ON FOUR-PARAMETER ODD GENERALIZED EXPONENTIAL-PARETO DISTRIBUTION: ITS PROPERTIES AND APPLICATIONS

By

JAMILA ABDULLAHI

DEPARTMENT OF STATISTICS

FACULTY OF PHYSICAL SCIENCES

AHMADU BELLO UNIVERSITY, ZARIA-NIGERIA

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Jamila ABDULLAHI

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DECLARATION

I declare that the work in this dissertation entitled "On Four-Parameter Odd Generalized Exponential-Pareto Distribution: it's Properties and Applications " has been carried out by me in the Department of Statistics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other institution.

______________________ ______________

Jamila Abdullahi Date

CERTIFICATION

This dissertation titled "ON FOUR-PARAMETER ODD GENERALIZED EXPONENTIAL-PARETO DISTRIBUTION: IT'S PROPERTIES AND APPLICATIONS" by Jamila ABDULLAHI meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

______________________________ ____________
Dr. A. Yahaya Date
Chairman, Supervisory Committee

______________________________ ____________
Dr. S. I. S. Doguwa Date
Member, Supervisory Committee

______________________________ ____________
Dr. O. I. Shittu Date
External Examiner

______________________________ ____________
Dr. H. G. Dikko Date
Head of Department

______________________________ ____________
Prof. S. Z. Abubakar Date
Dean Postgraduate School

DEDICATION

This dissertation is dedicated to my family.

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Firstly, our praise goes to Almighty Allah (S.W.T) for sparing my life, giving me the health and opportunity to carry out this dissertation successfully. May He continue to guide us through all our endeavors.

I am truly grateful to my parents for their support, love and care through out this program. I pray to Allah to reward them with Al-jannatul Firdusi, Ameen.

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ABSTRACT

The Pareto Distribution has received some sizeable attention in the academia especially by adding some elements of flexibility to it through introduction of one or more parameters using generalization approaches. It plays a vital role in analyzing skewed dataset especially in reliability analysis. In this research, we proposed and study a four-parameter Odd Generalized Exponential-Pareto Distribution (OGEPD). Some statistical properties comprising moments, moment generating function, quantile function, reliability analysis, distribution of order statistics and limiting behaviour of the new distribution were derived. We also provide plots of the CDF, pdf, survival function and hazard function for various values of distribution parameters. The plots for the pdf indicated that it is positively skewed and therefore more appropriate for fitting positively skewed datasets. The parameters of the distribution were estimated using method of maximum likelihood. Finally, the proposed distribution is applied to two real datasets to illustrate its fit as compared to other distributions.

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CHAPTER ONE

INTRODUCTION

1.1 Background of the study

The Pareto distribution is a widely known distribution in applied sciences as well as in Economics. It was introduced in order to explain the distribution of income in the society (Pareto, 1896). It was first proposed by a Professor of Economics, Vilfredo Pareto (1843-1923). The distribution was found while studying various distributions for modeling income in Switzerland. The various forms of the Pareto distribution are versatile and can usually be used to model uncertainties. Since that time its applicability spans diverse areas of human endeavour comprising Biology, Physics, Actuarial Science, Geography, etc. Pareto made several important contributions to Economics, mostly in the study of income distribution and in the analysis of individuals choices.

Pareto found out that income approximately follows a Pareto distribution, which is considered as power law probability distribution. The Pareto principle was named after him and noted that 80% of the land in Italy was owned by 20% of the population. One of Pareto's equations attained special importance and argument. He was captivated by problems of power and wealth. How do people get it? How is it spread around society? How do those who have it use it? The gap between rich and poor has always been part of the human condition, but Pareto resolved to measure it. He collected piles of data on wealth and income through different centuries, across different countries: the tax records of Basel, Switzerland, from 1454 and from Augsburg, Germany, in 1471, 1498 and 1512; contemporary rental income from Paris; personal income from Britain, Prussia, Saxony, Ireland, Italy, and Peru. What he discovered or thought he discovered was striking. When he plotted the data on a graph sheet, with income on one axis, and number of people with that income on the other, he observed similar scenario nearly everywhere in every era. Society was not a "social pyramid" with the percentage of rich to poor sloping gently from one class to the next. Instead it was more of a "social arrow" the bottom was very fat indicating where the mass of men live, and at the top was very thin indicating where the wealthy elite reside. Nor was this effect by chance; the data did not remotely fit a bell curve, as one would anticipate if wealth were randomly distributed. "It is a social law", he wrote: something "in the nature of man". At the bottom of the Wealth curve, he wrote, Men and Women starve and children die young. This reason makes Pareto to develop model for distribution of wealth.

The Pareto distribution was used to model prevalence of earthquakes, forest fire areas, oil and gas field sizes (Burroughs and Tebbens, 2001), as well as in online analytical processing (OLAP) by (Nadeau and Teorey, 2003) purposely to obtain meaningful information easily from large amount of data residing in a data warehouse. The Pareto distribution is a combination of exponential distribution with gamma mixing weights, some properties of the Pareto distribution shows that the distribution is a heavy tailed distribution. In insurance application, heavy tailed distribution are important tools for modelling extreme loss, principally for the more risky types of insurance like medical insurance. In financial applications, the study of heavy tailed distributions offers information about the potential for financial fiasco or financial ruin (Klugman *et al.,* 2004). Schroeder *et al.,* (2010) utilized it in modelling risk drive sector errors. Ever since, it plays a vital role in analyzing and dealing with skewed dataset as well as in reliability analysis.

The Pareto distribution has received more attention in the sense that many authors studied and added some elements of flexibility to it, by introducing one or more parameters to the distribution using some generalization approaches.

1.2 Pareto Distribution

If X is a random variable that follows a Pareto distribution, then the probability that X is larger than some number x, that is the survival function (also called tailed function) is given by:

$$\overline{G}\,(x;\theta,k) = \left(\frac{\theta}{x}\right)^{k} \qquad for \quad \theta \le x < \infty; k, \theta > 0 \tag{1.2.1}$$

where θ is the (necessarily positive) minimum possible value of x and k is a positive parameter. The Pareto distribution is characterized by a scale parameter θ and a shape parameter k, which is also called the tail index. When this distribution is used to model the distribution of wealth, then the parameter k is called the Pareto index.

1.2.1 Properties of Pareto Distribution

From the definition provided in section (1.2), The Cumulative Distribution Function (CDF) of a Pareto random variable with parameters θ and k is given by:

$$G(x;\theta,k) = 1 - \left(\frac{\theta}{x}\right)^{k} \qquad for \quad \theta \le x < \infty; k, \theta > 0 \tag{1.2.2}$$

The Probability Density Function (pdf) was obtained by differentiating the CDF in equation (1.2.2) with respect to x.

That is,

$$g(x;\theta,k) = \frac{dG(x;\theta,k)}{dx} = -\theta^{k}\left(-kx^{-k-1}\right) = k\theta^{k}x^{-(k+1)}$$

$$g(x;\theta,k) = \frac{k\theta^{k}}{x^{k+1}} \qquad for \quad \theta \le x < \infty; k, \theta > 0 \tag{1.2.3}$$

where $\theta > 0$ is a scale parameter and $k > 0$ is the shape parameter.

The mean and variance of the Pareto distribution are respectively given as:

$$E(X) = \frac{k\theta}{k-1} \quad for \quad k > 1$$

and

$$Var(X) = \frac{k\theta^2}{(k-1)^2 (k-2)} \quad for \quad k > 2$$

1.3 Generalized Exponential Distribution

Gupta and Kundu (1999) introduced the Generalized Exponential Distribution (GED) also known as Exponentiated Exponential Distribution (EED). It is a probability distribution having two parameters, and is very good in analyzing (especially) positively skewed data because its Probability density function is positively skewed. The CDF of GED is given by:

$$G(x;\alpha,\lambda) = \left(1 - e^{-\lambda x}\right)^{\alpha} \quad for \quad x,\alpha,\lambda > 0 \tag{1.3.1}$$

where α is the shape parameter and λ is the scale parameter.

The corresponding pdf is:

$$g(x;\alpha,\lambda) = \alpha\lambda\left(1 - e^{-\lambda x}\right)^{\alpha-1} e^{-\lambda x} \quad for \quad x,\alpha,\lambda > 0 \tag{1.3.2}$$

The survival function is:

$$S(x;\alpha,\lambda) = 1 - G(x;\alpha,\lambda) = 1 - \left(1 - e^{-\lambda x}\right)^{\alpha} \tag{1.3.3}$$

The hazard function is;

$$h(x;\alpha,\lambda) = \frac{g(x;\alpha,\lambda)}{S(x;\alpha,\lambda)} = \frac{\alpha\lambda\,(1 - e^{\lambda x})^{\alpha-1}\, e^{-\lambda x}}{1 - (1 - e^{-\lambda x})^{\alpha}} \tag{1.3.4}$$

The distribution is found to be mostly applicable in analyzing and modeling life time data. Gupta *et al.,* (2000) studied it and estimated the parameters of the GED using different method of estimation and compare their performance through numerical simulation.

1.4 Statement of the Research Problem

Many lifetime models have been proposed or developed by different reseachers using different generalization procedures to analyze data for certain purposes like the flexibility of properties and performance of new distribution compared to the original ones, but still there are some real datasets that cannot be fitted with the already existing standard distributions. Several researchers proposed some generalizations to the existing distributions to fill in this gap, however, this served as room for us to propose a new distribution called the Odd Generalized Exponential-Pareto Distribution (OGEPD) using a probability distribution generator introduced by Tahir *et al.,* (2015) known as the Odd Generalized Exponential (OGE) family of distribution.

1.5 Motivation

Recently Tahir *et al.,* (2015) introduced some special distributions that can be obtained from their generator which include the Odd Generalized Exponential Weibull (OGE-W), the Odd Generalized Exponential Frechet (OGE-fr) and the Odd Generalized Exponential Normal (OGE-N) distributions. Also, El-Damcese (2015) used this same generator to introduce a new Odd Generalized Exponential Gompertz (OGE-G) distribution. Tahir *et al.,*(2015) highlighted that the OGE family generates distributions whose shapes are symmetrical, left-skewed, right-skewed and reversed-J. Also the kurtosis is more flexible. Due to this fact, the OGE family can be very essential to fit real dataset under various shapes.

In view of the foregoing, we were motivated to study OGEPD based on the information available to us, that no any study was conducted on it since the introduction of the generator by Tahir *et al.,*(2015). More so, the available literature studied earlier highlighted that almost all generalized distributions (in which one or more parameters were added) performed well and have better presentation of data than their counterparts with less number of parameters.

1.6 Significance of the Study

The main importance of this research is to increase the flexibility of the Pareto distribution, there by making it an appropriate model for modelling data of various shapes that cannot be adequately fitted with the conventional probability distributions. Also the proposed distribution can be used in reliability analysis as well as in analyzing skewed datasets. Comparison between the proposed distribution and the existing distributions by using a real-life data will identify a better distribution.

1.7 Aim and Objectives of the Study

The aim of this research is to propose and study an Odd Generalized Exponential Pareto Distribution (OGEPD), its properties and as well application to real datasets.

In order to achieve the stated aim, the following objectives shall be attained; by

i. deriving the cumulative density function (CDF) as well as the probability density function (pdf) of the proposed distribution;

ii. studying some of the statistical properties of the new distribution comprising limit of the pdf, moments, moment generating function, quantile function, order statistics and reliability analysis;
iii. estimating the parameters of the new distribution using method of Maximum likelihood;
iv. comparing the performance of the OGEPD with other related distributions using real datasets.

1.8 Scope of the Study

The study is focused only on extending research on Pareto distribution, by proposing a new four-parameter distribution (OGEPD) and then deriving its pdf, CDF, the asymptotic behaviour, survival function, hazard function, quantile function (which is useful for calculating the median, skewness as well as kurtosis and for generation of random number), density functions for the minimum and maximum order statistics, moment generating function. The parameters of the distribution are estimated using the method of Maximum Likelihood Estimation (MLE) procedure.

