A STUDY ON EXPONENTIATED-NEW WEIGHTED EXPONENTIAL DISTRIBUTION

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Declaration

I declare that the work in this dissertation entitled **"A Study on Exponentiated-New Weighted Exponential Distribution"** has been performed by me in the Department of Statistics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree in this or any other Institution.

_____	_____	_____
Name of Student	Signature	Date

Certification

This Dissertation entitled "A STUDY ON EXPONENTIATED-NEW WEIGHTED EXPONENTIAL DISTRIBUTION" by Kemi Victoria DADA meets the regulations governing the award of the degree of Master of Science in Statistics of the Ahmadu Bello University Zaria, and is approved for its' contribution to knowledge and literary presentation.

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Dean, School of Postgraduate Studies	Signature	Date

Dedication

This work is dedicated to my late mother, Mrs. Racheal Funmilayo Dada, by means of your faith, you still speak even though you are dead.

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I give thanks to God, the Almighty, who has shown me grace, mercy and favour throughout this period of study.

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Finally, I must acknowledge the valuable efforts of my friends and course mates who assisted me in the collection of data and materials used for the research work.

ABSTRACT

Statistical probability distributions are vital in explaining real world situations. It has been shown overtime that extreme value distributions are found to be useful in various field of studies such as engineering, financial sector, biological sciences etc. Generalizing existing distributions is being studied frequently so as to enhance and improve on the flexibility of an existing distribution. In this work, a three-parameter distribution known as Exponentiated New Weighted Exponential distribution was proposed by introducing an exponentiated (shape) parameter to an existing distribution, it can be used to model real life situations. Structural statistical properties of the proposed distribution such as the hazard, survival, moments and moment generating function were derived. The order statistics and quantile function were also studied. The method of maximum likelihood estimation procedure was used in estimating the parameters of the proposed distribution. The kernel density estimates and the classical Kolmogorov-Smirnov two sample test was also utilized to see how the estimated kernel density function from the proposed distribution mimic the parent distribution and also to check if the data sets used (waiting time of bank customers and accelerated life test) and the sample drawn from the fitted ENWED are indeed from the same distribution. Finally, to prove the flexibility and performance of the proposed distribution, we applied two real life data sets to the proposed distribution, exponentiated-new weighted exponential distribution and to competing distributions such as weighted exponential distribution, new-weighted exponential distribution and exponentiated weighted exponential distribution. The results indicate that the newly proposed distribution outperformed the competing distributions.

Table of Contents

DECLARATION II

CERTIFICATION IV

DEDICATION V

ACKNOWLEDGEMENTS VI

ABSTRACT VII

CHAPTER ONE 1

INTRODUCTION 1

1.1 BACKGROUND OF THE STUDY 1
1.2 NEW WEIGHTED EXPONENTIAL DISTRIBUTION 2
1.3 STATEMENT OF PROBLEM 5
1.4 SIGNIFICANCE OF THE STUDY 6
1.5 AIM AND OBJECTIVES 6
1.6 LIMITATION OF THE RESEARCH 6

CHAPTER TWO 8

LITERATURE REVIEW 8

2.1 INTRODUCTION 8
2.2 ON EXPONENTIATED DISTRIBUTION 8

CHAPTER THREE 13

METHODOLOGY 13

3.1 INTRODUCTION 13
3.2 THE EXPONENTIATED-NEW WEIGHTED EXPONENTIAL DISTRIBUTION (ENWED) 13
3.2.1 PROBABILITY DENSITY FUNCTION OF ENWED 14
3.2.2 THE CUMULATIVE DISTRIBUTION FUNCTION OF ENWED 15
3.3 RELIABILITY ANALYSIS OF EXPONENTIATED-NEW WEIGHTED EXPONENTIAL DISTRIBUTION 16
3.3.1 SURVIVAL FUNCTION 16

3.3.2 ASYMPTOTIC BEHAVIORS 17
3.4 STATISTICAL PROPERTIES OF THE EXPONENTIATED-NEW WEIGHTED EXPONENTIAL DISTRIBUTION 19
3.4.1 THE QUANTILE FUNCTION AND THE MEDIAN 19
3.4.2 MOMENTS AND MOMENT GENERATING FUNCTION 21
3.4.3 DISTRIBUTION OF ORDER STATISTICS 23
3.5 PARAMETER ESTIMATION AND APPLICATION 24
3.5.1 MAXIMUM LIKELIHOOD ESTIMATION 24
3.6 DATA ANALYSIS PROCEDURE 27
3.6.1 THE KERNEL-BASED DENSITY ESTIMATION 28
3.6.2 KOLMOGROV-SMIRNOV TWO SAMPLE TEST 29
3.7 APPLICATION 30

CHAPTER FOUR 31

ANALYSIS AND DISCUSSIONS 31

CHAPTER FIVE 41

SUMMARY, CONCLUSION AND RECOMMENDATION 41

5.1 SUMMARY 41
5.2 CONCLUSION 41
5.3 RECOMMENDATION 42
5.4 CONTRIBUTION TO 42

REFRENCE 43

List of Tables

Table 4.1a: Waiting Time in Minutes of Bank Customer for n=100

Table 4.1b: Random Sample of Size 100 drawn from ENWED

Table 4.2: Performance of Distribution ENWED and NWED using Data on Waiting Time in Minutes of Bank Customers

Table 4.3a: Accelerated Life Test Conductors for n=59

Table 4.3b: Random Sample of Size 59 drawn from ENWED

Table 4.4: Performance of Distribution ENWED and NWED using Data on Accelerated Life Test Conductors

List of Figures

Figure 1: The Behavior of PDF of ENWED and NWED for Some Selected Parameter Values.

Figure 4.1 Estimated PDF for the Data on Waiting Time of Bank Customers.

Figure 4.2 Kolmogrov-Smirnov Two-Sample Test: Graph of the Empirical Distribution Function for Data on Waiting Time of Bank Customers.

.Figure 4.3: Estimated Survival and Hazard Function for the Data on Waiting Time of Bank Customers.

Figure 4.4 Estimated PDF for the Data on Accelerated Life Test of Conductors.

Figure 4.5 Kolmogrov-Smirnov Two-Sample Test: Graph of the Empirical Distribution Function for Data on Accelerated Life Test of Conductors.

Figure 4.6: Estimated Survival and Hazard Function for the Data on Accelerated Life Test of Conductors.

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Statistical probability distributions are very important in explaining real world situations. It is known that extreme value distributions have been found to be very useful in the field of engineering, biological sciences and financial sector. In statistical modeling, the quality of the test used heavily relies on the assumed probability distribution being adopted. Generalized distributions have been studied extensively in statistics, although numerous statisticians have developed different types of generalizations, there is still need to develop new probability distribution by generalizing existing distributions. For instance, the Kumaraswamy Generalized (KW-G) proposed by Cordeiro and de Castro (2011), the Beta Generalized (B-G) introduced by Eugene *et al.*,(2002) and the Kumaraswamy Odd log-logistic Generalized (KWOLL-G) introduced by Alisa del *et al* (2015).

In reliability engineering, the Weibull and exponential distribution are the mostly used distributions due to their simplicity and easy mathematical manipulations. Also in reliability engineering and life testing, it has been asserted that the exponential distribution is a suitable alternative to both the Gamma and Weibull as shown by Gupta *et al.*, (2001). Numerous researchers have used the exponential distribution as a better alternative to some related distributions.

Gupta *et al.,* (1998) introduced a class of exponentiated distribution which is based on the cumulative distribution function, it is derived by raising the cdf of an arbitrary parent distribution to an additional positive parameter, say, "θ".

Let X denotes a random variable from an arbitrary parent distribution G, the cdf of the resulting exponentiated distribution is given by:

$$F(x) = \left[G(x)\right]^{\theta}; \theta > 0 \qquad (1.1)$$

where: G(x) is the cdf of the parent distribution. Therefore, the corresponding pdf is obtained by differentiating equation (1.1) with respect to x to get;

$$f(x) = \theta\left[G(x)\right]^{\theta-1} g(x) \qquad (1.2)$$

where: $g(x) = \frac{dG(x)}{dx}$

1.2 New Weighted Exponential Distribution

According to Badmus *et al.,* (2014), the weighted distributions are introduced to an existing distribution to adjust the likelihood of the events as observed and recorded. Following Nasiru (2015), the Azzalini's method was used and applied to the exponential distribution by Oguntunde *et al.*, (2016) to produce a new weighted exponential distribution, n-WED(α, λ), which is simpler and mathematically easier to handle than the weighted exponential distribution proposed by Gupta and Kundu (2009) as a generalization of the exponential distribution.

Let $g(x)$ be a pdf and $\bar{G}(x)$ be the corresponding survival (or reliability) function such that the cdf $G(x)$ exists, then the new weighted family of distribution is given by:

$$f(x) = kg(x)\bar{G}(\lambda x), x > 0, \lambda \neq 0 \qquad (1.3)$$

Where k is a constant such that

$$\int_0^{\infty} f(x)dx = 1 \tag{1.4}$$

The pdf $g(x)$ and survival function $\bar{G}(x)$ of an exponential distribution are given as

$$g(x) = \alpha e^{-\alpha x}; x > 0, \alpha > 0 \quad \textit{and} \tag{1.5}$$

$$\bar{G}(x) = 1 - G(x)$$

$$= 1 - (1 - e^{-\alpha x})$$

$$= e^{-\alpha x} \tag{1.6}$$

Where α is a shape parameter

Hence substituting equations (1.5) and (1.6) into (1.3) gives the new weighted exponential distribution

$$f(x) = k\alpha e^{-\alpha x} e^{-\alpha\lambda x}$$

$$= k\alpha e^{-(\alpha x + \alpha\lambda x)}$$

$$= k\alpha e^{-\alpha x(1+\lambda)}; \text{ or}$$

$$f(x) = k\alpha e^{-\alpha(1+\lambda)x}$$

By equation (1.4)

$$\int_0^{\infty} k\alpha e^{-\alpha(1+\lambda)x} dx = 1$$

Let $t = \alpha(1+\lambda)x$

$$dt = \alpha(1+\lambda)dx$$

$$dx = \frac{1}{\alpha(1+\lambda)}dx$$

$$k\int_0^{\infty} \frac{\alpha e^{-t}}{\alpha(1+\lambda)}dt = -\frac{k}{(1+\lambda)}e^{-t}\Big| = -\frac{k}{(1+\lambda)}[0-1]$$

$$= \frac{k}{(1+\lambda)} = 1$$

$$\Rightarrow k = 1+\lambda$$

Therefore $f(x) = (1+\lambda)\alpha e^{-\alpha(1+\lambda)x}; x > 0, \lambda > 0, x > 0$ (1.7)

Where both α and λ are shape parameters. Note that if $\beta = \alpha(1+\lambda)$ or $\lambda = 0$, equation (1.7) reduces to a simple exponential density function with parameters β or α, that is

$$f(x) = \begin{cases} \beta e^{-\beta x} & if\ \beta = (1+\lambda)\alpha \\ \alpha e^{-\alpha x} & if\ \lambda = 0 \end{cases}, \quad (1.8)$$

And β and α are all shape parameters. Therefore from equation (1.7), both α and λ are shape parameters. The corresponding cdf of the new weighted exponential distribution is given as

$$F(x) = 1 - e^{-\alpha(1+\lambda)x} \quad (1.9)$$

Unlike in Oguntunde *et al.,* (2016), both α and λ are shape parameters; implying that a product of two shape parameters is also a shape parameter. Oguntunde *et al.,* (2016) erred by saying that α is a scale parameter.

The new weighted exponential distribution whose pdf and cdf are defined in equations (1.7) and (1.9) are much easier to handle mathematically than the ones proposed by Gupta and Kundu (2009):

$$f(x)=\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left[1-e^{-\lambda\alpha x}\right];x>0,\alpha>0,\lambda>0 \tag{1.10}$$

and

$$F(x)=\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda(1+\alpha)x}\right)\right] \tag{1.11}$$

Where; α is a shape parameter and λ is a scale parameter.

The hazard function of the new weighted exponential distribution is given by:

$$h(x)=\frac{(1+\lambda)\alpha e^{-\alpha(1+\lambda)x}}{e^{-\alpha(1+\lambda)x}} \tag{1.12}$$

$$=(1+\lambda)\alpha$$

In literature, other weighted distributions have also been defined, for instance, the weighted Weibull distribution by Mhady (2013), the weighted Inverted Exponential distribution by Hussain (2013), a weighted three parameter Weibull by Essam and Mohammed (2013).

1.3 Statement of Problem

Oguntunde (2015) generalized the weighted exponential distribution in equation (1.10) using the exponentiated family of distributions to propose the exponentiated Weighted Exponential Distribution.

The NWED(α,λ) cannot be used in fitting some real-life data set because some existing real datasets do not follow NWED and therefore cannot be appropriately described using it, as a result of this there is a need to generalize the NWED(α,λ) by adding an additional shape parameter which will increase its flexibility, applicability and provides a better fit.

