding definitions hold for harmonic functions in two dimensions with a circle replacing a sphere.

Harmonic functions have many interesting properties, for example, if a function U is harmonic in a sphere, the value of U at the centre of the sphere is the arithmetic mean of its values on the surface. Also if a function U that is harmonic in a region $\mathbb{R}$ is not a constant in that region, then U attains its maximum and minimum values only on the boundary S of $\mathbb{R}$. The two properties of the harmonic functions mentioned above are called mean value and extrema principle properties respectively.

Solutions of Laplace's equations are also called potential functions because of the physical significance of the solution.

Laplace's equation (1.2.1) is satisfied by the velocity potential in hydrodynamics, electrostatic potential in electrostatics and gravitational potential in gravitational theory. It is also satisfied by the temperature U of a body with uniform thermal conductivity properties when the body conditions are not time-dependent and a steady state has been attained.

1.3 Solution of Laplace's Equation associated with Boundary Value Problems in Cylinderical Coordinates

In section 1.2, different forms of Laplace's equations were mentioned. In this section we shall mention briefly the solution of Laplace's equation suitable for the boundary value problems in cylinderical polar coordinates while the solutions for other coordinates follow the same principle.

Consider the partial differential equation

$$\nabla^2 U = a^2 \frac{\partial^2 U}{\partial t^2} + b \frac{\partial U}{\partial t} + c\,U \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3.1)$$

where ∇^2 is the Laplacian operator, t is the time and a,b and c are either given constants or functions of space coordinates. A variety of important differential equations occuring in mathematical physics (e.g. electrodynamics, the theory of vibrations, the theory of heat conductions) are special cases of equation (1.3.1). The boundary conditions imposed on the function U many times require the use of a system of cylinderical polar coordinates ρ,ϕ,z related to the rectangular cartesian coordinates x,y,z, by the relations (1.2.2) where

$$0 \leq \rho \leq \infty, \quad 0 \leq \phi < 2\pi \quad \text{and} \quad -\infty < x < \infty \quad \ldots\ldots\ldots\ldots (1.3.2)$$

In cylinderical polar coordinates, equation (1.3.1) takes the form

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} = a^2 \frac{\partial^2 U}{\partial t^2} + b\frac{\partial U}{\partial t} + cU \quad \ldots\ldots\ldots\ldots\ldots (1.3.3)$$

In solving the partial differential equation (1.3.3) the method of separation of variables could be used which is one of the most fruitful sources of their solutions. There is more behind this approach than merely the establishment of the mechanism for producing a set of random functions which happen to satisfy the given equation.

By looking for a solution of the form $R(\rho)\Phi(\phi)Z(z)T(t)$ for example, where each of the functions depends on only one variable, we are well placed to apply a boundary condition along a boundary on which ρ,ϕ,z or t is a constant. For non-polar coordinates; for example, rectangular coordinate system, the same approach applies.

Equation (1.3.3) has infinitely many solutions of the form

$$U(\rho,\phi,z,t) = R(\rho)\Phi(\phi)Z(z)T(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3.4)$$

Substituting (1.3.4) into equation (1.3.3) and dividing through by $R\Phi ZT$, we have

$$\frac{1}{R\rho}\frac{d}{d\rho}(\rho\frac{dR}{d\rho}) + \frac{1}{\Phi\,\rho^2}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} - c = \frac{1}{T}\left(a^2\frac{d^2T}{dt^2} + b\frac{dT}{dt}\right)$$

Since variables ρ,ϕ,z,t are independent, both sides of thelast equation above must be equal to a constant which we denote by $-n^2$ say. This leads us to getting two equations

$$a^2\frac{d^2T}{dt^2} + b\frac{dT}{dt} + n^2T = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.3.5)$$

and $$\frac{1}{R\rho}\frac{d}{d\rho}(\rho\frac{dR}{d\rho}) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} + n^2 = c - \frac{1}{Z}\frac{d^2Z}{dz^2}$$

The same reasoning as above shows that both sides of equation above must be equal to a constant, which at this time we denote by $-m^2$, thus obtaining

$$\frac{d^2Z}{dz^2} - (m^2 + c)z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3.6)$$

and $$\rho^2\frac{1}{R\rho}\frac{d}{d\rho}[(\rho\frac{dR}{d\rho}) + (m^2 + n^2)] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

Again, both sides of the last equation must be equal to a constant, which, we denote by μ^2 and we have

$$\frac{d^2\Phi}{d\phi^2} + \mu^2\Phi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.3.7)$$

and $$\frac{1}{\rho}\frac{d}{d\rho}(\rho\frac{dR}{d\rho}) + (m^2 + n^2 - \frac{\mu^2}{\rho^2})R = 0 \quad \ldots\ldots\ldots\ldots\ldots (1.3.8)$$

The process just described is called separation of variables and it leads to infinitely many solutions of the form Of (1.3.4) depending on the parameters m,n and μ which could take real or complex values.

The first three of equations from (1.3.5) to (1.3.8) above can be solved in terms of elementary functions subject to the boundary conditions prescribed on U and the fourth equation is a Bessel equation of order μ whose solution is

$$R(\rho) = A\,J_\mu(\lambda\rho) + BY\mu(\lambda\rho) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.3.9)$$

where $\lambda = \sqrt{m^2 + n^2}$, $J_\mu(\lambda\rho)$ and $Y_\mu(\lambda\rho)$ are Bessel functions of order μ; A and B are arbitrary constants. The solutions of Bessel equations are generally called cylinder functions or Bessel functions. Bessel equations are encountered in the study of boundary value problems of potential theory

for cylinderical domain which explains the origin of cylinder functions.

Having mentioned how the use of the method of separation of variables can be used in solving the general partial differential equation (1.3.3) we shall look into how equation (1.2.1) which is obtained from equation (1.3.1) by putting a,b and c each equal to zero, can be solved.

A problem is axisymmetric if it is symmetric about z-axis, that is, it is independent of the polar angle ϕ; otherwise it is asymmetric.

In the case where U is independent of the polar angle ϕ, we have

$$\nabla^2 U = \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial z^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.3.10)$$

and then $U(\rho,z) = R(\rho)Z(z)$, and using the method of separation of variables we have

$$\frac{1}{R}\left(\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right) = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -\lambda^2$$

where λ is the separation parameter

From equation above we have the following equations

$$\left.\begin{aligned} &\frac{d^2Z}{dz^2} - \lambda^2 Z = 0 \\ \text{and}\quad &\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \lambda^2 R = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3.11)$$

The first equation of (1.3.11) can be solved in terms of elementary functions to get

$Z(z) = Ce^{\lambda z} + De^{-\lambda z}$ where C and D are arbitrary constants. The second equation of (1.3.11) is an example of Bessel equation of order zero whose solution is of the form $R(\rho) = A\,J_o(\lambda\rho) + BY_o(\lambda\rho)$ where $J_o(\lambda\rho)$ and $Y_o(\lambda\rho)$ are Bessel functions of the first and second kind respectively and of order zero; A and B are arbitrary constants. A solution of equation (1.3.10) is

$$U_\lambda(\rho,z) = [C(\lambda)e^{\lambda z} + D(\lambda)e^{-\lambda z}][A(\lambda)J_o(\lambda\rho) + B(\lambda)Y_o(\lambda\rho)] \quad \ldots\ldots\ldots (1.3.11a)$$

The complete solution can be written either as an integral

$$U(\rho,z) = \int_0^\infty U_\lambda(\rho,z)\,d\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3.11b)$$

or if U vanishes on a boundary $\rho = a$ say, then in series form

$$U(\rho,z) = \sum_{n=1}^{\infty} A_n\, J_o(\lambda_n\rho)[C_n\, e^{\lambda n z} + D_n e^{-\lambda n z}] \quad \ldots\ldots\ldots\ldots\ldots (1.3.11c)$$

where $\{\lambda n\}$, n = 1,2,3,.... is the sequence of positive zeros of $J_o(\lambda a)$

If in equation (1.3.11), $-\lambda^2$ is replaced by λ^2 we have different solutions from the previous ones of the form

$$Z(z) = C(\lambda)\sin\lambda z + D(\lambda)\cos\lambda z$$

$$R(\rho) = [A(\lambda)I_o(\lambda\rho) + B(\lambda)K_o(\lambda\rho)]$$

and $U_\lambda(\rho,z) = [C(\lambda)\sin\lambda z + D(\lambda)\cos\lambda z][A(\lambda)I_o(\lambda\rho) + B(\lambda)K_o(\lambda\rho)]$.....(1.3.12)

where $I_o(\lambda\rho)$ and $K_o(\lambda\rho)$ are modified Bessel functions of the first and third kind respectively and of order zero. The complete can be written as an integral

$$U(\rho,z) = \int_o^\infty U_\lambda(\rho,z)d\lambda \quad \text{.....(1.3.12a)}$$

For the solution of Laplace's equation in cylinderical coordinates in asymmetric case, that is, the solution depending on the polar angle ϕ, using the method of separation of variables on equation (1.2.3), that is,

$$\nabla^2 U = \frac{\partial^2 U}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial\phi^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

we have

$$\frac{1}{R}\left(\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2z}{dz^2} = 0 \quad \text{.....(1.3.13)}$$

For R,Φ,Z to satisfy the Laplace's equation (1.2.3) we have the following equations

$$\frac{d^2Z}{dz^2} - \lambda^2 Z = 0 \quad \text{..... (1.3.13a)}$$

$$\frac{d^2\Phi}{d\phi^2} + n^2\Phi = 0 \quad \text{..... (1.3.13b)}$$

and $$\frac{d^2R}{\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(\lambda^2 - \frac{n^2}{\rho^2}\right)R = 0 \quad \text{..... (1.3.13c)}$$

where λ and n are separation parameters. Equations (1.3.13a) and (1.3.13b) can be solved separately to obtain

$$Z(z) = c(\lambda)e^{\lambda z} + D(\lambda)e^{-\lambda z} \quad \text{..... (1.3.13d)}$$

and $$\Phi(\phi) = E_n \sin n\phi + F_n \cos n\phi \quad \text{..... (1.3.13e)}$$

Equation (1.3.13c) is a Bessel equation of order n whose solution is

$$R_n(\lambda) = A_n(\lambda)J_n(\lambda\rho) + B_n(\lambda)Y_n(\lambda\rho) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3.13f)$$

Combining solutions (1.3.13d),(1.3.13e) and (1.3.13f) we have

$$U_{\lambda_n}(\rho,\phi,z) = [C(\lambda)e^{\lambda z} + D(\lambda)e^{-\lambda z}][E_n \sin n\phi + F_n \cos n\phi][A_n(\lambda)J_n(\lambda\rho)+B_n(\lambda)Y_n(\lambda\rho)] \quad \ldots\ldots\ldots (1.3.14)$$

and the complete solution can be written either as an integral sum series of the form

$$U(\rho,\phi,z) = \sum_{n=1}^{\infty}\left(\int_0^{\infty} U_{\lambda_n}(\rho,z)d\lambda\right)[E_n \sin n\phi + F_n \cos n\phi] \quad \ldots\ldots\ldots\ldots\ldots (1.3.14a)$$

or in a corresponding double series form

$$U(\rho,\phi,z) = \sum_{n=o}^{\infty}\sum_{m=1}^{\infty}(E_{nm}\sin n\phi + F_{nm}\cos n\phi)A_{n_m}J_n(\lambda_{nm}\rho)[c(x)e^{\lambda z} + D(\lambda)e^{-\lambda z})] \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.3.14b)$$

where $\{\lambda_{nm}\}$; $m = 1,2,3,\ldots\ldots$ is the sequence of positive zeros of $J_n(\lambda a)$ if $U(\rho,\phi,z)$ vanishes on $\rho = a$.

If we replace $-\lambda^2$ in equations (1.3.13a) and (1.3.13c) by λ^2 we have a new set of solutions of the form

$$U_{\lambda_n}(\rho,\phi,z) = [c(\lambda)\sin\lambda z + D(\lambda)\cos\lambda z][E_n \sin n\phi + F_n \cos n\phi] [A_n(\lambda)I_n(\lambda\rho) + B_n(\lambda)K_n(\lambda\rho)] \quad \ldots\ldots\ldots (1.3.15)$$

The complete solution can be written in integral form, that is,

$$U(\rho,\phi,z) = \int_0^{\infty} U_{\lambda_n}(\rho,\phi,z)d\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3.15a)$$

where $I_n(\lambda\rho)$ and $K_n(\lambda\rho)$ are modified Bessel functions of the first and third kind respectively and of order n.

1.4. Integral Equations. Dual Integral Equations, Dual Series Relations, and the reduction of the mixed boundary value problem of the Electrified Disc to Dual Integral Equations and Dual series Relations

When the method of separation of variables is used for the solution of a partial differential equation, generally it reduces to a system of Sturm-Liouville problem whose solution can be expressed in terms of the eigen-functions of the related homogeneous problems. These eigenfunctions form a complete set over the domain considered and therefore the variable separable solution of the partial differential equation under consideration may be

expanded as a series of linear product of eigenfunctions of the derived Sturm-Liouville problem. This series expansion may be described as "Generalised" Fourier series solution of the corresponding partial differential equation. The coefficients of such Fourier series may be got by using the boundary conditions of the problem.

Partial differential equations and their Fourier series solutions have been discussed by Courant and Hilbert [8], Titchmarsh [34], Carslaw [3], Sneddon [20,21,26], churchill [4] etc. However, the methods given are not suitable for the solution of the mixed boundary value problems.

In the case of mixed boundary value problems, the method of dual Integral equations and dual series relations is one of the most effective tools.

The term "integral equations" is used for an equation in which the unknown function occurs at least once under one or more integral signs viz:

$$\left.\begin{aligned}
&\int_a^b K(x,\xi)\phi(\xi)d\xi = \phi(x); \quad a \leqslant x \leqslant b \\
&\int_a^b K(x,\xi)\phi(\xi)d\xi + f(x) = \phi(x); \quad a \leqslant x \leqslant b \\
&\int_a^b \int_a^b K(x,\xi,u)\phi(\xi)\phi(u)d\xi du = \phi(x); \quad a \leqslant x \leqslant b \\
&\int_a^b K(x,\xi)[\phi(x)]^2 d\xi = \phi(x); \quad a \leqslant x \leqslant b
\end{aligned}\right\} \quad \ldots\ldots\ldots(1.4.1)$$

in which function $\phi(\xi)$ in each case is the unknown function and all other functions are regarded as given, are examples of integral equations.

In equations of (1.4.1), $k(x,\xi)$ is called the kernel of the integral. The functions $k(x,\xi)$, $\phi(x)$, $\frac{f(x)}{\lambda}$ and the limits of integral are known and the function $\phi(\xi)$ satisfying each integral equation is to be determined.

The first two equations of (1.4.1) are examples of linear integral equations while the last two are non-linear. Integral equations generally arise from the solutions of boundary value problems. For example, the complete solution of the Laplace's equation (1.3.10) is put in integral form in equation (1.3.11b), that is,

$$U(\rho,z) = \int_0^\infty U_\lambda(\rho,z)d\lambda$$

$$= \int_0^\infty [C(\lambda)e^{\lambda z} + D(\lambda)e^{-\lambda z}][A(\lambda)J_0(\lambda\rho) + B(\lambda)Y_0(\lambda\rho)]d\lambda$$

where the right hand side of the above equation is got by using equation (1.3.11a).

Let the solution above satisfy the boundary conditions.

$$\begin{aligned} &\text{(i) } U(o,z) \text{ is finite} \\ &\text{(ii) } U(\rho,z) \neq o \text{ as } \sqrt{\rho^2 + z^2} \to \infty \\ &\text{(iii) } U(\rho,o) = f(\rho) \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.2)$$

then we have

$$\int_o^\infty P(\lambda) J_o(\lambda\rho) d\lambda = f(\rho) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.2a)$$

where $P(\lambda) = A(\lambda)D(\lambda)$ is unknown function that is to be determined and the kernel in this case is $J_o(\rho\lambda)$; and the above equation (1.4.2a) is an integral equation whose solution is got to be

$$P(\lambda) = \lambda \int_o^\infty \rho\, f(\rho)\, J_o(\lambda\rho) d\rho \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.2b)$$

by taking the inverse Hankel transform of equation (1.4.2a).

In linear integral equations, if the upper limit of integral is a variable, we call such linear integral equations Volterra integral equations which are of two kinds. If $k(x,\xi)$ is put equal to zero in the first two equations (1.4.1) when $x < \xi < b$ the two integral equations become

$$\int_a^x k(x,\xi)\phi(\xi)\, d\xi = \phi(x); \quad a \leqslant x \leqslant b$$

and

$$\int_a^x k(x,\xi)\phi(\xi) d\xi + f(x) = \phi(x); \quad a \leqslant x \leqslant b$$

and the integral equations above are examples of the Volterra integral equations of the first and second kind respectively.

General integral equations of the type of the first two equations of (1.4.1) are called Fredholm's integral equations of the first and second kind respectively.

We now come to the reduction of the mixed boundary value problem in potential theory to dual integral equations and dual series relations separately. Many problems of mathematical physics; potential theory and the theory of elasticity etc. lead to the consideration of the dual integral equations of the type

$$\int_0^\infty G(\xi)\phi(\xi)\, K(\rho,\xi)d\xi = f_1(\rho), \qquad 0 \leq \rho < 1$$

$$\int_0^\infty \phi(\xi)K(\rho,\xi)d\xi = f_2(\rho), \quad \rho > 1 \qquad \text{.....(1.4.3)}$$

where $G(\xi)$, $f_1(\rho)$, $f_2(\rho)$ and $K(\rho,\xi)$ are given functions of the variables indicated and the unknown function $\phi(\xi)$ satisfying the dual integral equations above is to be determined. Dual integral equations arise from the solutions of mixed boundary value problems.

Some cases in which solutions of dual integral equations (1.4.3) have been attempted are those in which the kernels are Bessel functions, Sine functions, Cosine functions, Conical functions, Legendre functions etc. In chapters 2 and 3 we shall consider the solution of dual integral equations with Bessel kernels. Dual integral equations with sine or cosine kernels can be reduced to a system of equations with Bessel kernels remembering that sine and cosine functions can be expressed in terms of Bessel functions. But independent methods of solutions also exist which are discussed in Srivastav [32] and Sneddon [22].

A specific example of a physical problem which we shall reduce to a dual integral equation is the problem of finding the electrostatic potential U due to the presence of an electrified disc of finite radius. We take the plane ~~centre~~ of the disc as the xy-plane. Mathematically we formulate the problem as follows:

The problem is symmetrical with respect to the plane z = o and therefore we consider the problem only for semi-infinite space z > o. Also there is no loss of generality if we assume the radius of the disc to be unity.

We want to find the potential U which is also a solution of Laplace's equation $\nabla^2 U = 0$ subject to the boundary conditions

(a) On the face z = 0

(i) $U = F(\rho, \phi)$; $0 \leq \rho \leq 1$, $0 \leq \phi < 2\pi$ .....(1.4.4)

(ii) $\frac{\partial U}{\partial z} = 0$, $\rho > 1$ .....(1.4.4a)

(b) $U \to 0$ as $\sqrt{\rho^2 + z^2} \to \infty$ ..... (1.4.4b)

Boundary condition (1.4.4) can be expressed in the trigonometric Fourier series of the form

$$U = \sum_{n=o}^{\infty} [f_n(\rho)\cos n\phi + g_n(\rho)\sin n\phi] \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.4c)$$

We can use the solution (1.3.14) of Laplace's equation (1.2.3). Alternatively we can obtain the solution by another procedure using the technique of Hankel transform as follows:

Hankel transform, (Sneddon [20]) statesthat

If $f(\xi_i) = \int_o^\infty xf(x)J_\mu(x\xi_i)dx$ then

$$f(x) = \int_o^\infty \xi_i \; f(\xi_i)\; J_\mu(x\xi_i)d\xi_i \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.5)$$

Instead of putting the solution $U(\rho,\phi,z)$ equal to $R(\rho)\Phi(\phi)Z(z)$, we shall put it equal to $F(\rho,z)\; G_n(\phi)$ where

$$G_n(\phi) = E_n \sin n\phi + F_n \cos n\phi \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.6)$$

which is got by using the method of separation of variables in the previous section.