1.9 Terminologies

1.9.1 Probability distribution

A probability distribution or model is a mathematical description that approximately agrees with the frequencies or probabilities of possible events of a random variable. Probability distribution for a random variable describes how the probabilities are distributed over the values of a random variable.

1.9.2 Distribution Function

This is a function that gives the total probability of a random variable X which extend over all values within its ranges that are less than or equal to x. It is otherwise known as cumulative distribution function (CDF). If X is a continuous random variable with pdf $f(x)$, then the distribution function is defined as;

$$F(x) = p(X \leq x) = \int_{-\infty}^{x} f(t)\, dt \quad for \quad -\infty < x < \infty \tag{1.9.1}$$

The following properties must be satisfy for any continuous distribution function, properties (2) , (3) and (4) uniquely characterized the CDF and violating any one or more of these three properties regard the cdf invalid.

1. $0 \leq F(x) \leq 1 \qquad -\infty < x < \infty$

2. $F'(x) = \frac{dF(x)}{dx} = f(x) \geq 0 \Rightarrow F(x)$ is non-decreasing function of x

3. $F(-\infty) = \lim\limits_{x \to -\infty} F(x) = \lim\limits_{x \to -\infty} \int_{-\infty}^{x} f(x)\, dx = \int_{-\infty}^{-\infty} f(x)\, dx = 0$ and $F(+\infty) = \lim\limits_{x \to \infty} F(x) = \lim\limits_{x \to \infty} \int_{-\infty}^{x} f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx = 1$

4. $F(x)$ is a continuous function of x on the right.

5. The discontinuity of $F(x)$ are at the most countable.

6. $P(a \leq X < b) = \int_{a}^{b} f(x)\, dx = \int_{-\infty}^{b} f(x)\, dx - \int_{-\infty}^{a} f(x)\, dx = F(b) - F(a)$

1.9.3 The Probability Density Function

The probability density function or simply density function is an opposite of distribution function. For any continuous random variable X, the Probability Density

Function (pdf) denoted by $f(x)$ is defined as follows;

$$f(x) = F'(x) = \frac{dF(x)}{dx} \tag{1.9.2}$$

with the following properties

1. $f(x) \geq 0 \qquad -\infty < x < \infty$

2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

3. For any event A, the probability $P(A)$is given by

$$P(A) = \int_E f(x)\,dx$$

1.9.4 Moment

Moment plays a vital role in statistics, most especially in applications were it can be used to derive the most important features and characteristics of a distribution such as measures of location, spread, skewness and kurtosis.
For a continuous random variable X with pdf $f(x)$, the r^{th} moment about the origin can be defined as:

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx \tag{1.9.3}$$

1.9.5 Moment Generating Function

The process or technique used in obtaining or generating all the moments of a probability distribution into one mathematical function is called the moment generating function. The moment generating function (mgf) of a continuous random variable X having the pdf $(f(x))$ can be obtained as;

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \tag{1.9.4}$$

where t is a real parameter and it is being assumed that the right-hand side of (1.9.4) is absolutely convergent for some positive numbers h such that $-h < t < h$ .Thus

$$M_X(t) = E(e^{tX}) = E\left[1 + tX + \frac{t^2X^2}{2!} + \ldots + \frac{t^rX^r}{r!} + \ldots\right] \tag{1.9.5}$$

In other words, the mgf generates the moments of X by differentiation i.e., for any real number say r, the r^{th} derivative of $M_X(t)$ evaluated at $t = 0$ is the r^{th} moment μ_r' of X . And its only exist if the integral in equation (1.9.4) converges

1.9.6 Order Statistics

Order Statistics are used in detection of outliers, robust statistical estimation, and characterization of probability distributions, goodness of fit tests, entropy estimation, and analysis of censored samples, reliability analysis, quality control and strength of materials. It can also be used in determining the distribution of the smallest (minimum) order statistic X_1 and largest (maximum) order statistic X_n of distribution respectively.

Definition; let $X_1, X_2, ..., X_n$ be n random variables from a continuous distribution with pdf $f(x)$ and CDF $F(x)$. Then the *pdf* $f_{i:n}(x)$ of the i^{th} order statistic is defined as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) \left[F(x)\right]^{i-1} \left[1 - F(x)\right]^{n-i} \tag{1.9.6}$$

1.9.7 Quantile Function

Quantile function is used for finding the median, skewness, kurtosis and for simulation of random numbers.

Definition; let $Q(u) = F^{-1}(u)$ be the Quantile Function (qf) of CDF $F(x)$ for $0 < u$

< 1. By solving $F(x) = u$. The qf of X is given as;

$$x = Q(u) = F^{-1}(u) \tag{1.9.7}$$

1.9.8 Scale Parameter

Define how spreads out the data are. A large scale value stretches the distribution, while a smaller scale value shrinks the distribution.

1.9.9 Shape Parameter

Define how data are distributed but does not affect the location or scale of a distribution. A large shape value gives a left-skewed curve, whereas a small shape values gives a right-skewed curve.

1.9.10 Reliability Analysis

Survival or reliability analysis is the process of modelling time-to-event information, also known as transition data (or survival time data or duration data). It is also seen as a statistical techniques used to describe and quantify time-to-event data. This analysis was originally developed or introduced for the purpose of evaluating the treatment efficacy of fatal condition like cancer. But also used in many other situation such as time for hand fracture to heal, excessive breast feeding and time to another pregnancy, time to exercise to maximum tolerance, time to breakdown for a machine, survival of patient after surgery, length of stay in hospital and so on. In survival analysis, we are interested in the time interval between entry into the study and an event. The outcome of interest is time-to-event. It has applicability in many areas of human endeavor including engineering, medicine, sciences, industry and etc.

1.9.11 Survival Time

This shows the length of time taken for failure to occur. The survival curve can be used to study times required to reach any well-defined endpoint.

1.9.12 Survival Function

Survival function $S(t)$ is defined as the proportion of the population that has survived to time t or the probability that a system will survive beyond a given time. However, $S(t)$ can be plotted as a function of time to produce a survival curve, and the area $S(t)$ under the curve to the right of time t will indicate the proportion of individuals in the population who have survived to time t. At time t equals to zero there will be no failure so $S(t)$ will be equals to one. Mathematically, the survival function is defined as:

$$S(t) = 1 - F(t) \tag{1.9.8}$$

where $F(t)$ is the failure function (CDF) indicating the cumulative proportion of the population that has died up to time t.

1.9.13 Hazard Function

Hazard function $h(t)$ is also known as conditional failure rate or instantaneous hazard and is defined as the instantaneous rate at which a randomly-selected individual known to be alive at time (t-1) will die at time t. In order word it is the probability that a system or individual will fail or die for an interval of time. Its gives the proportion of the population present at time t that fails per unit time. The hazard function is defined mathematically as;

$$h(t) = \frac{f(t)}{1 - F(t)} \tag{1.9.9}$$

where $f(t)$ and $F(t)$ are the instantaneous failure rate (pdf) and failure function (CDF), respectively of any baseline distribution when the variable under consideration is the length of time taken for an event to occur e.g. death.

1.9.14 Maximum Likelihood Estimation

The method of maximum likelihood estimation was introduced by R.A. Fisher in 1922. It is the method that allows us to estimate the parameter that maximizes the likelihood function (joint probability density function). Definition; Let $X_1, X_2, ..., X_n$ be n random variables from a population with sample values $x_1, x_2, ..., x_n$ having joint probability density function as $f(x_1, x_2, ..., x_n; \Theta)$ where Θ is an unknown parameter. Then the likelihood function$L(\Theta)$, of the random samples is defined as:

$$L(\Theta; x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n; \Theta) = L(\Theta) = \prod_{i=1}^{n} f(x_i, \Theta) \qquad (1.9.10)$$

The sample statistic that maximizes the likelihood function $L(\Theta)$ is called the maximum likelihood estimator of Θ and is denoted by $\widehat{\Theta}$.

CHAPTER TWO

LITERATURE REVIEW

The Pareto distribution is a well-known distribution used in analyzing and modelling skewed data in different field of human endeavor. It has been studied by different researchers with the aim of obtaining a better distribution. The first researcher that generalize or extend the Pareto distribution was Pickand in 1975, were he proposed a distribution called the Generalized Pareto Distribution (GPD). Some of the extended Pareto distributions are the Generalized Pareto distribution (Pickand, 1975), Exponentiated Pareto distribution (Gupter *et al.,* 1998), Beta Pareto distribution (Akinsete *et al.,* 2008), Beta Generalized Pareto distribution (Mahmoudi, 2011), Kumaraswamy Pareto distribution (Bourguignon *et al.,* 2012), Gamma Pareto distribution and its applications (Alzaatreh *et al.,* 2012), Weibull Pareto distribution (Alzaatreh *et al.,* 2013), Exponential Pareto distribution (Kareem *et al.,* 2013), Transmuted Pareto distribution (Merovci *et al.,* 2014), New Weibull Pareto distribution (Tahir *et al.,* 2014), Exponentiated Weibull Pareto distribution (Afify *et al.,* 2016), Odds Generalized Exponential Pareto distribution (Maiti and Pramanik, 2016), etc.

Akinsete *et al.,* (2008) introduced a four-parameter continuous distribution known as Beta-Pareto distribution (BPD), generated from the logit of a beta random variable. The study highlighted that the Pareto distribution is mainly used in modeling the heavy-tailed distributions that include data on income distribution, city population size and size of firms. Some other quantities measured in the physical, biological, technological and social systems of various kinds have been found to follow the Pareto distribution. The CDF as well as the pdf were well defined, also some statistical properties were determined including the moments, mean, mean deviation from the mean, the mean deviation from the median, variances, the Renyi entropy and the Shannon entropy . They discover that when $\alpha = \beta = 1$ the BPD reduces to the Pareto distribution with parameter k. The method of maximum likelihood was employed to estimate the parameters.

Shawky *et al.,* (2009) studied the Exponentiated Pareto distribution with the aim of comparing different methods of estimations. The different methods of estimations used are maximum likelihood (MLE's), method of moment (MME's), percentile-based estimation (PCE's), least square (LSE's) and weighted least square (WLSE's). They presented results of some numerical experiments to compare the performance of different estimators proposed mainly with respect to their biases and root mean squared errors (RMSE's) for different sample sizes and for different parameter values. When comparing the performance of all the estimators, it was observed that, as far as biases or RMSE's are concerned, the MLE performed better in almost all the cases.

Mierlus-Mazilu (2010) studied the GPD and the set of algorithms for numerical simulation of the distribution and presented the histogram of the generated data. The results revealed that GPD has applications in a number of fields including reliability studies in the modelling of large insurance claims as a failure time distribution. Also it plays an important role in modelling extreme events. A model is frequently used in the study of income distribution and in the analysis of extreme events e.g. for the analysis of precipitation data in the flood frequency analysis, in analysis of data of the greatest wave heights or sea levels, maximum winds loads on buildings, in the maximum rain fall analysis, in analysis of the greatest values of yearly floods, breaking strength of materials, aircraft loads, etc . The GPD has been quite popular not only for flood frequency analysis but for fitting the distribution of extreme natural events in general.

Mahmoudi (2011) introduced the Beta Generalized Pareto Distribution (BGPD) which served as an extension of the well-known GPD applied in modeling extreme value data due to its long tail feature. The study highlighted that the BGPD is more flexible and has some interesting properties. The probability density, cumulative distribution and hazard rate function were derived. The mathematical properties of the BGPD were discussed. Fisher's information matrix was also derived. However, the parameters were estimated using the method of maximum likelihood. Lastly, the new model was fitted to three real datasets and compared the result were compared with other models. The result showed that the BGPD provides a better fit than those it was compared with.