1.4 Significance of the Study

The addition of another shape parameter into the NWED(α, λ) will increase its flexibility in analyzing real data sets and will also improve the statistical test associated with the distribution.

1.5 Aim and Objectives

The aim of this research work is to propose a new distribution called Exponentiated-New Weighted Exponential Distribution and derive some of its structural properties which will be attained through the following specific objectives; to:

1. Derive the exponentiated-new weighted exponential distribution using the class of exponentiated distribution proposed by Gupta *et al.,* (1998).
2. Determine the reliability function (hazard and survival function) and obtain various structural properties (moments, moment generating function, quantile and order statistics) of the proposed distribution.
3. Estimate the parameters of the proposed exponentiated-new weighted distribution using the maximum likelihood estimation,
4. Compare the performance of the proposed exponentiated-new weighted exponential distribution with competitive alternatives, such as the new weighted exponential distribution, exponentiated weighted exponential distribution and weighted exponential distribution using real life datasets; and
5. Apply the kernel based density estimation method and the Kolmogorov-Smirnov two sample test method to give an enhancement to outlook of the proposed model.

1.6 Limitation of the Research

This research work is limited to using only Alkaike Information Criterion (AIC) for testing the goodness of fit for the best model

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

It has been of great interest among statisticians and applied researchers in constructing flexible lifetime models to aid better modeling of various data such as survival data, reliability and so on. This chapter covers quite a number of researches which has been carried on exponentiated distributions.

2.2 On Exponentiated Distribution

Oguntunde (2015) proposed a new distribution known as the exponentiated weighted exponential distribution which has three parameters. This was achieved by generalizing the weighted exponential distribution using the exponentiated class/family of distributions. The basic statistical properties of the model were derived clearly. The method of maximum likelihood was used to estimate the parameters of the proposed distribution. The exponentiated weighted exponential distribution was not applied on any real life data set.

Badmus *et al.*, (2014) carried out a research on Some Statistical Properties of Exponentiated Weighted Weibull (EWW) distribution. Important properties of the distribution such as the hazard rate function, survival rate function, moments generating function, asymptotic behaviors of the model was studied and the parameters of the distribution was estimated using maximum likelihood estimation. It was ascertained on the basis of application of real data sets on the model that the proposed distribution was more flexible than the weighted Weibull and the Weibull distribution.

Ebraheim (2014) carried out a research on exponentiated transmuted Weibull Distribution which is a new generalization of the two parameter Weibull. The quadratic rank transmutation map was used for generalizing of distribution. It was shown that the proposed distribution has the advantage of being able to model different shapes of aging and failure criteria. Statistical properties which includes; the moment of the distribution, survival, hazard, order statistics were derived for the new distribution. Explicit expression was derived for the quantiles. The method of maximum likelihood was used to estimate the parameter of the exponentiated transmuted Weibull Distribution.

Flaih *et al.*, (2012) proposed the exponentiated inverted Weibull distribution which is a generalization to the standard inverted Weibull distribution; it has two shape parameters and was applied to analyzing bathtub failure rates lifetime data. Also some important statistical properties was studied and also derived for the proposed distribution which includes; its moments, median, survival function, hazard function and its asymptotic confidence intervals. Maximum likelihood estimation and least squares estimators was used for estimating of parameters. It was discovered that the exponentiated inverted Weibull distribution gives a better fit than the inverted Weibull after a real data set was analyzed with its model.

Mahmoudi and Shiran (2012) introduced the Exponentiated Weibull-Geometric distribution and its application. It is a four parameter distribution which includes as special sub-models many other distributions. The hazard function was found to be decreasing, increasing, bathtub shape and unimodal. The quantiles, residual life function, order statistics are part of the properties derived. The maximum likelihood estimation procedure was used to estimate the parameters. The model of the new distribution was finally used to analyze two real data sets so as to show the flexibility of the proposed distribution.

Chung and Kang (2014) also introduced the Exponentiated Weibull-Geometric distribution. This was obtained by compounding the exponentiated Weibull with the geometric distribution. Using EM-algorithm and by using Gibbs sampler with Metropolis-Hastings algorithm, the maximum likelihood estimation and the Bayesian estimation was obtained respectively. The cumulative density function, hazard function, moments of the order statistics and its other basic statistical properties was derived. Finally, the flexibility of the exponentiated Weibull-geometric distribution was shown by applying two different real data sets and also to compare fit of the EWG model to exponentiated Weibull using AIC and BIC.

Ashour and Eltehiwy (2014) introduced a three parameter distribution which is known as the exponentiated power Lindley distribution. It presents more flexibility to analyze complex real data sets and also contains as special sub-models some well known distributions. Three different algorithms was used for generating random data from the proposed distribution, the basic statistical properties for the exponentiated power Lindley distribution was studied and for evaluation and estimating of the parameters, the least squares estimators and maximum likelihood estimation was used respectively. The exponentiated power Lindley distribution was shown to be quite effective and more flexible in analyzing real lifetime data after an application of the model to a real data set was analyzed using the introduced distribution.

A new five parameter was presented by Al-Zahrani *et al.,* (2015) known as the exponentiated transmuted Weibull geometric distribution; it consists of twenty two (22) lifetime distributions as special cases. The statistical properties of the new distribution were discussed extensively; explicit expression was use for deriving the moments and also for examining the order statistics. The maximum likelihood estimation is used for the evaluation of the parameters. The performance of the new model was examine by fitting two real data sets of data (that is, censored

and uncensored) to compare the fitting of the proposed model with some nested and non-nested models. The exponentiated transmuted Weibull geometric distribution gives the best fit to all of the data and it is capable for modeling different forms of aging and failure criteria.

Rodrigues and Silva (2015) extends the exponential distribution and some other distribution to propose the Exponentiated Kumaraswamy Exponential distribution. The moments, hazard functions, mean deviation and Renyi entropy are properties of the new distribution that was investigated. It was compared with other existing models like the exponentials, inverse exponential and the Weibull. The estimation of the model parameter was discussed and derived using maximum likelihood estimation and methods of moment. The model was said to give a better fit after its application to real datasets.

Elbatal and Muhammed (2014) introduced the Exponentiated Generalized Inverse Weibull Distribution and gave a comprehensive mathematical treatment for the distribution which includes: the hazard, survival, order of the moments, the distribution function and the probability density function inclusive. For estimating the parameters, the maximum likelihood estimation method was employed and the flexibility of model was determined using three life datasets. Using AIC, BIC and CAIC as a criteria for comparison, the model was compared to candidates distribution (Inverse Weibull and Generalized Inverse Weibull) and it was observed that the exponentiated Generalized Inverse Weibull provides a better fit and therefore gives a more flexible model.

Bakouch *et al.,* (2012) carried out a research on an exponentiated exponential binomial distribution with its application. This was carried out by compounding exponentiated exponential and binomial distribution. Properties of the new distribution was looked at and it was asserted

that the distribution has the ability to model lifetime data with either increasing, decreasing or upside-down bathtub shaped failure rates. After its application to real datasets, it was concluded that the EEB distribution provides a better fit than other related lifetime distribution.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

In this chapter, we will apply a class of exponentiated distribution which is based on the cumulative distribution function proposed by Gupta *et al.,* (1998) to derive the cumulative distribution function and probability density function of the proposed distribution, exponentiated-new weighted exponential distribution**.**

According to Gupta *et al.,* (1998) a random variable X is said to have an exponentiated class of distribution function if its probability density function (pdf) and cumulative distribution function (cdf) is given, respectively as;

$$f(x) = \theta g(x)\left[G(x)\right]^{\theta-1} \; x > 0, \theta > 0 \qquad (3.1)$$

and

$$F(x) = \left[G(x)\right]^{\theta}; x > 0, \theta > 0 \qquad (3.2)$$

where: $G(x)$ is the cdf of the baseline distribution and $g(x)$ is the pdf of the baseline distribution and θ is the shape parameter.

3.2 The Exponentiated-New Weighted Exponential Distribution (ENWED)

We introduced the exponentiated-new weighted exponential distribution by generalizing the new weighted exponential distribution proposed by Oguntunde *et al.,* (2016) using the exponentiated class of distribution which was introduced by Gupta *et al.,* (1998).

3.2.1 Probability Density function of ENWED

The probability distribution function of the exponentiated-new weighted exponential distribution will be obtained by applying the class of exponentiated distribution introduced by Gupta *et al.,* (1998).

Oguntunde *et al.,* (2016) proposed a distribution known as the new weighted exponential distribution (NWED). A random variable X is said to follow the new weighted exponential distribution, if the pdf and cdf can be expressed as in equation (1.8) and (1.9) respectively

The proposed probability density function of the exponentiated-new weighted exponential distribution is derived by substituting equation (1.8) and (1.9) into (3.1), we have

$$f(x) = \theta(1+\lambda)\alpha e^{-(1+\lambda)\alpha x}\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta-1}; x>0, \alpha>0, \lambda>0, \theta>0 \tag{3.3}$$

Where: α and λ are the shape parameters and θ is the exponentiated parameter.

3.2.1.1 Validation of the Probability Density Function

To test for the validity of the pdf, now we use

$$\int_{-\infty}^{\infty} f(x)dx = 1 \tag{3.4}$$

$$\int_{0}^{\infty} f(x)dx = \theta\alpha(1+\lambda)\int_{0}^{\infty} e^{-\alpha(1+\lambda)x}\left[1-e^{-\alpha(1+\lambda)x}\right]^{\theta-1} dx \tag{3.5}$$

Let $u = 1-e^{-\alpha(1+\lambda)x}$ (3.6)

$$\Rightarrow du = \alpha(1+\lambda)e^{-\alpha(1+\lambda)x}dx$$

$$dx = \frac{du}{\alpha(1+\lambda)e^{-\alpha(1+\lambda)x}} \tag{3.7}$$

as $x \to 0, u \to 0, x \to \infty, u \to 1$

Putting equation (3.6) and (3.7) into (3.5), we have

$$= \theta\alpha(1+\lambda)\int_0^1 e^{-\alpha(1+\lambda)x}\left[u\right]^{\theta-1}\frac{du}{\alpha(1+\lambda)e^{-\alpha(1+\lambda)x}}$$

$$\theta\int_0^1 u^{\theta-1}du = \left[u^{\theta}\right]_0^1$$

$$1^{\theta} - 0^{\theta} = 1$$

3.2.2 The Cumulative Distribution Function of ENWED

According to Gupta *et al.*, (1998), the cdf of a random variable X with an exponentiated class of distribution is given by equation (3.2).

By substituting $G(x)$ which is the cdf of the new weighted exponential distribution, the proposed cdf of exponentiated weighted exponential distribution will be given as:

$$F(x) = \left[1 - e^{-(1+\lambda)\alpha x}\right]^{\theta}; x > 0, \alpha > 0, \lambda > 0, \theta > 0 \qquad (3.8)$$

`where: α and λ are shape parameters and θ is the exponentiated parameter.

Figure 3.1 and 3.2 below presents illustrations of possible shapes of the pdf of Exponentiated-New Weighted Exponential distribution and the New Weighted Exponential distribution for selected values of parameters θ, λ, α.

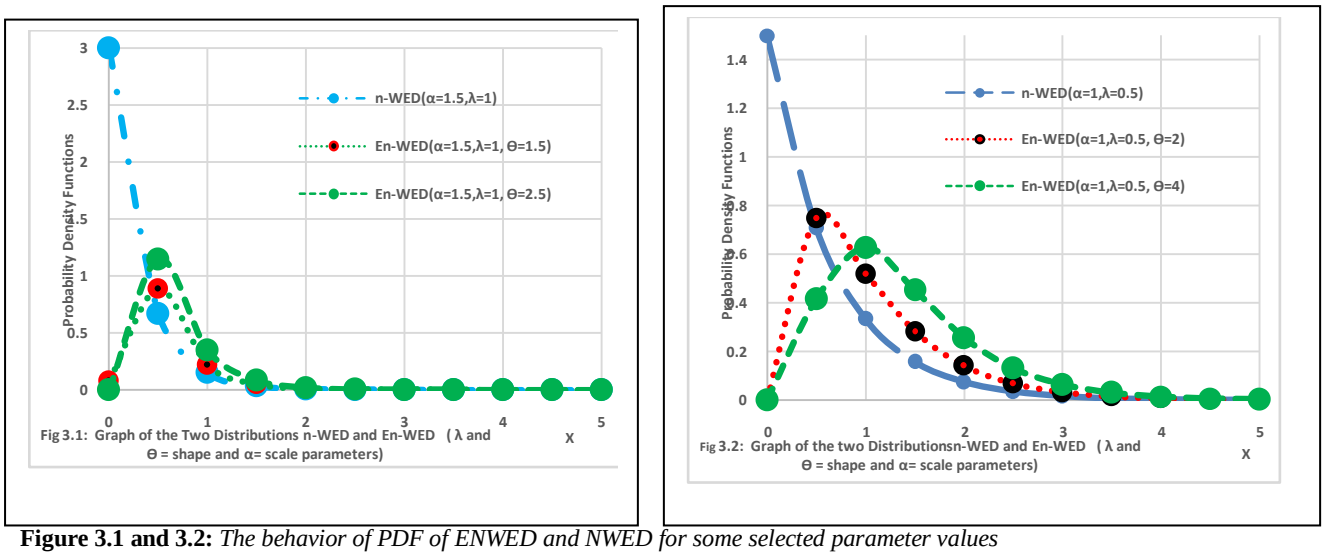


Figure 3.1 and 3.2: *The behavior of PDF of ENWED and NWED for some selected parameter values*

Note:

The blue dotted curve for fig 3.1 &3.2 represent the NWED with two parameters (α, λ) while the green and red dotted curves represent the ENWED with three parameters $(\alpha, \lambda, \theta)$

From the above figures, it can be deduced that addition of a shape parameter enhances the distribution, thereby making the model more flexible. As an exponentiated parameter (θ) is introduced into NWED, the plot gives a bell-like shape meaning there is an enhancement to what the model can do previously.