The function $F(\rho,z)$ can be obtained by solving the partial differential equation

$$\frac{\partial^2 F}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial \rho} - \frac{n^2}{\rho^2} F + \frac{\partial^2 F}{\partial z^2} = 0$$

Let $\bar{F}(z,\xi) = \int_o^\infty F(\rho,Z)\rho J_n(\rho\xi)d\rho \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.7)$

and let $M(F) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial F}{\partial \rho}\right) - \frac{n^2}{\rho^2} F$

We have, integrating by parts

$$\int_o^\infty \frac{\partial}{\partial \rho}\left(\rho\frac{\partial F}{\partial \rho}\right) J_n(\rho\xi)d\rho = \left[\rho\frac{\partial F}{\partial \rho} J_n(\rho\xi)\right]_b^\infty - \xi\int_o^\infty \rho\frac{\partial F}{\partial \rho} J_n'(\rho\xi)d\rho$$

$$= -\xi\int_o^\infty \rho\frac{\partial F}{\partial \rho} J_n'(p\xi)d\rho \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.7a)$$

Then $I = \int_o^\infty M(F)\rho\, J_n(\rho\xi)d\rho + \frac{\partial^2}{\partial z^2}\int_o^\infty F\rho J_n(\rho\xi)d\rho$

$$= \int_o^\infty \frac{\partial}{\partial \rho}\left(\rho\frac{\partial F}{\partial \rho}\right)J_n(\rho\xi)d\rho - n^2\int_o^\infty \frac{F}{\rho} J_n(\rho\xi)d\rho. + \frac{\partial^2}{\partial z^2}\int_o^\infty F\rho J_n(\rho\xi)d\rho$$

$$= -\xi\int_o^\infty \rho\frac{\partial F}{\partial \rho} J_n'(\rho\xi)d\rho - n^2\int_o^\infty \frac{F}{\rho} J_n(\rho\xi)d\rho + \frac{\partial^2}{\partial z^2}\int_o^\infty F\rho J_n(\rho\xi)d\rho \quad \ldots\ldots (1.4.8)$$

where the right hand side of the above equation is got by using the right hand side of equation (1.4.7a).

Integrating by parts the first term of right hand side of equation (1.4.8) we have

$$-\xi\int_o^\infty \rho\frac{\partial F}{\partial \rho} J_n'(\rho\xi)d\rho = -\xi[F\rho J_n'(\rho\xi)]_o^\infty + \xi\int_o^\infty F[J_n'(\rho\xi) + \rho\xi\, J_n''(\rho\xi)]d\rho$$

$$= \xi\int_o^\infty F[\rho\xi\, J_n''(\rho\xi) + J_n'(\rho\xi)]d\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.9)$$

Using equation (1.4.9) in equation (1.4.7a) we have

$$I = \xi\int_o^\infty F\{\rho\xi\, J_n''(\rho\xi) + J_n'(\rho\xi) - n^2\rho^{-1}\xi^{-1}J_n(\rho\xi)\}d\rho +$$

$$+ \frac{\partial^2}{\partial z^2}\int_o^\infty F\rho\, J_n(\rho\xi)d\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.10)$$

But the Bessel equation

$$\rho\xi\, J_n''(\rho\xi) + J_n'(\rho\xi) - n^2\rho^{-1}\xi^{-1}J_n(\rho\xi) + \rho\xi\, J_n(\rho\xi) = 0$$

Therefore

$$\rho\xi\, J_n''(\rho\xi) + J_n'(\rho\xi) - n^2\rho^{-1}\xi^{-1}J_n(\rho\xi) = -\rho\xi\, J_n(\rho\xi) \quad \ldots\ldots\ldots\ldots (1.4.11)$$

Using equation (1.4.11) in equation (1.4.10) we have

$$I = -\xi^2\int_o^\infty F\,\rho\, J_n(\rho\xi)d\rho + \frac{\partial^2}{\partial z^2}\int_o^\infty F\,\rho\, J_n(\rho\xi)d\rho = 0$$

Therefore, using equation (1.4.7) we have

$$I = \frac{d^2\bar{F}}{dz^2} - \xi^2\bar{F} = 0$$

The solution of the above equation is

$$\bar{F}(z,\xi) = r(\xi)e^{-\xi z} + t(\xi)e^{\xi z} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.12)$$

For $\bar{F}(z,\xi)$ to vanish as $z \to \infty$ then $F(\rho,z)$ should also vanish, therefore

$$\bar{F}(z,\xi) = r(\xi)\, e^{-\xi z} = \int_o^\infty F(\rho,z)\rho J_n(\rho\xi)d\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.13)$$

Taking inverse Hankel transform of equation (1.4.13) we have

$$F(\rho,z) = \int_o^\infty \xi\, r(\xi)e^{-\xi z}\, J_n(\rho\xi)d\xi$$

After finding $F(\rho,z)$ we have

$$U(\rho,\phi,z) = \sum_{n=o}^{\infty}\int_o^\infty \xi\; r(\xi)e^{-\xi z}J_n(\rho\xi)d\rho[E_n \sin n\phi + F_n \cos n\phi] \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.4.14)$$

For $U(\rho,\phi,z)$ to satisfy the boundary conditions from (1.4.4) to (1.4.4b) we have a sequence of pairs of dual integral equations of the form

$$\left.\begin{aligned} &\int_0^\infty \xi^{-1} A_n(\xi)\, J_n(\rho\xi)\,d\xi = f_n(\rho) \ , \quad 0 \leqslant \rho \leqslant 1 \\ &\int_0^\infty A_n(\xi)\, J_n(\rho\xi)\,d\xi = 0 \quad , \quad \rho > 1 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (1.4.15)$$

where $\xi^2 F_n\, r(\xi) = A_n(\xi)$, $n = o,1,2,\ldots$, F_n is an unknown constant and $r(\xi)$ is a function of ξ. Equation (1.4.15) is called a pair of dual integral equations with Bessel kernel.

In the analysis of certain mixed boundary value problems in mathematical physics we are sometimes required to solve the dual equations of the form

$$\left.\begin{aligned} &\sum_{n=1}^{\infty} \lambda_n^{-p} a_n J_\nu(\lambda_n\rho) = f_n(\rho) \ , \quad 0 \leqslant \rho \leqslant 1 \\ &\sum_{n=1}^{\infty} a_n J_\nu(\lambda_n\rho) = h_n(\rho) \quad , \quad \rho > 1 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (1.4.16)$$

where p and ν are real numbers satisfying the inequalities $-1 < p < 1$, $\nu > o$ and $\{\lambda_n\}$ is the sequence of positive zeros of $J_\nu(\lambda_a)$ Relations (1.4.16) are called dual series relations, $f_n(\rho)$ and $h_n(\rho)$ are prescribed and we wish to determine the sequence of real constants $\{a_n\}$ satisfying the dual series relations above.

A specific example of a mixed boundary value problem which we shall reduce to dual series relations is the problem of finding an electrostatic potential due to the presence of an electrified disc $\rho \leqslant 1$ as mentioned earlier but this time placed symmetrically on the axis of an infinite cylinder $\rho = a$, $a > 1$. The cylinder is earthed so that in addition to the boundary conditions from (1.4.4) to (1.4.4b) we have an additional boundary condition

$$U(\rho,\phi,z) = o \text{ on } \rho = a \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.4.17)$$

For the solution (1.3.14b) got earlier to satisfy the boundary conditions from 1.4.4) to (1.4.4b) and (1.4.17) we have

$$\left.\begin{aligned} &\sum_{m=1}^{\infty} A_{nm} J_n(\rho\lambda_{nm}) = f_n(\rho), \quad 0 \leqslant \rho \leqslant 1 \\ &\sum_{m=1}^{\infty} \lambda_{nm} Anm\, J_n(\rho\lambda_{nm}) = 0 \quad , \quad \rho > 1 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (1.4.18)$$

where $\{\lambda_{nm}\}$ is the sequence of positive zones of $J_n(\lambda_n a)$, and the above is an example of dual series relations, where $f_n(\rho)$ are given and the sequence of Fourier - Bessel constants satisfying the dual series relations (1.4.18) is to be determined.

1.5 Some Results involving Bessel Functions and Fourier-Dini Series:

In this section, we give some less known results involving Bessel functions which are frequently used in the following charters. A detailed account is given in works by Watson [36], Sneddon [20] and Erdelyi [11].

Theorem of Fourier-Dini Series expansion

If f(x) satisfies the Dirichlet condition in the closed interval [o ,a] and if its finite Hankel transform is defined by

$$f(\xi_i) = \int_o^a xf(x)\, J_\mu(x\xi_i)dx \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.5.1)$$

in which ξ_i is a root of the transcendental equation

$$\xi\, J_\mu'(a\xi) + h\, J_\mu(a\xi) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.5.2)$$

then at each point of the interval where f(x) is continuous

$$f(x) = \frac{2}{a^2} \sum_{i=1}^{\infty} \frac{\xi_i^2 f(\xi_i)}{\left[h^2+(\xi_i^2 - \frac{\mu^2}{a^2})\right]} \frac{J_\mu(x\xi_i)}{[J_\mu'(a\xi_i)]^2} \qquad \ldots\ldots\ldots\ldots(1.5.3)$$

where the summation is taken over all positive roots of the equation (1.5.2)

Some Integrals involving Bessel Functions

$$(a)\quad \int_o^\infty x^{\nu-\mu+1} J_\mu(ax)J_\nu(xy)dx = \frac{2^{\nu-\mu+1} y^\nu (a^2-y^2)^{\mu-\nu-1}\, H(a-y)}{a^\mu \Gamma(\mu-\nu)}$$

$$o < y < a,\ a > o,\ -1 < Re\nu < Re\,\mu \qquad \ldots\ldots\ldots\ldots(1.5.4)$$

where H(x) is the Heavy-side unit function

$$H(x) = \begin{cases} 0 & , x < o \\ 1 & , x > o \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.5.4a)$$

(b) $$\int_0^\infty x^{\mu-\nu+1} J_\mu(ax) J_\nu(xy)\,dx = \frac{2^{\mu-\nu+1} a^\mu (y^2-a^2)^{\nu-\mu-1}}{\Gamma(\nu-\mu) y^\nu}$$

$$a < y < \infty, \quad a > 0, \quad \mathrm{Re}\,\nu > \mathrm{Re}\,\mu > -1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.5.5)$$

(c) $$\int_0^t \frac{x^{\nu+1} J_\nu(xy)\,dx}{(t^2-x^2)^{\frac{1}{2}}} = \pi^{-\frac{1}{2}} 2^{-\nu} \Gamma(\tfrac{1}{2}-\nu) y^{\nu-\frac{1}{2}} \sin(ty)$$

$$|\,\mathrm{Re}\,\nu\,| < \tfrac{1}{2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.5.6)$$

(d) $$\int_0^t x^{\nu+1} (t^2-x^2)^\mu J_\nu(xy)\,dx = 2^\mu \Gamma(1+\mu) y^{-1-\mu} t^{\nu+\mu+1} J_{\nu+\mu+1}(ty)$$

$$\mathrm{Re}\,\mu > -1, \quad \mathrm{Re}\,\nu > -1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.5.7)$$

(e) $$\int_0^t \frac{\rho^{m+1} I_m(\rho y)\,d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} = (\pi/2)^{\frac{1}{2}} y^{-\frac{1}{2}} t^{m+\frac{1}{2}} I_{m+\frac{1}{2}}(ty) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.5.8)$$

(f) $$\int_0^t \frac{\rho^{m+2} I_{m+1}(\rho y)\,d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} = (\pi/2)^{\frac{1}{2}} t^{m+\frac{1}{2}} \{ y^{-\frac{1}{2}} I_{m+\frac{1}{2}}(ty) - \tfrac{1}{2} y^{-\frac{3}{2}} I_{m+\frac{1}{2}}(ty) \} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.5.9)$$

When m = 0, integrals (1.5.8) and (1.5.9) reduce respectively to

$$\left.\begin{aligned} \int_0^t \frac{\rho\, I_0(\rho y)\,d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} &= \frac{\sinh(ty)}{y} \\ \int_0^t \frac{\rho^2 I_1(\rho y)\,d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} &= \frac{ty\cosh(ty) - \mathrm{Sinh}(ty)}{y^2} \end{aligned}\right\} \qquad \ldots\ldots\ldots (1.5.10)$$

All integrals for which no references are given are to be found in Erdelyi et-al Tables of integral transforms, volumes I and II.

1.6 Integral representations for infinite series involving Bessel Functions

In subsequent chapters, we come across the function as a sum of the infinite series (wherever the series converges)

$$\therefore \quad S = 2 \sum_{n=1}^{\infty} \frac{J_\beta(\rho\lambda_n)J_\gamma(t\lambda_n)\lambda_n^{\delta+2}}{(a^2\lambda_n^2 - \nu^2 + H^2)\,[J_\gamma(a\lambda_n)]}$$

where $\{\lambda_n\}$ is the sequence of positive zeros of the equation

$$a\lambda\, J_\gamma'(a\lambda_n) + HJ_\nu(a\lambda_n) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.6.2)$$

arranged in ascending order of increasing magnitude; ν, H, ρ, t are real such that $\rho > 0$, $1 \leq t < a$, $t > -\frac{1}{2}$, and $-(\beta + \nu + 1) < \delta \leq 0$. We can give an integral representation to such a function by considering the contour integral

$$\int_C J_\beta(\rho z)J_\gamma(tz)z^{\delta+1}\phi(z)dz \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.6.3)$$

where

$$\phi(z) = \frac{z\,J_\nu'(az) + i\,Y_\nu(az)] + H[J_\nu(az) + i\,Y_\nu(az)]}{z\,J_\nu'(az) + HJ_\nu(az)} \qquad \ldots\ldots\ldots\ldots(1.6.4)$$

over the contour C consisting of

(i) The real axis from $x = \delta$ to $x = R$ which has been idented by means of small semi-circular arcs of radii δ_s $(s = 1,2,\ldots,\beta)$ with centres at $z = \lambda_s$ $(s = 1,2,\ldots,p)$ respectively where δ's are small and $\lambda_p < R < \lambda_{p+1}$

(ii) A large circular quadrant of radius R with centres at origin extending from $Z = R$ to $Z = Re^{\pi i/2} = Ri$

(iii) The positive imaginary axis from $Z = Ri$ to $Z = \delta i$

(iv) A small circular quadrant with centre at origin and extending from $z = \delta i$ to $Z = \delta$.

The point $z = 0$ is a branch point of the integrand in general. Each branch is however an analytic function. We choose a branch for which real parts of the integrand is $J_\beta(\rho z)J_\gamma(tz)z^{\delta+1}$ for real values of z.

Using the method of calculus of residue it could be easily found that

$$S = 2 \sum_{n=1}^{\infty} \frac{J_\beta(\rho_{\lambda_n}(J_\gamma(t_{\lambda_n})\,\lambda_n^{z+\delta}}{(a^2\lambda_n^2 - \nu^2 + H^2][J_\nu'(a\lambda_n)]^2} = \int_0^\infty J_\beta(\rho x)J_\gamma(tx)x^{\delta+1}dx +$$

$$+ \frac{2}{\pi}\sin\{\pi/2(\delta+\beta+\gamma-2\nu)\} \int_0^\infty \frac{[yk_\nu'(ay) + HK_\nu(ay)]I_\beta(\rho y)I_\gamma(ty)y^{\delta+1}dy}{[y\,I_\nu'(ay) + H\,I_\nu(ay)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(1.6.5)$$

Using equation (1.6.5) the following hold:

$$S = \frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_1(\rho\lambda n)\sin(t\lambda n)}{\lambda n\, J_o^{\,2}(a\lambda n)} = \int_0^{\infty} J_1(px)\sin(tx)dx -$$

$$-\frac{2}{\pi}\int^{\infty}\frac{K_1(ay)}{I_1(ay}\ I_1(\rho y)\sinh(ty)dy \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.6.6)$$

$$S = \frac{2}{a^2}\sum_{n=1}^{\infty}\frac{\lambda n\,\sin\,(t\lambda n)}{J_o(a\lambda n)(\lambda_n^{\,2}+\xi^2)^2} = \frac{I_1(a\xi)[t\xi\cosh(t\xi) + \sinh(t\xi)] - I_o\,(a\xi)\sinh(t\xi)}{2\,\xi^2 I_1^2(a\xi)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(1.6.7)$$

1.7 Solution of Abel's Integral Equation

The solution of Abel's integral equation

$$\int_o^x \frac{g(t)dt}{(x^2-t^2)^{\frac{1}{2}}} = f(x) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.7.1)$$

Is given to be

$$g(t) = \frac{2}{\pi}\frac{d}{dt}\int_o^t \frac{u\, f(u)du}{(t^2-u^2)^{\frac{1}{2}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.7.2)$$

$$= \frac{2}{\pi}\left[f(o) + t\int_o^t \frac{f'(u)du}{(t^2-u^2)^{\frac{1}{2}}}\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.7.3)$$

provided $f(x)$ is differentiable in $o < u < t$.

1.8 Summary of the Work

This thesis is concerned only with solution of mixed boundary value problems in cylinderical polar coordinates which have been reduced to either dual integral equations or dual series relations. Generally, we consider the problem as solved if we reduce the problem to the solution of an integral equation in terms of an auxilliary function, in whose terms the unknowns in a solution can be expressed. These integral equations are either Abel's integral equation in case of dual integral equations or Fredholm integral equation of the second kind in case of dual series relations which are easily amenable to numerical methods of solution.

In chapter 2 of the thesis, we look into the solutions of the general dual integral equations (2.1.4) and (2.1.2) for both the axisymmetric and the asymmetric cases respectively.

In the same chapter, we derive the solution of the general dual series relations (2.3.2) and (2.3.1) for both the axisymmetric and asymmetric cases respectively.

In chapter 3, we define biharmonic functions and biharmonic equations and we mention briefly how solutions of many problems in mathematical theory of elasticity reduce to the solution of biharmonic equations. We solve the problem of the stress distribution in an infinite cylinder due to the presence of a penny-shaped crack inside the cylinder and situated symmetrically on the axis of the cylinder.

As will be seen from going through the thesis we have followed a unified approach given by Sneddon and Srivastav while assuming the form of the unknowns in terms of an auxilliary function. The analysis is straight forward and use of the operators of fractional integration (Sneddon [22]), which lend an air of mystery to the problems, has been avoided. Analysis is purely formal and the existence of conditions for interchange in the order of integrations or integration and summation or summations have been preserved.

SOLUTIONS OF DUAL INTEGRAL EQUATIONS AND DUAL SERIES RELATIONS AS APPLIED TO POTENTIAL THEORY

0 Introduction

In this chapter we deal with dual integral equations and dual series relations which arise from some mixed boundary value problems in potential theory for cylinderical boundaries as mentioned in the previous chapter.

In section 1 of the chapter, we derive the solution of the dual integral equations (2.1.4) while in section 2 we derive the solution of the dual integral equations (2.1.2).

Section 3 and 4 of the chapter are devoted for the derivation of the solutions of the dual series relations (2.3.2) and (2.3.1) respectively.

As mentioned earlier, we have followed an unified approach in attempting to solve dual integral equations and dual series relations. We reduce the problem to an integral equation in terms of an auxilliary function $g(t)$ in whose terms the potential function can be expressed.

It will be seen that while the dual integral equations are usually reduced to the Abel's integral equations which are easily inverted, dual series relations have been finally reduced to Fredholm's integral equation of the second kind.

Solution of the Dual integral Equations in the Axisymmetric Case

By employing the technique of Hankel transform in section 1.4 we obtained solution (1.4.14) of the Laplace's equation (1.2.3). And applying the boundary conditions from (1.4.4) to (1.4.4b) on the solution we obtained the dual integral equations (1.4.15) for the asymmetric case of the problem of the Electrified disc, that is,

$$\int_0^\infty \xi^{-1} A_n(\xi) J_n(\rho\xi)\, d\xi = f_n(\rho) \quad , \quad 0 \leq \rho \leq 1$$

$$\int_0^\infty A_n(\xi)\, J_n(\rho\xi)\, d\xi = 0 \quad , \quad \rho > 1$$

$$n = 0,1,2,3,\ldots\ldots$$

But if we replace the boundary condition (1.4.4a) by

a new boundary condition

$$\frac{\partial U}{\partial z} = H(\rho, \phi) = \sum_{n=0}^{\infty} h_n(\rho) \cos n\phi + l_n(\rho) \sin n\phi, \quad \rho > 1 \qquad (2.1.1)$$

on $z = 0$ while the boundary conditions (1.4.4) and (1.4.4b) remain the same and applying these boundary conditions on solution (1.4.14) we have two similar dual integral equations of the form

$$\begin{aligned} &\int_0^{\infty} \xi^{-1} A_n(\xi)\, J_n(\rho\xi)\, d\xi = f_n(\rho), \quad 0 \le \rho \le 1 \\ &\int_0^{\infty} A_n(\xi)\, J_n(\rho\xi)\, d\xi = h_n(\rho), \quad \rho < 1 \\ &n = 0, 1, 2, 3, \text{---------} \end{aligned} \qquad (2.1.2)$$

Equations (2.1.2) is the general form of dual integral equations with Bessel Kernel in the asymmetric case.