Bourguignon *et al.,* (2012) studied a new distribution called the Kumaraswany Pareto distribution. The study defined and expressed the pdf of the distribution as an infinite linear combination of the Pareto densities which allowed them to derive some of its mathematical properties like the moments, generating function, density function of the order statistics and obtained their moments. The method of maximum likelihood was used for estimating the model parameters and the observed information matrix was derived. An application to a real dataset shows that the fit of the new model is greater than the fit of its main sub-model.

Alzaatreh *et al.,* (2012) studied a special case of the gamma-X family and proposed the gamma-Pareto distribution. Different properties of the distribution were examined such as moments, mean deviations and median, hazard function, uni-modality, entropies and Fisher information matrix. Results of the uniformly minimum variance unbiased estimator was obtained for one of the shape parameters of the distribution. Three real datasets were fitted to the gamma-Pareto distribution and compared with other known distributions. The results showed that the gamma-Pareto distribution provides a good fit to each data set and they suggested that the gamma-Pareto distribution can be a good model to fit data with a reversed J-shape, approximately symmetric and long right tail characteristics.

Kareema *et al.,* (2013) introduced a new distribution called Exponential Pareto distribution and derived some statistical properties including; the moment generated function, mean, mode, median , variance , the rth moment about the mean, the rth moment about the origin, reliability, hazard functions, coefficient of variation, skewness and kurtosis. Finally, the parameters of the distribution were estimated using the method of maximum likelihood estimation.

Shams (2013) developed a new generalization of the Pareto distribution called the kumaraswamy generalized exponentiated Pareto distribution. The researcher defined and expressed the pdf of the new distribution as an infinite linear combination of Pareto densities. Thus, some of its statistical properties can be derived directly from those properties of the Exponentiated Pareto distribution. Some properties of the distribution were studied. The method of maximum likelihood was used for estimating the model parameters and the observed information matrix was derived. A real data set was used to compare the fits of the Kum-GEP distribution and those of other sub-models. The result reveals a better fit compared to the principals sub-models.

Merovci *et al.,* (2014) proposed a new model called the transmuted Pareto distribution which extends the Pareto distribution in the analysis of data with real support. One of the reason for extending standard distribution is due to the fact that the extended form provides greater flexibility in analysing real life data. They derived expansions for the expectation, variance, moments and the moment generating function. The estimation of parameters was approached by the method of maximum likelihood and the information matrix was derived. They considered the likelihood ratio statistics to compare the model with its baseline model. An application of the transmuted Pareto distribution to real data showed that the new distribution provides better fit than the Pareto distribution.

James (2015) studied the Bayesian inference of the Weibull-Pareto distribution. A desirable property of the Weibull-Pareto distribution is its capability to analysed skewed data. This is applicable for proposing models in human longevity and actuarial science. In their study, a hierarchical Bayesian model was developed using the Weibull-Pareto distribution. He explored how Bayes estimates performed for the parameters of the Weibull-Pareto Distribution through simulations.

Tahir *et al.,* (2015) recently introduced a new family of distribution named as Old Generalized Exponential (OGE) family. The research proposed and studied three special models in this family. The models include the Odd Generalized Exponential Weibull (OGE – W) distribution, the Odd Generalized Exponential Frechet (OGE – Fr) distribution and the Odd Generalized Exponential Normal (OGE – N) distribution. The OGE family served as a special case of the widely known exponential Weibull distribution. Its hazards rate could be increasing, decreasing, J, reversed-J, bath tub and upside down bath tub. They also highlighted that their motivation for using the OGE family are to make the kurtosis more flexible compared to the base line distribution and possible to construct heavy tailed for modelling real data.

However, the authors studied the structural properties of the family such as expressing the density function as an exponentiated-G density function by means of expansion, derived the ordinary and incomplete moment, quantile and generating functions, mean deviation, Bonferroni and Lorenz curves, Shannon and Renyi entropies and order statistics of the new distribution. They also estimated the parameters using maximum likelihood method as well as comparing the performance of the new models to the existing sub models using two real dataset and concluded that the new models fit better in all the cases.

El-Damcase *et al.,* (2015) proposed a four parameter continuous distribution called the Odd Generalized Gxponential Gompertz Distribution (OGEGD) by applying the OGE family proposed by Tahir *et al.,* (2015). The researchers extended or generalized the Gompertz distribution. The structural properties including moment, mean, quantile, median, order statistic and fisher information matrix of the distribution was studied. The method of maximum likelihood was used in estimating the parameters and its usefulness by applying to real data, and the result showed that the model performed well or better than its sub-models.

Afify *et al.,* (2016) generalized the Weibull- Pareto Distribution (WPD) to a new four-parameter continuous distribution called the Exponentiated Weibull Pareto

Distribution (EWPD). According to them, the main purpose for generalizing a distribution is the fact that the generalization provides more flexibility to analyze real life data. Some mathematical properties of the distribution were studied. Explicit expressions for moments, generating function, Renyi entropies were also derived. The density function of the order statistic and their moments were also obtained. Maximum likelihood estimation was also discussed. The proposed distribution was applied to a real data set and discovered that it performed better than the other models.

Maiti and Pramanik (2016) proposed a three- parameter probability distribution known as Odds Generalised Exponential Pareto Distribution by applying the generator proposed by Alzaatreh *et al.,* (2013) called T-X family. The study obtained the pdf and CDF of the new distribution and as well studied the reliability properties. Some mathematical properties such as the descriptive statistics, moments, moment generating function, characteristic function, cumulant generating function, Quantile function, Bonferroni curve, Lorenz curve, Order statistics and Entropy were derived. The parameters were estimated using the method of maximum likelihood. The dataset has been analysed to illustrate its usefulness and applicability.

With this understanding, this research seeks to increase the flexibility of the Pareto distribution using the odd generalized exponential family of distribution proposed by Tahir *et al.,* (2015). Hence, this study proposed a Four-Parameter Odd Generalized Exponential-Pareto Distribution based on the probability generator proposed by Tahir *et al.,* (2015).

CHAPTER THREE

RESEARCH METHODOLOGY

3.1 Definition of the Odd Generalized Exponential-Pareto Distribution (OGEPD)

According to Tahir *et al.,* (2015), the OGE family is defined as follows; if the baseline distribution depends on a parameter vector ϕ having CDF, pdf and survival functions denoted by $G(x;\phi)$, $g(x;\phi)$ and $\overline{G} = 1 - G(x;\phi)$ respectively, then the CDF of the OGE family is defined by replacing x in equation (1.3.1) with $\frac{G(x;\phi)}{\overline{G}(x;\phi)}$.

we obtained;

$$F(x;\alpha,\lambda,\phi) = \left(1 - e^{-\lambda\frac{G(x;\phi)}{\overline{G}(x;\phi)}}\right)^{\alpha} \tag{3.1.1}$$

where $\alpha > 0, \lambda > 0$ are two additional parameters.

To obtain the corresponding pdf, we differentiate(3.1.1) with respect to x, and making use of the fact that;

$$f(x;\alpha,\lambda,\phi) = \frac{dF(x;\alpha,\lambda,\phi)}{dx} \tag{3.1.2}$$

now, let

$$y = F(x;\alpha,\lambda,\phi) = \left(1 - e^{-\lambda\frac{G(x;\phi)}{\overline{G}(x;\phi)}}\right)^{\alpha}$$

$$u = 1 - e^{-\lambda\frac{G(x;\phi)}{\overline{G}(x;\phi)}}$$

$$t = \lambda\frac{G(x;\phi)}{\overline{G}(x;\phi)}$$

this implies

$$y = u^{\alpha}$$

$$u = 1 - e^{-t}$$

then equation (3.1.2) can be written as;

$$f(x;\alpha,\lambda,\phi) = \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dt}.\frac{dt}{dx} \tag{3.1.3}$$

then

$$\frac{dy}{du} = \alpha u^{\alpha-1}$$

$$\frac{du}{dt} = e^{-t}$$

$$\frac{dt}{dx} = \lambda\left[\frac{\overline{G}(x;\phi)g(x;\phi) - G(x;\phi)(-g(x;\phi))}{\left[\overline{G}(x;\phi)\right]^2}\right]$$

$$= \lambda\left[\frac{(1 - G(x;\phi))g(x;\phi) + G(x;\phi)g(x;\phi)}{\left[\overline{G}(x;\phi)\right]^2}\right]$$

$$= \frac{\lambda(g(x;\phi)}{\left[\overline{G}(x;\phi)\right]^2}$$

$$\Rightarrow f(x;\alpha,\lambda,\phi) = \frac{dy}{dx} = \alpha\lambda u^{\alpha-1}e^{-t}\frac{g(x;\phi)}{\overline{G}(x;\phi)} \tag{3.1.4}$$

substituting the value of u and t in equation (3.1.4) we obtained the pdf as;

$$f(x;\alpha,\lambda,\phi) = \frac{\lambda\alpha g(x;\phi)}{\overline{G}\,(x;\phi)^2}e^{-\lambda\frac{G(x;\phi)}{\overline{G}(x;\phi)}}\left(1 - e^{-\lambda\frac{G(x;\phi)}{\overline{G}(x;\phi)}}\right)^{\alpha-1} \tag{3.1.5}$$

where $g\,(x;\phi)$, $G(x;\phi)$ and $\overline{G}(x;\phi)$ are the respective pdf, CDF and survival functions of the baseline distribution. Now equation (3.1.1) and (3.1.5) are the CDF and pdf of the OGE family of distribution, respectively.

To obtained the CDF and pdf of OGEPD, we now substitute the CDF (1.2.2) and pdf (1.2.3) of the baseline (Pareto) distribution in the CDF (3.1.1) and pdf (3.1.5) of the Odd Generalized Exponential Family of distribution. We obtained the CDF and pdf of the proposed distribution (OGEPD) as follows;

$$F(x;\alpha,\lambda,\theta,k) = \left(1 - e^{-\lambda \frac{\left(1-\left(\frac{\theta}{x}\right)^{k}\right)}{1-\left(1-\left(\frac{\theta}{x}\right)^{k}\right)}}\right)^{\alpha} = \left(1 - e^{\left(-\lambda\left(1-\left(\frac{\theta}{x}\right)^{k}\right)\left(\frac{\theta}{x}\right)^{-k}\right)}\right)^{\alpha} \tag{3.1.6}$$

by simplifying equation (3.1.6), it gave;

$$F(x;\alpha,\lambda,\theta,k) = \left(1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha} \quad for \;\; x \geq \theta; \alpha,\lambda,\theta,k > 0 \tag{3.1.7}$$

the corresponding pdf to (3.1.7) is;

$$f(x;\alpha,\lambda,\theta,k) = \frac{\left(\frac{\alpha\lambda k\theta^{k}}{x^{k+1}}\right)}{\left(1-\left(1-\left(\frac{\theta}{x}\right)^{k}\right)\right)^{2}} e^{-\lambda \frac{\left(1-\left(\frac{\theta}{x}\right)^{k}\right)}{1-\left(1-\left(\frac{\theta}{x}\right)^{k}\right)}} \left(1 - e^{-\lambda \frac{\left(1-\left(\frac{\theta}{x}\right)^{k}\right)}{1-\left(1-\left(\frac{\theta}{x}\right)^{k}\right)}}\right)^{\alpha-1} \tag{3.1.8}$$

by simplifying equation (3.1.8), we have

$$f(x;\alpha,\lambda,\theta,k) = \frac{\left(\frac{\alpha\lambda k\theta^{k}}{x^{k+1}}\right)}{\left(\frac{\theta}{x}\right)^{2k}} e^{-\lambda\left(\left(1-\left(\frac{\theta}{x}\right)^{k}\right)\left(\frac{\theta}{x}\right)^{-k}\right)} \left(1 - e^{-\lambda\left(\left(1-\left(\frac{\theta}{x}\right)^{k}\right)\left(\frac{\theta}{x}\right)^{-k}\right)}\right)^{\alpha-1} \tag{3.1.9}$$

$$f(x;\alpha,\lambda,\theta,k) = \alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} \left(1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1} \quad for \;\; x \geq \theta; \alpha,\lambda,\theta,k > 0 \tag{3.1.10}$$

Now equation (3.1.7) and (3.1.10) are the respective CDF and pdf of the new distribution (OGEPD).