3.3 Reliability Analysis of Exponentiated-New Weighted Exponential Distribution

3.3.1 Survival Function

Survival function can be defined as the probability that an event happened after time, t. It models time to-an-event and takes note of the time it takes for an event to occur. Examples of such event are; Time to get a job after graduation, time it takes for a bank customer to wait before service is being rendered. Given the cdf of a probability function, the survival function is the probability of survival beyond time x and is defined by;

$$S(x) = 1 - F(x) \tag{3.9}$$

For the exponentiated-new weighted exponential distribution, the survival function is given as:

$$S(x) = 1 - \left[1 - e^{-(1+\lambda)\alpha x}\right]^{\theta};\ x > 0, \alpha > 0, \lambda > 0, \theta > 0 \tag{3.10}$$

Where: α and λ are the parameters and θ is the exponentiated parameter.

3.3.1 Hazard Function

The hazard function is also known as the failure rate is defined as the conditional probability of failure of an item/device given that the item has survived to the time, t. It is use in reliability applications to describe the instantaneous failure rate at any point in time. The hazard rate

function of a random variable X is the ratio of the pdf, $f(x)$ to the survival function, $S(x)$ and is given by:

$$H(x) = \frac{f(x)}{S(x)}; \tag{3.11}$$

By substituting equation (3.3) and (3.9) into equation (3.11), the hazard function for the exponentiated-new weighted exponential distribution is given by

$$h(x) = \frac{\theta(1+\lambda)\alpha e^{-(1+\lambda)\alpha x}\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta-1}}{1-\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta}}; x>0, \alpha>0, \lambda>0, \theta>0 \tag{3.12}$$

3.3.2 Asymptotic Behaviors

To investigate the behavior of the hazard function as $x \to 0$ and $x \to \infty$

$$\lim_{x\to 0} h(x) = \lim_{x\to 0}\left\{\frac{\theta(1+\lambda)\alpha e^{-(1+\lambda)\alpha x}\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta-1}}{1-\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta}}\right\} = 0 \tag{3.13}$$

$$\lim_{x\to \infty} h(x) = \lim_{x\to \infty}\left\{\frac{\theta(1+\lambda)\alpha e^{-(1+\lambda)\alpha x}\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta-1}}{1-\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta}}\right\} = (1+\lambda)\alpha \tag{3.14}$$

Proof of equation 3.14

$$\lim_{x\to \infty} h(x) = \lim_{x\to \infty}\left\{\frac{\theta(1+\lambda)\alpha e^{-(1+\lambda)\alpha x}\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta-1}}{1-\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta}}\right\}$$

let $k(x) = e^{-(1+\lambda)\alpha x}$ (3.15)

$$\lim_{x\to \infty} k(x) = e^{-(1+\lambda)\alpha x} = 0$$

Substituting equation (4.15) into (4.14); we have,

$$h(k) = \frac{\theta(1+\lambda)\alpha k(1-k)^{\theta-1}}{1-(1-k)^{\theta}} \tag{3.16}$$

Therefore $\lim_{x\to \infty} h(x) = \lim_{k\to 0} h(k)$

Recall by binomial expansion,

$$(1-k)^n = 1-\binom{n}{1}k^1+\binom{n}{2}k^2-\binom{n}{3}k^3+...$$

$$(1-k)^\theta = 1-\binom{\theta}{1}k^1+\binom{\theta}{2}k^2-\binom{\theta}{3}k^3+...$$

$$(1-k)^{\theta-1} = 1-\binom{\theta-1}{1}k^1+\binom{\theta-1}{1}k^2-\binom{\theta-1}{1}k^3+...$$

Therefore equation (4.16) can be re-written as

$$h(k)=\frac{\theta(1+\lambda)\alpha k[1-\binom{\theta-1}{1}k^1+\binom{\theta-1}{1}k^2-\binom{\theta-1}{1}k^3+...]}{1-[1-\binom{\theta}{1}k^1+\binom{\theta}{2}k^2-\binom{\theta}{3}k^3+...]}$$

$$h(k)=\frac{\theta(1+\lambda)\alpha k[1-(\theta-1)k+\frac{1}{2}(\theta-1)(\theta-2)k^2-\frac{1}{6}(\theta-1)(\theta-2)(\theta-3)k^3+...]}{1-[1-(\theta-1)k+\frac{1}{2}(\theta-1)(\theta-2)k^2-\frac{1}{6}(\theta-1)(\theta-2)(\theta-3)k^3+...]}$$

$$h(k)=\frac{\theta(1+\lambda)\alpha k[1-(\theta-1)k+\frac{1}{2}(\theta-1)(\theta-2)k^2-...]}{\theta k-\frac{1}{2}\theta(\theta-1)k^2+\frac{1}{6}\theta(\theta-1)(\theta-2)k^3-...]}$$

$$h(k)=\frac{\theta(1+\lambda)\alpha k[1-(\theta-1)k+\frac{1}{2}(\theta-1)(\theta-2)k^2-...]}{k[\theta-\frac{1}{2}\theta(\theta-1)k+\frac{1}{6}\theta(\theta-1)(\theta-2)k^2-...]}$$

$$=\frac{\theta(1+\lambda)\alpha[1-(\theta-1)k+\frac{1}{2}(\theta-1)(\theta-2)k^2-...]}{\theta-\frac{1}{2}\theta(\theta-1)k+\frac{1}{6}\theta(\theta-1)(\theta-2)k^2-...]}$$

$$\lim_{k\to 0} h(k)=\frac{\theta(1+\lambda)\alpha[1-0+0-0.....]}{\theta-0+0-....}$$

$$=(1+\lambda)\alpha$$

$$\therefore \lim_{x\to\infty} h(x)=\lim_{k\to 0} h(k)=(1+\lambda)\alpha$$

Going by the above result, the constant in the hazard function after taking the limit of $h(x)$ as $x\to\infty$ shows that the hazard function is independent of the shape parameter, θ

Following the steps above in hazard function, taking $\lim_{x\to 0} f(x;\alpha,\lambda,\theta)$ and $\lim_{x\to\infty} f(x;\alpha,\lambda,\theta)$ of the pdf of exponentiated-new weighted exponential distribution is examine as follows; we have from equation 3.3,

$$\lim_{x\to 0} f(x) = \theta(1+\lambda)\alpha e^{-(1+\lambda)\alpha x}\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta-1} = 0$$

and

$$\lim_{x\to \infty} f(x) = \theta(1+\lambda)\alpha e^{-(1+\lambda)\alpha x}\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta-1} = 0$$

From the above results as $x\to\infty$, $f(x)=0$ and $x\to 0$, $f(x)=0$ this indicates that the distribution has a unique mode.

From equation 3.4, $\lim_{x\to 0} F(x;\alpha,\lambda,\theta)$ and $\lim_{x\to\infty} F(x;\alpha,\lambda,\theta)$ of the cdf of the exponentiated-new weighted exponential distribution is given by:

$$\lim_{x\to 0} F(x) = \lim_{x\to 0}\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta} = 0$$

and

$$\lim_{x\to \infty} F(x) = \lim_{x\to \infty}\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta} = 1$$

3.4 Statistical Properties of the Exponentiated-New Weighted Exponential Distribution

3.4.1 The Quantile Function and the Median

The quantile function, q of a distribution is the inverse of its probability distribution function. It is used in the generation of random realizations from a given distribution.

Let $Q(q)=$ be the quantile function of $F(x)$ for $0<q<1$, solving for $F(x)=q$, the quantile function of x gives:

$$x = Q(q) = F^{-1}(q) \qquad (3.17)$$

Therefore the quantile function of the Exponentiated-New Weighted Exponential Distribution is obtained through a solution to the following equation x_q

$$F(x_q) = q \tag{3.18}$$

Where q follows a uniform distribution with interval (0, 1) and $F(x_q)$ is the cdf of Exponentiated-New Weighted Exponential Distribution, therefore substituting equation (3.4) into equation (3.18), we have:

$$\left[1 - e^{-\alpha(1+\lambda)x_q}\right]^{\theta} = q \tag{3.19}$$

Solving for x_q in equation (3.19)

$$\left[1 - e^{-\alpha(1+\lambda)x_q}\right] = q^{\frac{1}{\theta}}$$

$$\left(1 - q^{\frac{1}{\theta}}\right) = e^{-\alpha(1+\lambda)x_q}$$

$$\ell n\left(1 - q^{\frac{1}{\theta}}\right) = -\alpha(1+\lambda)x_q$$

A random sample $x_1, x_2, ..., x_n$ can be generated from the ENWED using the quantile function by assuming q to be uniformly distributed in the interval (0, 1) and is given by:

$$x_q = \frac{1}{-\alpha(1+\lambda)} \ell n\left(1 - q^{\frac{1}{\theta}}\right) \tag{3.20}$$

The median is derived when $q = 0.5$

$$x_{0.5} = \frac{1}{-\alpha(1+\lambda)} \ell n\left(1 - (0.5)^{\frac{1}{\theta}}\right) \tag{3.21}$$

3.4.2 Moments and Moment Generating Function

In statistical analysis, applied work to be precise, emphasizes on the necessity and importance of the moments because significant attributes (features) of a distribution can be studied through the moments, example of such includes the measures of central tendency, dispersion, skewness and kurtosis.

For any continuous distribution with a pdf $f(x)$, the rth moment μ_r is given by:

$$\mu_r = E\left[X^r\right] = \int_{-\infty}^{\infty} X^r f(x)dx \tag{3.22}$$

The Moments is therefore given by;

$$u_r^i = E\left[X^r\right] = \int_{0}^{\infty} x^r f(x)dx \tag{3.23}$$

$$E[X^r] = \int_{0}^{\infty} x^r \theta(1+\lambda)\alpha e^{-\alpha(1+\lambda)x}\left[1-e^{-\alpha(1+\lambda)x}\right]^{\theta-1} dx \tag{3.24}$$

By binomial expansion,

$$\left[1-e^{-\alpha(1+\lambda)x}\right]^{\theta-1} = \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i e^{-i\alpha(1+\lambda)x} \tag{3.25}$$

By substituting equation (3.25) into (3.24), we have

$$E[X^r] = \theta(1+\lambda)\alpha\sum\binom{\theta-1}{i}(-1)^i \int_{0}^{\infty} x^r e^{-\alpha(1+i)(1+\lambda)x}dx \tag{3.26}$$

Let $m = \alpha(1+i)(1+\lambda)x$ (3.27)

$$x = \frac{m}{\alpha(1+i)(1+\lambda)}$$

$$dm = \alpha(1+i)(1+\lambda)dx \tag{3.28}$$

$$dx = \frac{dm}{\alpha(1+i)(1+\lambda)} \tag{3.29}$$

Putting eqn. (3.27), (3.28) and (3.29) into (3.26) will lead to as $x \to 0, m \to 0, x \to \infty, m \to 0$

$$u_r^i = \theta(1+\lambda)\alpha\sum\binom{\theta-1}{i}(-1)^i\int_0^\infty \frac{m^r}{\alpha^r(1+i)^r(1+\lambda)^r}e^{-m}\frac{dm}{\alpha(1+i)(1+\lambda)}$$

$$= \frac{\theta\sum\binom{\theta-1}{i}(-1)^i}{\alpha^r(1+i)^{r+1}(1+\lambda)^r}\int_o^m m^r e^{-m}dm = \theta\Gamma(r+1)\sum\frac{\sum\binom{\theta-1}{i}(-1)^i}{(\alpha(1+\lambda))^r(1+i)^{r+1}}$$

But $\binom{\theta-1}{i} = \frac{(\theta-1)i}{(\theta-1-i)!i!}$

$$= \frac{\Gamma\theta}{(\theta-i-1)!i!}$$

Therefore the moment is given as:

$$u_r^i = \frac{\theta\Gamma\theta\Gamma(r+1)}{(\alpha(1+\lambda))^r}\sum\frac{(-1)^i}{(1+i)^{r+1}}$$

$$= \frac{\Gamma(\theta+1)\Gamma(r+1)}{\alpha^r(1+\lambda)^r}\sum\frac{(-1)^i}{(1+i)^{r+1}(\theta-i-1)!}$$