In the axisymmetric case, the potential function U is independent of the polar angle ϕ and it corresponds to n equals zero and we have the Laplace's equation (1.3.10), that is,

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial z^2} = 0$$

Using an alternative procedure by employing the technique of Hankel transform and all the assumptions mentioned in section 1.4 we can obtain a solution of the Laplace's equation (1.3.10) above of the form

$$U(\rho, z) = \int_0^{\infty} \xi\, r(\xi) e^{-\xi z}\, J_0(\rho\xi)\, d\xi \qquad (2.1.3)$$

Applying the boundary condition (1.4.4), (1.4.4b) and (2.1.1) bearing in mind that n is equal to zero, we have the equations

$$\begin{aligned} &\int_0^{\infty} \xi^{-1} A(\xi)\, J_0(\rho\xi)\, d\xi = f(\rho), \quad 0 \le \rho \le 1 \\ &\int_0^{\infty} A(\xi)\, J_0(\rho\xi)\, d\xi = h(\rho), \quad \rho > 1 \end{aligned} \qquad (2.1.4$$

Equations (2.1.4) is the general form of a dual integral equations with Bessel Kernel in the Axisymmetric case.

Dual integral equations (2.14) for the axisymmetric case can be obtained from that of the asymmetric case, that is, from dual integral equations (2.1.2) above by putting n equal to zero in the equations; and $A_o(\xi)$ is replaced by $A(\xi)$, $f_o(\rho)$ by $f(\rho)$ and $h_o(\rho)$ by $h(\rho)$.

Some mathematicians solved the pair of dual integral equations above by inspection using some well known equations to "spot" the wanted function when the functions $f(\rho)$ and $h(\rho)$ are given. It is the object of this chapter to develop a formal solution of the dual integral equations above, while the solution given by Conson [7], Beltrami [2], Cooke [5] etc are sufficiently general to be useful in many physical problems. We shall first give the method of solving the dual integral equation (2.1.4) and the solution of the dual integral equations (2.1.2) will follow.

Our problem is to determine the function $A(\xi)$ that satisfies the dual integral equations (2.1.4). To simplify the calculations involved, we split the problem posed by solving the pair of integral equations (2.1.4) into two parts viz:

Problem (a): Determine the function $A(\xi)$ satisfying the dual integral equations

$$\int_0^\infty \xi^{-1} A(\xi) J_o(\rho\xi)d\xi = f(\rho), \quad o \leq \rho \leq 1 \text{ --- } (2.1.5a)$$

$$\int_0^\infty A(\xi) J_o(\rho\xi)d\xi = 0 \quad , \rho > 1 \qquad (2.1.5b)$$

and

Problem (b): Determine the function $A(\xi)$ satisfying the dual integral equations

$$\int_0^\infty \xi^{-1} A(\xi) J_o(\rho\xi)d\xi = 0 \quad , \quad o \leq \rho \leq 1 \qquad (2.1.6a)$$

$$\int_0^\infty A(\xi) J_o(\rho\xi)d\xi = h(\rho) \quad , \rho > 1 \qquad (2.1.6b)$$

and problem (b) will be called the companion problem to problem (a). The solution of the dual integral equations (2.1.4) will be obtained by superpositioning the solutions of problems (a) and (b).

For the solution of problem (a), we assume that for $0 \leq \rho \leq 1$

$$\int_0^\infty A(\xi) J_0(\rho\xi)\,d\xi = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\int_\rho^1 \frac{r(t)\,dt}{(t^2-\rho^2)^{\frac{1}{2}}} J_0(\rho\xi)\,d\rho \qquad (2.1.7)$$

Using Hankel inversion formula on equation (2.1.7) we have

$$\frac{A(\xi)}{\xi} = -\int_0^1 \rho \frac{1}{\rho}\frac{\partial}{\partial\rho}\int_\rho^1 \frac{r(t)\,dt}{(t^2-\rho^2)^{\frac{1}{2}}} J_0(\rho\xi)\,d\rho$$

$$= -\int_0^1 \frac{\partial}{\partial\rho}\int_\rho^1 \frac{r(t)\,dt}{(t^2-\rho^2)^{\frac{1}{2}}} J_0(\rho\xi)\,d\rho \qquad (2.1.8)$$

Integrating by parts right hand side of equation (2.1.8) we have

$$\frac{A(\xi)}{\xi} = \int_0^1 \frac{g(t)}{t}dt - \int_0^1 \xi J_1(\rho\xi)\int_\rho^1 \frac{r(t)\,dt}{(t^2-\rho^2)^{\frac{1}{2}}} \qquad (2.1.9)$$

Changing the order of integrations of the second term of right hand side of equation (2.1.9) we have

$$\frac{A(\xi)}{\xi} = \int_0^1 \frac{g(t)}{t}dt - \int_{t=0}^1 g(t)\int_{\rho=0}^t \frac{\xi J_1(\rho\xi)\,d\rho\,dt}{(t^2-\rho^2)^{\frac{1}{2}}} \qquad (2.1.10)$$

But $$\int_0^t \frac{\xi J_1(\rho\xi)\,d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} = \frac{1-\cos(t\xi)}{t} \qquad (2.1.11)$$

Substituting the value of the integral above in equation (2.1.10) we have

$$\frac{A(\xi)}{\xi} = \int_0^1 \frac{g(t)}{t}\cos(t\xi)\,dt$$

Substituting the value of $A(\xi)$ above into equation (2.1.5a) we have

$$\int_0^\infty \int_0^1 \frac{r(t)}{t}\cos(t\xi)\,J_0(\rho\xi)\,d\xi\,dt = f(\rho)$$

Changing the order of integrations of the above equation we have

$$\int_o^1 \frac{g(t)}{t} \int_o^\infty J_o(\rho\xi)\cos(t\xi)d\xi dt = f(\rho) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1.13)$$

Using equation (1.5.5) when $\mu=-\frac{1}{2}$, $\nu = o$, $\rho = y$, $x = \xi$ and $a = t$ we have

$$\int_o^\infty J_o(\rho\xi)\cos(t\xi)d\xi = (\rho^2 - t^2)^{-\frac{1}{2}}, \quad t<\rho<\infty \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1.14)$$

Using integral (2.1.14) in equation (2.1.13) we have

$$\int_o^\rho \frac{g(t)dt}{t(\rho^2-t^2)^{\frac{1}{2}}} = f(\rho) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1.15)$$

Inverting the Abel's integral equation (2.1.15) we have

$$\frac{g(t)}{t} = \frac{2}{\pi}\frac{d}{dt}\int_o^t \frac{\rho\, f(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1.16)$$

Where $f(\rho)$ is prescribed. After determining the function $g(t)$, we substitute it into equation (2.1.12) to determine the function $A(\xi)$ satisfying the dual integral equations (2.1.5a) and (2.1.5b) and this gives the solution of problem (a).

We ..... now proceed to solve problem (b) represented by equations (2.1.6a) and (2.1.6b). We assume that for $o \leq \rho \leq 1$

$$\int_o^\infty A(\xi)J_o(\rho\xi)d\xi = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\int_\rho^1 \frac{g(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}}$$

Therefore for the interval $o\leq\rho\leq\infty$ we have

$$\int_o^\infty A(\xi)J_o(\rho\xi)d\xi = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\int_\rho^1 \frac{g(t)dt\ H(1-\rho)}{(t^2-\rho^2)^{\frac{1}{2}}} + h(\rho)H(\rho-1) \qquad \ldots\ldots\ldots\ldots(2.1.17)$$

Where $H(x)$ is the Heavyside's unit function as mentioned in (1.5.4a)
Applying the same method as in equations (2.1.8) to (2.1.12) we have

$$\frac{A(\xi)}{\xi} = \int_o^1 \frac{g(t)}{t}\cos(t\xi)dt + F(\xi) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1.18)$$

$$\text{where } F(\xi) = \int_1^\infty h(\rho)J_o(\rho\xi)d\rho \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1.19)$$

Substituting the value of $A(\xi)$ from equation (2.1.18) into equation (2.1.6a) we have

$$\int_o^\infty \int_o^1 \frac{g(t)}{t}\cos(t\xi)J_o(\rho\xi)d\xi dt + \int_o^\infty F(\xi)J_o(\rho\xi)d\xi = 0$$

Therefore

$$\int_o^\infty \int_o^1 \frac{g(t)}{t}\cos(t\xi)J_o(\rho\xi)d\xi dt = G(\rho) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1.20)$$

$$\text{where } G(\rho) = \int_o^\infty F(\xi)J_o(\rho\xi)d\xi \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1.21)$$

Changing the order of integrations of equation (2.1.20) we have

$$\int_0^1 \frac{g(t)}{t} \left(\int_0^\infty J_0(\rho\xi)\cos(t\xi)d\xi\right)dt = -G(\rho) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1.22)$$

Applying the result in equation (2.1.14) in equation (2.1.22) we have

$$\int_0^\rho \frac{g(t)dt}{t(\rho^2-t^2)^{\frac{1}{2}}} = -G(\rho) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.1.23)$$

Inverting the Abel's integral equation (2.1.23) we have

$$\frac{g(t)}{t} = \frac{-2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho\, G(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$$

$$= -\frac{2}{\pi}\int_0^t \frac{\rho\left(\int_0^\infty F(\xi)J_0(\rho\xi)d\xi\right)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.1.24)$$

where the right hand side of equation (2.1.24) is obtained by using result (2.1.21).

Changing the order of integrations of equation (2.1.24) we have

$$\frac{g(t)}{t} = \frac{-2}{\pi}\int_0^\infty F(\xi)\left(\frac{d}{dt}\int_0^t \frac{\rho\, J_0(\rho\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right)d\xi \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1.25)$$

Using equation (1.5.6) where $\nu = 0$ we have

$$\int_0^t \frac{\rho\, J_0(\rho\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} = \frac{\sin(t\xi)}{t} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.1.26)$$

Using result (2.1.26) in equation (2.1.25) we have

$$\frac{g(t)}{t} = -\frac{2}{\pi}\int_0^\infty F(\xi)\cos(t\xi)d\xi$$

$$= -\int_0^\infty\int_1^\infty \rho\, h(\rho)J_0(\rho\xi)\cos(t\xi)d\xi d\rho$$

on using equation (2.1.19) $\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.1.27)$

Changing the order of integrations of equations (2.1.27) we have

$$\frac{g(t)}{t} = -\frac{2}{\pi}\int_1^\infty \rho h(\rho).\left(\int_0^\infty J_0(\rho\xi)\cos(t\xi)d\xi\right)d\rho$$

$$= -\frac{2}{\pi}\int_1^\infty \frac{\rho\, h(\rho)d\rho}{(\rho^2-t^2)^{\frac{1}{2}}}, \qquad t < \rho < \infty \qquad \ldots\ldots\ldots\ldots\ldots(2.1.28)$$

Combining the values of $A(\xi)$ got in the solutions of problems (a) and (b) we have

$$A(\xi) = \frac{\xi}{\pi}\left\{\int_0^1 \left(\frac{d}{dt}\int_0^t \frac{\rho f(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right) \cos(t\xi)dt \right.$$

$$\left. \int_0^1 \left(\int_1^\infty \frac{\rho h(\rho)d\rho}{(\rho^2-t^2)^{\frac{1}{2}}}\right) \cos(t\xi)dt + \frac{\pi}{2}\int_1^\infty \rho h(\rho) J_0(\rho\xi)d\rho\right\} \quad .....(2.1.29)$$

Equation (2.1.29) gives the expression for $A(\xi)$ that satisfies the dual Integral equations (2.1.4). The solution (2.1.29) is obtained by superpositioning the individual solutions (2.1.16) and (2.1.28).

The method of the solution of the dual integral equations (2.1.4) derived in this section can be applied to find the potential functions in the axisymmetric cases; where the boundary conditions are of the mixed type.

For example, if the function $f(\rho)$ is prescribed, and the function $h(\rho)$ is put equal to zero in the dual integral equations (2.1.4) as we have in problem (a) represented by the dual integral equations (2.1.5a) and (2.1.5b), the problem is that of the axisymmetric case of the problem of the electrified disc and the solution of the problem (a) alone derived earlier is sufficient to give the solution of the dual integral equations (2.1.4).

But if the function $f(\rho)$ is given to be zero while the function $h(\rho)$ is prescribed say, the dual integral equations (2.1.4) reduces to the type of problem (b) represented by the dual integral equations (2.1.6a) and (2.1.6b) and the solution of the problem(b) is sufficient to give the solution of the dual integral equation (2.1.4).

Weber [37] solved the dual integral equations

$$\int_0^\infty \xi^{-1}A(\xi)J_0(\rho\xi)d\xi = 1 \ , \quad 0 \leq \rho \leq 1$$

$$\int_0^\infty A(\xi)J_0(\rho\xi)d\xi = 0, \qquad \rho > 1 \qquad .....(2.1.30)$$

which is an example of problem (a) above by noticing the similarity between the dual integral equations above and the discontinuous integrals

$$\int_0^\infty \frac{\sin\xi}{\xi} J_0(\rho\xi)d\xi = \begin{cases} \pi/2 , & 0 \leq \rho \leq 1 \\ \sin^{-1}(\frac{1}{\rho}), & \rho > 1 \end{cases}$$

$$\int_0^\infty \sin\xi\, J_0(\rho\xi)d\xi = \begin{cases} (1-\rho^2)^{-\frac{1}{2}} & 0 \leq \rho \leq 1 \\ 0 & \rho > 1 \end{cases}$$

to arrive at the conclusion that the function $A(\xi)$ satisfying the dual integral equations (2.1.30) is found to be $\frac{2}{\pi}\sin\xi$. We arrive at the same value by using the method of solution we derived and the method provides useful indications for procedures which might be used in more complicated situations.

2.2. Solution of the Dual Integral Equations in the Asymmetric Case

In this section, we shall derive the method of solving the dual integral equations (2.1.2) of the asymmetric case which we mentioned in section 2.1, that is

$$\int_0^\infty \xi^{-1} A_n(\xi) J_n(\rho\xi)d\xi = f_n(\rho) , \quad 0 \leq \rho \leq 1$$

$$\int_0^\infty A_n(\xi) J_n(\rho\xi)d\xi = h_n(\rho) , \quad \rho > 1$$

We solve the dual integral integrations above by determining the functions $A_n(\xi)$ that satisfy them. To simplify the problem posed in solving the dual integral equations above, we split the general problem into two parts as we did in section 2.1, viz:

Problem (a): Determine the sequence of functions $\{A_n(\xi)\}$ satisfying the dual integral equations

$$\int_0^\infty \xi^{-1} A_n(\xi) J_n(\rho\xi)d\xi = f_n(\rho), \quad 0 \leq \rho \leq 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2.2.1a)$$

$$\int_0^\infty A_n(\xi) J_n(\rho\xi)d\xi = 0 \qquad \rho > 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2.2.1b)$$

and

Problem (b): Determine the sequence of functions $\{A_n(\xi)\}$ satisfying the dual integral equations

$$\int_0^\infty \xi^{-1} A_n(\xi) J_n(\rho\xi)d\xi = 0 , \quad 0 \leq \rho \leq 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2.2.2a)$$

$$\int_0^\infty A_n(\xi)\, J_n(\rho\xi)d\xi = h_n(\rho) \qquad \rho > 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2.2.2b)$$

and problem (b) above will be called the companion problem to problem(a). After solving the problems (a) and (b), the solution of the general problem will be got by superposing the solutions of problems (a) and (b).

We first derive the solution of problem (a)

Solution

We assume that for $o \leq \rho \leq 1$

$$\int_o^\infty A_n(\xi)J_n(\rho\xi)d\xi = -\rho^{n-1}\frac{\partial}{\partial\rho}\int_\rho^1 \frac{g_n(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}}H(\rho-1) \quad .....(2.2.3)$$

Applying Hankel inversion formula on equation (2.2.3) we have

$$\frac{A_n(\xi)}{\xi} = -\int_o^1 \rho^n \frac{\partial}{\partial\rho}\int_\rho^1 \frac{g_n(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} J_n(\rho\xi)d\rho \quad .....(2.2.4)$$

Integrating by parts we have from equation (2.2.4)

$$\frac{A_n(\xi)}{\xi} = \left[-\rho^n J_n(\rho\xi)\int_\rho^1 \frac{g_n(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}}\right]_o^1 + \int_o^1 \xi\,\rho^n\, J_{n-1}(\rho\xi)\int_\rho^1 \frac{g_n(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}}\,d\rho$$

$$= \int_o^1 \xi\,\rho^n\, J_{n-1}(\rho\xi)\int_\rho^1 \frac{g_n(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}}\,d\rho \quad .....(2.2.5)$$

Changing the order of integrations of equation (2.2.5) we have

$$\frac{A_n(\xi)}{\xi} = \int_o^1 g_n(t)\int_{\rho=o}^t \frac{\xi\rho^n J_{n-1}(\rho\xi)d\rho\,dt}{(t^2-\rho^2)^{\frac{1}{2}}} \quad .....(2.2.6)$$

Using equation (1.5.7) where $\nu = n-1$, $\mu = -\frac{1}{2}$, $a = t$, $y = \xi$ and $\rho = x$ we have

$$\int_o^t \frac{\xi\rho^n J_{n-1}(\rho\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} = \sqrt{\frac{\pi}{2}}\,\xi^{\frac{1}{2}}\,t^{n-\frac{1}{2}}\,J_{n-\frac{1}{2}}(t\xi) \quad .....(2.2.7)$$

Using result (2.2.7) in equation (2.2.6) we have

$$\frac{A_n(\xi)}{\xi} = \sqrt{\frac{\pi}{2}}\int_o^1 g_n(t)\xi^{\frac{1}{2}}\,t^{n-\frac{1}{2}}\,J_{n-\frac{1}{2}}(t\xi)dt \quad .....(2.2.8)$$

Substituting the value of $A_n(\xi)$ into equation (2.2.1a) we have

$$\frac{\pi}{2}\int_o^1 g_n(t)\left(\int^\infty \xi^{\frac{1}{2}}t^{n-\frac{1}{2}}J_{n-\frac{1}{2}}(t\xi)J_n(\rho\xi)d\xi\right)dt = f_n(\rho) \quad .....(2.2.9)$$

Using equations (1.5.5) where $\mu = n-\frac{1}{2}$, $\nu = n$, $x = \xi$, $y = \rho$ and $a = t$ we have

$$\int_o^\infty \xi^{\frac{1}{2}}\,t^{n-\frac{1}{2}}J_{n-\frac{1}{2}}(t\xi)J_n(\rho\xi)d\xi = \frac{\sqrt{\frac{2}{\pi}}(\rho^2-t^2)^{-\frac{1}{2}}}{t^{\frac{1}{2}-n}}\rho^{-n};\ t < \rho < \infty \quad .....(2.2.10)$$

Using result (2.2.10) in equation (2.2.9) we have

$$\int_o^\rho \frac{t^{2n-1} g_n(t)dt}{\rho^n(\rho^2 - t^2)^{\frac{1}{2}}} = f_n(\rho)$$

Therefore

$$\int_o^\rho \frac{g_n(t)dt}{t^{1+2n}(\rho^2-t^2)^{\frac{1}{2}}} = \rho^n f_n(\rho) \qquad \ldots\ldots\ldots\ldots(2.2.11)$$

Inverting Abel's integral equation (2.2.11) we have

$$\frac{g_n(t)}{t^{1-2n}} = \frac{2}{\pi}\frac{d}{dt}\int_o^t \frac{\rho^{1+n} f_n(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} \qquad \ldots\ldots\ldots\ldots(2.2.12)$$

Substituting the expression obtained in equation (2.2.12) for function $g(t)$ into equation (2.2.8) the functions $A_n(\xi)$ satisfying the dual integral equation (2.2.1a) and (2.2.1b) are obtained and

$$A_n(\xi) = \sqrt{\frac{2}{\pi}}\,\xi^{\frac{1}{2}}\frac{d}{dt}\left[\int_o^1 t^{\frac{1}{2}-n} J_{n-\frac{1}{2}}(t\xi)\left(\int_o^t \frac{\rho^{1+n} f_n(\rho\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}dt\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\,\xi^{\frac{1}{2}}\int_o^1 t^{\frac{1}{2}-n}\left(\frac{d}{dt}\int_o^t \frac{\rho^{1+n} f_n(\rho\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right) J_{n-\frac{1}{2}}(t\xi)dt \qquad \ldots\ldots\ldots(2.1.13)$$