The plot of the CDF and pdf for different values of the parameters is provided in figure (3.1.1) and (3.1.2) respectively, where $a = \alpha$, $b = \lambda$, $c = \theta$ and $d = k$.

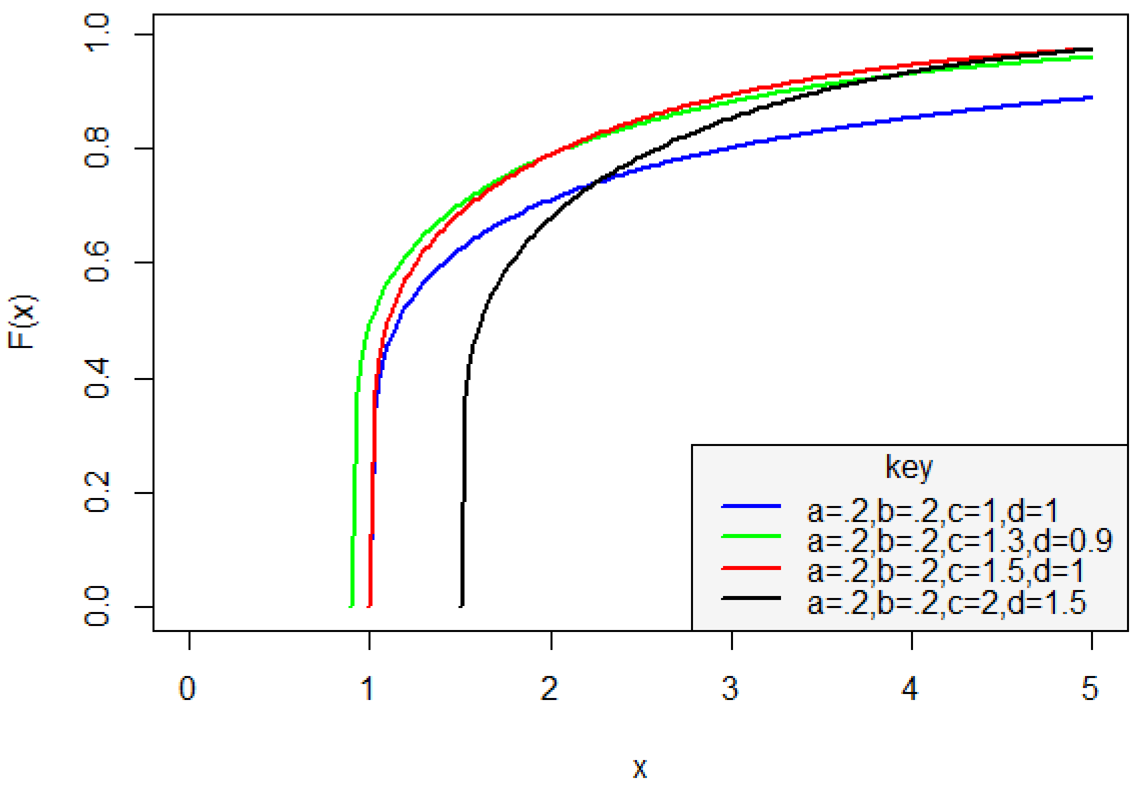

Figure 3.1.1: CDF plot of OGEPD

3.1.1 Model Validity Check

To show whether the pdf (3.1.10) is a valid pdf , it must satisfied the fact that;

$$\int_{\theta}^{\infty} f(x;\alpha,\lambda,\theta,k)dx = 1 \tag{3.1.11}$$

The proof

$$\int_{\theta}^{\infty} f(x;\alpha,\lambda,\theta,k)dx = \int_{\theta}^{\infty} \alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1}dx \tag{3.1.12}$$

let

$$y = \left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1}$$

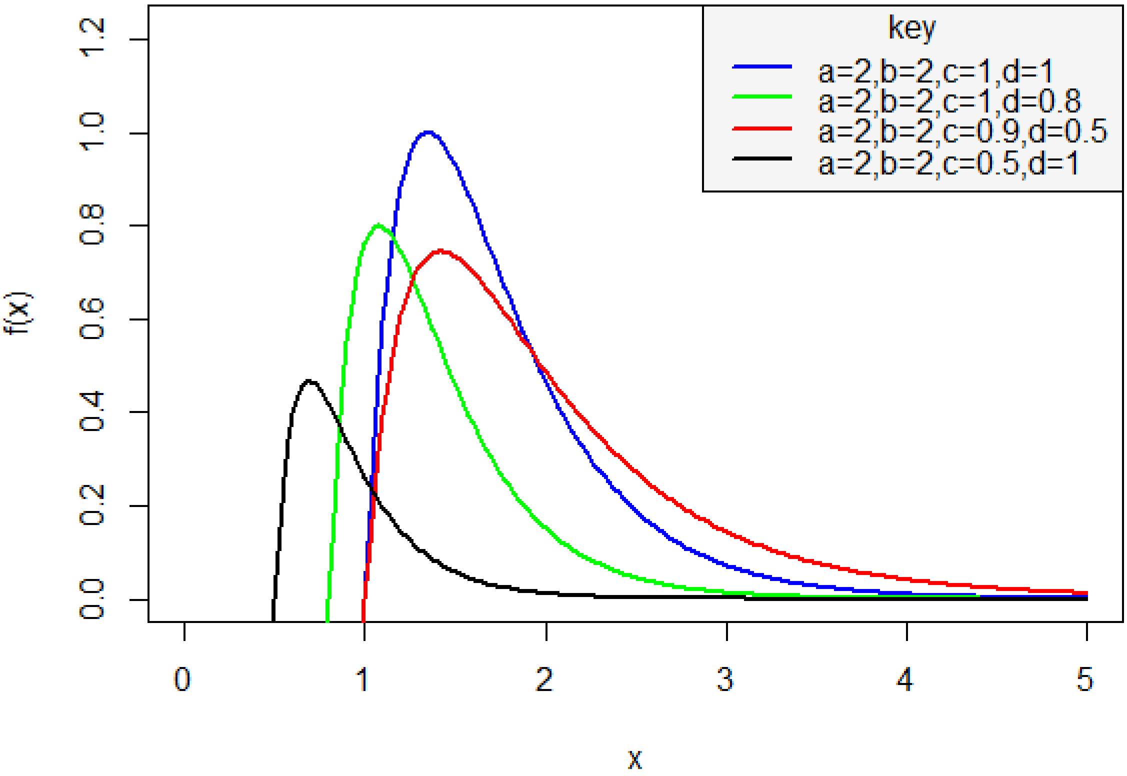

Figure 3.1.2: pdf plot of OGEPD

$$u = 1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1\right)}$$

and

$$t = \lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1\right)$$

Implies,

$$y = u^{\alpha-1}$$

$$u = 1 - e^{-t}$$

then

$$\frac{dy}{du} = (\alpha - 1)u^{\alpha-2}$$

$$\frac{du}{dt} = e^{-t}$$

$$\frac{dt}{dx} = k\lambda\theta^{-k}x^{k-1}$$

using chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dt}.\frac{dt}{dx} = k(\alpha-1)\lambda\theta^{-k}x^{k-1}e^{-t}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-2}$$

then

$$dx = \frac{\theta^k dy}{k(\alpha-1)\lambda x^{k-1}e^{-t}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-2}} \tag{3.1.13}$$

substituting (3.1.13) in (3.1.12), we write

$$\begin{aligned}&= \int_\theta^\infty \alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1} \times \\ &\frac{\theta^k dy}{k(\alpha-1)\lambda x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-2}}\end{aligned} \tag{3.1.14}$$

$$\int_\theta^\infty f(x;\alpha,\lambda,\theta,k)dx = \int_0^\infty \tfrac{\alpha}{\alpha-1}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)dy \tag{3.1.15}$$

recall that

$$y = \left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1}$$

$$\Rightarrow y^{\frac{1}{\alpha-1}} = \left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)$$

Putting the value of y in (3.1.15), we write

$$\int_\theta^\infty f(x;\alpha,\lambda,\theta,k)dx = \tfrac{\alpha}{\alpha-1}\int_\theta^\infty y^{\frac{1}{\alpha-1}}dy$$

$$= \tfrac{\alpha}{\alpha-1}\left[\frac{y^{\frac{1}{\alpha-1}+1}}{\frac{1}{\alpha-1}+1}\right]_\theta^\infty = \left[y^{\frac{1}{\alpha-1}+1}\right]_\theta^\infty$$

$$= \left[\left(\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1}\right)^{\frac{\alpha}{\alpha-1}}\right]_\theta^\infty$$

$$= \left[\left(\left(1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1 \right)} \right)^{\alpha - 1} \right)^{\frac{\alpha}{\alpha - 1}} \right]_{\theta}^{\infty}$$

$$= \left[\left(1 - e^{-\lambda\left(\left(\frac{x}{\theta}\right)^{k} - 1 \right)} \right)^{\alpha} \right]_{\theta}^{\infty}$$

$$= \left[\left(1 - e^{-\lambda\left(\left(\frac{\infty}{\theta}\right)^{k} - 1 \right)} \right)^{\alpha} - \left(1 - e^{-\lambda\left(\left(\frac{\theta}{\theta}\right)^{k} - 1 \right)} \right)^{\alpha} \right]$$

$$\int_{\theta}^{\infty} f(x, \alpha, \lambda, \theta, k) dx = [(1 - 0) - (1 - 1)] = 1 \tag{3.1.16}$$

This shows that the proposed distribution (OGEPD) is a valid probability distribution.

3.1.2 Some Useful Expansions for the CDF and pdf of OGEPD

Here, we provide simple expansions for the CDF and pdf of OGEPD by making use of the concept of generalized Binomial expansion and power series expansion. This will aid in deriving some of the properties of the proposed distribution.

Consider the CDF (3.1.7) given as;

$$F(x; \alpha, \lambda, \theta, k) = \left(1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1 \right)} \right)^{\alpha}$$

using Binomial expansion for the term $\left(1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1 \right)} \right)^{\alpha}$, we obtain

$$F(x, \alpha, \lambda, \theta, k) = \sum_{i=0}^{\infty} \binom{\alpha}{i} (-1)^{i} e^{-i\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1 \right)} \tag{3.1.17}$$

using Maclaurin's series expansion for the exponential, we have;

$$e^{-i\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1 \right)} = \sum_{j=0}^{\infty} \frac{(-1)^{j} \left(i\lambda \left(\left(\frac{\theta}{x}\right)^{-k} - 1 \right) \right)^{j}}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^{j} (i)^{j} (\lambda)^{j} \left(\left(\frac{\theta}{x}\right)^{-k} - 1 \right)^{j}}{j!} \tag{3.1.18}$$

substituting (3.1.18) in to (3.1.17) we obtained;

$$F(x;\alpha,\lambda,\theta,k) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(i\lambda)^j \binom{\alpha}{i}\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j}{j!} \tag{3.1.19}$$

Now consider the pdf (3.1.10) given as;

$$f(x;\alpha,\lambda,\theta,k) = \alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1} \tag{3.1.20}$$

Since we have $0 < 1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} < 1$, then using binomial expansion we obtained;

$$\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1} = \sum_{i=0}^{\infty}\begin{pmatrix}\alpha-1\\ i\end{pmatrix}(-1)^i e^{-i\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} \tag{3.1.21}$$

Putting (3.1.21) in to (3.1.20) we have;

$$f(x;\alpha,\lambda,\theta,k) = \alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\sum_{i=0}^{\infty}\begin{pmatrix}\alpha-1\\ i\end{pmatrix}(-1)^i e^{-i\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}$$

$$f(x;\alpha,\lambda,\theta,k) = \alpha\lambda k\theta^{-k}x^{k-1}\sum_{i=0}^{\infty}\begin{pmatrix}\alpha-1\\ i\end{pmatrix}(-1)^i e^{-\lambda(1+i)\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} \tag{3.1.22}$$

using series expansion for the exponential we have;

$$e^{-\lambda(1+i)\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} = \sum_{j=0}^{\infty}\frac{(-1)^j\left(\lambda(1+i)\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)\right)^j}{j!}$$

$$= \sum_{j=0}^{\infty}\frac{(-1)^j(1+i)^j(\lambda)^j\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j}{j!} \tag{3.1.23}$$

Putting (3.1.23) in to (3.1.22) it becomes;

$$f(x;\alpha,\lambda,\theta,k) = \alpha k\theta^{-k}x^{k-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(\lambda)^{1+j}(1+i)^{j}\begin{pmatrix}\alpha-1\\ i\end{pmatrix}\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^{j}}{j!} \tag{3.1.24}$$

Equation (3.1.19) and (3.1.24) are the respective CDF and pdf of the OGEPD in binomial and power series form.