The moment generating function is given by:

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx}f(x)dx \qquad (3.30)$$

using series expansion $e^{tx} = \sum_{m=1}^{\infty}\frac{t^m x^m}{m!}$

$$\Rightarrow M_x(t) = \sum_{m=0}^{\infty}\frac{t^m}{m!}\int x^m f(x)dx = \sum_{m=0}^{\infty}\frac{t^m}{m!}E(X^m)$$

As in the case of moment above,

$$E(X^r) = E(X^m)\,\text{For}\,m = r$$

$$\Rightarrow E(X^m) = \frac{\Gamma(\theta+1)\Gamma(m+1)}{\alpha^m(1+\lambda)^m}\sum_{i=0}^{\infty}\frac{(-1)^i}{(1+i)^{m+1}(\theta-i-1)!i!}$$

Hence, the moment generating function of ENWED is;

$$M_x(t) = \frac{\Gamma(\theta+1)\Gamma(m+1)}{\alpha^m(1+\lambda)^m}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^m(-1)^i}{m!i!(\theta-i-1)!(1+i)^{m+1}}$$

3.4.3 Distribution of Order Statistics

In non-parametric statistics and inference, the order statistics is one of the most essential and of course fundamental tools. Let $x_1, x_2, ..., x_{n,}$ denotes n-independent random variable from a distribution function $F(x)$ with probability density function, $f(x)$, then $x_1, x_2, ..., x_{n,}$ represent the order sample arrangement and the pdf of $x_{(n)}$ is given by:

$$f_{i(n)}(x) = \frac{n!}{(i-1)!(n-i)!}f(x)\left[F(x)\right]^{i-1}\left[1-F(x)\right]^{n-i}; \quad \text{For i=1,2,…,n.} \tag{3.31}$$

where: $f(x)$ and $F(x)$ is the pdf and cdf of ENWED as defined in 3.3 and 3.8 respectively Then the pdf of the order statistics $x_{(i)}$ from the exponentiated-new weighted exponential distribution is given by:

$$f_{i(n)}(x) = \tfrac{n!}{(i-1)!(n-i)!}\theta(1+\lambda)\alpha e^{-(1+\lambda)\alpha x}\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta-1} X\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta(i-1)} X\left[1-\left[1-e^{-(1+\lambda)\alpha x}\right]\right]^{\theta(n-i)} \tag{3.32}$$

The pdf of the largest order statistic $x_{(n)}$ and the smallest order statistic $x_{(1)}$ are therefore given by:

$$f_{(n)}(x) = n\theta(1+\lambda)\alpha e^{-(1+\lambda)\alpha x}\left[1-e^{-(1+\lambda)\alpha x)}\right]^{\theta-1} X\left[1-e^{-(1+\lambda)\alpha x}\right]^{\theta(n-1)} \tag{3.33}$$

and

$$f_{(1)}(x) = n\theta(1+\lambda)\alpha e^{-(1+\lambda)\alpha x}\left[1-e^{-(1+\lambda)\alpha x)}\right]^{\theta-1} X\left[1-\left(1-e^{-(1+\lambda)\alpha x}\right)\right]^{\theta(n-1)} \tag{3.34}$$

respectively.

3.5 Parameter Estimation and Application

In this work, we will be using the method of maximum likelihood estimation for estimating the parameters of the Exponentiated-New Weighted Exponential distribution and its competitors.

3.5.1 Exponentiated-New Weighted Exponential Distribution

Let $X_1, X_2, ..., X_n$ be a random sample of n independently and identically distributed random variables each having the Exponentiated-New Weighted Exponential distribution as defined in equation (3.3), the likelihood function L is given by

$$L = \Pi_{i=1}^{n} f(x_i) = \theta^n (1+\lambda)^n \alpha^n \prod_{i=1}^{n} e^{-(1+\lambda)\alpha \sum x_i} \left[1 - e^{-(\alpha x + \alpha\lambda x)}\right]^{\theta-1} \tag{3.35}$$

Taking the log of the likelihood function, we have

$$\log L = n\log\theta + n\log(1+\lambda) + n\log\alpha - (1+\lambda)\alpha\sum x_i + (\theta-1)\sum \log\left[1 - e^{-(\alpha x + \alpha\lambda x)}\right] \tag{3.36}$$

Therefore the MLEs of θ, α, λ can be derived from the derivatives of eqn. (3.36) with respect to the parameters and equating them to zero, they should satisfy the following equations:

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \sum \log\left[1 - e^{-(\alpha x + \alpha\lambda x)}\right] \tag{3.37}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{(1+\lambda)} - \alpha\sum x_i - (\theta-1)\sum_{i=1}^{n} \frac{\alpha x e^{-\alpha\lambda x}}{\left[1 - e^{-(\alpha x + \alpha\lambda x)}\right]} \tag{3.38}$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - (1+\lambda)\sum x_i + (\theta-1)\sum_{i=1}^{n} \frac{(x+\lambda x)e^{-(\alpha x + \alpha\lambda x)}}{\left[1 - e^{-(\alpha x + \alpha\lambda x)}\right]} \tag{3.39}$$

Setting equations 3.37, 3.38 and 3.39 to zero, solutions can be derived. The MLEs of the non-linear equation can only be estimated using any iterative method such as Newton-Raphson algorithm.

3.5.2 Weighted Exponential Distribution

Let $x_1, x_2, ..., x_n$ be a random sample of n independently and identically distributed random variables each having the Weighted Exponential distribution. Given the pdf as

$$f(x) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left[1-e^{-\lambda\alpha x}\right]; x>0, \alpha>0, \lambda>0 \tag{3.40}$$

the likelihood function L is given by;

$$L(x_i|\alpha,\lambda) = \prod_{i=1}^{n} f(x_i)$$

$$= \prod_{i=1}^{n} \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left[1-e^{-\alpha\lambda x}\right] \tag{3.41}$$

$$= (\alpha+1)^n \alpha^n \lambda^n e^{-\lambda\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left[1-e^{-\alpha\lambda x}\right]$$

$$\log L = n\log(\alpha+1) - n\log\alpha + n\log\lambda - \lambda\sum_{i=1}^{n} x_i + \sum\log\left[1-e^{-\alpha\lambda x}\right] \tag{3.42}$$

Therefore the MLEs of α, λ can be derived from the derivatives of eqn. (3.42) with respect to the parameters and equating them to zero, they should satisfy the following equations:

$$\frac{\partial l}{\partial\alpha} = \frac{n}{(1+\alpha)} - \frac{n}{\alpha} + \sum_{i=1}^{n}\frac{\lambda x e^{-\lambda\alpha x}}{\left[1-e^{-\lambda\alpha x}\right]} \tag{3.43}$$

$$\frac{\partial l}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{\alpha x e^{-\lambda\alpha x}}{\left[1-e^{-\lambda\alpha x}\right]} \tag{3.44}$$

Setting equations 3.43 and 3.44 to zero, solutions can be derived. The MLEs of the non-linear equation can only be estimated using any iterative method such as Newton-Raphson algorithm.

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3.5.3 Exponentiated Weighted Exponential Distribution

Let $x_1, x_2, ..., x_n$ be a random sample of n independently and identically distributed random variables each having the Exponentiated Weighted Exponential distribution. The pdf as defined by Oguntunde P.E (2015) is given as below:

$$f(x) = \theta\left[\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-e^{-\lambda\alpha x}\right)\right]\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta-1} \tag{3.45}$$

$$L(x_i|\theta,\lambda,\alpha) = \prod_{i=1}^{n}\theta\left[\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-e^{-\lambda\alpha x}\right)\right]\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta-1} \tag{3.46}$$

$$= \prod_{i=1}^{n}\theta\left[\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-e^{-\lambda\alpha x}\right)\right]\left(\frac{\alpha+1}{\alpha}\right)^{\theta-1} - \left(\frac{\alpha+1}{\alpha}e^{-\lambda x}\right)^{\theta-1} - \left(\frac{1}{\alpha}\right)^{\theta-1} + \left(\frac{1}{\alpha}e^{-\lambda x(1+\alpha)}\right)^{\theta-1}$$

$$L = \theta^n\left[\left(\frac{\alpha+1}{\alpha}\right)^n \lambda^n e^{-\lambda\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-e^{-\lambda\alpha x}\right)\right]\left(\frac{\alpha+1}{\alpha}\right)^{n(\theta-1)} - \left(\frac{\alpha+1}{\alpha}e^{-\lambda x}\right)^{n(\theta-1)} - \left(\frac{1}{\alpha}\right)^{n(\theta-1)} + \left(\frac{1}{\alpha}e^{-\lambda x(1+\alpha)}\right)^{n(\theta-1)}$$

Taking the log likelihood of the equation (3.46), gives

$$\log L = n\log\theta + n\log(\alpha+1) - n\log\alpha + n\log\lambda - \lambda\sum_{i=1}^{n}x_i + \sum\log\left(1-e^{-\lambda\alpha x}\right) + n(\theta-1)\log(\alpha+1)$$
$$-n(\theta-1)\log(\alpha) - n(\theta-1)\log\left(\frac{\alpha+1}{\alpha}e^{-\lambda\sum_{i=1}^{n}x_i}\right) - n(\theta-1)\log\left(\frac{1}{\alpha}\right) + n(\theta-1)\log\left(\frac{1}{\alpha}e^{-\lambda\sum_{i=1}^{n}x_i(1+\alpha)}\right) \tag{3.47}$$

Solving the resulting simultaneous equation of

$$\frac{\partial l}{\partial\alpha} = 0, \frac{\partial l}{\partial\lambda} and \frac{\partial l}{\partial\theta} = 0$$

Gives the maximum likelihood estimates of α, λ, θ

The MLEs of the non-linear equation can only be estimated using any iterative method such as Newton-Raphson algorithm.

3.5.4 New Weighted Exponential Distribution

Let $x_1, x_2, ..., x_n$ be a random sample of n independently and identically distributed random variables each having the New Weighted Exponential distribution. The pdf as defined in equation (1.7) is given as:

$$f(x) = (1+\lambda)\alpha e^{-(1+\lambda)x}$$

$$L(x|\alpha,\lambda) = \prod_{i=1}^{n}(1+\lambda)\alpha e^{-(1+\lambda)x} \tag{3.48}$$

$$= (1+\lambda)^n \alpha^n e^{-(1+\lambda)\sum_{i=1}^{n} x_i}$$

$$\log L = n\log(1+\lambda) + n\log\alpha - \alpha\sum_{i=1}^{n} x_i(1+\lambda) \tag{3.49}$$

Therefore the MLEs of α, λ can be derived from the derivatives of eqn. (3.49) with respect to the parameters and equating them to zero, they should satisfy the following equations:

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i(1+\lambda) \tag{3.50}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{(1+\lambda)} - \alpha\sum_{i=1}^{n} x_i \tag{3.51}$$

Setting equations 3.50 and 3.51 to zero, solutions can be derived. The MLEs of the non-linear equation can only be estimated using any iterative method such as Newton-Raphson algorithm.

3.6 Data Analysis Procedure

The Kernel based density estimation technique and the classical Kolmogrov-Smirnov two sample test procedure is hereby used to give a clearer picture on the usefulness of the Exponentiated-New Weighted Exponential distribution during the analysis of the real life datasets.

3.6.1 The Kernel-based Density Estimation

The Kernel-based Density Estimation is a non-parametric density estimation which requires no assumption that the underlying density function is from a parametric family. Kernel Density Estimation also known as Parzen's Window (Parzen, 1962), is a popular way to estimate the underlying probability density function of a dataset. It learns the shape of the density from the data automatically. Kernel density estimation is mostly used as a tool to explore the pattern of a data.

Using the kernel approach, the estimator of the probability density function $f(x)$ is given by:

$$\hat{f}(x) = \frac{1}{n\xi}\sum_{i=1}^{n} k\left(\frac{x - x_i}{\xi}\right), x > 0 \qquad (3.52)$$

Where: $k(.)$ is the kernel, ξ is the smoothing parameter and n is the sample size of the data. The Epanechnikov (1969)'s kernel, $k(.)$ is an appropriate kernel function and is defined by:

$$k(r) = \begin{cases} \dfrac{0.75(1-0.2r^2)}{\sqrt{5}} & if\ |r| \le \sqrt{5} \\ 0 \quad elsewhere \end{cases} \qquad (3.53)$$

The parameter ξ determines the smoothness of the estimator; whenever ξ increases the mean square error is being reduced but there will be an increase in the biasness of the estimate. For computational studies, we used the value of ξ as given by Feksel (1988) as:

$$\xi = \frac{25}{\sqrt{\psi}}, \qquad (3.54)$$

Where: ψ is the mean of the data set being considered.

The estimated ENWED and NWED probability density functions will be compared graphically with $\hat{f}(x)$ (eqn(3.47)).