We now derive the method of solving Problem (b)

Solution of problem (b)

We assume that for $o \leq \rho \leq 1$

$$\int_o^\infty A_n(\xi)J_n(\rho\xi)d\xi = -\rho^{n-1}\frac{\partial}{\partial\rho}\int_\rho^1 \frac{g_n(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} H(1-\rho)$$

then for the interval $o \leq \rho \leq \infty$ we have

$$\int_o^\infty A_n(\xi)J_n(\rho\xi)d\xi = -\rho^{n-1}\frac{\partial}{\partial\rho}\int_\rho^1 \frac{g_n(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} H(1-\rho) + h_n(\rho)\,H(\rho-1)\ldots\ldots(2.2.14)$$

and applying the same methods as given in equations (2.2.4) to (2.2.8) we have

$$\frac{A_n(\xi)}{\xi} = \sqrt{\frac{\pi}{2}} \int_0^1 g_n(t)\xi^{\frac{1}{2}}\, t^{n-\frac{1}{2}}\, J_{n-\frac{1}{2}}(t\xi)dt + \sigma_n(\xi) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.2.15)$$

where $\sigma_n(\xi) = \int_1^\infty \rho\, h_n(\rho)\, J_n(\rho\xi)d\rho$ $\qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.2.16)$

Substituting the value of $A_n(\xi)$ from equation (2.2.15) into equation (2.2.2a) we have

$$\sqrt{\frac{\pi}{2}} \int_\rho^\infty \int_0^1 g_n(t)\xi^{\frac{1}{2}} t^{n-\frac{1}{2}} J_{n-\frac{1}{2}}(t\xi)J_n(\rho\xi)d\,dt + \int_0^\infty \sigma_n(\xi)J_n(\rho\xi)d\xi = 0$$

Changing the order of integration of the first term of the above equation we have

$$\sqrt{\frac{\pi}{2}} \int_0^1 g_n(t)\left(\int_0^\infty \xi^{\frac{1}{2}} t^{n-\frac{1}{2}}\, J_{n-\frac{1}{2}}(t\xi)J_n(\rho\xi)d\xi\right)dt = -\ T_n(\rho) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.2.17)$$

where $T_n(\rho) = \int_0^\infty \sigma_n(\xi)\, J_n(\rho\xi)d\xi$ $\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (2.2.18)

Using result (2.2.10) in equation (2.2.17) we have

$$\int_0^\rho \frac{t^{2n-1}\, g_n(t)dt}{\rho^n\,(\rho^2-t^2)^{\frac{1}{2}}} = -\ T_n(\rho)$$

Therefore

$$\int_0^\rho \frac{g_n(t)dt}{t^{1-2n}(\rho^2-t^2)^{\frac{1}{2}}} = -\ \rho^n\, T_n(\rho) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.2.19)$$

Inverting the Abel's integral equation (2.2.19) we have

$$\frac{g_n(t)}{t^{1-2n}} = -\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho^{1+n}\, T_n(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$$

$$= -\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho^{1+n}\int_0^\infty \sigma_n(\xi)J_n(\rho\xi)d\xi}{(t^2-\rho^2)^{\frac{1}{2}}}\, d\rho$$

on using equation (2.2.18). $\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.2.20)$

Changing the order of integration in equation (2.2.20) we have

$$\frac{g_n(t)}{t^{1-2n}} = -\frac{2}{\pi}\int_0^\infty \sigma_n(\xi)\left(\frac{d}{dt}\int_0^t \frac{\rho^{1+n}J_n(\rho\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right)d\xi \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.2.21)$$

Using equation (1.5.7) where $\mu = -\frac{1}{2}$, $\nu = n$, $x = \rho$, $y = \xi$

we have

$$\int_0^t \frac{\rho^{1+n} J_n(\rho\xi)\,d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} = \sqrt{\frac{\pi}{2}}\,\xi^{\frac{1}{2}}\,t^{n+\frac{1}{2}}\,J_{n+\frac{1}{2}}(t\xi) \qquad \text{.....(2.2.22)}$$

Using result (2.2.22) in equation (2.2.21) we have

$$\frac{g_n(t)}{t^{1-2n}} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \xi^{\frac{1}{2}}\,t^{n+\frac{1}{2}}\,\sigma_n(\xi) J_{n-\frac{1}{2}}(t\xi)\,d\xi$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty \xi^{\frac{1}{2}}\,t^{n+\frac{1}{2}}\,J_{n-\frac{1}{2}}(t\xi)\left(\int_1^\infty \rho h_n(\rho) J_n(\rho\xi)\,d\rho\right)d\xi$$

on using equation (2.2.16) .....(2.2.23)

Changing the order of integrations in equation (2.2.23) we have

$$\frac{g_n(t)}{t^{1-2n}} = -\sqrt{\frac{2}{\pi}} \int_1^\infty \rho h_n(\rho) \left(\int_0^\infty \xi^{\frac{1}{2}} t^{n+\frac{1}{2}}\,J_{n-\frac{1}{2}}(t\xi) J_n(\rho\xi)\,d\xi\right)d\rho \qquad \text{.....(2.2.24)}$$

Using result (2.2.10) in equation (2.2.24) we have

$$\frac{g_n(t)}{t^{1-2n}} = -\frac{2}{\pi} \int_1^\infty \frac{t^{2n} \rho^{1-n} h_n(\rho)\,d\rho}{(\rho^2-t^2)^{\frac{1}{2}}}$$

That is,

$$\frac{g_n(t)}{t} = -\frac{2}{\pi} \int_1^\infty \frac{\rho^{1-n} h_n(\rho)\,d\rho}{(\rho^2-t^2)^{\frac{1}{2}}}, \quad t < \rho < \infty \qquad \text{.....(2.2.25)}$$

$$n = 0, 1, 2, 3, \ldots\ldots\ldots$$

Substituting the expression for the function $g(t)$ from equation (2.2.25) into equation (2.2.15) the functions $A_n(\xi)$ satisfying the dual integral equations (2.2.2a) and (2.2.2b) are obtained. We have

$$\frac{A_n(\xi)}{\xi} = -\sqrt{\frac{2}{\pi}} \int_0^1 \left(\int_0^\infty \frac{\rho^{1-n} h_n(\rho)\,d\rho}{(\rho^2-t^2)^{\frac{1}{2}}}\right) \xi^{\frac{1}{2}} t^{n-\frac{1}{2}} J_{n-\frac{1}{2}}(t\xi)\,dt +$$

$$+ \int_1^\infty \rho h_n(\rho) J_n(\rho\xi)\,d\rho$$

The solution of the dual integral equations (2.1.2) of the asymmetric case is obtained by superpositioning the solutions of problems (a) and (b).

For n = o, we can clearly see that the solutions we derived for problems (a) and (b) in asymmetric case are identical to those of the problems (a) and (b) in the axisymmetric case.

2.3 Solution of the Dual Series Relations in the Axisymmetric case

In section 1.4, we obtained the dual series relations (1.4.18), that is

$$\sum_{m=1}^{\infty} A_{nm} J_n(\rho\lambda_{nm}) = f_n(\rho) \quad , \quad o \leq \rho \leq 1$$

$$\sum_{m=1}^{\infty} \lambda_{nm} A_{nm} J_n(\rho\lambda_{nm}) = 0 \quad , \quad \rho > 1$$

$n = 0, 1, 2, \ldots\ldots\ldots$ But if we apply the boundary conditions (1.4.4), (1.4.4b), (1.4.17) and (2.1.1) on the solution (1.3.14b) of the Laplace's equati (1.2.3). We obtain dual series relations (that are different from the above), of the form

$$\sum_{m=1}^{\infty} A_{nm} J_n(\rho\lambda_{nm}) = f_n(\rho) \quad , \quad o \leq \rho \leq 1$$

$$\sum_{m=1}^{\infty} \lambda_{nm} A_{nm} J_n(\rho\lambda_{nm}) = h_n(\rho), \quad \rho > 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2.3.1)$$

In sections 2.1 and 2.2 we derived the solutions of the dual integral equations (2.1.2) and (2.1.4). In this section and the section that follows we derive the solution of the dual series relations in the Axisymmetric and Asymmetric cases respectively.

Cooke [5], Tranter [35] solved the dual series relations (2.3.1) above by reducing the problem to an infinite system of linear algebraic equations. But Cooke and Tranter [6] discussed the solution of certain dual series relations by using a method similar to that discussed by Tranter [35] while another alternative method was discussed by Sneddon and Srivastav [30] and Srivastav [32].

We shall derive the solution of the dual series relations by giving the integral representation of the constants A_{nm}. This integral involves a new unknown function g(t) which we can show to be a solution of a Fredholm integral equation of the second kind. The advantage of the present method lies in

the fact that it is much easier to derive a numerical solution of an integral equation of this kind than it is to find one for an infinite set of algebraic equations.

Dual series relations (2.3.1) is called the general form of dual series relations in the asymmetric case since it is the one we obtained by solving the Laplace's equation (1.2.3) where the potential function U is dependent on the polar angle ϕ.

In the axisymmetric case, this corresponds to a case when n = 0 and the general form of the dual series relations for the axisymmetric case can be obtained by applying the boundary conditions (1.4.4), (1.4.4b), (1.4.17) and (2.1.1) where this time n = o, on the solution (1.3.11c) of the Laplace's equation (1.3.10) and we have

$$\sum_{n=1}^{\infty} A_n J_o(\rho\lambda_n) = f(\rho) \quad , \quad o \leq \rho \leq 1$$

$$\sum_{n=1}^{\infty} \lambda_n A_n J_o(\rho\lambda_n) = h(\rho) \quad , \quad \rho > 1 \qquad \text{.....(2.3.2)}$$

In order to simplify the problem posed in solving the dual series relations (2.3.2) we split problem into two new problems (a) and (b) viz·

Problem (a): Determine the constants $\{A_n\}$ satisfying the dual series relations

$$\sum_{n=1}^{\infty} A_n J_o(\rho\lambda_n) = f(\rho) \quad , \quad o \leq \rho \leq 1 \qquad \text{.....(2.3.3a)}$$

$$\sum_{n=1}^{\infty} \lambda_n A_n J_o(\rho\lambda_n) = 0 \quad , \quad \rho > 1 \qquad \text{.....(2.3.3b)}$$

and

Problem (b): Determine the constants $\{A_n\}$ satisfying the dual series relations

$$\sum_{n=1}^{\infty} A_n J_o(\rho\lambda_n) = 0 \quad , \quad o \leq \rho \leq 1 \qquad \text{.....(2.3.4a)}$$

$$\sum_{n=1}^{\infty} \lambda_n A_n J_o(\rho\lambda_n) = h(\rho) \quad \rho > 1 \qquad \text{.....(2.3.4b)}$$

and problem (b) will be called the companion problem to problem (a). The solution of the problem posed by solving the dual series relations (2.3.2) is obtained by superpositioning the solutions of problems (a) and (b).

We now start to derive the solution of problem (a):

Solution:

We assume that for $0 \leq \rho \leq 1$

$$\sum_{n=1}^{\infty} \lambda_n A_n J_o(\rho\lambda_n) = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\int_\rho^1 \frac{g(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} \qquad \ldots\ldots(2.3.5)$$

Taking Hankel transform of both sides of equation (2.3.5) we have

$$\int_o^a \lambda_n A_n \rho J_o^2(\rho\lambda_n)d\rho = -\int_o^1 \frac{\partial}{\partial\rho}\int_\rho^1 \frac{g(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} J_o(\rho\lambda_n)d\rho$$

But $\int_o^a \rho J_o^2(\rho\lambda_n)d\rho = \frac{a^2}{2} J_1^2(a\lambda_n)$ $\qquad \ldots\ldots(2.3.6)$

Therefore

$$\lambda_n A_n = \frac{-2}{a^2 J_1^2(a\lambda_n)}\int_o^1 \frac{\partial}{\partial\rho}\int_\rho^1 \frac{g(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} J_o(\rho\lambda_n)d\rho \qquad \ldots\ldots(2.3.7)$$

Following the methods used in equations (2.1.9) to (2.1.12) we have

$$\lambda_n A_n = \frac{2}{a^2 J_1^2(a\lambda_n)}\int_o^1 \frac{g(t)}{t}\cos(t\lambda_n)dt \qquad \ldots\ldots(2.3.8)$$

Substituting the value of A_n from above equation into equation (2.3.3a) we have

$$\sum_{n=1}^{\infty} \frac{2}{\lambda_n a^2 J_1^2(a\lambda_n)}\int_o^1 \frac{g(t)}{t}\cos(t\lambda_n)J_o(\rho\lambda_n)dt = f(\rho) \qquad \ldots\ldots(2.3.9)$$

Changing the order of summation and integration in equation (2.3.9) we have

$$\int_o^1 \frac{g(t)}{t}\left\{\frac{2}{a^2}\sum_{n=1}^{\infty} \frac{J_o(\rho\lambda_n)\cos(t\lambda_n)}{\lambda_n J_1^2(a\lambda_n)}\right\}dt = f(\rho) \qquad \ldots\ldots(2.3.9a)$$

Using equation (1.6.5) where $H = 1$, $\nu = 1$, $\beta = 0$, $\delta = -\frac{1}{2}$, $\gamma = -\frac{1}{2}$ we have

$$\frac{2}{a^2}\sum_{n=1}^{\infty} \frac{J_o(\rho\lambda_n)\cos(t\lambda_n)}{\lambda_n J_1^2(a\lambda_n)} = \int_o^{\infty} J_o(\rho x)\cos(tx)dx -$$

$$- \frac{2}{\pi}\int_o^{\infty} \frac{K_o(ay)}{I_o(ay)} I_o(\rho y)\cosh(ty)dy \qquad \ldots\ldots(2.3.10)$$

Using result (2.1.14) in the left hand side of equation (2.3.11) we have

$$\int_o^1 \frac{g(t)}{t} \cdot \left(\int_o^\infty J_o(\rho x)\cos(tx)dx\right)dt = f(\rho) + \frac{2}{\pi}\int_o^1 \frac{g(t)}{t} * \left(\int_o^\infty \frac{K_o(ay)}{I_o(ay)} I_o(\rho y)\cosh(ty)dy\right)dt \quad \ldots\ldots(2.3.11)$$

Using result (2.1.14) in the left hand side of equation (2.3.11) we have

$$\int_o^\rho \frac{g(t)\,dt}{(\rho^2-t^2)^{\frac{1}{2}}} = f(\rho) + \frac{2}{\pi}\int_o^1 \frac{g(t)}{t}\cdot\left(\int_o^\infty \frac{K_o(ay)}{I_o(ay)} I_o(\rho y)\cosh(ty)dy\right)dt \quad \ldots\ldots(2.3.12)$$

Inverting the Abel's integral equation (2.3.12) we have

$$\frac{g(t)}{t} = \frac{2}{\pi}\frac{d}{dt}\int_o^t \frac{\rho f(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} + \frac{4}{\pi^2}\frac{d}{dt}\int_o^t \rho \left[\int_o^1 \frac{g(u)}{u}\left(\int_o^\infty \frac{K_o(ay)I_o(\rho y)\cosh(uy)dy}{I_o(ay)(t^2-\rho^2)^{\frac{1}{2}}}\right)du\right]d\rho$$

Changing the order of integrations in the second term of the right hand side of equation above we have

$$\frac{g(t)}{t} = \frac{2}{\pi}\frac{d}{dt}\int_o^t \frac{\rho f(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} + \frac{4}{\pi^2}\int_o^1 \frac{g(u)}{u}\left[\int_o^\infty \frac{K_o(ay)}{I_o(ay)}\left(\frac{d}{dt}\int_o^t \frac{\rho I_o(\rho y)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right)\cosh(uy)dy\right]du \quad \ldots(2.3.13)$$

Using the first equation of (1.5.10) in the second term of the right hand side of equation (2.3.13) we have

$$\frac{g(t)}{t} = \frac{2}{\pi}\frac{d}{dt}\int_o^t \frac{\rho f(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} + \frac{4}{\pi^2}\int_o^1 \frac{g(u)}{u}\int_o^\infty \frac{K_o(ay)}{I_o(ay)}\cosh(ty)\cosh(uy)dydu \quad \ldots\ldots(2.3.$$

$$= \chi(t) + \int_o^1 \frac{g(u)}{u}\mathbb{K}(t,u)du \quad \ldots\ldots(2.3.$$

where $\chi(t) = \frac{2}{\pi}\frac{d}{dt}\int_o^t \frac{\rho f(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$ and

$$\mathbb{K}(t,u) = \mathbb{K}(u,t) = \frac{4}{\pi^2}\int_o^\infty \frac{K_o(ay)}{I_o(ay)}\cosh(ty)\cosh(uy)dy$$

Equation (2.3.14a) is a Fredholm integral equation of the second kind which gives the expression for the function $g(t)$. Evaluating the integral equation (2.3.14a) will enable us to determine the constants A_n satisfying the dual series relations (2.3.3a) and (2.3.3b).

For the solution of the above problem, we assume that for $o \leqslant \rho \leqslant 1$

$$\sum_{n=1}^{\infty} \lambda_n A_n J_o(\rho\lambda_n) = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\int_{\rho}^{1}\frac{g(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}}$$

Therefore for $o \leqslant \rho \leqslant a$, we have

$$\sum_{n=1}^{\infty} \lambda_n A_n J_o(\rho\lambda_n) = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\int_{\rho}^{1}\frac{g(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} H(1-\rho) + h(\rho)\,H(\rho-1) \quad \ldots\ldots\ldots(2.3.15)$$

Following the methods used in equation (2.1.9) to (2.1.12) we have

$$A_n = \frac{2}{\lambda_n a^2 J_1^2(a\lambda_n)}\left[\int_o^1 \frac{g(t)}{t}\cos(t\lambda_n)dt + \int_1^a rh(r)J_o(r\lambda_n)dr\right] \quad \ldots\ldots\ldots(2.3.16)$$

The second term of right hand side of equation (2.3.16) is obtained by changing the dummy variable ρ to r.