3.2 Statistical Properties

Some statistical properties considered in this section comprises of the moments, moment generating function, limiting behavior, quantile function, distribution of order statistics, reliability, etc.

3.2.1 Limit of the CDF and pdf

Lemma I. The limit of the pdf of the OGEPD, $f(x;\alpha,\lambda,\theta,k)$, as $x \to \infty$ and as $x \to \theta$ is *0*. This can be proved by taking the limit of $f(x;\alpha,\lambda,\theta,k)$as $x \to \infty$ and $x \to \theta$.

That is,

$$\lim_{x\to\infty} f(x;\alpha,\lambda,\theta,k) = \lim_{x\to\theta} f(x;\alpha,\lambda,\theta,k) = 0 \tag{3.2.1}$$

The Proof:

$$\lim_{x\to\infty} f(x;\alpha,\lambda,\theta,k) = \lim_{x\to\infty} \alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1} \tag{3.2.2}$$

$$= \alpha\lambda k\theta^{-k} \lim_{x\to\infty} x^{k-1} e^{-\lambda\left(\left(\frac{x}{\theta}\right)^{k}-1\right)} \left(1 - e^{-\lambda\left(\left(\frac{x}{\theta}\right)^{k}-1\right)}\right)^{\alpha-1}$$

$$= \alpha\lambda k\theta^{-k} \left(\infty\right)^{k-1} e^{-\lambda\left(\left(\frac{\infty}{\theta}\right)^{k}-1\right)} \left(1 - e^{-\lambda\left(\left(\frac{\infty}{\theta}\right)^{k}-1\right)}\right)^{\alpha-1}$$

$$= \alpha\lambda k\theta^{-k} \times \infty \times 0 \times (1-0)^{\alpha-1} = 0$$

$$\lim_{x\to\theta} f(x;\alpha,\lambda,\theta,k) = \lim_{x\to\theta} \alpha\lambda k\theta^{-k} x^{k-1} e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} \left(1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1} \tag{3.2.3}$$

$$= \alpha\lambda k\theta^{-k} \lim_{x\to\theta} x^{k-1} e^{-\lambda\left(\left(\frac{x}{\theta}\right)^{k}-1\right)} \left(1 - e^{-\lambda\left(\left(\frac{x}{\theta}\right)^{k}-1\right)}\right)^{\alpha-1}$$

$$= \alpha\lambda k\theta^{-k} \left(\theta\right)^{k-1} e^{-\lambda\left(\left(\frac{\theta}{\theta}\right)^{k}-1\right)} \left(1 - e^{-\lambda\left(\left(\frac{\theta}{\theta}\right)^{k}-1\right)}\right)^{\alpha-1}$$

$$= \alpha\lambda k\theta^{-1} \times 1 \times (1-1)^{\alpha-1}$$

$$= \alpha\lambda k\theta^{-1} \times 1 \times 0 = 0$$

From the proved provided in equation (3.2.2) and (3.2.3) we concluded that the OGEPD have a mode (unimodal).

Lemma II. The limit of the CDF of the OGEPD, $F(x;\alpha,\lambda,\theta,k)$, as $x \to \infty$ is *1* and as $x \to \theta$ is *0*. This can be proved by taking the limit of $F(x;\alpha,\lambda,\theta,k)$ as $x \to \infty$ and $x \to \theta$.

That is,

$$\lim_{x\to\infty} F(x;\alpha,\lambda,\theta,k) = 1 \tag{3.2.4}$$

$$\lim_{x\to\theta} F(x;\alpha,\lambda,\theta,k) = 0 \tag{3.2.5}$$

The Proof:

$$\lim_{x\to\infty} F(x,\alpha,\lambda,\theta,k) = \lim_{x\to\infty} \left(1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha} \tag{3.2.6}$$

$$= \lim_{x\to\theta} \left(1 - e^{-\lambda\left(\left(\frac{x}{\theta}\right)^{k}-1\right)}\right)^{\alpha}$$

$$= \left(1 - e^{-\lambda\left(\left(\frac{\infty}{\theta}\right)^{k}-1\right)}\right)^{\alpha-1}$$

$$= (1 - e^{-\infty})^{\alpha} = (1-0)^{\alpha} = 1$$

$$\lim_{x\to\theta} F(x,\alpha,\lambda,\theta,k) = \lim_{x\to\theta}\left(1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha} \tag{3.2.7}$$

$$= \left(1 - e^{-\lambda\left(\left(\frac{\theta}{\theta}\right)^{k}-1\right)}\right)^{\alpha-1}$$

$$= \left(1 - e^{-0}\right)^{\alpha} = (1-1)^{\alpha} = 0$$

From the proved provided in equation (3.2.6) and (3.2.7) we concluded that the OGEPD is a valid CDF.

3.2.2 The Moments

For a continuous random variable X following *OGEPD*, the r^{th} moment can be defined as;

$$\mu_r' = E(X^r) = \int_{\theta}^{\infty} x^r f(x;\alpha,\lambda,\theta,k)dx \tag{3.2.8}$$

where $f(x;\alpha,\lambda,\theta,k)$ is the pdf of the *OGEPD* in (3.1.2).

$$E(X^r) = \int_{\theta}^{\infty} x^r \alpha k \theta^{-k} x^{k-1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{j+1}(1+i)^j \binom{\alpha-1}{i}\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j}{j!} dx \tag{3.2.9}$$

$$E(X^r) = \alpha k \theta^{-k} \sum_{i=o}^{\infty}\sum_{j=o}^{\infty} \binom{\alpha-1}{i} \frac{(-1)^{i+j}(\lambda)^{j+1}(1+i)^j}{j!} \int_{\theta}^{\infty} x^{k+r-1}\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j dx \tag{3.2.10}$$

But

$$\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j = \sum_{l=0}^{\infty}\binom{j}{l}\left(\frac{\theta}{x}\right)^{-kl}(-1)^{j-l} \tag{3.2.11}$$

substituting (3.2.11) into (3.2.10) we have

$$E(X^r) = \alpha k \theta^{-k} \sum_{i=0}^{\infty}\sum_{j=o}^{\infty}\sum_{l=o}^{\infty} \binom{\alpha-1}{i}\binom{j}{l} \frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!} \int_{\theta}^{\infty} x^{k+r-1}\left(\frac{\theta}{x}\right)^{-kl} dx \tag{3.2.12}$$

further simplification of (3.2.12) yield;

$$E(X^r) = \alpha k\theta^{-k(l+1)} \sum_{i=0}^{\infty}\sum_{j=o}^{\infty}\sum_{l=o}^{\infty} \begin{pmatrix} \alpha-1 \\ i \end{pmatrix} \begin{pmatrix} j \\ l \end{pmatrix} \frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!} \int_{\theta}^{\infty} x^{k+kl+r-1} dx \tag{3.2.13}$$

$$E(X^r) = \alpha k\theta^{-k(l+1)} \sum_{i=0}^{\infty}\sum_{j=o}^{\infty}\sum_{l=o}^{\infty} \begin{pmatrix} \alpha-1 \\ i \end{pmatrix} \begin{pmatrix} j \\ l \end{pmatrix} \frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!} \left[\frac{x^{k+kl+r-1+1}}{k+kl+r-1+1}\right]_{\theta}^{\infty} \tag{3.2.14}$$

$$E(X^r) = \alpha k\theta^{-k(l+1)} \sum_{i=0}^{\infty}\sum_{j=o}^{\infty}\sum_{l=o}^{\infty} \begin{pmatrix} \alpha-1 \\ i \end{pmatrix} \begin{pmatrix} j \\ l \end{pmatrix} \frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!\,(k\,(l+1)+r)} \left[x^{(k(l+1)r)}\right]_{\theta}^{\infty} \tag{3.2.15}$$

$$E(X^r) = \alpha k\theta^{-k(l+1)} \sum_{i=0}^{\infty}\sum_{j=o}^{\infty}\sum_{l=o}^{\infty} \begin{pmatrix} \alpha-1 \\ i \end{pmatrix} \begin{pmatrix} j \\ l \end{pmatrix} \frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!\,(k\,(l+1)+r)} \left[\infty^{k(l+1)+r} - \theta^{k(l+1)+r}\right] \tag{3.2.16}$$

Equation (3.2.16) is the r^{th} moment of OGEPD.

3.2.3 The Moment Generating Function

The moment generating function of any random variable X that follows OGEPD with pdf $f(x;\alpha,\lambda,\theta,k)$ is defined as;

$$M_x(t) = E(e^{tx}) = \int_{\theta}^{\infty} e^{tx} f(x;\alpha,\lambda,\theta,k)dx \qquad \forall x \tag{3.2.17}$$

where $f(x;\alpha,\lambda,\theta,k$ is the pdf (3.1.24)

$$E\left(e^{tx}\right) = \int_{\theta}^{\infty} e^{tx}\alpha k\theta^{-k}x^{k-1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{j+1}(1+i)^{j} \begin{pmatrix} \alpha-1 \\ i \end{pmatrix} \left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^{j}}{j!} dx \tag{3.2.18}$$

$$E\left(e^{tx}\right) = \alpha k\theta^{-k} \sum_{i=o}^{\infty}\sum_{j=o}^{\infty} \begin{pmatrix} \alpha-1 \\ i \end{pmatrix} \frac{(-1)^{i+j}(\lambda)^{j+1}(1+i)^{j}}{j!} \int_{\theta}^{\infty} e^{tx}x^{k-1}\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^{j} dx \tag{3.2.19}$$

using binomial expansion we have;

$$\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^{j}=\sum_{l=0}^{\infty}\begin{pmatrix} j \\ l \end{pmatrix}\left(\frac{\theta}{x}\right)^{-kl}(-1)^{j-l} \tag{3.2.20}$$

substituting (3.2.20) in (3.2.19) we have;

$$E\left(e^{tx}\right)=\alpha k\theta^{-k}\sum_{i=0}^{\infty}\sum_{j=o}^{\infty}\sum_{l=o}^{\infty}\begin{pmatrix} \alpha-1 \\ i \end{pmatrix}\begin{pmatrix} j \\ l \end{pmatrix}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!}\int_{\theta}^{\infty}e^{tx}x^{k-1}\left(\frac{\theta}{x}\right)^{-kl}dx \tag{3.2.21}$$

using Maclaurin's series expansion for e^{tx}

$$e^{tx}=\sum_{m=0}^{\infty}\frac{t^{m}x^{m}}{m!} \tag{3.2.22}$$

substituting (3.2.22) in (3.2.21) we get;

$$E\left(e^{tx}\right)=\alpha k\theta^{-k(l+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=o}^{\infty}\sum_{m=o}^{\infty}\begin{pmatrix} \alpha-1 \\ i \end{pmatrix}\begin{pmatrix} j \\ l \end{pmatrix}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}t^{m}}{j!m!}\int_{\theta}^{\infty}x^{k(l+1)+m-1}dx \tag{3.2.23}$$

$$E\left(e^{tx}\right)=\alpha k\theta^{-k(l+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=o}^{\infty}\sum_{m=o}^{\infty}\begin{pmatrix} \alpha-1 \\ i \end{pmatrix}\begin{pmatrix} j \\ l \end{pmatrix}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}t^{m}}{j!m!}\left[\frac{x^{k(l+1)+m}}{k(l+1)+m}\right]_{\theta}^{\infty} \tag{3.2.24}$$

$$=\alpha k\theta^{-k(l+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=o}^{\infty}\sum_{m=o}^{\infty}\begin{pmatrix} \alpha-1 \\ i \end{pmatrix}\begin{pmatrix} j \\ l \end{pmatrix}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}t^{m}}{j!m!\left(k(l+1)+m\right)}\left[\infty^{k(l+1)+m}-\theta^{k(l+1)+m}\right] \tag{3.2.25}$$

Equation (3.2.25) is the mgf of OGEPD.