3.6.2 Kolmogrov-Smirnov Two Sample Test

Let X and Y be populations with cumulative distribution functions denoted by F_x and F_y respectively. A random sample of size m is drawn from population X and another random sample of size n is drawn independently from population Y

$$X_1, X_2, ..., X_m \text{ and } Y_1, Y_2, ..., Y_n$$

In K-S test the hypotheses of concern is whether the two samples comes from the same population.

$$H_0 : F_Y(x) = F_X(x) \ \forall x$$

For the two random samples from populations F_X and F_Y, the order statistics is given by:

$$X_{(1)}, X_{(2)}, ..., X_{(m)} \text{ and } Y_{(1)}, Y_{(2)}, ..., Y_{(n)}$$

$S_m(x)$ and $S_n(x)$ is the empirical distribution functions for each population and is given by:

$$S_m(x) = \begin{cases} 0 & if \quad x < X_{(1)} \\ \dfrac{k}{m} & if \quad X_k \le x < X_{k+1} \\ 1 & if \quad x \ge X(m) \end{cases} \qquad k = 1, 2, ..., m-1$$

and

$$S_n(x) = \begin{cases} 0 & if \quad x < Y_{(1)} \\ \dfrac{k}{n} & if \quad Y_k \le x < Y_{k+1} \\ 1 & if \quad x \ge Y(n) \end{cases} \qquad k = 1, 2, ..., n-1$$

In a combined ordered arrangement of the $N = m + n$ sample observations, $S_m(x)$ and $S_n(x)$ are the respective proportions of X and Y observations which do not exceed the specified value of x. If the null hypothesis is true, the population distributions are identical. The two-sided Kolmogrov-Smirnov two-sample Test criterion $D_{m,n}$ is based on the maximum absolute difference between the two empirical distributions

$$D_{m,n} = \underset{x}{Sup} \left| S_m(x) - S_n(x) \right| \tag{3.55}$$

and the rejection region is the upper tail given as:

$$D_{m,n} \geq C_\alpha$$

where

$$P\left(D_{m,n} \geq C_\alpha \,/\, H_0 \right) = \alpha \tag{3.56}$$

For m and n large, right-tail critical values based on the asymptotic distribution can be calculated as:

α	0.2	0.1	0.05	0.01
C_α	$1.07\sqrt{\frac{N}{nm}}$	$1.22\sqrt{\frac{N}{nm}}$	$1.36\sqrt{\frac{N}{nm}}$	$1.63\sqrt{\frac{N}{nm}}$

3.7 Application

In the next chapter we will use real life data sets to illustrate the usefulness of the Exponentiated-New Weighted Exponential distribution for modeling reliability data. We will make a comparison of the results between the Exponentiated-New Weighted Exponential distribution and other competing distributions. To give a clearer picture of the Exponentiated-New Weighted Exponential distribution, the kernel based estimation technique and the classical Kolmogorov-Smirnov two sample test procedure will be utilize during the analysis of the real life data sets.

CHAPTER FOUR

ANALYSIS AND DISCUSSIONS

In this section, the ENWED is fitted to two real life datasets, Oguntunde *et al.*, (2016) used them to examine the flexibility of the NWED over the Weighted Exponential distribution. The first dataset has previously been utilized by Ghitany *et al.*, (2008), it represents the waiting time (measured in minutes) before service is being rendered to 100 bank customers. The data is given below in table 4(a)

Table 4.1 (a): Waiting Time in Minutes of Bank Customers

0.8	8.8	1.9	9.6	4.4
3.3	11.2	4.2	12.5	6.2
4.7	14.1	5.3	18.1	8
6.3	21.4	7.1	33.1	10.7
8.6	1.5	8.9	2.6	13.1
11	4	11.9	4.3	18.9
13.7	4.9	17.3	5.7	3.1
20.6	7.1	27	7.6	4.6
0.8	8.8	1.9	9.7	6.2
3.5	11.2	4.2	12.9	8.2
4.8	15.4	5.5	18.2	10.9
6.7	21.9	7.1	38.5	13.3
8.6	1.8	9.5	2.7	19
11.1	4.1	12.4	4.4	3.2
13.9	5	17.3	6.1	4.7
21.3	7.1	31.6	7.7	6.2
1.3	8.9	2.1	9.8	8.6
3.6	11.5	4.3	13	11
4.9	15.4	5.7	18.4	13.6
6.9	23	7.4	2.9	19.6

Source: Ghitany, et. al. (2008)

Table 4.1(b): Random sample of size 100 drawn from ENWED(α=0.331, λ=-0.520,Θ=2.183)

8.9	18.0	8.6	4.8	11.8
11.3	8.5	6.7	16.4	19.8
5.2	20.9	11.8	5.7	7.2
5.5	2.1	8.5	2.4	29.0
9.1	6.6	1.5	3.2	3.9
20.4	31.6	15.8	6.8	9.2
12.4	8.9	6.1	11.1	6.2
4.9	7.9	9.2	11.7	21.4
7.1	12.3	14.3	6.7	16.0
4.0	2.8	8.4	4.1	5.4
8.9	2.7	17.8	5.0	4.7
12.6	4.6	26.0	1.3	10.6
8.4	20.7	10.8	12.5	13.1
19.1	2.2	8.7	8.7	1.6
7.7	4.4	8.7	7.9	4.5
4.9	3.5	22.0	13.9	4.9
10.5	12.9	20.3	25.6	13.5
5.5	17.3	5.7	9.8	9.6
14.1	8.7	1.2	1.2	6.8
5.8	2.0	34.4	8.9	12.2

Simulated Data from ENWED

Table 4.2 presents the estimates of the parameters, log-likelihood and the Alkaike Information Criterion (AIC) generated based on the data for waiting time of 100 bank customers.

To compare the performances of these distributions, the AIC will be used as a basis and the model with the lowest value of AIC will be considered as the best of model to fit the dataset. The statistics of the AIC is given by:-

$$AIC = -2ll + 2k$$

where the ll represents the log-likelihood and k is the number of parameters to be estimated. The R software is used to determine the MLE and the log-likelihood for the ENWED in Table 4.2

Table 4.2: Performance of distributions (Maximum Likelihood Estimates, Log-likelihood and AIC for data on waiting time of 100 bank customers fitted on ENWED and NWED.

Distributions	**Estimates**	**Log-likelihood**	**AIC**
New Weighted Exponential	$\hat{\alpha}$ =0.004735 $\hat{\lambda}$ =20.383945	-329.0209	662.0418
Weighted Exponential	$\hat{\alpha}$ =0.030411 $\hat{\lambda}$ =1.000434	-635.7945	1275.589
Exponentiated Weighted Exponential	$\hat{\alpha}$ =418.995 $\hat{\lambda}$ =137.5429 $\hat{\theta}$ =7.0227e-07	-62504.25	125014.5
Exponentiated-New Weighted Exponential	$\hat{\alpha}$ =0.3311753 $\hat{\lambda}$ =-0.5195202 $\hat{\theta}$ =2.182741	-317.0953	640.1906

Source: Software: R, AIC: Alkaike Information Criteria

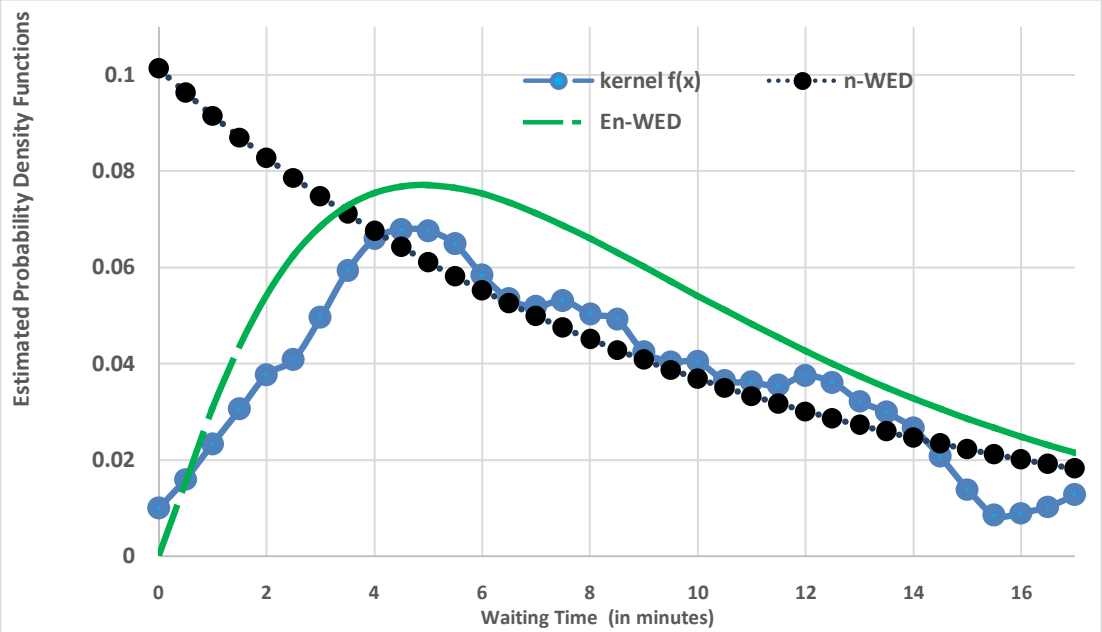


Fig 4.1: Estimated Probability Density Function for the Data on Waiting Time of Bank Customers

Table 4.2 shows the maximum likelihood estimates (MLE) associated to each distribution fitted on the first dataset. It also shows the log-likelihood and the AIC for each model. The AIC of the ENWED can be seen to be lower when compared to the NWED, WED, EWED indicating that the ENWED outperforms the NWED, WED, EWED for fitting the data set. Also the kernel density approach was used to estimate the unknown probability density function $f(x)$ of the real life data. Using the same data set for both the Exponentiated-New Weighted Exponential distribution (ENWED) and the New-Weighted Exponential distribution (NWED), the plots of the $\hat{f}(x)$ for the real life data and the estimated density function is also given in figure 4.1. The estimated density function from the proposed distribution can be seen to closely imitate that of the estimated kernel based density function than the baseline distribution.

Having estimated the ENWED parameters from the data based on waiting time of bank customers, the quantile function is used to generate a sample of size 100 observations; the generated sample is given in Table 4.1b.Using the Kolmogrov-Smirnov test, we now test the null hypothesis that the data on waiting time of 100 bank customers before service is being rendered and the sample drawn from the fitted ENWED are both from the same distribution.

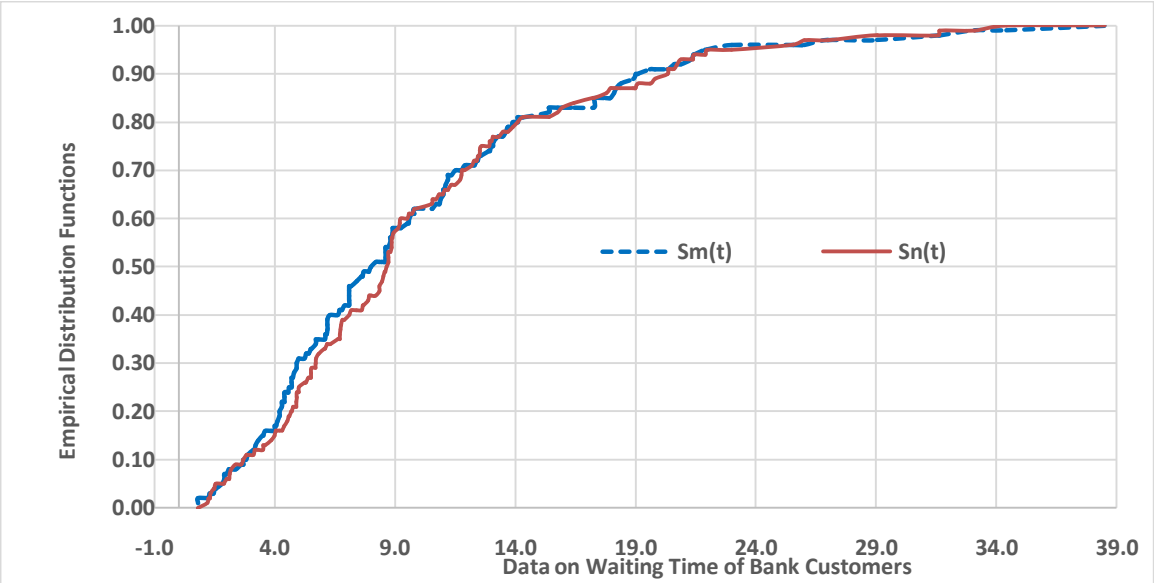


Fig 4.2: Kolmogrov-Smirnov two sample test: Graph of the empirical distribution Functions

The computed Kolmogrov-Smirnov statistic based on both empirical distribution functions $S_m(x)$ and $S_n(x)$ is:

$$D_{100,100} = \underset{x}{Sup} \mid S_m(x) - S_n(x) \mid$$
$$= 0.0900$$

The null hypothesis that the data on waiting time of 100 bank customers and the one drawn from the proposed distribution are from the same distribution can only be rejected if $D_{100,100} \geq C_\alpha$ with $P\left(D_{100,100} \geq C_\alpha\right) = 0.01$ but $C_\alpha = 0.230517$ but since $D_{100,100} = 0.0900$ is less than the critical value of 0.230517 at the 1% level of significance, we thereby accept the hypothesis and come to the conclusion that both dataset are from the same distribution. It can be clearly seen from Figure 4.2