Substituting the value of A_n from equation (2.3.16) into equation (2.3.4a) we have

$$\frac{2}{a^2}\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_1^2(a\lambda_n)}\int_o^1\frac{g(t)}{t}\cos(t\lambda_n)J_o(\rho\lambda_n)dt =$$

$$-\frac{2}{a^2}\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_1^2(a\lambda_n)}\int_1^a rh(r)J_o(r\lambda_n)J_o(\rho\lambda_n)dr \quad \ldots\ldots\ldots(2.3.17)$$

Changing the order of summation and integration in equation (2.3.17) we have

$$\int_o^1\frac{g(t)}{t}*\left(\frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_o(\rho\lambda_n)\cos(t\lambda_n)}{\lambda_n J_1^2(a\lambda_n)}\right)dt = -\int_1^a rh(r)\mathbf{H}_o(\rho,r)dr \quad \ldots\ldots(2.3.17a)$$

where $$\mathbf{H}_o(\rho,r) = \frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_o(\rho\lambda_n)J_o(r\lambda_n)}{\lambda_n J_1^2(a\lambda_n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.3.17b)$$

Using result (2.3.10) in the left hand side of equation (2.3.17a) we have

$$\int_o^1\frac{g(t)}{t}*\left(\int_o^{\infty}J_o(\rho x)\cos(tx)dx\right)dt -$$

$$-\frac{2}{\pi}\int_o^1\frac{g(t)}{t}\left(\int_o^{\infty}\frac{K_o(ay)I_o(\rho y)\cosh(ty)dy}{I_o(ay)}\right)dt = Q(\rho) \quad \ldots\ldots\ldots\ldots(2.3.18)$$

where $$Q(\rho) = -\int_1^a r\,h(r)\,\mathbf{H}_o(\rho,r)dr \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.3.18a)$$

Therefore

$$\int_0^1 \frac{g(t)}{t} \left(\int_0^\infty J_0(\rho x)\cos(tx)dx\right)dt = Q(\rho) + \frac{2}{\pi}\int_0^1 \frac{g(t)}{t}\left(\int_0^\infty \frac{K_0(ay)}{I_0(ay)} I_0(\rho y)\cosh(ty)dy\right)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.3.18b)$$

Using result (2.1.14) in the left hand side of equation (2.3.18b) we have

$$\int_0^\rho \frac{g(t)dt}{t(\rho^2-t^2)^{\frac{1}{2}}} = \frac{2}{\pi}\int_0^1 \frac{g(t)}{t}\left(\int_0^\infty \frac{K_0(ay)}{I_0(ay)} I_0(\rho y)\cosh(ty)dy\right)dt + Q(\rho) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.3.19)$$

Inverting Abel's integral equation (2.3.19) we have

$$\frac{g(t)}{t} = \frac{4}{\pi^2}\frac{d}{dt}\int_0^t \rho\left[\int_0^1 \frac{g(u)}{u}\left(\int_0^\infty \frac{K_0(ay)}{I_0(ay)} I_0 \frac{(\rho y)\cosh(uy)dy}{(t^2-\rho^2)^{\frac{1}{2}}}\right)du\right]d\rho + \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho\, Q(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.3.20)$$

Changing the order of integrations in the first term of right hand side of equation (2.3.20) we have

$$\frac{g(t)}{t} = \frac{4}{\pi^2}\int_0^1 \frac{g(u)}{u}\left[\int_0^\infty \frac{K_0(ay)}{I_0(ay)}\left\{\frac{d}{dt}\int_0^t \frac{\rho\, I_0(\rho y)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right\}\cosh(uy)dy\right]du + \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho\, Q(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.3.20a)$$

Using the first equation of (1.5.10) in the first term of right hand side of equation (2.3.20a) we have

$$\frac{g(t)}{t} = \frac{4}{\pi^2}\int_0^1 \frac{g(u)}{u}\left(\int_0^\infty \frac{K_0(ay)}{I_0(ay)}\cosh(ty)\cosh(uy)dy\right)du + \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho\, Q(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.3.21)$$

Dealing with the second term of the right hand side of equation (2.3.21) we have

$$I_1 = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho\, Q(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$$

$$= -\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\left(\delta_1^a rh(r)\, \mathbf{H}_0(\rho r)dr\right)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} \quad \text{on using equation (2.3.18a)} \quad \ldots\ldots\ldots\ldots(2.3.22)$$

Changing the order of integrations in equation (2.3.22) we have

$$\mathbb{I}_1 = -\frac{2}{\pi}\int_1^a rh(r)\left[\frac{d}{dt}\int_0^t \rho^*\left(\frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_0(\rho\lambda_n)J_0(r\lambda_n)}{\lambda_n J_1^2(a\lambda_n)}\,d\rho\right)\right]dr$$

$$= -\frac{2}{\pi}\int_1^a rh(r)*\left\{\frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_0(r\lambda_n)}{\lambda_n J_1^2(a\lambda_n)}\left(\frac{d}{dt}\int_0^t \frac{J_0(\rho\lambda_n)\,d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right)\right\}dr$$

Therefore

$$\mathbb{I}_1 = -\frac{2}{\pi}\int_1^a rh(r)*\left\{\frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_0(r\lambda_n)\cos(t\lambda_n)}{\lambda_n J_1^2(a\lambda_n)}\right\}dr \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.3.23)$$

Using result (2.3.10) in equation (2.2.23) we have

$$\mathbb{I}_1 = -\frac{2}{\pi}\int_1^a rh(r)*\left(\int_0^\infty J_0(rx)\cos(tx)\,dx\right)dr -$$

$$+\frac{4}{\pi^2}\int_1^a rh(r)*\left(\int_0^\infty \frac{K_0(ay)}{I_0(ay)}I_0(ry)\cosh(ty)\,dy\right)dr \qquad \ldots\ldots\ldots\ldots(2.3.24)$$

Using result (2.1.14) in the first λ of right hand side of equation (2.3.24) we have

$$\mathbb{I}_1 = -\frac{2}{\pi}\int_1^a \frac{\rho h(\rho)\,d\rho}{(\rho^2-t^2)^{\frac{1}{2}}} + \frac{4}{\pi^2}\int_1^a \rho h(\rho)\left(\int_0^\infty \frac{K_0(ay)}{I_0(ay)}I_0(\rho y)\cosh(ty)\,dy\right)d\rho$$

on changing dummy variable r back to ρ $\qquad \ldots\ldots\ldots\ldots\ldots(2.3.25)$

Using result (2.3.25) in equation (2.3.21) we have

$$\frac{g(t)}{t} = \frac{4}{\pi^2}\int_0^1 \frac{g(u)}{u}\left(\int^\infty \frac{K_0(ay)}{I_0(ay)}\cosh(ty)\cosh(uy)\,dy\right)du +$$

$$+\frac{4}{\pi^2}\int_1^a \rho\, h(\rho)\left(\int_0^\infty \frac{K_0(ay)}{I_0(ay)}I_0(\rho y)\cosh(ty)\,dy\right)d\rho -$$

$$-\frac{2}{\pi}\int_1^a \frac{\rho h(\rho)\,d\rho}{(\rho^2-t^2)^{\frac{1}{2}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.3.26)$$

Therefore

$$\frac{g(t)}{t} = B(t) + \int_0^1 \frac{g(u)}{u}\mathbb{K}(t,u)\,du \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.3.26a)$$

where

$$\mathbb{K}(t,u) = \mathbb{K}(t,u) = \frac{4}{\pi^2}\int_0^\infty \frac{K_0(ay)}{I_0(ay)}\cosh(tv)\cosh(uy)\,dy$$

and $B(t) = \frac{4}{\pi^2}\int_1^a \left\{-\frac{\pi}{2}\frac{\rho h(\rho)}{(\rho^2-t^2)^{\frac{1}{2}}} + \rho h(\rho)\,K(\rho,t)\right\}d\rho$

where $\mathbb{K}(\rho,t) = \int_0^\infty \frac{K_0(ay)}{I_0(ay)} I_0(\rho y)\cosh(ty)dy$

Applying the analysis involved in the derivations, we find that the solution of the problem is reduced to determining the function $g(t)$ in the Fredholm's integral equation (2.3.26) of the second kind.

Substituting the expression for $g(t)$ above into equation (2.3.16), the Fourier-Bessel coefficients A_n satisfying the dual series relations (2.3.4a) and (2.3.4b) can be determined. To determine the Fourier-Bessel coefficients satisfying the dual series relations (2.3.2) we superpose the solutions of problems (a) and (b) derived in this section.

We can apply the method of solution mentioned above to solve mixed boundary value problems that are encountered in mathematical physics. The solution of the problem (a) derived in this section is applicable to the problem of the electrified disc that was mentioned in section 1.4.

2.4 Solution of the Dual Series Relations in the Asymmetric Case

In this section we want to derive the method of solving the dual series relations (2.3.1) that is

$$\sum_{m=1}^{\infty} A_{nm} J_n(\rho\lambda_{nm}) = f_n(\rho), \quad 0 \leq \rho \leq 1$$

$$\sum_{m=1}^{\infty} \lambda_{nm} A_{nm} J_n(\rho\lambda_{nm}) = h_n(\rho), \quad \rho > 1$$

Similar to what we had in the previous section, we want to determine the constants (Fourier-Bessel coefficients) $\{A_{nm}\}$ satisfying the dual series relations above. In order to simplify the problem posed in solving dual series relations (2.3.1) we split the problem into two parts viz:

Problem (a): Determine the constants $\{A_{nm}\}$ satisfying the dual series relations

$$\sum_{m=1}^{\infty} A_{nm} J_n(\rho\lambda_{nm}) = f_n(\rho), \quad 0 \leq \rho \leq 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2.4.1a)$$

$$\sum_{m=1}^{\infty} \lambda_{nm} A_{nm} J_n(\rho\lambda_{nm}) = 0, \quad \rho > 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2.4.1b)$$

$n = 0, 1, 2, 3, \ldots\ldots$ and

Problem(b): Determine the constants $\{A_{nm}\}$ satisfying the dual series relations

$$\sum_{m=1}^{\infty} A_{nm} J_n(\rho\lambda_{nm}) = 0 \quad , \quad 0 \leq \rho \leq 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.4.2a)$$

$$\sum_{m=1}^{\infty} \lambda_{nm} A_{nm} J_n(\rho\lambda_{nm}) = h_n(\rho), \ \rho > 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.4.2b)$$

$n = 0,1,2,\ldots\ldots\ldots$ The solution of the dual series relations (2.3.1) is obviously got by superpositioning the solutions of problems (a) and (b) in this section.

We first derive the method of solving problem (a):

Solution:

We assume that for $0 \leq \rho \leq 1$

$$\sum_{m=1}^{\infty} \lambda_{nm} A_{nm} J_n(\rho\lambda_{nm}) = -\rho^{n-1} \frac{\partial}{\partial\rho} \int_{\rho}^{1} \frac{g_n(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.4.3)$$

$n = 0, 1, 2,\ldots\ldots\ldots$.

Taking Hankel transform on both sides of equation (2.4.3) we have

$$\lambda_{nm} A_{nm} \int_0^a \rho J_n^{\,2}(\rho\lambda_{nm})d\rho = -\int_0^1 \rho^n \frac{\partial}{\partial\rho} \frac{g_n(t)\,dt}{(t^2-\rho^2)^{\frac{1}{2}}} J_n(\rho\lambda_{nm})d\rho \qquad \ldots\ldots\ldots\ldots(2.4.4)$$

$$\text{But} \quad \int_0^a \rho J_n^{\,2}(\rho\lambda_{nm})d\rho = \frac{a^2}{2} J_{n+1}^{\,2}(a\lambda_{nm}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.4.5)$$

Using result (2.4.5) in equation (2.4.4) we have

$$A_{nm} = \frac{2}{\lambda_{nm} a^2 J_{n+1}^{\,2}(a\lambda_{nm})} \int_0^1 \rho^n \frac{\partial}{\partial\rho} \int_{\rho}^{1} \frac{g_n(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} J_n(\rho\lambda_{nm})d\rho \qquad \ldots\ldots\ldots\ldots(2.4.6)$$

Following the methods used in equations (2.2.5) to (2.2.8) we have

$$A_{nm} = \frac{2\sqrt{\pi/2}}{\lambda_{nm}^{\frac{1}{2}} a^2 J_{n+1}^{\,2}(a\lambda_{nm})} \int^1 g_n(t) t^{n-\frac{1}{2}} J_{n-\frac{1}{2}}(t\lambda_{nm})dt \qquad \ldots\ldots\ldots\ldots(2.4.7)$$

Substituting the value of A_{nm} from equation (2.4.7) into equation (2.4.1a) we have

$$\sqrt{\pi}/2^{\kappa}\frac{2}{a^2} \sum_{m=1}^{\infty} \frac{1}{\lambda_{nm}^{\frac{1}{2}} J_{n+1}^{\,2}(a\lambda_{nm})} \int_0^1 g_n(t) t^{n-\frac{1}{2}} J_{n-\frac{1}{2}}(t\lambda_{nm}) J_n(\rho\lambda_{nm})dt = f_n(\rho) \ldots (2.4.8)$$

Changing the order of summation and integration in equation (2.4.8) we have

$$\sqrt{\pi}/2 \int_o^1 g_n(t)t^{n-\frac{1}{2}}\left\{\frac{2}{a^2}\sum_{m=1}^{\infty}\frac{J_n(\rho\lambda_{nm})\,J_{n-\frac{1}{2}}(t\lambda_{nm})\lambda_{nm}^{-\frac{1}{2}}}{J_{n+1}{}^2(a\lambda_{nm})}\right\}dt = f_n(\rho) \quad .....(2.4.9)$$

Using equation (1.6.5) where $\delta = \frac{3}{2}$, $\beta = n$, $\nu = n+1$ and $\gamma = n-\frac{1}{2}$

we have

$$\frac{2}{a^2}\sum_{m=1}^{\infty}\frac{J_n(\rho\lambda_{nm})J_{n-\frac{1}{2}}(t\lambda_{nm})\lambda_{nm}^{-\frac{1}{2}}}{J_{n+1}{}^2(a\lambda_{nm})} = \int_o^{\infty} J_n(\rho x)J_{n-\frac{1}{2}}(tx)x^{\frac{1}{2}}dx -$$

$$- \frac{2}{\pi}\int_o^{\infty}\frac{K_n(ay)}{I_n(ay)}I_n(\rho y)I_{n-\frac{1}{2}}(ty)y^{\frac{1}{2}}dy \quad .....(2.4.10)$$

Using result (2.4.10) in equation (2.4.9) we have

$$\sqrt{\frac{\pi}{2}}\int_o^1 g_n(t)t^{n-\frac{1}{2}}\left(\int_o^{\infty} J_n(\rho x)J_{n-\frac{1}{2}}(tx)x^{\frac{1}{2}}dx\right)dt = f_n(\rho) +$$

$$+ \frac{2}{\pi}\sqrt{\frac{\pi}{2}}\int_o^1 g_n(t)t^{n-\frac{1}{2}}\left(\int_o^{\infty}\frac{K_n(ay)}{I_n(ay)}I_n(\rho y)I_{n-\frac{1}{2}}(ty)y^{\frac{1}{2}}dy\right)dt \quad .....(2.4.11)$$

Using result (2.2.10) in the left hand side of equation (2.4.11) we have

$$\int_o^{\rho}\frac{g_n(t)dt}{t^{1-2n}\rho^n(\rho^2-t^2)^{\frac{1}{2}}} = f_n(\rho) +$$

$$+ \frac{2}{\pi}\sqrt{\frac{\pi}{2}}\int_o^1 g_n(t)t^{n-\frac{1}{2}}\left(\int_o^{\infty}\frac{K_n(ay)}{I_n(ay)}I_n(\rho y)I_{n-\frac{1}{2}}(ty)y^{\frac{1}{2}}dy\right)dt$$

Inverting the Abel's integral equation above we have

$$\frac{g_n(t)}{t^{1-2n}} = \frac{2}{\pi}\frac{d}{dt}\int_o^t \frac{\rho^{1+n}f_n(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$$

$$+ \frac{2}{\pi}\sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_o^t \rho\left[\int_o^1 g_n(u)U^{n-\frac{1}{2}}\left(\int_o^{\infty}\frac{K_n(ay)\rho^n I_n(\rho y)I_{n-\frac{1}{2}}(uy)y^{\frac{1}{2}}dy}{I_n(ay)\,(t^2-\rho^2)^{\frac{1}{2}}}\right)du\right]d\rho \quad .(2.4.12$$

Changing the order of the integrations of the second term of the right hand side of equation (2.4.12) we have

$$\frac{g_n(t)}{t^{1-2n}} = \chi(t) + \frac{2}{\pi}\sqrt{\frac{2}{\pi}} \int_0^1 \frac{K_n(ay)}{I_n(ay)} \left(\frac{d}{dt}\int_0^t \frac{\rho^{1+n} I_n(\rho y)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right) I_{n-\frac{1}{2}}(uy)y^{\frac{1}{2}}dy\,du$$

Using equation (1.5.8) in the second term of the right hand side of the equation above we have

$$\frac{g_n(t)}{t^{1-2n}} = \chi(t) + \frac{2}{\pi}\int_0^1 \frac{g_n(u)}{U^{\frac{1}{2}-n}}\left(\int_0^\infty \frac{K_n(ay)}{I_n(ay)}\, t^{n+\frac{1}{2}} I_{n-\frac{1}{2}} I_{n-\frac{1}{2}}(ty) I_{n-\frac{1}{2}}(uy)dy\right)du$$

.....(2.4.13)

Therefore

$$\frac{g_n(t)}{t^{1-2n}} = \chi(t) + \frac{2}{\pi}\int_0^1 \frac{g_n(u)}{U^{\frac{1}{2}-n}}\, \mathbb{K}(t,u)du \qquad .....(2.4.14)$$

where $\chi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho^{1+n} f_n(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$

and $\mathbb{K}_n(t,u) = \mathbb{K}_n(u,t) =$

$$= \frac{2}{\pi}\, t^{n+\frac{1}{2}} \int_0^\infty \frac{K_n(ay)}{I_n(ay)} I_{n-\frac{1}{2}}(ty) I_{n-\frac{1}{2}}(uy)\, ydy$$

Equation (2.4.14) is a sequence of Friedholm integral equations of the second kind which when solved gives the expression for the unknown function $g(t)$. Substituting the expression for $g_n(t)$ into equation (2.4.7), the value of Fourier-Bessel coefficients A_{nm} satisfying the dual series relation (2.4.1a) and (2.4.1b) are determined. For $n = 0$ we can clearly see that the solution (2.4.14) we got above is identical to solution (2.3.14) that we got for problem(a) in section 2.3.

We proceed to the solution of problem (b) which is called the companion problem to problem(a).

Solution of Problem (b):

We assume that for $o \leqslant \rho \leqslant 1$

$$\sum_{m=1}^{\infty} \lambda_{nm} A_{nm} J_n(\rho\lambda_{nm}) = - \rho^{n-1} \frac{\partial}{\partial\rho} \int_{\rho}^{1} \frac{g_n(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}}$$

Then for $o \leqslant \rho \leqslant a$

$$\sum_{m=1}^{\infty} \lambda_{nm} A_{nm} J_n(\rho\lambda_{nm}) = - \rho^{n-1} \frac{\partial}{\partial\rho} \int_{\rho}^{1} \frac{g_n(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} + h_n(\rho)H(\rho-1) \quad \ldots\ldots\ldots(2.4.15)$$

Following the same principles as in equations (2.2.4) to (2.2.8) we have

$$A_{nm} = \frac{2\sqrt{\pi}/2}{nm^{\frac{1}{2}}a^2 J_{n+1}{}^2(a\lambda_{nm})} \left[\int_o^1 g_n(t) t^{n-\frac{1}{2}} J_{n-\frac{1}{2}}(t\lambda_{nm})dt + \right.$$

$$\left. + \int_1^a r h_n(r) J_n(r\lambda_{nm})dr \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.4.16)$$

where the second term of the right hand side of equation (2.4.16) is got by changing the dummy variable ρ to r.

