ANALYSIS OF FOUR-BAR MECHANISMS

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NOTATION

The symbols used in this study are defined as follows:

A, B, C, ...	point positions.
1, 2, 3, ...	links of the mechanism; 1 denotes the fixed link.
A_2, B_3, C_1, ...	numeral subscript denotes the body to which the point belongs.
AB, AC, BD, ...	lengths of directed line segments from the first point to the second.
$\underline{V}$ ...	linear velocity.
$\underline{V}_{ba}$ ...	velocity of B relative to A
$\nrightarrow$, $\longrightarrow$ ...	vector addition, vector subtraction $\underline{V}_{ba} = \underline{V}_b \longrightarrow \underline{V}_a$ or $\underline{V}_b = \underline{V}_a \nrightarrow \underline{V}_{ba}$.
ω ...	angular velocity.
ω_3...	angular velocity of the body indicated by subscript.
ω_{23}...	angular velocity of link 2 relative to link 3; $\omega_{23} = \omega_2 - \omega_3$.
$\underline{a}$ ...	linear acceleration.
$\underline{a}_a$, $\underline{a}_b$, $\underline{a}_c$ ...	acceleration of position indicated by subscript.

$\underline{a}_{ba}$ ...	acceleration of point B relative to point A, $\underline{a}_{ba} = \underline{a}_b \longrightarrow \underline{a}_a$
$\underline{a}_a^n$, $\underline{a}_{ba}^n$, ...	acceleration component perpendicular (normal) to the corresponding velocity.
$\underline{a}_a^t$, $\underline{a}_{ba}^t$, ...	acceleration component parallel (tangential) to the corresponding velocity.
α ...	angular acceleration.
α_1, α_2, ...	angular acceleration of the body indicated.
α_{21} ...	angular acceleration of link 2 relative to link 1. $\alpha_{21} = \alpha_2 - \alpha_1$.
$\overline{A}$, $\overline{B}$, $\overline{C}$, ...	fixed points about which A, B, C, rotate respectively.
a, b, c, ...	lengths of the links denoted by 1, 2, 3, ...

CHAPTER ONE

INTRODUCTION

A most important part of machines is that set of systems which controls the motion of the various machine parts; these systems are the mechanisms around which the machines are built.

The simplicity and versatility of the 4-bar linkage makes it a preferred device for the mechanisation and control of motion.

A 4-bar linkage is a closed chain of four rigid members coupled by lower pairs (turning or sliding) in such a manner that the motion is prescribed and periodic (fig. 1.01).

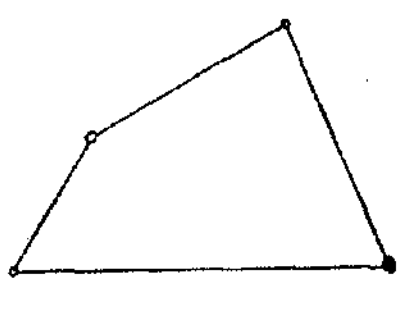


fig. 1.01 : 4-bar linkage.

The 4-bar linkage becomes a mechanism when one of its members is fixed. A 4-bar mechanism is analogous to a 4-bar chain whose motion characteristics (displacement, velocity, acceleration) are considered relative to one of its four links.

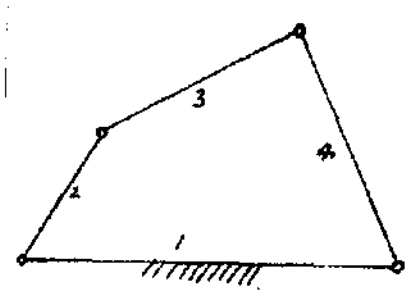


fig. 1.02 : 4-bar mechanism.

A 4-bar mechanism has, functionally, an input crank (2), an output crank (4), and a connecting rod or coupler (3); (fig. 1.02).

The determination of suitable dimensions for each of the four members is important in the kinematic design of a mechanism. The motion characteristics of a 4-bar mechanism depend on the relative lengths of its links and the study of this has been embarked upon over the years as the need for more accurately designed mechanisms has increased. This work concerns study in the same line and the theory discussed will be for plane motion.