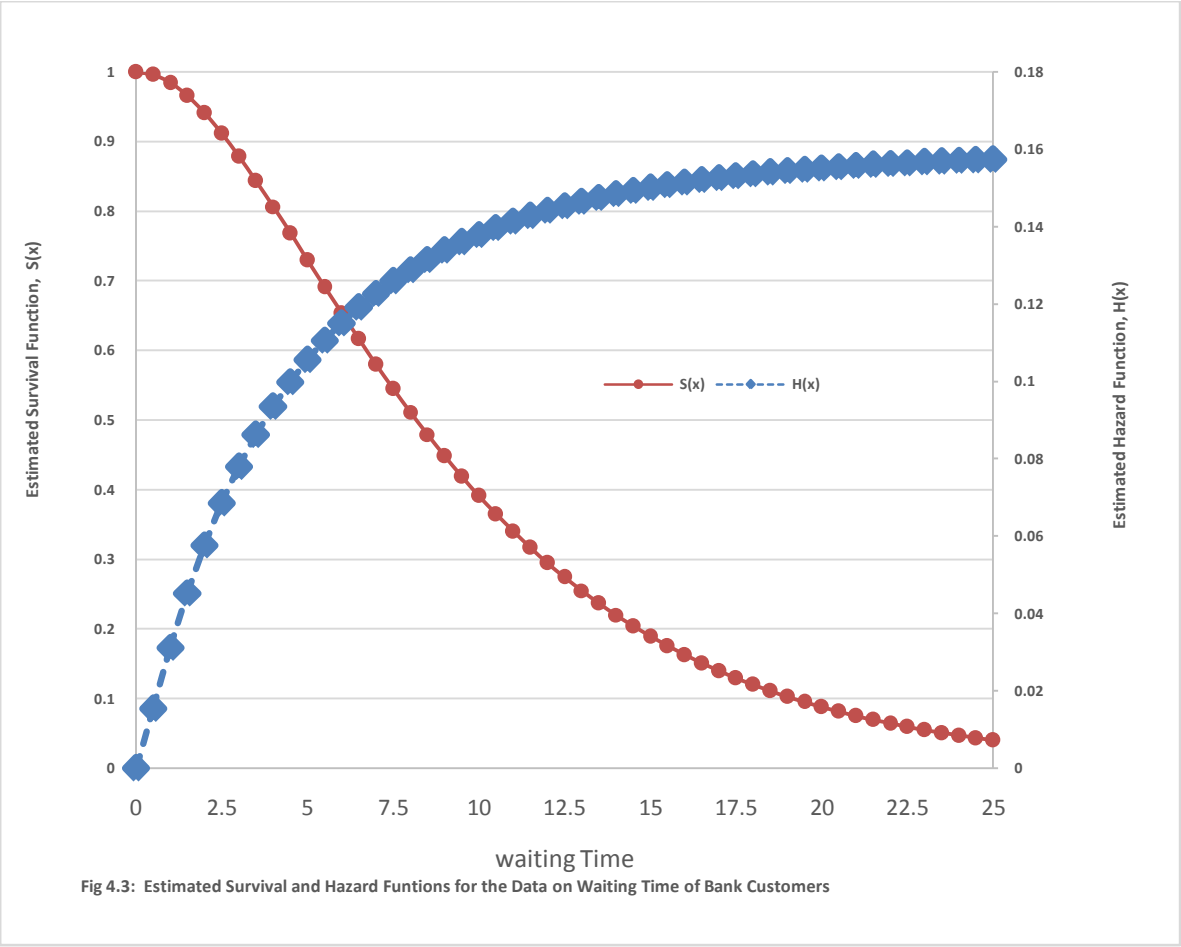


Fig 4.3: Estimated Survival and Hazard Funtions for the Data on Waiting Time of Bank Customers

Figure 4.3 represents the hazard and survival function based on the data on waiting time of bank customers. On X-axis represent the time in minutes at which it takes for an event (in this case event denotes waiting time before a bank customer is serviced) to happen after a given time t, and at the same time also describe the instantaneous failure rate at any point in time. While Y-axis represents the probability of survival and hazard function.

From the curve in fig. 4.3, at $s(x) = 1$ and at time (0.1mins), all bank customers were not attended to and thereafter customers are being attended to gradually as time increase, thereby reducing the percentage of bank customers that have been serviced as time increases.

Also from time, 0mins, the hazard function is seen to be on a steady increase as time increases to approximately 6mins when the rate of increase reduced, this implies that the customers are now being attended to at a regular pace.

Data 2:

The second real life dataset was used by Nasiri *et al.*,(2011); it is a data from an accelerated life test of 59 conductors. The data is given below:

Table 4.3 (a): Accelerated Life Test of 59 Conductors

6.545	6.573	7.398	9.289	6.538
6.033	7.543	5.589	10.092	6.956
6.087	7.496	6.492	5.087	4.531
5.459	6.725	7.974	8.120	8.532
8.799	4.706	6.663	7.683	8.687
6.369	7.224	2.997	7.024	7.365
8.591	8.336	6.923	6.129	9.218
5.640	11.038	7.945	5.434	5.381
6.869	7.937	6.958	6.352	6.515
4.288	4.700	6.476	6.522	6.948
6.071	4.137	9.254	10.491	7.459
5.009	5.923	7.495	7.489	

Source: Nasir, *et. al*. (2011)

Table 4.3(b): Random sample of size 59 drawn from ENWED(α=0.005, λ= 129.97,Θ=55.83)

6.902	6.761	6.256	7.996	8.355
7.577	9.556	12.635	9.094	6.761
5.826	6.560	6.901	6.845	9.225
5.911	5.722	6.625	4.582	11.261
6.975	7.356	7.829	6.823	7.442
9.869	5.917	4.954	6.289	6.862
7.866	8.295	4.901	7.693	6.862
5.717	6.009	5.603	6.808	10.267
6.397	9.269	9.961	4.302	9.858
5.405	6.790	4.677	8.719	5.974
6.901	10.001	5.546	6.091	4.176
7.901	4.645	5.232	6.998	

Simulated Data from ENWED

The estimates of the parameters, log-likelihood and the Alkaike Information Criterion (AIC) is represented in Table 4.4 and is generated based on the data for accelerated life test of 59 conductors. The R software is used to determine the MLE and the log-likelihood for the ENWED in Table 4.4.

Table 4.4: Performance of distributions (Maximum Likelihood Estimates, Log-likelihood and AIC) for data on accelerated life test of 59 conductors fitted on ENWED and NWED

Distributions	**Estimates**	**Log-likelihood**	**AIC**
New Weighted Exponential	$\hat{\alpha}$ **=34.558133**	**-173.2091**	**350.4182**
	$\hat{\lambda}$ **=-0.995824**		
Weighted Exponential		**-149.4845**	**302.969**
	$\hat{\alpha}$ **=0.012466**		
Exponentiated Weighted Exponential	$\hat{\lambda}$ **=-0.999969**		
		-148.4845	**302.969**
	$\hat{\alpha}$ **=418.955**		
	$\hat{\lambda}$ **=137.5429**		
	$\hat{\theta}$ **=7.02279e-07**		
Exponentiated-New Weighted Exponential	$\hat{\alpha}$ **=0.005006**	**-113.6047**	**233.2094**
	$\hat{\lambda}$ **=129.970600**		
	$\hat{\theta}$ **=55.830050**		

Source: Software: R, AIC: Alkaike Information Criteria

Table 4.4 shows the maximum likelihood estimates (MLE) associated to each distribution fitted on the second dataset. It also shows the log-likelihood and the AIC for each model. The AIC of the ENWED can be seen to be lower when compared to the NWED, WED and EWED, indicating that the ENWED outperforms the NWED, WED and EWED for fitting the data set. Also the kernel density approach was used to estimate the unknown probability density function $f(x)$ of the real life data. Using the same data set for both the Exponentiated-New Weighted Exponential distribution (ENWED) and the New Weighted Exponential distribution (NWED), the plots of the $\hat{f}(x)$ for the real life data and the estimated density function is given below in figure 4.4.

The estimated density function from the proposed distribution can be seen to closely imitate that of the estimated kernel based density function than the baseline distribution.

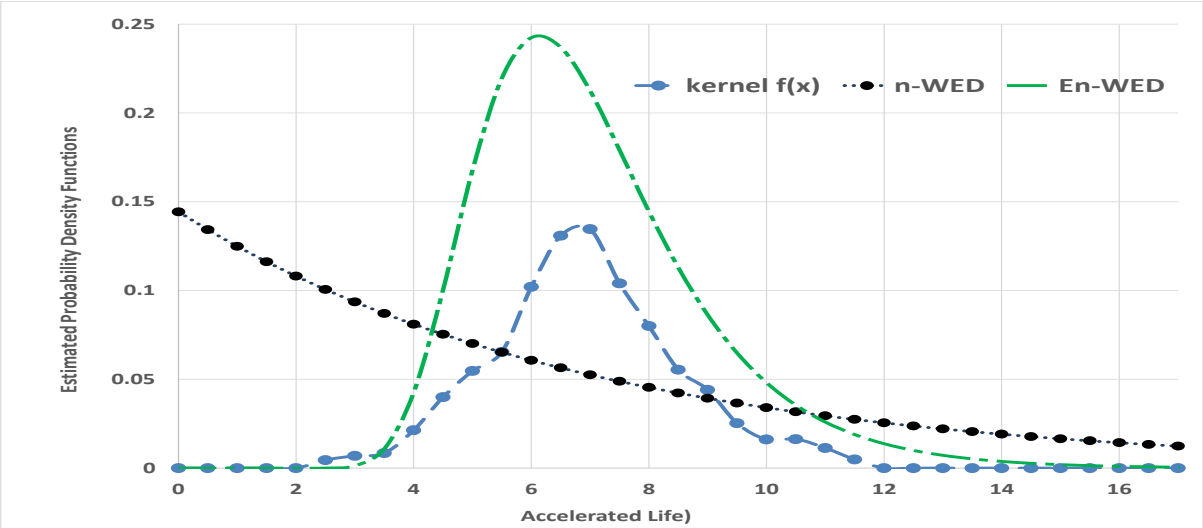


Fig 4: Estimated Probability Density Function for the Data on Accelerated Life Test of Conductors

Having estimated the ENWED parameters based on the data from accelerated life test of 59 conductors, the quantile function was used to generate a sample of size 59 observations; the generated sample is given in Table 4.3(b).

Using the Kolmogrov-Smirnov test, we now test the null hypothesis that the data on accelerated life test of 59 conductors and the sample drawn from the fitted ENWED are both from the same distribution. The computed Kolmogrov-Smirnov statistic based on both empirical distribution functions $S_m(x)$ and $S_n(x)$ is:

$$D_{59,59} = \underset{x}{Sup} \mid S_m(x) - S_n(x) \mid$$
$$= 0.084745763 \approx 0.0847$$

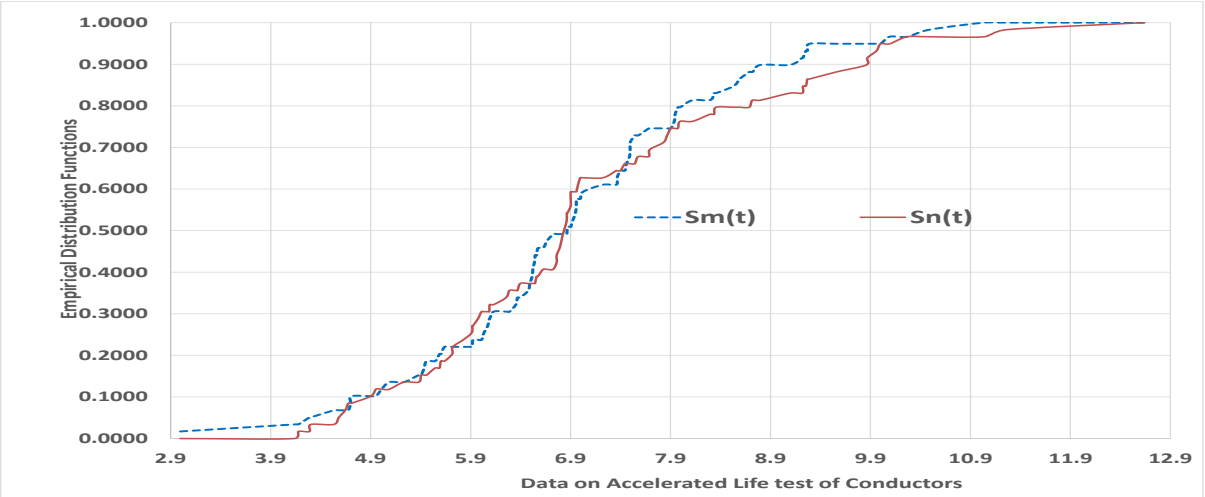


Fig 4(b): Kolmogrov-Smirnov two sample test: Graph of the empirical distribution Functions