Substituting the value of A_{nm} from equation (2.4.16) into equation (2.4.2b) we have

$$\sqrt{\frac{\pi}{2}} \frac{2}{a^2} \int_o^1 g_n(t) \sum_{m=1}^{\infty} \frac{t^{n-\frac{1}{2}} J_{n-\frac{1}{2}}(t\lambda_{nm}) J_n(\rho\lambda_{nm})dt}{\lambda_{nm}^{\frac{1}{2}} J_{n+1}{}^2(a\lambda_{nm})} =$$

$$- \sqrt{\frac{\pi}{2}} \frac{2}{a^2} \sum_{m=1}^{\infty} \frac{J_n(\rho\lambda_{nm})}{\lambda_{nm} J_{n+1}{}^2(a\lambda_{nm})} \int_1^a r h_n(r) J_n(r\lambda_{nm})dr \quad \ldots\ldots\ldots(2.4.17)$$

Changing the order of summation and integration in equation (2.4.21) we have

$$\sqrt{\frac{\pi}{2}} \int_o^1 g_n(t) \left\{ \frac{2}{a^2} \sum_{m=1}^{\infty} \frac{J_n(\rho\lambda_{nm}) J_{n-\frac{1}{2}}(t\lambda_{nm}) \lambda_{nm}^{-\frac{1}{2}}}{J_{n+1}{}^2(a\lambda_{nm})} \right\} t^{n-\frac{1}{2}} dt =$$

$$- \sqrt{\frac{\pi}{2}} \int_1^a r h_n(r) \left(\frac{2}{a^2} \sum_{m=1}^{\infty} \frac{J_n(r\ _{nm}) J_n(\ _{nm})}{nm\, J_{n+1}{}^2(a\lambda_{nm})} \right) dr \quad \ldots\ldots\ldots(2.4.18)$$

Using result (2.4.10) in the left hand side of equation (2.4.18) we have

$$\sqrt{\frac{\pi}{2}} \int_o^1 g_n(t) t^{n-\frac{1}{2}} \left(\int_o^{\infty} J_n(\rho x) J_{n-\frac{1}{2}}(tx) x^{\frac{1}{2}} dx - \sqrt{\frac{2}{\pi}} \int_o^{\infty} \frac{K_n(ay)}{I_n(ay)} I_n(\rho y) I_{n-\frac{1}{2}}(ty) y^{\frac{1}{2}} dy \right) dt =$$

$$- Q_n(\rho) \ldots\ldots\ldots(2.4.19)$$

where $Q_n(\rho) = \int_1^a r h_n(r) \left(\frac{2}{a^2} \sum_{m=1}^{\infty} \frac{J_n(r\lambda_{nm}) J_n(\rho\lambda_{nm})}{\lambda_{nm} J_{n+1}^2(a\lambda_{nm})} \right) dr$ .....(2.4.20)

Therefore

$$\sqrt{\frac{\pi}{2}} \int_0^1 g_n(t) t^{n-\frac{1}{2}} \left(\int_0^\infty J_n(\rho x) J_{n-\frac{1}{2}}(tx) x^{\frac{1}{2}} dx \right) dt =$$

$$\sqrt{\frac{2}{\pi}} \int_0^1 g_n(t) t^{n-\frac{1}{2}} \left(\int_0^\infty \frac{K_n(ay)}{I_n(ay)} I_n(\rho y) I_{n-\frac{1}{2}}(ty) y^{\frac{1}{2}} dy \right) dt - Q_n(\rho) \quad ....(2.4.21)$$

Using result (2.2.10) in the left hand side of equation (2.4.21) we have

$$\int_0^\rho \frac{g_n(t)dt}{t^{1-2n} \rho^n (\rho^2 - t^2)^{\frac{1}{2}}} = \sqrt{\frac{2}{\pi}} \int_0^1 g_n(t) t^{n-\frac{1}{2}} \left(\int_0^\infty \frac{K_n(ay)}{I_n(ay)} I_n(\rho y) I_{n-\frac{1}{2}}(ty) y^{\frac{1}{2}} dy \right) dt -$$

$$- Q_n(\rho) \quad .....(2.4.22)$$

Inverting Abel's integral equation (2.4.22) we have

$$\frac{g_n(t)}{t^{1-2n}} = \frac{2}{\pi} \sqrt{\frac{2}{\pi}} \frac{d}{dt} \int_0^t \rho \left[\int_0^1 \frac{g_n(u)}{u^{\frac{1}{2}-n}} \left(\int_0^\infty \frac{K_n(ay)}{I_n(ay)} \frac{\rho^n I_n(\rho y) I_{n-\frac{1}{2}}(uy) y^{\frac{1}{2}} dy}{(t^2-\rho^2)^{\frac{1}{2}}} \right) du \right] d\rho$$

$$- \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{\rho^{1+n} Q_n(\rho) d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} \quad .....(2.4.23)$$

Changing the order of integrations in the first term of the right hand side of equation (2.4.23) we have

$$\frac{g_n(t)}{t^{1-2n}} = \frac{2}{\pi} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{g_n(u)}{u^{\frac{1}{2}-n}} \left[\int_0^\infty \frac{K_n(ay)}{I_n(ay)} \left(\frac{d}{dt} \int_0^t \frac{\rho^{1+n} I_n(\rho y) d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} \right) I_{n-\frac{1}{2}}(uy) y^{\frac{1}{2}} dy \right] du -$$

$$- \mathbb{I}_n \quad .....(2.4.24)$$

where $\mathbb{I}_n = \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{\rho^{1+n} Q_n(\rho) d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$ .....(2.4.25)

Using equation (1.5.8) in the first term of the right hand side of equation (2.4.24) we have

$$\frac{g_n(t)}{t^{1-2n}} = -\frac{2}{\pi}\int_0^1 \frac{g_n(u)}{u^{\frac{1}{2}-n}}\left(\int_0^\infty \frac{K_n(ay)}{I_n(ay)}\, t^{n+\frac{1}{2}}\, I_{n-\frac{1}{2}}(ty) I_{n-\frac{1}{2}}(uy) y dy\right) du - \mathbb{I}_n$$

Dealing with the second term of the right hand side of of equation (2.4.26) we have

$$\mathbb{I}_n = \frac{2}{\pi}\,\frac{d}{dt}\int_0^t \frac{\rho^{1+n} Q_n(\rho) d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$$

$$= \frac{2}{\pi}\,\frac{d}{dt}\int_0^t \frac{\rho^{1+n}\left\{\int_1^a r h_n(r)\left[\frac{2}{a^2}\sum_{m=1}^{\infty} \frac{J_n(r\lambda_{nm}) J_n(\rho\lambda_{nm})}{\lambda_{nm} J_{n+1}^2(a\lambda_{nm})}\right] dr\right\} d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$$

Changing the order of summation and integration of the above equation we have

$$\mathbb{I}_n = \frac{2}{\pi}\int_1^a r h_n(r)\left[\frac{2}{a^2}\sum_{m=1}^{\infty} \frac{J_n(r\lambda_{nm})}{\lambda_{nm} J_{n+1}^2(a\lambda_{nm})}\left(\frac{d}{dt}\int_0^t \frac{\rho^{1+n} J_n(\rho\lambda_{nm}) d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right)\right] dr \quad \ldots(2.4.27)$$

Using result (2.2.22) in equation (2.4.27) we have

$$\mathbb{I}_n = \sqrt{\frac{2}{\pi}}\int_1^a r h_n(r)\left(\frac{2}{a^2}\sum_{m=1}^{\infty} \frac{J_n(r\lambda_{nm}) J_{n-\frac{1}{2}}(t\lambda_{nm}) \lambda_{nm}^{-\frac{1}{2}}}{J_{n+1}^2(a\lambda_{nm})}\right) t^{n+\frac{1}{2}} dr \quad \ldots\ldots\ldots\ldots(2.4.28)$$

Using result (2.4.10) in equation (2.4.28) we have

$$\mathbb{I}_n = \sqrt{\frac{2}{\pi}}\int_1^a r h_n(r)\left(\int_0^\infty J_n(rx) J_{n-\frac{1}{2}}(tx) x^{\frac{1}{2}}\, dx - \right.$$

$$\left. - \frac{2}{\pi}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{K_n(ay)}{I_n(ay)} I_n(ry)\, I_{n-\frac{1}{2}}(ty) y^{\frac{1}{2}} dy\right) t^{n+\frac{1}{2}} dr \quad \ldots\ldots\ldots\ldots(2.4.28a)$$

Using result (2.2.10) in the first term of the right hand side of equation (2.4.28a) we have

$$\mathbb{I}_n = \frac{2}{\pi}\int_1^a \frac{r^{1-n} h_n(r) dr}{t^{-2n}(r^2-t^2)^{\frac{1}{2}}}$$

$$- \frac{2}{\pi}\sqrt{\frac{2}{\pi}}\, t^{n+\frac{1}{2}} \int_1^a r h_n(r)\int_0^\infty \frac{K_n(ay)}{I_n(ay)} I_n(ry) I_{n-\frac{1}{2}}(ty) y^{\frac{1}{2}} dr$$

Therefore

$$\Pi_n = \frac{2}{\pi}\int_1^a \frac{\rho^{1-n}h_n(\rho)d\rho}{t^{-2n}(\rho^2-t^2)^{\frac{1}{2}}} -$$

$$-\frac{2}{\pi}\sqrt{\frac{2}{\pi}}\,t^{n+\frac{1}{2}}\int_1^a \rho h_n(\rho)\left(\int_0^\infty \frac{K_n(ay)}{I_n(ay)} I_n(\rho y)I_{n-\frac{1}{2}}(ty)y^{\frac{1}{2}}dy\right)d\rho$$

where the right hand side of the equation above is obtained by changing back from dummy variable r to ρ. .....(2.4.29)

Using equation (2.4.29) in equation (2.4.26) we have

$$\frac{g_n(t)}{t^{1-2n}} = \frac{2t^{n+\frac{1}{2}}}{\pi}\int_0^1 \frac{g_n(u)}{u^{\frac{1}{2}-n}}\left(\int_0^\infty \frac{K_n(ay)}{I_n(ay)} I_{n-\frac{1}{2}}(uy)\, I_{n-\frac{1}{2}}(uy)ydy\right)du$$

$$+\frac{2}{\pi}\sqrt{\frac{2}{\pi}}\,t^{n+\frac{1}{2}}\int_1^a \rho h_n(\rho)\left(\int_0^\infty \frac{K_n(ay)}{I_n(ay)} I_n(\rho y)I_{n-\frac{1}{2}}(ty)y^{\frac{1}{2}}dy\right)d\rho$$

$$-\frac{2}{\pi}\int_1^a \frac{\rho^{1-n}\,h_n(\rho)d\rho}{t^{1-2n}(\rho^2-t^2)^{\frac{1}{2}}} \qquad .....(2.4.30)$$

Therefore

$$\frac{g_n(t)}{t^{1-2n}} = B_n(t) + \int_0^1 \frac{g_n(u)}{u^{\frac{1}{2}-n}}\mathbb{K}_n(t,u)du \qquad .....(2.4.30a)$$

where

$$\mathbb{K}_n(t,u) = \mathbb{K}_n(u,t) = \frac{4}{\pi^2}\,t^{n+\frac{1}{2}}\int_0^\infty \frac{K_n(ay)}{I_n(ay)} I_{n-\frac{1}{2}}(uy)I_{n-\frac{1}{2}}(uy)ydy$$

and

$$B_n(t) = \frac{2}{\pi}\int_1^a \rho h_n(\rho)\left[\sqrt{\frac{2}{\pi}}\,t^{n+\frac{1}{2}}\int_0^\infty \frac{K_n(ay)}{I_n(ay)} I_n(\rho y)I_{n-\frac{1}{2}}(ty)y^{\frac{1}{2}}dy -\right.$$

$$\left. - \frac{t^{2n}}{\rho^n(\rho^2-t^2)^{\frac{1}{2}}}\right]d\rho$$

Equation (2.4.30) gives the expression for the unknown function $g_n(t)$ in the Fredholm integral equation of the second kind. Substituting the expression for the function $g(t)$ above into equation (2.4.16), the values of the Fourier Bessel coefficients A_{nm} satisfying the dual series relations (2.4.2a) and (2.4.2b) can be determined.

If n is put equal to zero in the Fredholm integral equation (2.4.30), it is identical to the Fredholm integral equation (2.3.26) which we got as the solution of problem (b) in the axisymmetric case.

The method of solution derived can be applied to the mixed boundary value problems in potential theory e.g. for finding the potential function or the electrostatic field due to the presence of an electrified disc of infinite radius placed inside a hollow circular cylinder whose outer surface is earthed (Sneddon [22] and (Sneddon and Snivastav [30]).

3. CERTAIN MIXED BOUNDARY VALUE PROBLEMS IN ELASTOSTATICS FOR CYLINDRICAL SOLIDS

3.0 Introduction

The boundary value problems in mathematical theory of elasticity usually pose greater mathematical difficulties than the boundary value problems in potential theory. One obvious reason for this difficulty is the higher order of the partial differential equations governing the elastic phenomenon. While in domains other than elasticity, the governing partial differential equations usually are of second order, In elasticity, a fourth order partial differential equation involving a biharmonic differential operator usually occurs.

We are not going into the depth of the theory of elasticity, but we are going to concern ourselves with how to transform some mixed boundary value problems in the theory of elasticity to dual series relations by employing the techniques of Hankel transform and Fourier transform. Some authors in the field of the theory of elasticity are Love [15], Sokolonikoff [31], Timoshenko and Goodier [33]

In this chapter we shall be concerned only with crack problems in infinite cylinders. Cragg [9] and Atkinson [1] have worked on the problem of the expanding circular crack. Mal [17] discussed the response of a penny-shaped crack to a loading in the form of a plane harmonic dilation wave propagating along the axis of the crack while Mal [18] further investigated the response to an incident plane harmonic shear wave polarised in a plane normal to the plane of the crack and propagating along the axis of the crack. Cragg (9) also discussed the solution of the problem of the stress distribution in a long circular cylinder when a discontinuous pressure is applied to the curved surface. A good account of crack problems in various elastic bodies has been given by Sneddon and Lovengrub [27]

In section 1 of the chapter, we define biharmonic functions and biharmonic equations, where the former are the solutions of the latter. We also mention some examples of biharmonic functions· while in section 2, we mention the occurence of biharmonic functions in the mathematical theory of elasticity.

In section 3, we formulate the mixed boundary value problems that arise from two cases (a) and (b) of the problem of the stress distribution in an infinite cylinder due to the presence of a penny-shaped crack inside the cylinder and situated on the axis of the cylinder symmetrically.

In sections 4 and 5 of the chapter, we derive the solutions of the cases (a) and (b) of the problem above.

These problems have been solved by Sneddon and Tait [29] and Sneddon and Welch [28] by reducing the problems to dual integral equations. In the present analysis, the problems have been reduced to dual series relations and then following the method of Srivastavs [32], they are reduced to Fredholm integral equations of the second kind in an auxilliary function g(t) in whose terms other quantities can be expressed.

3.1 Biharmonic Functions and Equations

A function $f(x,y,z)$ is biharmonic if the relation

$$\nabla^4 f(x,y,z) = \nabla^2[\nabla^2 f(x,y,z)] = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.1.1)$$

where (3.1.1) is called a biharmonic equation.
For two-dimensional function $f(x,y)$

$$\nabla^4 f(x,y) = \left(\frac{\partial^4 f}{\partial x^4} + \frac{2\,\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4} \right) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3.1.2)$$

If $U = U(\rho,z)$ then

$$\nabla^4 U(\rho,z) = \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}\right)^2 U(\rho,z) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3.1.3)$$

Trivially, a harmonic function $f(x,y,z)$ is biharmonic since

$$\nabla^4 f = \nabla^2 (\nabla^2 f) = \nabla^2 (0) = 0$$

Lemma:

If f_1 and f_2 are two biharmonic functions then $\nabla^4 (f_1 + f_2) = 0$; that is, $f_1 + f_2$ is also biharmonic.

$$\nabla^4 (f_1 + f_2) = \nabla^2 ([\nabla^2 (f_1 + f_2)]$$
$$= \nabla^2 [\nabla^2 f_1 + \nabla^2 f_2]$$

$$= \nabla^2(\nabla^2 f_1) + \nabla^2(\nabla^2 f_2)$$

$= 0$ which is the required result.

If ψ_1, ψ_2 and ψ_3 are biharmonic functions where each of the ψ's is a function of three independent variables x,y,z, then $x\psi_1$, $y\psi_2$ and $z\psi_3$ are biharmonic. We have

$$\nabla^2(x\psi_1) = \frac{\partial^2}{\partial x^2}(x\psi_1) + \frac{\partial^2}{\partial y^2}(x\psi_1) + \frac{\partial^2}{\partial x^2}(x\psi_1)$$

$$\frac{\partial}{\partial x}(x\psi_1) = \psi_1 + \frac{\partial \psi_1}{\partial x}$$

$$\frac{\partial^2}{\partial x^2}(x\psi_1) = \frac{2\partial\psi_1}{\partial x} + \frac{x\partial^2\psi_1}{\partial x^2}$$

Similarly $\frac{\partial^2}{\partial y^2}(x\psi_1) = x\frac{\partial^2\psi_1}{\partial y^2}$

and $\frac{\partial^2}{\partial z^2}(x\psi_1) = \frac{x\partial^2\psi_1}{\partial z^2}$

But $\nabla^2\nabla^2(x\psi_1) = \nabla^2\left(\frac{2\partial\psi_1}{\partial x}\right) = 2\frac{\partial}{\partial x}(\nabla^2\psi_1) = 0$

therefore $x\psi_1$ is biharmonic. Similarly $y\psi_2$ and $z\psi_3$ are also biharmonic functions.

Using the lemma above, then the function $x\psi_1 + y\psi_2 + 2\psi_3$ which we get by adding the three biharmonic functions together is also biharmonic and this is a result which is useful when solving boundary value problems in elasticity with cartesian boundaries.

Also if $U(\rho,z)$ is a harmonic function, then $\rho\frac{\partial u}{\partial \rho}$ is biharmonic since

$$\nabla^2\left(\rho\frac{\partial u}{\partial \rho}\right) = \frac{\partial^2}{\partial \rho^2}\left(\rho\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + \frac{\partial^2}{\partial z^2}\left(\rho\frac{\partial u}{\partial \rho}\right)$$

$$= \rho\frac{\partial}{\partial \rho}\left(\frac{\partial^2 u}{\partial \rho^2}\right) + 2\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\left[\frac{\partial u}{\partial \rho} + \rho\frac{\partial}{\partial \rho}\frac{\partial u}{\partial \rho}\right] + \rho\frac{\partial}{\partial \rho}\frac{\partial^2 u}{\partial z^2}$$

$$= \rho \frac{\partial}{\partial \rho}\left[\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2}\right] + 2\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho}$$

$$= 2\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho}\right) = -2\frac{\partial^2 u}{\partial z^2}$$

But $\nabla^4\left(\rho\frac{\partial u}{\partial \rho}\right) = \nabla^4\left(\nabla^2\left(\rho\frac{\partial u}{\partial \rho}\right)\right) = \nabla^2\left(-2\frac{\partial^2 u}{\partial z^2}\right)$

$$= -2\frac{\partial^2}{\partial z^2}(\nabla^2 U) = 0$$

which is the required result. Since U, zU and $\rho\frac{\partial u}{\partial p}$ are biharmonic functions then using the lemma above $U + zU + \rho\frac{\partial u}{\partial \rho}$ is also a biharmonic function and this is a result useful when solving boundary value problems with cylinderical symmetry.

Another interesting property we shall be using is that a derivative of a biharmonic function is also biharmonic.

3.2 Occurence of Biharmonic Functions in the Mathematical Theory of Elasticity

Many problems in classical theory of elasticity reduce one way or the other to the solution of biharmonic functions.

The basic equations in the theory of elasticity are

$$\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho F_1 = 0 \quad \left(=\rho\frac{\partial^2 u}{\partial t^2}\right)\\
&\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho F_2 = 0 \quad \left(=\rho\frac{\partial^2 v}{\partial t^2}\right) \qquad \text{.....(3.2.1)}\\
&\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho F_3 = 0 \quad \left(=\rho\frac{\partial^2 w}{\partial t^2}\right)
\end{aligned}$$

and are called Lames of nations where u,v,w are displacement components in the directions x,y,z axes respectively and σ_{xx}, σ_{yy}, σ_{zz} are normal stress; $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, $\sigma_{yz} = \sigma_{zy}$ are sharing stresses: F_1, F_2, F_3 are body forces per unit mass in the directions of x,y and z-axes respectively.

The equations above could be transformed to express stresses in terms of displacements by first expressing stresses in terms of strains and the strains in terms of displacements. Then equations of (3.2.1) could be written as

$$\left.\begin{aligned}
(\lambda + \mu)\frac{\partial \theta}{\partial x} + \mu \nabla^2 u + \rho F_1 = 0 \quad \left(= \rho \frac{\partial^2 u}{\partial t^2}\right) \\
(\lambda + \mu)\frac{\partial \theta}{\partial y} + \mu \nabla^2 v + \rho F_2 = 0 \quad \left(= \rho \frac{\partial^2 v}{\partial t^2}\right) \\
(\lambda + \mu)\frac{\partial \theta}{\partial z} + \mu \nabla^2 w + \rho F_3 = 0 \quad \left(= \rho \frac{\partial^2 w}{\partial t^2}\right)
\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (3.2.2)$$

where $\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ is the dilation, λ and μ are physical factors and ρ is the density of a body.

In the absence of body forces, each equation of (3.2.2) becomes homogeneous. For static equilibrium conditions, the right hand side of equations of (3.2.2) becomes zero. Dividing each equation of (3.2.2) by μ and introducing an elastic constant k, equations of (3.2.2) become

$$\left.\begin{aligned}
\nabla^2 u + (k+1)\frac{\partial \theta}{\partial x} = 0 \\
\nabla^2 v + (k+1)\frac{\partial \theta}{\partial y} = 0 \\
\nabla^2 w + (k+1)\frac{\partial \theta}{\partial z} = 0
\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.2.3)$$

For Lames equation, in the absence of body forces, the dilation strain θ satisfies Laplace's equation

$$\nabla^2 \theta = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.2.4)$$

If θ is harmonic, so also will $\frac{\partial \theta}{\partial x}$, $\frac{\partial \theta}{\partial y}$ and $\frac{\partial \theta}{\partial z}$ be harmonic. Under this condition equation (3.2.3) becomes

$$\left.\begin{aligned}
\nabla^2 \nabla^2 u = 0 \\
\nabla^2 \nabla^2 v = 0 \\
\nabla^2 \nabla^2 w = 0
\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.2.5)$$

Of particular importance is the fact that the equations above are independent, that is, only one of the unknowns u,v and w enters each equation and each of them is the solution of a biharmonic equation.