3.2.4 Quantile Function

Let $Q(u) = F^{-1}(u)$ be the quantile function (qf) of $F(x)$ for $0 < u < 1$. Solving $F(x; \alpha, \lambda, \theta, k) = u$, the qf of X is;

$$x = Q(u) = F^{-1}(u) \tag{3.2.26}$$

Let

$$F(x, \alpha, \lambda, \theta, k) = \left(1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1\right)}\right)^{\alpha} = u \tag{3.2.27}$$

this implies

$$u^{\frac{1}{\alpha}} = 1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1\right)} \tag{3.2.28}$$

further simplification of (3.2.28) yield;

$$1 - u^{\frac{1}{\alpha}} = e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1\right)} \tag{3.2.29}$$

$$1 - u^{\frac{1}{\alpha}} = \frac{1}{e^{\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1\right)}} \tag{3.2.30}$$

$$1 = \left(1 - u^{\frac{1}{\alpha}}\right) e^{\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1\right)} \tag{3.2.31}$$

divide both sides of (3.2.31) by $\left(1 - u^{\frac{1}{\alpha}}\right)$

$$e^{\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1\right)} = \frac{1}{1 - u^{\frac{1}{\alpha}}} \tag{3.2.32}$$

taking the natural *log* of both sides of (3.2.32), we get

$$\lambda\left(\left(\frac{\theta}{x}\right)^{-k} - 1\right) = \ln\left(\frac{1}{1 - u^{\frac{1}{\alpha}}}\right) \tag{3.2.33}$$

$$\left(\frac{\theta}{x}\right)^{-k} - 1 = \frac{\ln\left(\frac{1}{1 - u^{\frac{1}{\alpha}}}\right)}{\lambda} \tag{3.2.34}$$

$$x^k\theta^{-k} = \frac{\ln\left(\frac{1}{1-u^{\frac{1}{\alpha}}}\right)}{\lambda} + 1 \tag{3.2.35}$$

multiply both sides of (3.2.35) by θ^k and take the k^{th} root gave

$$x = \left[\theta^k\left(\frac{\ln\left(\frac{1}{1-u^{\frac{1}{\alpha}}}\right)+\lambda}{\lambda}\right)\right]^{\frac{1}{k}} \tag{3.2.36}$$

equation (3.2.35) can be written as;

$$x_u = \left[\frac{\theta^k}{\lambda}\left(\lambda + \ln\left(\frac{1}{1-u^{\frac{1}{\alpha}}}\right)\right)\right]^{\frac{1}{k}} \tag{3.2.37}$$

Now , equation (3.2.36) is the quantile function of the OGEPD and setting $u = 0.5$, we obtained the median of the distribution.

That is,

$$x_{0.5} = \left[\frac{\theta^k}{\lambda}\left(\lambda + \ln\left(\frac{1}{1-0.5^{\frac{1}{\alpha}}}\right)\right)\right]^{\frac{1}{k}} \tag{3.2.38}$$

3.2.5 Distribution of Order Statistics

The pdf $f_{i:n}(x)$ of the i^{th} order statistic for a random sample $(x_1, x_{2,}...n_n)$ from the OGEPD can be obtained by:

$$f(i:n) = \frac{n!}{(i-1)!(n-i)!}f(x)\left[F(x)\right]^{i-1}\left[1-F(x)\right]^{n-i} \tag{3.2.39}$$

now equation (3.2.39) can further be simplify using binomial expansion that is,

$$\left[1-F(x)\right]^{n-i} = \sum_{m=0}^{n-i}\begin{pmatrix} n-i \\ m \end{pmatrix}(-1)^m\left[F(x)\right]^m \tag{3.2.40}$$

Putting (3.2.40) in (3.2.39) we obtained;

$$f(i:n)=\sum_{m=0}^{n-i}\begin{pmatrix} n-i \\ m \end{pmatrix}\frac{(-1)^{m}\,n!}{(i-1)!\,(n-i)!}f\left(x\right)\left[F\left(x\right)\right]^{i+m-1} \tag{3.2.41}$$

but,

$$\begin{pmatrix} n-i \\ m \end{pmatrix}=\frac{(n-i)!}{(n-i-m)!m!} \tag{3.2.42}$$

substituting (3.2.42) in (3.2.41), it becomes

$$f(i:n)=\sum_{m=0}^{n-i}\frac{(-1)^{m}\,n!}{(n-i-m)!\,(i-1)!m!}f\left(x\right)\left[F\left(x\right)\right]^{i+m-1} \tag{3.2.43}$$

substituting CDF (3.1.7) and pdf (3.1.10) in (3.2.43) we get;

$$f(i:n)=\sum_{m=0}^{n-i}\frac{(-1)^{m}\,n!}{(n-i-m)!\,(i-1)!m!}\alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}$$
$$\times\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha(i+m-1)} \tag{3.2.44}$$

$$f(i:n)=\sum_{m=0}^{n-i}\frac{(-1)^{m}\,n!}{(n-i-m)!\,(i-1)!m!}\alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha(i+m)-1} \tag{3.2.45}$$

Hence, the pdf (3.2.45) can be used to find the distribution of the smallest (minimum) order statistic X_1 and largest (maximum) order statistic X_n of the proposed distribution respectively.

Substituting i=1 and i=n in (3.2.45) we obtained the pdf of the the smallest (minimum) order statistic X_1 and largest (maximum) order statistic X_n as provided in (3.2.46) and (3.2.47) respectively.

$$f(1:n)=\sum_{m=0}^{n-i}\frac{(-1)^{m}\,n!}{(n-1-m)!m!}\alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha(1+m)-1} \tag{3.2.46}$$

$$f(n:n)=n\alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha n-1} \tag{3.2.47}$$

3.2.6 Reliability Analysis

Survival Function

Survival function is the probability that a system or an individual will survive beyond a given time.

Mathematically, the survival function of OGEPD is given by:

$$S(x;\alpha,\lambda,\theta,k) = 1 - F(x;\alpha,\lambda,\theta,k) \tag{3.2.48}$$

where $F(x;\alpha,\lambda,\theta,k)$ is the CDF (3.1.7).

now substituting (3.1.7) in (3.2.48) we get;

$$S(x;\alpha,\lambda,\theta,k) = 1 - \left(1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha} \quad for \quad x \geq \theta; \alpha,\lambda,\theta,k > 0 \tag{3.2.49}$$

Equation (3.2.49) is the survival function of OGEPD.

The plot of the survival function for different values of the parameters is provided in figure (3.2.1), where $a = \alpha$, $b = \lambda$, $c = \theta$ and $d = k$.

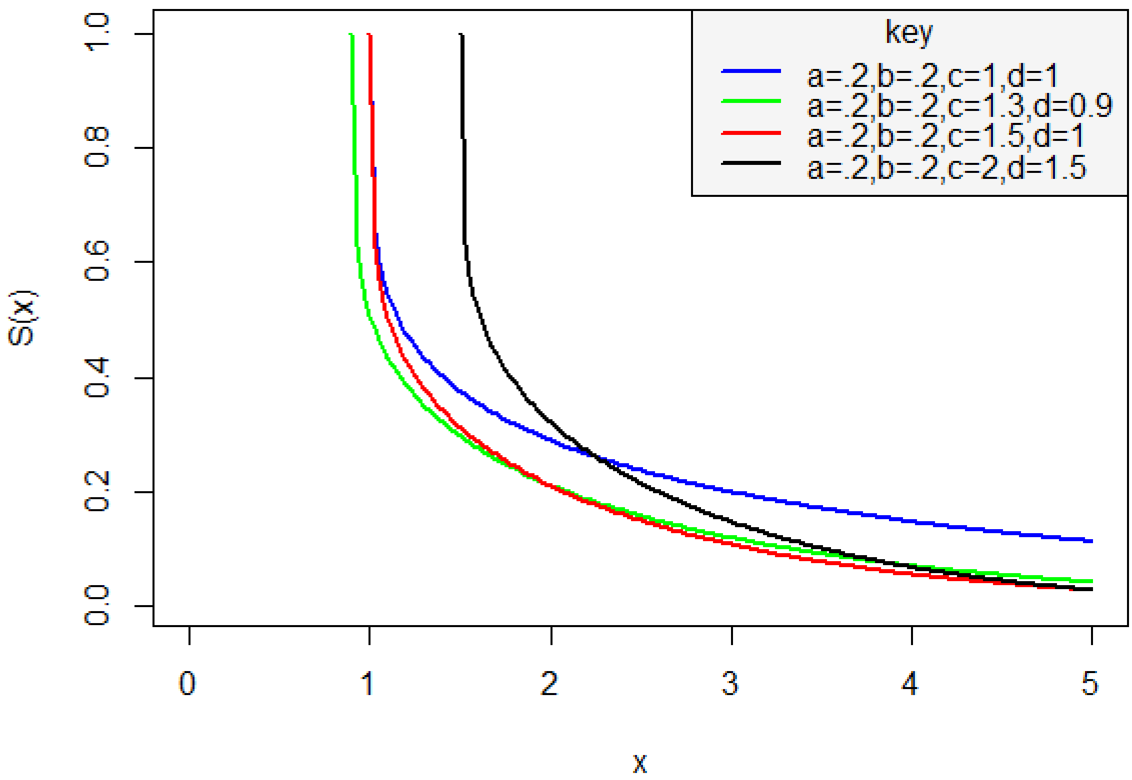

Figure 3.2.1: Survival function of OGEPD

Hazard Function

Hazard function is also known as the failure or risk function and is the probability that a component will fail or die over an interval of time. The hazard function of OGEPD is defined as;

$$H(x;\alpha,\lambda,\theta,k) = \frac{f(x;\alpha,\lambda,\theta,k)}{1 - F(x;\alpha,\lambda,\theta,k)} = \frac{f(x;\alpha,\lambda,\theta,k)}{S(x;\alpha,\lambda,\theta,k)} \tag{3.2.50}$$

where $f(x;\alpha,\lambda,\theta,k)$ and $F(x;\alpha,\lambda,\theta,k)$ are the pdf and CDF, of the proposed distribution respectively.

substituting CDF (3.1.7) and pdf (3.1.10) in (3.2.50) we get

$$H(x;\alpha,\lambda,\theta,k) = \frac{\alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1}}{1-\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha}} \quad for \;\; x \geq \theta, \alpha, \lambda, \theta, k > 0 \tag{3.2.51}$$

Equation (3.2.51) is the hazard function of OGEPD.

The plot of the hazard function for different values of the parameters is provided in figure (3.2.2), where $a = \alpha$, $b = \lambda$, $c = \theta$ and $d = k$.