The null hypothesis that the data on accelerated life test of 59 conductors and the one drawn from the proposed distribution are from the same distribution can only be rejected if $D_{59,59} \geq C_{\alpha}$ with $P\left(D_{59,59} \geq C_{\alpha}\right) = 0.01$ but $C_{\alpha} = 0.230517$ but since $D_{59,59} = 0.0847$ is less than the critical value of 0.300107 at the 1% level of significance, we thereby accept the hypothesis and come to the conclusion that both dataset are from the same distribution. That can be clearly seen from Figure4.5

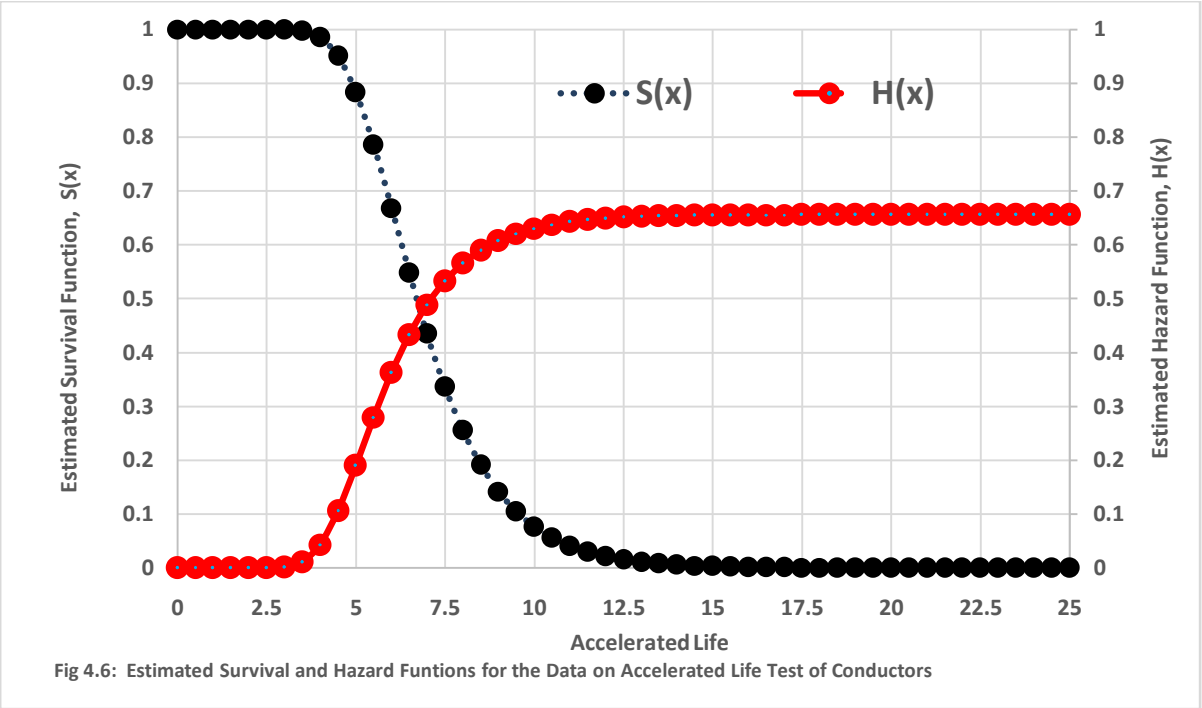


Fig 4.6: Estimated Survival and Hazard Funtions for the Data on Accelerated Life Test of Conductors

Figure 4.6 represents the hazard and survival function based on the data on accelerated life-test of conductors. On X-axis represent the time in minutes at which it takes for an event to happen after a given time t, and at the same time also describe the instantaneous failure rate at any point in time. While Y-axis represents the probability of survival and hazard function.

From the curve of the survival function in Fig.4.6, at about time 0-4mins, 100% of the conductors have their life span intact (accelerated). At approximately 5mins, 90% of the conductors start experiencing a decline in their performances. As time increases, the curve shows steady decrease until at about 12.5mins, when it remains at a constant level.

The curve of the hazard function in Fig. 4.6 indicates that at about time 0-3mins, there is no event but at time 4mins probability of event occurring starts rising for 10% of the conductors up to time 10mins where 60% of the conductors have experience an event, after which it remains at a constant.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 Summary

In this research work, we proposed a three parameter distribution known as the Exponentiated-New Weighted Exponential Distribution. It generalizes the New Weighted Exponential Distribution by adding an exponentiated (i.e. a shape) parameter into the NWED. Adding a shape parameter to the NWED increases the flexibility in such a way that the model can accommodate different kind of data sets in practice which outperforms the NWED based on two real life data sets that were used.

Properties for the proposed distribution such as the order statistics, quantiles, reliability analysis, moment generating function (mgf) were derived and the maximum likelihood estimation method was used for estimating the parameters. The kernel density estimation method and Kolmogrov-Smirnov test was also explored to present a clearer picture on the proposed distribution.

The Exponentiated-New Weighted Exponential Distribution exhibit important features that are unique to it when applied to two different real life datasets and from the result of the analysis, it can be deduced that, the ENWED outperforms other competitive distribution on the two occasions.

5.2 Conclusion

We proposed a new distribution which generalizes the new Weighted Exponential distribution. Statistical properties such as the order statistics, moments and mgf were derived. The maximum likelihood estimation procedure was used in estimating the model parameters. The R software

was employed for analysis of the data for the proposed distribution known as ENWED. The kernel density estimates and the classical Kolmogorov-Smirnov two-sample test statistic also validates that both data on waiting time of 100 bank customers and accelerated life test of 59 conductors could have been drawn from the new proposed distribution. Finally, the usefulness of the exponentiated-new weighted exponential distribution is illustrated by means of analyzing two real datasets in which the exponentiated-new weighted exponential distribution gives a better representation (provide better fit) than its competitive (the exponentiated weighted exponential distribution, the new weighted exponential distribution and the weighted exponential distribution) distribution. The proposed distribution outperforms the baseline distribution.

5.3 Recommendation

We recommend based on the findings of this dissertation that the proposed distribution should be used to model data based on waiting time of any events. Examples, waiting time to get a job after graduation, how long a device have before it fails etc.

5.4 Contribution to Knowledge

Addition of a shape parameter to a distribution has been known overtime to upgrade and also presents a better model than the existing distribution. Therefore this was carried out on the New Weighted Exponential distribution. The proposed distribution, ENWED gives a better fit when compared to other competing distribution such as WED, EWED and NWED. Statistical properties concerning the proposed distribution were also derived.

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Appendix 3: Computation of the Kolmogrov-Smirnov Statistic for Bank Customers Waiting Time						
t	#(X≤t)	$S_m(t)$	#(Y≤t)	$S_n(t)$	Sm(t)-Sn(t)	Abs(Sm(t)-Sn(t))
0.800	1	0.0100	0	0.0000	0.0100	0.0100
0.800	2	0.0200	0	0.0000	0.0200	0.0200
1.189	2	0.0200	1	0.0100	0.0100	0.0100
1.244	2	0.0200	2	0.0200	0.0000	0.0000
1.300	3	0.0300	2	0.0200	0.0100	0.0100
1.329	3	0.0300	3	0.0300	0.0000	0.0000
1.455	3	0.0300	4	0.0400	-0.0100	0.0100
1.500	4	0.0400	4	0.0400	0.0000	0.0000
1.552	4	0.0400	5	0.0500	-0.0100	0.0100
1.800	5	0.0500	5	0.0500	0.0000	0.0000
1.900	6	0.0600	5	0.0500	0.0100	0.0100
1.900	7	0.0700	5	0.0500	0.0200	0.0200
1.987	7	0.0700	6	0.0600	0.0100	0.0100
2.100	8	0.0800	6	0.0600	0.0200	0.0200
2.119	8	0.0800	7	0.0700	0.0100	0.0100
2.185	8	0.0800	8	0.0800	0.0000	0.0000
2.370	8	0.0800	9	0.0900	-0.0100	0.0100
2.600	9	0.0900	9	0.0900	0.0000	0.0000
2.693	9	0.0900	10	0.1000	-0.0100	0.0100
2.700	10	0.1000	10	0.1000	0.0000	0.0000
2.818	10	0.1000	11	0.1100	-0.0100	0.0100
2.900	11	0.1100	11	0.1100	0.0000	0.0000
3.100	12	0.1200	11	0.1100	0.0100	0.0100
3.156	12	0.1200	12	0.1200	0.0000	0.0000
3.200	13	0.1300	12	0.1200	0.0100	0.0100
3.300	14	0.1400	12	0.1200	0.0200	0.0200
3.500	15	0.1500	12	0.1200	0.0300	0.0300
3.525	15	0.1500	13	0.1300	0.0200	0.0200
3.600	16	0.1600	13	0.1300	0.0300	0.0300
3.851	16	0.1600	14	0.1400	0.0200	0.0200
3.998	16	0.1600	15	0.1500	0.0100	0.0100
4.000	17	0.1700	15	0.1500	0.0200	0.0200
4.055	17	0.1700	16	0.1600	0.0100	0.0100
4.100	18	0.1800	16	0.1600	0.0200	0.0200
4.200	19	0.1900	16	0.1600	0.0300	0.0300
4.200	20	0.2000	16	0.1600	0.0400	0.0400
4.300	21	0.2100	16	0.1600	0.0500	0.0500
4.300	22	0.2200	16	0.1600	0.0600	0.0600
4.399	22	0.2200	17	0.1700	0.0500	0.0500
4.400	23	0.2300	17	0.1700	0.0600	0.0600
4.400	24	0.2400	17	0.1700	0.0700	0.0700
4.524	24	0.2400	18	0.1800	0.0600	0.0600
4.568	24	0.2400	19	0.1900	0.0500	0.0500
4.600	25	0.2500	19	0.1900	0.0600	0.0600
4.697	25	0.2500	20	0.2000	0.0500	0.0500
4.700	26	0.2600	20	0.2000	0.0600	0.0600
4.700	27	0.2700	20	0.2000	0.0700	0.0700
4.758	27	0.2700	21	0.2100	0.0600	0.0600

4.800	28	0.2800	21	0.2100	0.0700	0.0700
4.900	29	0.2900	21	0.2100	0.0800	0.0800
4.900	30	0.3000	21	0.2100	0.0900	0.0900
4.906	30	0.3000	22	0.2200	0.0800	0.0800
4.913	30	0.3000	23	0.2300	0.0700	0.0700
4.921	30	0.3000	24	0.2400	0.0600	0.0600
5.000	31	0.3100	24	0.2400	0.0700	0.0700
5.014	31	0.3100	25	0.2500	0.0600	0.0600
5.239	31	0.3100	26	0.2600	0.0500	0.0500
5.300	32	0.3200	26	0.2600	0.0600	0.0600
5.404	32	0.3200	27	0.2700	0.0500	0.0500
5.500	33	0.3300	27	0.2700	0.0600	0.0600
5.504	33	0.3300	28	0.2800	0.0500	0.0500
5.522	33	0.3300	29	0.2900	0.0400	0.0400
5.700	34	0.3400	29	0.2900	0.0500	0.0500
5.700	35	0.3500	29	0.2900	0.0600	0.0600
5.703	35	0.3500	30	0.3000	0.0500	0.0500
5.713	35	0.3500	31	0.3100	0.0400	0.0400
5.816	35	0.3500	32	0.3200	0.0300	0.0300
6.080	35	0.3500	33	0.3300	0.0200	0.0200
6.100	36	0.3600	33	0.3300	0.0300	0.0300
6.165	36	0.3600	34	0.3400	0.0200	0.0200
6.200	37	0.3700	34	0.3400	0.0300	0.0300
6.200	38	0.3800	34	0.3400	0.0400	0.0400
6.200	39	0.3900	34	0.3400	0.0500	0.0500
6.300	40	0.4000	34	0.3400	0.0600	0.0600
6.619	40	0.4000	35	0.3500	0.0500	0.0500
6.700	41	0.4100	35	0.3500	0.0600	0.0600
6.709	41	0.4100	36	0.3600	0.0500	0.0500
6.729	41	0.4100	37	0.3700	0.0400	0.0400
6.754	41	0.4100	38	0.3800	0.0300	0.0300
6.804	41	0.4100	39	0.3900	0.0200	0.0200
6.900	42	0.4200	39	0.3900	0.0300	0.0300
7.093	42	0.4200	40	0.4000	0.0200	0.0200
7.100	43	0.4300	40	0.4000	0.0300	0.0300
7.100	44	0.4400	40	0.4000	0.0400	0.0400
7.100	45	0.4500	40	0.4000	0.0500	0.0500
7.100	46	0.4600	40	0.4000	0.0600	0.0600
7.177	46	0.4600	41	0.4100	0.0500	0.0500
7.400	47	0.4700	41	0.4100	0.0600	0.0600
7.600	48	0.4800	41	0.4100	0.0700	0.0700
7.650	48	0.4800	42	0.4200	0.0600	0.0600
7.700	49	0.4900	42	0.4200	0.0700	0.0700
7.877	49	0.4900	43	0.4300	0.0600	0.0600
7.923	49	0.4900	44	0.4400	0.0500	0.0500
8.000	50	0.5000	44	0.4400	0.0600	0.0600
8.200	51	0.5100	44	0.4400	0.0700	0.0700