For static conditions, we have that Beltrami's equations reduce to the solution of biharmonic equations since in the absence of body forces, equation (3.2.4) holds and the Beltrami's equations

$$\left.\begin{aligned}
(1+k)\,\nabla^2\sigma_{xx} + \partial^2\theta/\partial x^2 &= 0\\
(1+k)\,\nabla^2\sigma_{yy} + \partial^2\theta/\partial y^2 &= 0\\
(1+k)\,\nabla^2\sigma_{zz} + \partial^2\theta/\partial z^2 &= 0\\
(1+k)\,\nabla^2\sigma_{xy} + \partial^2\theta/\partial x\partial y &= 0\\
(1+k)\,\nabla^2\sigma_{yz} + \partial^2\theta/\partial y\partial z &= 0\\
(1+k)\,\nabla^2\sigma_{zx} + \partial^2\theta/\partial z\partial x &= 0
\end{aligned}\right\} \qquad \text{.....(3.2.6)}$$

reduce to

$$\left.\begin{aligned}
\nabla^2\nabla^1\sigma_{xx} &= 0\,; \quad \nabla^2\nabla^2\sigma_{yz} = 0\\
\nabla^2\nabla^2\sigma_{yy} &= 0\,; \quad \nabla^2\nabla^2\sigma_{zx} = 0\\
\nabla^2\nabla^2\sigma_{zz} &= 0\,; \quad \nabla^2\nabla^2\sigma_{xy} = 0
\end{aligned}\right\} \qquad \text{.....(3.2.7)}$$

Each equation of (3.2.7) above is a biharmonic equation while the functions are to be biharmonic.

Hence, in the absence of body forces, Lamé's and Beltrami's equations reduce to biharmonic equations. A very important conclusion follows that, all the basic unknowns in the general problem of the theory of elasticity are biharmonic functions. In fluid dynamics, equation of motion of viscous fluid in the steady state, that is, when velocity is very small reduces to the solution of a biharmonic equation.

3.3 The problem of the effect of a penny-shaped crack on the distribution of stress in a long circular cylinder under tension

In this section, we shall give an analysis (based on Sneddon and Tait [29] and Sneddon and Welch [28]) of the state of stress in an infinitely long cylinder $o \leq \rho \leq a$, $-\infty < z < \infty$ containing a penny-shaped crack lying in the plane $z = 0$ and perpendicular to the axis of the cylinder. The state of stress is due to the prescribed tractions applied at crack faces.

For convenience, we take the unit of length in our problem to be the radius of the crack, so that $o \leq \rho \leq 1$, $z = o$ define the region occupied by the crack. We shall consider separately two sets of boundary conditions viz:

Case (a):

We assume that the cylinderical surface is free from shear and it is supported in such a way that the radial component of the displacement vector vanishes on the surface. Such a situation would arise physically if the elastic cylinder were resting in a smooth cylinderical hollow (of exactly the same radius) in a rigid body and were then deformed by the application of a known pressure to the surface of the crack. The problem of determining the distribution of stress in the vicinity of the crack is equivalent to that of finding the distribution of stress in the semi-infinite cylinder $z \geq o$, $o \leq \rho \leq a$ with the boundary condition.

(i) On the lateral surface $\rho = a$

$$U_\rho = 0 \qquad \ldots\ldots(3.3.1)$$

$$\sigma_{\rho z} = 0 \qquad \ldots\ldots(3.3.1)$$

(ii) On the plane of the crack $z = o$

$$\sigma_{\rho z} = 0 \; ; \quad o \leq \rho \leq a \qquad \ldots\ldots(3.3.3)$$

$$\sigma_{zz} = 2\mu f(\rho); \quad o \leq \rho \leq 1 \qquad \ldots\ldots(3.3.4)$$

$$U_z = 0 \; ; \quad 1 \leq \rho \leq a \qquad \ldots\ldots(3.3.5)$$

and

Case (b):

We assume that the cylinderical surface is stress free. Conditions (3.3.3) to (3.3.5) remain the same but (3.3.1) and (3.3.2) are replaced by the pair:

on the lateral surface:

$$\left.\begin{aligned}\sigma_{\rho\rho} &= 0\\ \sigma_{\rho z} &= 0\end{aligned}\right\}, \quad \rho = a, \quad 0 < z < \infty \qquad \text{.....(3.3.6)}$$

From the basic equations of the theory of elasticity, the stress components can easily be determined from the stress-strain relations. From Sneddon [23,24 27] we can express components of displacement vector and stress tensor in terms of a biharmonic function $\chi(\rho,z)$ as follows:

$$\left.\begin{aligned}U_\rho &= -\chi_{,\rho z}\\ U_z &= 2(1-\eta)\nabla^2\chi\\ \sigma_{\rho\rho} &= 2\mu\frac{\partial}{\partial z}\left[\eta\,\nabla^2\chi - \chi_{,\rho\rho}\right]\\ \sigma_{\theta\theta} &= 2\mu\,\frac{\partial}{\partial z}\left[\eta\nabla^2\chi - \frac{1}{\rho}\chi_{,\rho}\right]\\ \sigma_{zz} &= 2\mu\frac{\partial}{\partial z}\left[(2-\eta)\nabla^2\chi - \chi_{,zz}\right]\\ \sigma_{\rho z} &= \frac{\partial}{\partial\rho}\left[(1-\eta)\nabla^2\chi - \chi_{,zz}\right]\end{aligned}\right\} \qquad \text{.....(3.3.7)}$$

$\sigma_{\rho z} = \sigma_{\theta z} = 0$ due to axisymmetry. In (3.3.7) comma, followed by subscripts in the right hand side of χ stands for the partial derivative of $\chi(\rho,z)$ with respect to these variables. Now we assume that

$$\chi(\rho,z) = \sum_{n=1}^{\infty}\lambda_n^{-2}[a_n + \lambda_n b_n z]J_o(\rho\lambda_n)e^{-\lambda_n z} +$$

$$= \int_o^\infty \xi^{-2}[d_1(\xi)\, I_o(\rho\xi) + d_2(\xi)\rho\xi I_1(\rho\xi)]\sinh\xi z\,d\xi \qquad \text{.....(3.3.8)}$$

where $\chi(\rho,z)$ in a biharmonic function, $d_1(\xi)$ and $d_2(\xi)$ are functions of ξ alone and λ_n is the sequence of positive zeros of $J_o'(a\lambda)$ i.e. $J_1(a\lambda)$.

Using equation (3.3.7) and (3.3.8) we find that

$$\frac{U_\rho}{2\mu} = \sum_{n=1}^{\infty}(a_n - b_n + \lambda_n b_n z)J_1(\rho\lambda_n)e^{-\lambda_n z} +$$

$$+ \int_0^\infty [d_1(\xi)I_1(\rho\xi)+d_2(\xi)\rho\xi I_0(\rho\xi)]\cos\xi z d\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.3.9)$$

$$\frac{U_z}{2\mu} = \sum_{n=1}^{\infty}[2(1-2\eta)b_n + a_n + \lambda_n b_n z]J_0(\rho\xi_n)e^{-\lambda} -$$

$$- \int_0^\infty [\{2(1-\eta)d_2(\xi)+d_1(\xi)\}I_0(\rho\xi)+d_2(\xi)\rho\xi I_1(\rho\xi)]\sin\xi z d\xi \ldots\ldots\ldots\ldots(3.3.10)$$

$$\sigma_{zz} = \sum_{n=1}^{\infty}\lambda_n \quad (1-2\eta)b_n + a_n + \lambda_n b_n z]J_0(\rho\lambda_0)e^{-\lambda_n z} +$$

$$+ \int_0^\infty \xi[\{d_1(\xi)+2(2-\eta)d_2(\xi)\}I_0(\rho\xi)+d_2(\xi)\rho\xi I_1(\rho\xi)]\cos\xi z d\xi \ldots\ldots\ldots(3.3.11)$$

$$\frac{\sigma_{\rho z}}{2\mu} = \sum_{n=1}^{\infty}\lambda_n(a_n + \lambda_i b_n z - 2\eta b_n)J_1(\rho\xi_n)e^{-\lambda_n z} +$$

$$+ \int_0^\infty \xi[\{2(1-\eta)d_2(\xi)+d_1(\xi)\}I_1(\rho\xi)+d_2(\xi)\rho\xi I_0(\rho\xi)]\sin\xi z d\xi \ldots\ldots\ldots(3.3.12)$$

$$\frac{\sigma_{\rho\rho}}{2\mu} = \sum_{n=1}^{\infty}\lambda_n[(1+2\eta)b_n - a_n - \lambda_n b_n z]J_0(\rho\lambda_n)e^{-\lambda_n z} +$$

$$+ \sum_{n=1}^{\infty}\lambda_n(a_n + \lambda_n b_n z - b_n)\frac{J_1(\rho\lambda_n)e^{-\lambda_n z}}{\rho\lambda_n}$$

$$- \int_0^\infty \xi\{1-2\eta)d_2(\xi)+d_1(\xi)\}I_0(\rho\xi) + \{d_2(\xi)\rho^2\xi^2 - d_1(\xi)]\frac{I_1(\rho\xi)}{\rho\xi}]\cos\xi z\xi\xi$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(3.3.13)$$

$$\frac{\sigma_{\theta\theta}}{2\mu} = \sum_{n=1}^{\infty}\lambda_n[2\eta b_n J_0(\rho\lambda_n) - (a_n + \lambda_n b_n z - b_n)\frac{J_1(\rho\lambda_n)}{\rho\lambda_n}e^{-\lambda_n z} +$$

$$+ \int_0^\infty \xi[2\eta d_2(\xi)I_0(\rho\xi) - d_1(\xi)\frac{I_1(\rho\xi)}{\rho\xi} - d_2(\xi)I_0(\rho\xi)]\cos\xi z d\xi \ldots\ldots(3.3.14)$$

3.4 Solution of the Case (a)

In this section, we shall, using the technique of Hankel transform and Fourier transform obtain the dual series relations arising from case(a) of the problem mentioned in section 3.3 and then solve it.

Case (a):

$U_\rho = 0$ when $\rho = a$, and using equation (3.3.9) we have

$$\sum_{n=1}^{\infty}(a_n + \lambda_n b_n z - b_n)J_1(a\lambda_n)e^{-\lambda_n z} +$$

$$+ \int_0^\infty [d_1(\xi)I_1(a\xi) + d_2(\xi)a\xi I_0(a\xi)]\cos\xi z d\xi = 0$$

That is, the above equation becomes

$$\int_0^\infty [d_1(\xi)I_1(a\xi) + d_2(\xi)a\xi I_0(a\xi)]\cos\xi z d\xi = 0$$

since $\{\lambda_n\}$ is the sequence positive zeros of $J_0'(a\lambda)$ i.e. $J_1(a\lambda)$.....(3.4.1)

$\sigma_{\rho z} = 0$ at $\rho = a$; and using equation (3.3.12) we have

$$\int_0^\infty \xi[\{2(1-\eta)d_2(\xi)+d_1(\xi)\}I_1(a\xi)+d_2(\xi)a\xi I_0(a\xi)]\sin\xi z d\xi = 0 \quad \text{.....(3.4.2)}$$

$\sigma_{\rho z} = 0$ on $z = 0$, and using equation (3.3.12) we have

$$\sum_{n=1}^{\infty}\lambda_n(a_n - 2\eta b_n)J_1(\rho\lambda_n) = 0 \quad \text{.....(3.4.3)}$$

On $z = 0$, using equations (3.3.11), (3.3.10), (3.3.4) and (3.3.5) give

$$\frac{\sigma_{zz}}{2\mu} = f(\rho); \quad 0 \le \rho \le 1$$

that is

$$\sum_{n=1}^{\infty}[(1-2\eta)b_n + a_n]J_0(\rho\lambda_n) +$$

$$+ \int_0^\infty \xi[\{d_1(\xi) + 2(2-\eta)d_2(\xi)\}I_0(\rho\xi) + d_2(\xi)\rho\xi I_1(\rho\xi)]d\xi = f(\rho):$$

$$0 < \rho < 1 \quad \text{.....(3.4.4)}$$

and

$U_z = 0$ gives

$$\sum_{n=1}^{\infty} [2(1-2\eta)b_n + a_n]J_o(\rho\lambda_n) = 0; \quad 1 < \rho \leq a \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.4.5)$$

Taking Hankel transform of equation (3.4.3) we have

$$a_n - 2\eta b_n = 0$$

therefore $a_n = 2\eta b_n$. $\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.4.6)$

Taking inverse Fourier cosine transform of equation (3.4.1) we have

$$d_1(\xi)I_1(a\xi) + d_2(\xi)a\xi I_o(a\xi) = 0$$

that is

$$d_1(\xi)I_1(a\xi) = - d_2(\xi)a\xi I_o(a\xi) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.4.7)$$

Taking inverse Fourier sine transform of equation (3.4.2) we have

$$[\{2(1-\eta)d_2(\xi) + d_1(\xi)\}I_1(a\xi) + d_2(\xi)a\xi I_o(a\xi)] = 0 \qquad \ldots\ldots\ldots\ldots(3.4.8)$$

Using equation (3.4.7) in equation (3.4.8) we have

$$d_1(\xi) = d_2(\xi) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.4.8a)$$

Using equations (3.4.6) and (3.4.8a) in equations (3.4.4) and (3.4.5) gives

$$\sum_{n=1}^{\infty} \lambda_n b_n J_o(\rho\lambda_n) = f(\rho) \qquad 0 \leq \rho \leq 1 \qquad \ldots\ldots\ldots\ldots\ldots(3.4.9a)$$

$$\sum_{n=1}^{\infty} b_n J_o(\rho\lambda_n) = 0 \; ; \; 1 < \rho \leq a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.4.9b)$$

Equations (3.4.9a) and (3.4.9b) form a pair of series relations known as dual series relations.

We solve the dual series relations above.

Solution:

We assume that for $o \leq \rho \leq 1$

$$\sum_{n=1}^{\infty} b_n J_o(\rho\lambda_n) = \int_{\rho}^{1} \frac{g(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.4.10)$$

Taking Hankel's transform of both sides of equation (3.4.10) we have

$$b_n = \frac{2}{a^2 J_o^{\,2}(a\lambda_n)} \int_o^1 \rho \int_{\rho}^{1} \frac{g(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} J_o(\rho\lambda_n)d\rho \qquad \ldots\ldots\ldots\ldots(3.4.11)$$

Changing the order of integrations in equation (3.4.11) we have

$$b_n = \frac{2}{a^2 J_o^{\,2}(a\lambda_n)} \int_o^1 g(t) \int_{\rho=o}^{t} \frac{\rho J_o(\rho\lambda_n)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} dt$$

Using result (2.1.26) in the right hand. side of equation (3.4.11) we have

$$b_n = \frac{2}{a^2 J_o^{\,2}(a\lambda_n)} \int_o^1 \frac{g(t)\sin(t\lambda_n)}{\lambda_n} dt \qquad \ldots\ldots\ldots\ldots(3.4.12)$$

Multiply equation (3.4.9a) by ρ and integrate it from 0 to ρ we have

$$\sum_{n=1}^{\infty} b_n J_1(\rho\lambda_n) = \frac{1}{\rho} \int_o^{\rho} u\, f(u)du \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.4.13)$$

Substituting the value of b_n from equation (3.4.12) into equation (3.4.13) we have

$$\frac{2}{a^2} \sum_{n=1}^{\infty} \int_o^1 \frac{g(t)\sin(t\lambda_n)J_1(\rho\lambda_n)}{\lambda_n J_o^{\,2}(a\lambda_n)} dt = \frac{1}{\rho} \int_o^{\rho} u\, f(u)du \qquad \ldots\ldots\ldots\ldots(3.4.14)$$

Changing the order of summation and integration in the left hand side of equation (3.4.14) we have

$$\int_o^1 g(t) \left(\frac{2}{a^2} \sum_{n=1}^{\infty} \frac{J_1(\rho\lambda_n)\sin(t\lambda_n)}{\lambda_n J_o\,(a\lambda_n)}\right) dt = \frac{1}{\rho} \int uf(u)du \qquad \ldots\ldots\ldots(3.4.15)$$

Using equation (1.6.5) where $\beta = 1$, $\gamma = \frac{1}{2}$, $\Gamma = 0$, $\nu = 0$ and $\delta = -\frac{1}{2}$

we have

$$\frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_1(\rho\lambda_n)\sin(t\lambda_n)}{\lambda_n J_o^{\,2}(a\lambda_n)} = \int_o^{\infty} J_1(\rho x)\sin(tx)dx - \frac{2}{\pi}\int_o^{\infty}\frac{K_1(ay)}{I_1(ay)} I_1(\rho y)\sinh(ty)dy \quad \ldots\ldots\ldots\ldots(3.4.16)$$

Using result (3.4.16) in equation (3.4.15) we have

$$\int_o^1 g(t)\int_o^{\infty} J_1(\rho x)\sin(tx)\,dxdt = \frac{1}{\rho}\int_o^{\rho} uf(u)du + \frac{2}{\pi}\int_o^1 g(t)\left(\int_o^{\infty}\frac{K_1(ay)}{I_1(ay)} I_1(\rho y)\sinh(ty)dy\right)dt \quad \ldots\ldots\ldots\ldots(3.4.17)$$

Using equation (1.5.4) when $\mu = \frac{1}{2}$, $\nu = 1$, $a = t$, $y = \rho$ we have

$$\int_o^{\infty} J_1(\rho x)\sin(tx)dx = \frac{t}{\rho}(\rho^2 - t^2)^{-\frac{1}{2}}, \quad t < \rho < \infty \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.4.18)$$

Using result (3.4.18) in the right hand side of equation (3.4.17) we have

$$\int_o^{\rho}\frac{t\,g(t)}{(\rho^2-t^2)^{\frac{1}{2}}} = \int_o^{\rho} u\,f(u)du + \frac{2}{\pi}\int_o^1 g(t)\left(\int_o^{\infty}\frac{K_1(ay)}{I_1(ay)}\rho I_1(\rho y)\sinh(ty)dy\right)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.4.19)$$

Inverting Abel's integral equation (3.4.19) we have

$$tg(t) = \frac{2}{\pi}\frac{d}{dt}\int_o^t \frac{\rho\left(\int_o^{\rho} uf(u)du\right)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} + \frac{4}{\pi^2}\frac{d}{dt}\int_o^t \rho\left\{\int_o^1 g(u)\left[\int_o^{\infty}\frac{K_1(ay)}{I_1(ay)}\frac{\rho I_1(\rho y)\sinh(uy)dy\right]du\,d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$$

Changing the order of integrations in the second term of the right hand side of equation (3.4.20) we have

$$tg(t) = \frac{2}{\pi}\frac{d}{dt}\int_o^t \rho \frac{(\int_o^\rho uf(u)du)}{(t^2-\rho^2)^{\frac{1}{2}}}d\rho \quad +$$

$$+ \frac{4}{\pi^2}\int_o^1 g(u)\left(\int_o^\infty \frac{K_1(ay)}{I_1(ay)}\left[\frac{d}{dt}\int_o^t \frac{\rho^2 I_1(\rho y)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right]\sinh(uy)dy\right)du \quad .....(3.4.20a)$$

Using equation (1.5.8) when m = 1 we have

$$\frac{d}{dt}\int_o^t \frac{\rho^2 I_1(\rho y)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} = t\sinh(ty) \quad .....(3.4.21)$$

Using result (3.4.21) in equation (3.4.20a) we have

$$tg(t) = \frac{2}{\pi} t\int_o^t \frac{\rho f(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} + \frac{4}{\pi^2}\int_o^\infty \frac{K_1(ay)t\sinh(ty)\sinh(uy)dy}{I_1(ay)}du \quad .....(3.4.22)$$

We have

$$g(t) = \chi(t) + \int_o^1 g(u)\mathbb{K}(t,u)du$$

where $\chi(t) = \frac{2}{\pi}\int_o^t \frac{\rho f(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$ and

$$\mathbb{K}(t,u) = \mathbb{K}(u,t) = \frac{4}{\pi^2}\int_o^\infty \frac{K_1(ay)}{I_1(ay)}\sinh(ty)\sinh(uy)dy$$

Equation (3.4.22) is a Fredholm equation of the second kind when solved give. the auxilliary function $g(t)$. Components of physical interest can be expressed in terms of $g(t)$ using (3.4.12).

3.5 Solution of the case (b)

In this section, using the technique of Hankel transform and Fourier transform, we obtain the dual series relations arising from case (b) of the problem mentioned in section 3.3 and we shall also derive the method of solving the dual series.