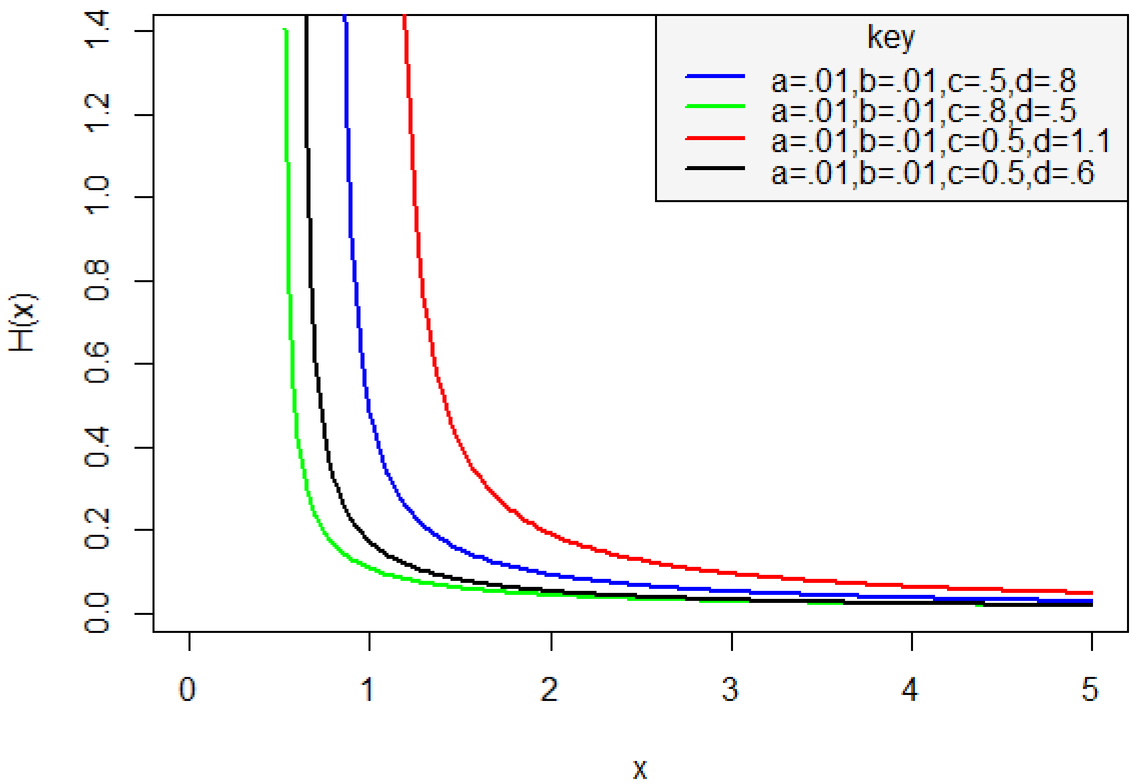

Figure 3.2.2: Hazard plot of OGEPD

3.3 Estimation of Distribution Parameters

The method of maximum likelihood estimation (MLEs) is employed here for estimation of parameters of the OGEPD. Let $X_1, X_2, ..., X_n$ be n random variables from a population that follows OGEPD with sample values $x_1, x_2, ..., x_n$, having joint probability density function as $f(x_1, x_2, ..., x_n; \Theta)$, where $\Theta = (\alpha, \lambda, \theta, k)^T$ is a vector of an unknown parameter. Then the likelihood function $L(\Theta)$, of the random samples is defined as:

$$L(\Theta; x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n; \Theta) = L(\Theta) = \prod_{i=1}^{n} f(x_i, \Theta) \tag{3.3.1}$$

recall that

$$f(x; \alpha, \lambda, \theta, k) = \alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1} \tag{3.3.2}$$

then the likelihood function for (3.3.2) is given by;

$$L(\Theta) = \prod_{i=1}^{n}\left[\alpha\lambda k\theta^{-k}x_i^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\left(1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\right)^{\alpha-1}\right] \tag{3.3.3}$$

$$L(\Theta) = \left(\alpha\lambda k\theta^{-k}\right)^n \prod_{i=1}^{n} x_i^{k-1}e^{\sum_{i=1}^{n} -\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)} \prod_{i=1}^{n}\left(\left(1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\right)^{\alpha-1}\right) \tag{3.3.4}$$

taking the natural logarithm of both side of (3.3.4), we have

$$ll = \log\left[\left(\alpha\lambda k\theta^{-k}\right)^n \prod_{i=1}^{n} x_i^{k-1}e^{\sum_{i=1}^{n} -\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)} \prod_{i=1}^{n}\left(\left(1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\right)^{\alpha-1}\right)\right] \tag{3.3.5}$$

where $\log L(\Theta)$ is denoted by ll.

$$ll = n\log\left(\alpha\lambda k\theta^{-k}\right)(k-1)\left(\log\prod_{i=1}^{n} x_i\right)\left(-\lambda\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)\right)(\alpha-1)\log\left(\prod_{i=1}^{n}\left(1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\right)\right) \tag{3.3.6}$$

$$ll = n\log\alpha + n\log\lambda + n\log k - kn\log\theta + (k-1)\textstyle\sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)$$

$$+ (\alpha - 1)\sum_{i=1}^{n}\log\left(1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}\right) \quad (3.3.7)$$

Equation (3.3.7) is the total log-likelihood function of the OGEPD.

Hence, differentiating (3.3.7) partially with respect to each of the parameter $(\alpha, \lambda, \theta, k)$ and setting the results equal to zero gives the maximum likelihood estimates of the respective parameters. The equations are given by:

$$\frac{\partial ll}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}\right) \quad (3.3.8)$$

$$\frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}\right) = 0 \quad (3.3.9)$$

$$\frac{\partial ll}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right) + (\alpha - 1)\sum_{i=1}^{n}\left(\frac{\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}}{1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}}\right) \quad (3.3.10)$$

$$\frac{n}{\lambda} - \sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right) + (\alpha - 1)\sum_{i=1}^{n}\left(\frac{\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}}{1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}}\right) = 0 \quad (3.3.11)$$

$$\frac{\partial ll}{\partial\theta} = \frac{-kn}{\theta} + k\lambda\theta^{-(k+1)}\textstyle\sum_{i=1}^{n}x_i^k - (\alpha - 1)\sum_{i=1}^{n}\left(\frac{k\lambda\theta^{-(k+1)}x_i^k e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}}{1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}}\right) \quad (3.3.12)$$

$$\frac{-kn}{\theta} + k\lambda\theta^{-(k+1)}\sum_{i=1}^{n}x_i^k - (\alpha - 1)\sum_{i=1}^{n}\left(\frac{k\lambda\theta^{-(k+1)}x_i^k e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}}{1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}}\right) = 0 \quad (3.3.13)$$

$$\frac{\partial ll}{\partial k} = \frac{n}{k} - n\log\theta + \sum_{i=1}^{n}\log x_i^k + \lambda\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{-k}\log\left(\frac{\theta}{x_i}\right) - (\alpha - 1)\sum_{i=1}^{n}\left(\frac{\lambda\left(\frac{\theta}{x_i}\right)^{-k}\log\left(\frac{\theta}{x_i}\right)e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}}{1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}}\right) \quad (3.3.14)$$

$$\frac{n}{k} - n\log\theta + \sum_{i=1}^{n}\log x_i^k + \lambda\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{-k}\log\left(\frac{\theta}{x_i}\right) - (\alpha - 1)\sum_{i=1}^{n}\left(\frac{\lambda\left(\frac{\theta}{x_i}\right)^{-k}\log\left(\frac{\theta}{x_i}\right)e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}}{1 - e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k} - 1\right)}}\right) = 0 \quad (3.3.15)$$

However, the above system of nonlinear equations (3.3.9), (3.3.11), (3.3.13) and

(3.3.15) cannot be solve analytically. In order to obtained the MLEs, we adopt one of the numerical methods. We used the most commonly method called Newton Raphson Algorithm, which is also available in R statistical software.

CHAPTER FOUR

RESULT AND DISCUSSION

4.1 Application to Real Life Data

To illustrate the importance of the proposed distribution (OGEPD), we fit it to two different real datasets and compared their performances with an already existing distributions such as: kumaraswamy Pareto distribution (KPD) proposed by Bourguignon *et. al.,* (2012), Exponentiatial Pareto distribution (EPD) proposed by Kareema *et. al.,* (2013) and three parameter Odd generalized Exponential Pareto Distribution (OGEPD) proposed by Maiti and Pramanik (2016). The pdf and total log likelihood (ll) of these distributions are given in the following equations respectively.

The Exponentiatial Pareto Distribution

$$f\left(x,\lambda,\theta,k\right)=\frac{\lambda k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}e^{-\lambda\left(\frac{x}{\theta}\right)^{k}} \tag{4.1.1}$$

$$ll=n\log\lambda+n\log k-n\log\theta+\left(k-1\right)\sum_{i=1}^{n}\log x_i-n\left(k-1\right)\sum_{i=1}^{n}\theta-\lambda\frac{\sum_{i=1}^{n}x_i^k}{\theta^k} \tag{4.1.2}$$

The kumaraswamy Pareto Distribution

$$f\left(x,\alpha,\lambda,\theta,k\right)=\frac{\alpha\lambda k\theta^k}{x^{k+1}}\left[1-\left(\frac{\theta}{x}\right)^k\right]^{\alpha-1}\left[1-\left(1-\left(\frac{\theta}{x}\right)^k\right)^{\alpha}\right]^{\lambda-1} \tag{4.1.3}$$

$$ll=n\log\alpha+n\log\lambda+n\log k+nk\log\lambda-\left(k+1\right)\sum_{i=1}^{n}\log x_i$$

$$+\left(\alpha-1\right)\sum_{i=1}^{n}\log\left(1-\left(\frac{\theta}{x}\right)^k\right)+\left(\lambda-1\right)\sum_{i=1}^{n}\log\left(1-\left(1-\left(\frac{\theta}{x}\right)^k\right)^{\alpha}\right) \tag{4.1.4}$$

The three- Parameter Odd Generalized Exponential Pareto Distribution

$$f(x, \lambda, \theta, k) = \lambda k \theta^{-k} x^{k-1} e^{\lambda} e^{-\lambda\left(\frac{\theta}{x}\right)^{-k}} \tag{4.1.5}$$

$$ll = n \log \lambda + n \log k + n\lambda - nk \log \theta + (k-1) \sum_{i=1}^{n} \log x_i - \lambda \theta^{-k} \sum_{i=1}^{n} x_i^k \tag{4.1.6}$$

4.2 Dataset

The first dataset considered for this analysis is that used by Maiti and Pramanik (2016). The real dataset was obtained from Linhart and Zucchini (1986). In 2007, Lee *et. al.,* also fitted it to a distribution known as Beta Weibull distribution. The dataset are provided in table (4.1).

The second dataset used represents the actual taxes dataset. The data consists of the monthly actual taxes revenue in Egypt from January 2006 to November 2010. The data (in 1000 million Egyptian pounds) are provided in table (4.2). Nassar and Nada (2011) and Mead (2014) used the data in their study.

The datasets are analyzed with the aid of an R software.

Table 4.1: Dataset I

1, 3, 5, 7, 11, 11, 11, 12, 14, 14, 14, 16, 16, 20, 21, 23, 42, 47, 52, 62, 71, 71, 87, 90, 95, 120, 120, 225, 246, 261

Table 4.2: Dataset II

5.9 ,20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17.0, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10.0, 4.1, 36.0, 8.5, 8.0, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7.0, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11.0, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8

The summary of the two datasets are also provided in table (4.3) and table (4.4) respectively.

Table 4.3: Summary of dataset I

Parameters	N	Minimum	Q1	Median	Q3	Mean	Maximum	Variance
Value	30	1.0	12.5	22.0	83.0	59.6	261	5167.421

Table 4.4: Summary of dataset II

Parameters	N	Minimum	Q1	Median	Q3	Mean	Maximum	Variance
Value	59	4.10	8.45	10.60	16.85	13.49	39.20	64.8266

We also plot the histograms for the two datasets, which is displayed in figure (4.2.1) and figure (4.2.2). Observing the histograms for the two datasets, it indicates that the two datasets are positively skewed and therefore suitable for skewed or asymmetric distributions. Since the proposed distribution (OGEPD) is skewed to the right, its will be appropriate.