12.941	74	0.7400	76	0.7600	-0.0200	0.0200
13.000	75	0.7500	76	0.7600	-0.0100	0.0100
13.074	75	0.7500	77	0.7700	-0.0200	0.0200
13.100	76	0.7600	77	0.7700	-0.0100	0.0100
13.300	77	0.7700	77	0.7700	0.0000	0.0000
13.490	77	0.7700	78	0.7800	-0.0100	0.0100
13.600	78	0.7800	78	0.7800	0.0000	0.0000
13.700	79	0.7900	78	0.7800	0.0100	0.0100
13.881	79	0.7900	79	0.7900	0.0000	0.0000
13.900	80	0.8000	79	0.7900	0.0100	0.0100
14.100	80	0.8000	80	0.8000	0.0000	0.0000
14.100	81	0.8100	80	0.8000	0.0100	0.0100
14.334	81	0.8100	81	0.8100	0.0000	0.0000
15.400	82	0.8200	81	0.8100	0.0100	0.0100
15.400	83	0.8300	81	0.8100	0.0200	0.0200
15.764	83	0.8300	82	0.8200	0.0100	0.0100
15.960	83	0.8300	83	0.8300	0.0000	0.0000
16.448	83	0.8300	84	0.8400	-0.0100	0.0100
17.254	83	0.8300	85	0.8500	-0.0200	0.0200
17.300	84	0.8400	85	0.8500	-0.0100	0.0100
17.300	85	0.8500	85	0.8500	0.0000	0.0000
17.774	85	0.8500	86	0.8600	-0.0100	0.0100
17.952	85	0.8500	87	0.8700	-0.0200	0.0200
18.100	86	0.8600	87	0.8700	-0.0100	0.0100
18.200	87	0.8700	87	0.8700	0.0000	0.0000
18.400	88	0.8800	87	0.8700	0.0100	0.0100
18.900	89	0.8900	87	0.8700	0.0200	0.0200
19.000	90	0.9000	87	0.8700	0.0300	0.0300
19.104	90	0.9000	88	0.8800	0.0200	0.0200
19.600	91	0.9100	88	0.8800	0.0300	0.0300
19.813	91	0.9100	89	0.8900	0.0200	0.0200
20.320	91	0.9100	90	0.9000	0.0100	0.0100
20.365	91	0.9100	91	0.9100	0.0000	0.0000
20.600	92	0.9200	91	0.9100	0.0100	0.0100
20.733	92	0.9200	92	0.9200	0.0000	0.0000
20.896	92	0.9200	93	0.9300	-0.0100	0.0100
21.300	93	0.9300	93	0.9300	0.0000	0.0000
21.400	94	0.9400	93	0.9300	0.0100	0.0100
21.430	94	0.9400	94	0.9400	0.0000	0.0000
21.900	95	0.9500	94	0.9400	0.0100	0.0100
21.972	95	0.9500	95	0.9500	0.0000	0.0000
23.000	96	0.9600	95	0.9500	0.0100	0.0100
25.551	96	0.9600	96	0.9600	0.0000	0.0000
26.018	96	0.9600	97	0.9700	-0.0100	0.0100
27.000	97	0.9700	97	0.9700	0.0000	0.0000
29.046	97	0.9700	98	0.9800	-0.0100	0.0100
31.600	98	0.9800	98	0.9800	0.0000	0.0000
31.643	98	0.9800	99	0.9900	-0.0100	0.0100
33.100	99	0.9900	99	0.9900	0.0000	0.0000
34.424	99	0.9900	100	1.0000	-0.0100	0.0100
38.500	100	1.0000	100	1.0000	0.0000	0.0000

		Max	**0.0900**			
	N	200	N/mn	0.02		
	mn	10000	sqrt(N/mn)	0.1414214		
P value		0.2	0.1	0.05	0.02	0.01
Critical level		0.151320851	0.1725341	0.192333	0.21496	0.230517

Appendix 4: Computation of the Kolmogrov-Smirnov Statistic for Accelerated Life Test of Conductors						
t	#(X≤t)	$S_m(t)$	#(Y≤t)	$S_n(t)$	Sm(t)-Sn(t)	Abs(Sm(t)-Sn(t))
2.997	1	0.0169	0	0.0000	0.0169	0.016949153
4.137	2	0.0339	0	0.0000	0.0339	0.033898305
4.176	2	0.0339	1	0.0169	0.0169	0.016949153
4.288	3	0.0508	1	0.0169	0.0339	0.033898305
4.302	3	0.0508	2	0.0339	0.0169	0.016949153
4.531	4	0.0678	2	0.0339	0.0339	0.033898305
4.582	4	0.0678	3	0.0508	0.0169	0.016949153
4.645	4	0.0678	4	0.0678	0.0000	0
4.677	4	0.0678	5	0.0847	-0.0169	0.016949153
4.7	5	0.0847	5	0.0847	0.0000	0
4.706	6	0.1017	5	0.0847	0.0169	0.016949153
4.901	6	0.1017	6	0.1017	0.0000	0
4.954	6	0.1017	7	0.1186	-0.0169	0.016949153
5.009	7	0.1186	7	0.1186	0.0000	0
5.087	8	0.1356	7	0.1186	0.0169	0.016949153
5.232	8	0.1356	8	0.1356	0.0000	0
5.381	9	0.1525	8	0.1356	0.0169	0.016949153
5.405	9	0.1525	9	0.1525	0.0000	0
5.434	10	0.1695	9	0.1525	0.0169	0.016949153
5.459	11	0.1864	9	0.1525	0.0339	0.033898305
5.546	11	0.1864	10	0.1695	0.0169	0.016949153
5.589	12	0.2034	10	0.1695	0.0339	0.033898305
5.603	12	0.2034	11	0.1864	0.0169	0.016949153
5.64	13	0.2203	11	0.1864	0.0339	0.033898305
5.717	13	0.2203	12	0.2034	0.0169	0.016949153
5.722	13	0.2203	13	0.2203	0.0000	0
5.826	13	0.2203	14	0.2373	-0.0169	0.016949153
5.911	13	0.2203	15	0.2542	-0.0339	0.033898305
5.917	13	0.2203	16	0.2712	-0.0508	0.050847458
5.923	14	0.2373	16	0.2712	-0.0339	0.033898305
5.974	14	0.2373	17	0.2881	-0.0508	0.050847458
6.009	14	0.2373	18	0.3051	-0.0678	0.06779661
6.033	15	0.2542	18	0.3051	-0.0508	0.050847458
6.071	16	0.2712	18	0.3051	-0.0339	0.033898305
6.087	17	0.2881	18	0.3051	-0.0169	0.016949153
6.091	17	0.2881	19	0.3220	-0.0339	0.033898305
6.129	18	0.3051	19	0.3220	-0.0169	0.016949153
6.256	18	0.3051	20	0.3390	-0.0339	0.033898305
6.289	18	0.3051	21	0.3559	-0.0508	0.050847458
6.352	19	0.3220	21	0.3559	-0.0339	0.033898305
6.369	20	0.3390	21	0.3559	-0.0169	0.016949153
6.397	20	0.3390	22	0.3729	-0.0339	0.033898305
6.476	21	0.3559	22	0.3729	-0.0169	0.016949153
6.492	22	0.3729	22	0.3729	0.0000	0
6.515	23	0.3898	22	0.3729	0.0169	0.016949153
6.522	24	0.4068	22	0.3729	0.0339	0.033898305
6.538	25	0.4237	22	0.3729	0.0508	0.050847458
6.545	26	0.4407	22	0.3729	0.0678	0.06779661

6.560	26	0.4407	23	0.3898	0.0508	0.050847458
6.573	27	0.4576	23	0.3898	0.0678	0.06779661
6.625	27	0.4576	24	0.4068	0.0508	0.050847458
6.663	28	0.4746	24	0.4068	0.0678	0.06779661
6.725	29	0.4915	24	0.4068	0.0847	0.084745763
6.761	29	0.4915	25	0.4237	0.0678	0.06779661
6.761	29	0.4915	26	0.4407	0.0508	0.050847458
6.790	29	0.4915	27	0.4576	0.0339	0.033898305
6.808	29	0.4915	28	0.4746	0.0169	0.016949153
6.823	29	0.4915	29	0.4915	0.0000	0
6.845	29	0.4915	30	0.5085	-0.0169	0.016949153
6.862	29	0.4915	31	0.5254	-0.0339	0.033898305
6.862	29	0.4915	32	0.5424	-0.0508	0.050847458
6.869	30	0.5085	32	0.5424	-0.0339	0.033898305
6.901	30	0.5085	33	0.5593	-0.0508	0.050847458
6.901	30	0.5085	34	0.5763	-0.0678	0.06779661
6.902	30	0.5085	35	0.5932	-0.0847	0.084745763
6.923	31	0.5254	35	0.5932	-0.0678	0.06779661
6.948	32	0.5424	35	0.5932	-0.0508	0.050847458
6.956	33	0.5593	35	0.5932	-0.0339	0.033898305
6.958	34	0.5763	35	0.5932	-0.0169	0.016949153
6.975	34	0.5763	36	0.6102	-0.0339	0.033898305
6.998	34	0.5763	37	0.6271	-0.0508	0.050847458
7.024	35	0.5932	37	0.6271	-0.0339	0.033898305
7.224	36	0.6102	37	0.6271	-0.0169	0.016949153
7.356	36	0.6102	38	0.6441	-0.0339	0.033898305
7.365	37	0.6271	38	0.6441	-0.0169	0.016949153
7.398	38	0.6441	38	0.6441	0.0000	0
7.442	38	0.6441	39	0.6610	-0.0169	0.016949153
7.459	39	0.6610	39	0.6610	0.0000	0
7.489	40	0.6780	39	0.6610	0.0169	0.016949153
7.495	41	0.6949	39	0.6610	0.0339	0.033898305
7.496	42	0.7119	39	0.6610	0.0508	0.050847458
7.543	43	0.7288	39	0.6610	0.0678	0.06779661
7.577	43	0.7288	40	0.6780	0.0508	0.050847458
7.683	44	0.7458	40	0.6780	0.0678	0.06779661
7.693	44	0.7458	41	0.6949	0.0508	0.050847458
7.829	44	0.7458	42	0.7119	0.0339	0.033898305
7.866	44	0.7458	43	0.7288	0.0169	0.016949153
7.901	44	0.7458	44	0.7458	0.0000	0
7.937	45	0.7627	44	0.7458	0.0169	0.016949153
7.945	46	0.7797	44	0.7458	0.0339	0.033898305
7.974	47	0.7966	44	0.7458	0.0508	0.050847458
7.996	47	0.7966	45	0.7627	0.0339	0.033898305
8.12	48	0.8136	45	0.7627	0.0508	0.050847458
8.295	48	0.8136	46	0.7797	0.0339	0.033898305
8.336	49	0.8305	46	0.7797	0.0508	0.050847458
8.355	49	0.8305	47	0.7966	0.0339	0.033898305
8.532	50	0.8475	47	0.7966	0.0508	0.050847458
8.591	51	0.8644	47	0.7966	0.0678	0.06779661

8.687	52	0.8814	47	0.7966	0.0847	0.084745763
8.719	52	0.8814	48	0.8136	0.0678	0.06779661
8.799	53	0.8983	48	0.8136	0.0847	0.084745763
9.094	53	0.8983	49	0.8305	0.0678	0.06779661
9.218	54	0.9153	49	0.8305	0.0847	0.084745763
9.225	54	0.9153	50	0.8475	0.0678	0.06779661
9.254	55	0.9322	50	0.8475	0.0847	0.084745763
9.269	55	0.9322	51	0.8644	0.0678	0.06779661
9.289	56	0.9492	51	0.8644	0.0847	0.084745763
9.556	56	0.9492	52	0.8814	0.0678	0.06779661
9.858	56	0.9492	53	0.8983	0.0508	0.050847458
9.869	56	0.9492	54	0.9153	0.0339	0.033898305
9.961	56	0.9492	55	0.9322	0.0169	0.016949153
10.001	56	0.9492	56	0.9492	0.0000	0
10.092	57	0.9661	56	0.9492	0.0169	0.016949153
10.267	57	0.9661	57	0.9661	0.0000	0
10.491	58	0.9831	57	0.9661	0.0169	0.016949153
11.038	59	1.0000	57	0.9661	0.0339	0.033898305
11.261	59	1.0000	58	0.9831	0.0169	0.016949153
12.635	59	1.0000	59	1.0000	0.0000	0

		Max	**0.084745763**			
	N	118	N/mn	0.033898		
	mn	3481	sqrt(N/mn)	0.184115		
P value		0.2	0.1	0.05	0.02	0.01
Critical level		0.197003	0.224620207	0.250396	0.2798547	0.300107