Case (b):

On $\rho = a$, using equations (3.3.6), (3.3.12) and (3.3.13) we have equation (3.4.2) that is

$$\int_o^\infty \xi[\{2(1-\eta)d_2(\xi) + d_1(\xi)\}I_1(a\xi) + d_2(\xi)a\xi I_o(a\xi)\sin\xi z d\xi = 0$$

and we have

$\sigma_{\rho\rho} = 0$, that is

$$\sum_{n=1}^{\infty} \lambda_n [\{1-2\eta)b_n - a_n - \lambda_n b_n z\}]J_o(a\lambda_n)e^{\lambda_n z} -$$

$$- \int_o^\infty \xi [\{(1-2\eta)d_2(\xi) + d_1(\xi)\}I_o(a\xi) + \{d_2(\xi)a^2\xi^2 - d_1(\xi)\}\frac{I_1(a\xi)}{a\xi}]*$$

$$*\cos\xi z d\xi = 0 \quad \ldots\ldots\ldots\ldots (3.5.1)$$

On $z = 0$, using equations (3.3.3) to (3.3.5); (3.3.10), (3.311) and (3.312) we have

$U_z = 0$ and this gives equation (3.4.5),

that is

$$\sum_{n=1}^{\infty} [2(1-2\eta)b_n + a_n]J_o(\rho\lambda_n) = 0; \quad 1 < \rho \leq a,$$

$\sigma_{\rho z} = 0$, and we have equation (3.4.3),

That is,

$$\sum_{n=1}^{\infty} \lambda_n(a_n - 2_\eta b_n)J_1(\rho\lambda_n) = 0$$

and for $\sigma_{zz} = 2\mu f(\rho)$; $o \leq \rho \leq 1$ we have equation (3.4.4), that is,

$$\sum_{n=1}^{\infty} \lambda_n [1-2\eta)b_n + a_n]J_o(\rho\lambda_n) +$$

$$+ \int_o^\infty \xi[\{d_1(\xi) + 2(2-\eta)d_2(\xi)\}I_o(\rho\xi) + d_2(\xi)\rho\xi I_1(\rho\xi)]d\xi = f(\rho)$$

Taking inverse Fourier sine transform of equation (3.4.2) we have equation (3.4.8), that is,

$$\{2\ (1-\eta)d_2(\xi) + d_1(\xi)\}I_1(a\xi) + d_2(\xi)a\xi I_o(a\xi) = 0$$

Taking Hankel transform of equation (3.4.3) we have equation (3.4.6), that is

$$a_n = 2\eta\ b_n$$

Using the above equation (3.4.6) in equation (3.4.4) we have

$$\sum_{n=1}^{\infty} \lambda_n b_n J_o(\rho\lambda_n) + \int_o^{\infty} \xi[\{d_1(\xi) + 2(2-\eta)d_2(\xi)\}I_o(\rho\xi) + d_2(\xi)\rho\xi I_1(\rho\xi)]d\xi = f(\rho) \quad \ldots\ldots\ldots\ldots\ldots(3.5.2)$$

Using equation (3.4.6) in equation (3.4.5) we have equation (3.4.9b), that is

$$\sum_{n=1}^{\infty} b_n J_n(\rho\lambda_n) = 0, \quad 1 < \rho \leq a$$

Taking inverse Fourier cosine transform of equation (3.5.1) we have

$$\xi[\{(1-2\eta)d_2(\xi) + d_1(\xi)\}\ I_o(a\xi) + \{d_2(\xi)a^2\xi^2 - d_1(\xi)\}\frac{I_1(a\xi)}{a\xi} =$$

$$\sum_{n=1}^{\infty} \lambda_n b_n \{\int_o^{\infty} (1-\lambda_n z)J_o(a\lambda_n)\cos\xi z\,dz$$

$$= \sum_{n=1}^{\infty} \frac{\lambda_n^2 b_n J_o(a\lambda_n)}{(\lambda_n^2 + \xi^2)^2} - \sum_n \frac{\lambda_n^2\ b_n J_o(a\lambda_n)(\lambda_n^2 - \xi^2)}{(\lambda_n^2 + \xi^2)^2}$$

$$= 2\xi^2 \sum_{n=1}^{\infty} \frac{\lambda_n^2\ b_n J_o(a\lambda_n)(\lambda_n^2 - \xi^2)}{(\lambda_n^2 + \xi^2)^2}$$

Therefore

$$d_2(\xi)\{(1-2\eta)I_o(a\xi) + a\ I_1(a\xi)\} + d_1(\xi)\{\ I_o(a\xi) - \frac{I_1(a\xi)}{a\xi}\ \}$$

$$= 2\xi \sum_{n=1}^{\infty} \frac{\lambda_n^2 b_n J_o(a\lambda_n)}{(\lambda_n^2 + \xi^2)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.5.3)$$

The result (3.4.8) gives

$$\{2(1-\eta)I_1(a\xi) + a\xi I_0(a\xi)\}d_2(\xi) + d_1(\xi)I_1(a\xi) = 0$$

The determinant of the coefficient matrix of equations (3.4.8) and (3.5.3) is

$$I_1^2(a\xi)[2(1-\eta) + a^2\xi^2] - a^2\xi^2 I_0^2(a\xi) = G(a\xi) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.5.4)$$

Using result (3.5.4) in equations (3.4.8) and (3.5.3) we have

$$d_2(\xi) = 2a\xi^2 \sum_{n=1}^{\infty} \frac{\lambda_n^2 b_n J_0(a\lambda_n) I_1(a\xi)}{(\lambda_n^2 + \xi^2)^2 G(a\xi)}$$

and

$$d_1(\xi) = -2a^2 \sum_{n=1}^{\infty} \frac{\lambda_n^2 b_n J_0(a\lambda_n)}{(\lambda_n^2 + \xi^2)^2 G(a\xi)} [2(1-\eta)I_1(a\xi) + a I_0(a\xi)] \ldots(3.5.6)$$

Using results (3.5.5) and (3.5.6) in equation (3.5.2) we have

$$\left.\begin{aligned} &\sum_{n=1}^{\infty} \lambda_n b_n J_0(\rho\lambda_n) = F(\rho) \; ; \quad 0 \le \rho \le 1 \\ &\sum_{n=1}^{\infty} b_n J_0(\rho\lambda_n) = 0 \; ; \quad 1 < \rho \le a \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(3.5.7)$$

where

$$F(\rho) = f(\rho) - 2a \sum_{n=1}^{\infty} \lambda_n^2 b_n J_0(\rho\lambda_n) \int_0^{\infty} \frac{N(\rho,\xi)d\xi}{G(a\xi)(\lambda_n^2 + \xi^2)^2} \quad \ldots\ldots\ldots\ldots(3.5.8)$$

and $N(\rho,\xi) = \xi^3\{I_1(a\xi)[2I_0(\rho\xi) + \rho\xi I_1(\rho\xi)] - a\xi I_0(a\xi)I_0(\rho\xi)\}$ .....(3.5.8a)

The dual series relations (3.5.7) are identical with (3.4.9a) and (3.4.9b) except that $f(\rho)$ in (3.4.9b) is replaced by $F(\rho)$.

Solution of Case (b):

We assume that for $0 \le \rho \le 1$

$$\sum_{n=1}^{\infty} b_n J_0(\rho\lambda_n) = \int_{\rho}^{1} \frac{g(t)dt}{(t^2-\rho^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots(3.5.9$$

Following the method used in equations (3.4.11) to (3.4.15) we have

$$\int_0^1 g(t)\left(\frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_1(\rho\lambda_n)\sin(t\lambda_n)}{\lambda_n J_o^{\,2}(a\lambda_n)}\right)dt = \frac{1}{\rho}\int_o^{\rho} F(r)dr \qquad \text{.....(3.5.10)}$$

Dealing with the right hand side of equation (3.5.10) we have

$$\frac{1}{\rho}\int_o^{\rho} r\,F(r)dr = \frac{1}{\rho}\int_o^{\rho} r\,f(r)dr - 2a\frac{1}{\rho}\int_o^{\rho} r\sum_{n=1}^{\infty}\lambda_n^{\,2}h_n J_o(a\lambda_n)\int_o^{\infty}\frac{(r,\xi)d\xi}{(\lambda_n^{\,2}+\xi^2)^2\,G(a\xi)}$$

$$\text{.....(3.5.11)}$$

$$= \frac{1}{\rho}\int_o^{\rho} r\,f(r)dr - 2a\frac{1}{\rho}\int_o^{\rho} r\sum_{n=1}^{\infty}\frac{2\lambda_n^{\,2}}{a^2 J_o^{\,2}(a\lambda_n)}\int_o^1\frac{g(t)\sin(t\lambda_n)}{\lambda_n}\,dt\,J_o(a\lambda_n)\cdot$$

$$\cdot\int_o^{\infty}\frac{r\,N(r,\xi)\,d\xi}{G(a\xi)(\lambda_n^{\,2}+\xi^2)^2} \qquad \text{.....(3.5.11a)}$$

Changing the order of summation and integration of the second term of right hand side of equation (3.5.11a) we have

$$\frac{1}{\rho}\int_o^{\rho} r\,F(r)dr = \frac{1}{\rho}\int_o^{\rho} rf(r)dr -$$

$$- 2a\frac{1}{\rho}\int_o^1 g(t)\left[\int_o^{\infty}\left(\frac{2}{a^2}\sum_{n=1}^{\infty}\frac{\lambda_n\sin(t\lambda_n)}{(\lambda_n^2+\xi^2)^2 J_o(a\lambda_n)}\int_o^{\rho}\frac{rN(r,\xi)dr)d\xi\right]dt}{G(a\xi)}\right. \qquad \text{.....(3.5.11b)}$$

But

$$\int_o^{\rho} rN(r,\xi)dr = \xi^2\{I_1(a\xi)[2\rho I_1(\rho\xi) + \xi\rho^2 I_2(\rho\xi)] - a\xi I_o(a\xi)\rho I_1(\rho\xi)\} = M(\rho,\xi)$$

$$\text{.....(3.5.12)}$$

Also from the result of using calculus of residue we have

$$\frac{2}{a^2}\sum_{n=1}^{\infty}\frac{\lambda_n\sin(t\lambda_n)}{(\lambda_n^2+\xi^2\,{}^2 J_o(a\lambda_n)} = \frac{I_1(a\xi)[t\xi\cosh(t\xi) + \sinh(t\xi)] - a\xi I_o(a\xi)\sinh(t\xi)}{\xi^2 I_1^{\,2}(a\xi)}$$

$$= \frac{V(t,\xi)}{\xi^2 I_1^{\,2}(a\xi)} \qquad \text{.....(3.15.13)}$$

where

$$\Psi(t,\xi) = I_1(a\xi)[t\xi\cosh(t\xi) + \sinh(t\xi)] - a\xi I_0(a\xi)\sinh(t\xi)$$

.....(3.5.13a)

Using results (3.5.12), (3.5.13) and (3.5.13a) in equation (3.5.11b) we have

$$-\frac{1}{\rho}\int_0^\rho r\,F(r)dr = \frac{1}{\rho}\int_0^\rho r\,f(r)dr - 2a\frac{1}{\rho}\int_0^1 g(t)\int_0^\infty \frac{M(\rho,\xi)\Psi(t,\xi)d\xi\,dt}{\xi^2 G(a\xi)I_1^2(a\xi)} \quad ....(3.5.14)$$

Using result (3.4.14) in equation (3.5.10) we have

$$\int_0^1 g(t)\left(\frac{\rho}{a^2}\sum_{n=1}^{\infty}\frac{J_1(\rho\lambda_n)\sin(t\lambda_n)}{\lambda_n J_0^2(a\lambda_n)}\right)dt = \frac{1}{\rho}\int_0^\rho rf(r)dr -$$

$$- 2a\,\frac{1}{\rho}\int_0^1 g(t)\left(\int_0^\infty \frac{M(\rho,\xi)\Psi(t,\xi)d\xi}{\xi^2 G(a\xi)I_1^2(a\xi)}\right)dt \quad .....(3.5.15)$$

Using result (3.4.16) in the left hand side of equation (3.5.15) we have

$$\int_0^1 g(t)\int_0^\infty J_1(\rho x)\sin(tx)dudt = \frac{1}{\rho}\int_0^\rho r\,f(r)dr +$$

$$+\frac{2}{\pi}\int_0^1 g(t)\int_0^\infty \frac{K_1(ay)}{I_1(ay)}I_1(\rho y)\sinh(ty)dydt -$$

$$- 2a\frac{1}{\rho}\int_0^1 g(t)\int_0^\infty \frac{M(\rho,\xi)\Psi(t,\xi)d\xi dt}{\xi^2 G(a\xi)I_1^2(a\xi)} \quad .....(3.5.16)$$

Using result (3.4.18) in equation (3.5.16) we have

$$\int_0^\rho \frac{t\,g(t)dt}{(\rho^2-t^2)^{\frac{1}{2}}} = \int_0^\rho rf(r)dr + \frac{2}{\pi}\int_0^1 g(t)\int_0^\infty \frac{K_1(ay)}{I_1(ay)}I_1(\rho y)\sinh(ty)dydt$$

$$- 2a\int_0^1 g(t)\int_0^\infty \frac{M(\rho,\xi)\Psi(t,\xi)d\xi dt}{\xi^2 G(a\xi)I_1^2(a\xi)} \quad .....(3.5.17)$$

Inverting Abel's integral equation (3.5.17) we have

$$tg(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho\int_0^\rho rf(r)dr)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} + \frac{4}{\pi^2}\frac{d}{dt}\int_0^t \rho\left[\int_0^1 g(u)\int_0^\infty \frac{K_1(ay)}{I_1(ay)}\frac{\rho I_1(\rho y)\sinh(uy)dyu]d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right.$$

$$- \frac{4}{\pi^2 a}\frac{d}{dt}\int_0^t \rho\left[\int_0^1 g(u)\left(\int_0^\infty \frac{V(u,\xi)\,M(\rho,\xi)d\rho}{\xi^2 I_1^2(a\xi)G(a\xi)(t^2-\rho^2)^{\frac{1}{2}}}\right)d\xi\right]du \quad \ldots\ldots\ldots\ldots(3.15.18)$$

Changing the order of integrations of equation (3.5.18) we have

$$tg(t) = \frac{2}{\pi}t\int_0^t \frac{\rho f(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} + \frac{4}{\pi^2}\int_0^1 g(u)\left[\int_0^\infty \frac{K_1(ay)}{I_1(ay)}\left(\frac{d}{dt}\right)\int_0^t \frac{\rho^2 I_1(\rho y)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right]\sinh(uy)dy\,du$$

$$- \frac{4}{\pi^2 a}\int_0^1 g(u)\left[\int_0^\infty \frac{V(u;\xi)}{\xi^2 G(a\xi)I_1^2(a\xi)}\left(\frac{d}{dt}\int_0^t \frac{\rho M(\rho,\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right)d\xi\right]du \quad \ldots\ldots(3.5.18a)$$

Using equation (1.5.8) where m = 2, we have

$$\frac{d}{dt}\int_0^t \frac{\rho^3 I_2(\rho\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} = \sqrt{\frac{\pi}{2}}\,\xi^{\frac{1}{2}}\,t^{5/2}I_{3/2}(t\xi)$$

$$= t^2\cosh(t\xi) - t\xi^{-1}\sinh(t\xi) \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.5.19)$$

Using results (3.4.21) and (3.5.19) we have

$$\frac{d}{dt}\int_0^t \frac{M(\rho,\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} = \xi^2\left[\frac{d}{dt}\int_0^t I_1(a\xi)\left\{\frac{2\rho^2 I_1(\rho\xi) + \xi\rho^3 I_2(\rho\xi)}{(t^2-\rho^2)^{\frac{1}{2}}}\right\}d\rho - \right.$$

$$\left. - a\xi\frac{d}{dt}\int_0^t I_0(a\xi)\frac{\rho^2 I_1(\rho\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right]$$

$$= \xi^2 I_1(a\xi)\left\{2\frac{d}{dt}\int_0^t \frac{\rho^2 I_1(\rho\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} + \xi\frac{d}{dt}\int_0^t \frac{\rho^3 I_2(\rho\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}\right\}$$

$$- a\xi I_0(a\xi)\frac{d}{dt}\int_0^t \frac{\rho^2 I_1(\rho\xi)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$$

$$= \xi^2\{I_1(a\xi)[2t\sinh(t\xi) + t^2\xi\cosh(t\xi) - t\sinh(t\xi)]$$

$$- aI_0(a\xi)t\xi\sinh(t\xi)\}$$

$$= \xi^2\{I_1(a\xi)[t^2\xi\cosh(t\xi) + t\sinh(t\xi)] - a\xi I_0(a\xi)t\sinh(t\xi)\}$$

$$= \xi^2 t\{I_1(a\xi)[t\xi\cosh(t\xi) + \sinh(t\xi)] - a\xi I_0(a\xi)\sinh(t\xi)\}$$

$$\ldots\ldots\ldots(3.5.20)$$

Using results (3.4.21) and (3.5.20) in the second and third terms of the right hand side of equation (3.5.18a) we have

$$tg(t) = \frac{2}{\pi}\int_0^t \frac{\rho f'(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} + \frac{4}{\pi^2} t \int_0^1 \frac{K_1(ay)}{I_1(ay)} \sinh(ty)\sinh(uy)dydu$$

$$- \frac{4}{\pi} at \int_0^1 g(u) \int_0^\infty \frac{\mathbf{V}(u,\xi)\,\mathbf{V}(t,\xi)d\xi du}{I_1^2(a\xi)G(a\xi)} \qquad \text{.....(3.5.21)}$$

that is

$$g(t) = \chi(t) + \int_0^1 g(u)\{\mathbb{K}(t,u) + W(t,u)\}du \qquad \text{.....(3.5.21a)}$$

where

$$\chi(t) = \frac{2}{\pi}\int_0^t \frac{\rho f(\rho)d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}$$

and $\mathbb{K}(t,u) = \mathbb{K}(u,t) = \frac{4}{\pi^2}\int_0^\infty \frac{K_1(ay)}{I_1(ay)} \sinh(ty)\sinh(uy)dy$

$$W(t,u) = \frac{4}{\pi} a \int_0^\infty \frac{\mathbf{V}(u,\xi)\,\mathbf{V}(t,\xi)d\xi}{I_1^2(a\xi)G(a\xi)}$$

Equation (3.5.21a) is a Fredholm equation of the second kind which gives the expression for the auxilliary function $g(t)$, in whose terms we can express the components of the displacement vector and the stress tensor.

Summary of the Thesis entitled "Some Mixed Boundary Value Problems in Cylinderical Coordinates"

By
M. O. Adewale

The aim of the study is to find the formal solutions of mixed boundary value problems in cylinderical coordinates by using the methods of dual integral equations and dual series relations.

The thesis is divided into three chapters. Chapter 1 serves as the introduction to our study. In this chapter a brief account of linear differential equations and boundary conditions is given. We describe briefly homogeneous and non-homogeneous linear differential equations and boundary conditions leading to the consideration of linear homogeneous boundary value problems.

Different types of boundary value problems that are encountered in mathematical physics are mentioned with particular emphasis on mixed boundary value problems in potential theory with cylinderical symmetry; and the method of solving the Laplace's equation in cylinderical coordinates is derived.

Since one of the ways of solving mixed boundary value problems is to reduce them to dual integral equations or dual series relations, but before going to dual integral equations, we briefly mention integral equation with references to Volterra and Fredholm integral equations and then a brief account of the dual integral equations and dual series relations is given.

An example of mixed boundary value problem for an electrified disc is formulated mathematically and then it is reduced to dual integral equations (by using the Hankel transform) and dual series relations seperately depending on the boundary conditions.

Chapter 2 is concerned with mixed boundary value problems in potential theory which have either been reduced to dual integral equations or dual series relations. The general forms of the dual integral equations is obtained by applying Hankel transform on Laplace's equation in cylinderical coordinates subject to some prescribed mixed boundary conditions on the integral equation obtained while the general form of dual series relations is obtained by applying the same mixed boundary conditions on the solution of the Laplace's equation derived by using the method of separation of variables.

By using the techniques developed by Sneddon and Srivastav, the solutions of the general forms of the dual integral equations both in the axisymmetric and asymmetric cases are reduced to the problem of solving Abel's integral equations in terms of an auxilliary function, in whose terms the unknowns in the solutions can be expressed. Also the solutions of the general forms of the dual series relations both in the axisymmetric and asymmetric cases are reduced to the problem of solving the Fredholm integral equations of the second kind where the integral equations are easily amenable to numerical methods of solutions. The infinite series encountered in the solutions of the above problems are expressed in a form of their integral representations by using the method of the calculus of residue.

Chapter 3 deals with certain mixed boundary value problems in the theory of elasticity. Boundary value problems in classicial theory of elasticity are usually reduced to find the solution of biharmonic equations subject to given boundary conditions. We give a brief account of biharmonic functions and we use it to get solutions of certain crack problems for infinite cylinders which are concerned with stress distribution inside an infinite cylinder containing a symmetrically situated penny-shaped crack on its axes.

The two cases of the problem of the effect of a penny-shaped crack on the distribution of a stress in a long circular cylinder under tension is formulated and the two cases of the problem are reduced to dual series relations. These are then reduced to Fredholm integral equations of second kind in whose terms stress tensor components can be found.

The analysis presented is only formal and we have preserved the conditions which allow change in the order of integrations or integration and summation.

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