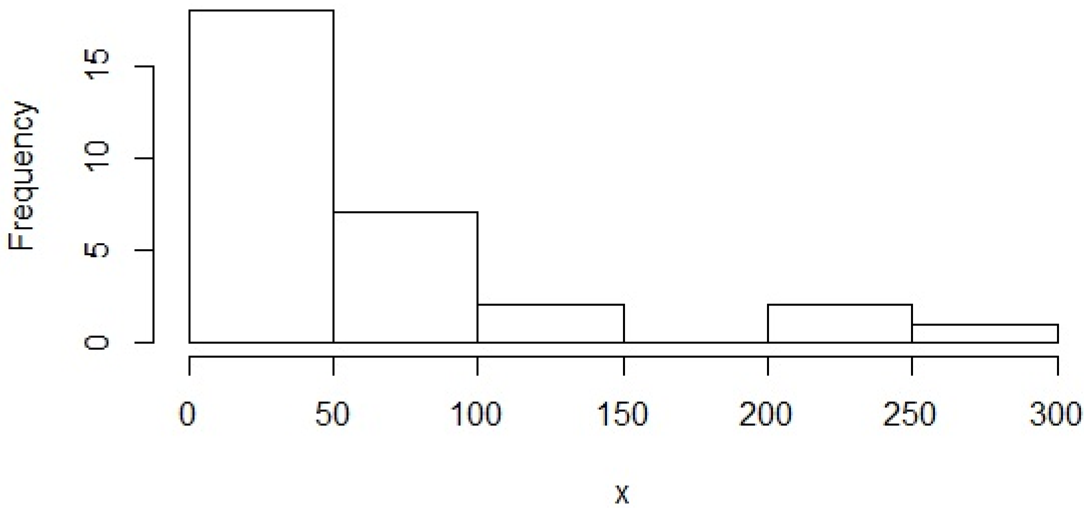

Figure 4.2.1: Histogram plot of dataset I

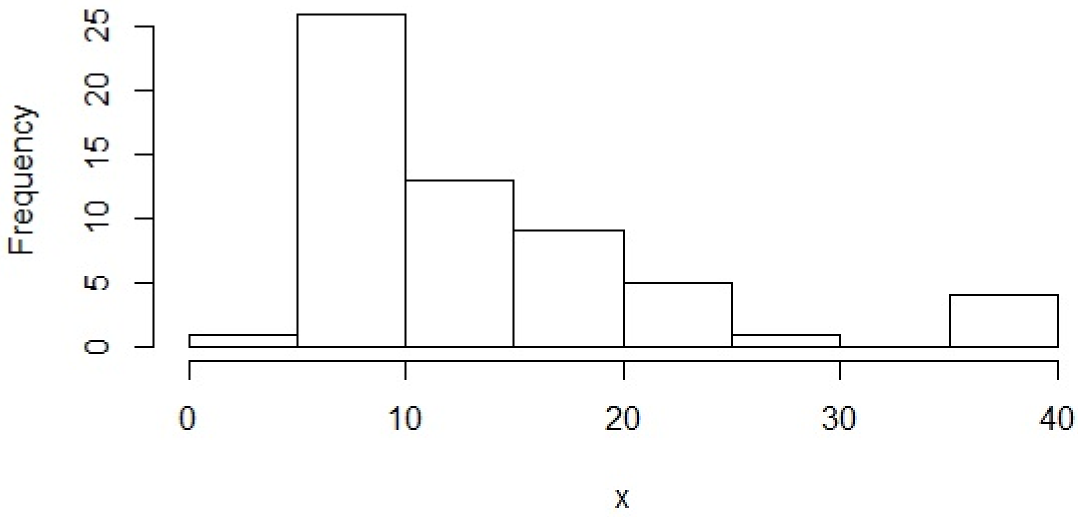

Figure 4.2.2: Histogram plot of dataset II

4.3 Information Criteria used for Comparing the Distributions

In order to compare these distributions, we considered some criteria: the ll (log-likelihood), AIC (Akaike Information Criterion), $CAIC$ (Consistent Akaike Information Criterion) and BIC (Bayesian Information Criterion). These statistics are given by the following equations:

$$AIC = -2ll + 2p \tag{4.3.1}$$

$$BIC = -2ll + p\log(n) \tag{4.3.2}$$

$$CAIC = -2ll + \frac{2pn}{n-p-1} \tag{4.3.3}$$

where ll denotes the log-likelihood function evaluated at the MLEs, p is the number of distribution parameters and n is the sample size.

Note: The distribution with the lowest values for these criteria is chosen as the best model that fitted the data.

Table 4.5: The MLE's of the models for dataset I

The Model	MLE's of the parameter(s)
$EPD\,(\lambda,\theta,k)$	$\widehat{\lambda}=0.00011$, $\widehat{\theta}=0.60458$, $\widehat{k}=0.60458$
$KPD\,(\alpha,\lambda,\theta,k)$	$\widehat{\alpha}=0.93726$, $\widehat{\lambda}=0.95000$, $\widehat{\theta}=0.00157$, $\widehat{k}=0.04977$
$OGEPD\,(\lambda,\theta,k)$	$\widehat{\lambda}=0.19323$, $\widehat{\theta}=0.00101$, $\widehat{k}=0.10420$
$OGEPD\,(\alpha,\lambda,\theta,k)$	$\widehat{\alpha}=2.00132$, $\widehat{\lambda}=0.19410$, $\widehat{\theta}=0.00932$, $\widehat{k}=0.10513$

Table 4.6: The *ll*, *AIC, BIC,* and *CAIC* of the models for dataset I

The Model	**Statistics**			
	$-ll$	AIC	BIC	$CAIC$
$EPD\,(\lambda,\theta,k)$	-238.8147	483.6294	482.0608	484.5525
$KPD\,(\alpha,\lambda,\theta,k)$	-187.7629	383.5258	381.4343	385.1258
$OGEPD\,(\lambda,\theta,k)$	-197.0826	400.1652	398.5966	401.0883
$OGEPD\,(\alpha,\lambda,\theta,k)$	**-187.0856**	**382.1712**	**380.0797**	**383.7712**

Table 4.7: The MLE's of the models for dataset II

The Model	MLE's of the parameter(s)
$EPD\,(\lambda,\theta,k)$	$\widehat{\lambda}=0.00017$, $\widehat{\theta}=0.98461$, $\widehat{k}=0.92684$
$KPD\,(\alpha,\lambda,\theta,k)$	$\widehat{\alpha}=0.17043$, $\widehat{\lambda}=0.89515$, $\widehat{\theta}=0.00795$, $\widehat{k}=0.0121$
$OGEPD\,(\lambda,\theta,k)$	$\widehat{\lambda}=0.28899$, $\widehat{\theta}=0.10301$, $\widehat{k}=0.00119$
$OGEPD\,(\alpha,\lambda,\theta,k)$	$\widehat{\alpha}=1.80107$, $\widehat{\lambda}=0.19466$, $\widehat{\theta}=0.00315$, $\widehat{k}=0.10727$

Table 4.8: The *ll*, *AIC, BIC,* and *CAIC* of the models for dataset II

The Model	**Statistics**			
	$-ll$	AIC	BIC	$CAIC$
$EPD\,(\lambda,\theta,k)$	-528.7221	1063.4442	1063.3012	1063.8806
$KPD\,(\alpha,\lambda,\theta,k)$	-317.9943	643.9886	6430720	644.7293
$OGEPD\,(\lambda,\theta,k)$	-323.5772	653.1544	652.4670	653.5908
$OGEPD\,(\alpha,\lambda,\theta,k)$	**-315.2932**	**638.5864**	**637.6698**	**639.3271**

4.4 Discussion

Table (4.5), shows the MLEs of different parameters of the distributions for dataset I, whereas table(4.6) indicate the performance of the distribution base on some selected distributions. we can see how well the models fit the data. Some numerical values of the log-likelihood (ll), *AIC, BIC* and *CAIC* are provided for each distribution. The distribution with the lowest value of *AIC, BIC* and *CAIC* correspond to the new distribution (A four parameter OGEPD), the new distribution seems to be a very competitive distribution to this data. Moreover, we can see that the distribution with four parameters (*KPD* and *OGEPD*) performed better to this dataset compared to the less parameter ones.

Table (4.7), shows the maximum likelihood estimates (MLEs) to each one of the four fitted distributions for dataset II. Then table (4.8) indicate the corresponding values of log-likelihood, *AIC, BIC* and *CAIC* for each model. The values in table (4.8) are evidence that the new distribution (*OGEPD*) performed better than the other three distributions for this data, since it's has the lowest values of the information criteria. Therefore it is chosen as the best distribution compared to the other three distributions. However, the four parameters *OGEPD* is still far better than the three parameters *OGEPD*. This proved the fact that generalizing probability distributions provides compound distributions that are sometimes more flexible compared to their baseline distribution.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

In this research, we aimed at introducing a new four-parameter continuous probability distribution called Odd Generalized Exponential-Pareto Distribution (OGEPD), by using the generator introduced by Tahir *et al.,* (2015) known as the Odd Generalized Exponential (OGE) family of distribution. This served as an extension of the well-known Pareto distribution. The process of extending probability distribution have received considerable great attentions by researchers, due to the fact that the extended distribution are more flexible and sometimes performed well and have better presentation of data than their counterparts with less number of parameter. The CDF and pdf of the new distribution were well defined and expanded to ease derivation of some properties of the distribution. More so, some Statistical and Mathematical properties including; limiting behaviour, moments, moment generating function, quantile function, order statistics and reliability analysis were derived. The parameters were estimated using method of maximum likelihood estimation. The usefulness or importance of the proposed distribution (OGEPD) was illustrated by fitting it to two different real dataset and compared their performances with some already existing distributions.

Lastly, the criteria employed for model selection are; log likelihood (ll), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion ($CAIC$) and Bayesian information criterion (BIC). The new OGEPD was found to be the best distribution that fitted the two different datasets, because it has the lowest value of the *ll, AIC, AICC* and *BIC*. The plot of the pdf shows that the distribution was positively skewed and also the two datasets are skewed to the right. R statistical software program was used to run the analysis.

5.2 Conclusion

We introduced a new four-parameter continuous probability distribution called the Odd Generalized Exponential-Distribution (OGEPD) and studied Some of its properties. Some statistical properties of the proposed distribution have been derived and discussed appropriately. We defined the CDF, the pdf, the survival and hazard function of this distribution. We provide explicit expressions for its moments, moment generating function, quantile function, ordered statistics and asymptotic behaviour. We also provide some plots of the distribution which indicate that it is a positively-skewed distribution and uni-modal. The distribution parameters were estimated using the method of maximum likelihood estimation. An application to two different datasets revealed that the fit of the new distribution is the best compared to the fit of other distributions consider in this research.

5.3 Recommendations

Based on the results and conclusions made earlier in this research, we recommended that the extended distribution (OGEPD) can be used to fit in data of various shapes that cannot be adequately fitted with the conventional probability distributions, It can also be appropriate in analyzing skewed datasets. The proposed distribution can also be applied in many areas of knowledge including Economics, Sociology and in reliability analysis etc.

5.4 Contribution to Knowledge

We have proposed a four- parameter distribution which is appropriate for modelling skewed data, and derived some of its statistical properties (which have never been derived before by any researcher/institution across the globe). It's performed better than some of the generalized distribution with less or equal number of parameters. We also increase the flexibility of the Pareto distribution. This research will also benefit others in one way or the other were its proved to us that generalized distribution are most a times better than the parent distributions.

5.5 Areas of Further Research

Other researchers can propose and study new distributions using this same generator, since there is always room for extending baseline distributions so as to fill in the gap. The remaining statistical properties other than the ones captured in this research can be derived and discussed. Comparison of different classical methods of estimation can be done for the distribution. finally, researchers can estimate the parameters of this (OGEPD) using Bayesian approach.

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