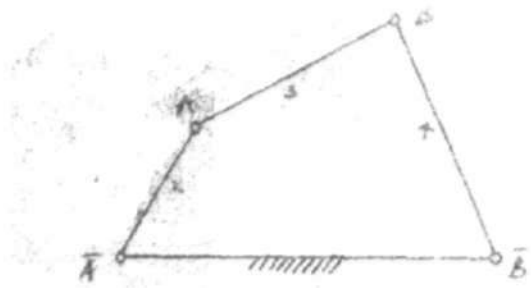
The 4-bar linkage serves as a useful guide for some other mechanisms that are not basically 4-bar mechanisms but

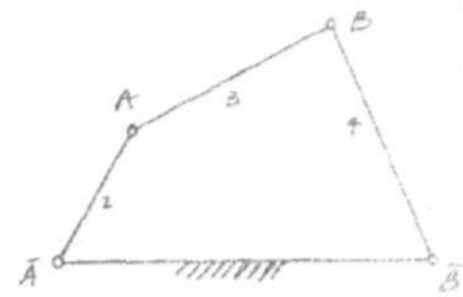
(a) have 4-bar basic layout;

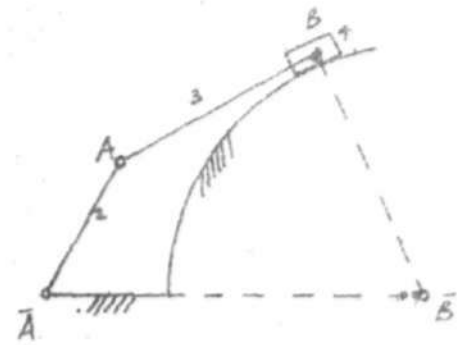
(b) are equivalent to the 4-bar chain in some aspect of their motion;

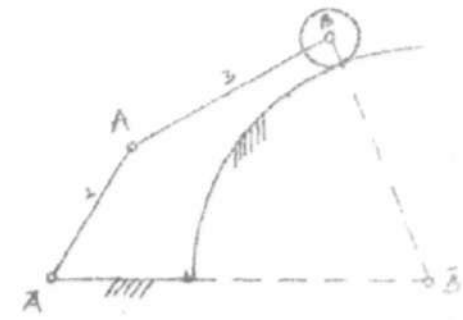
(c) have several 4-bar chains incorporated in their complex mechanism.

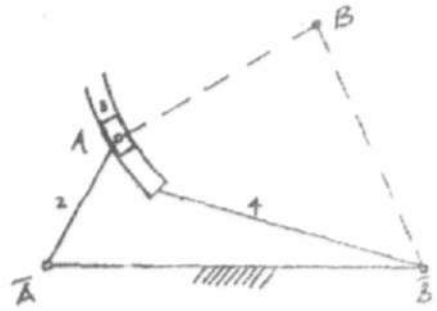
These are shown in figs. 1.03 to 1.07.











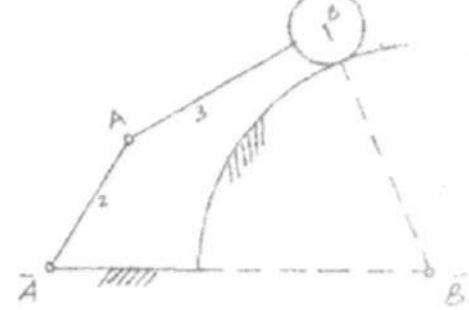
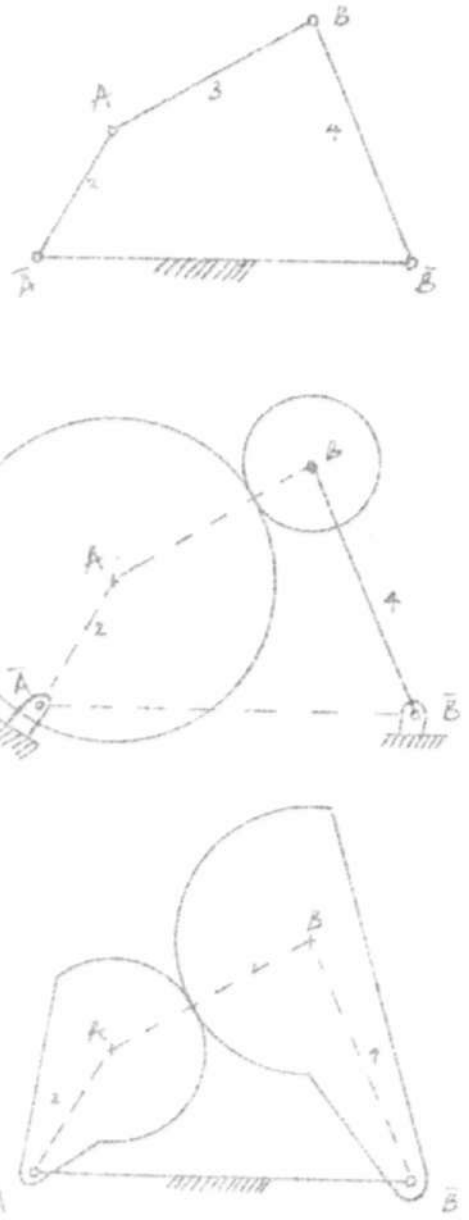
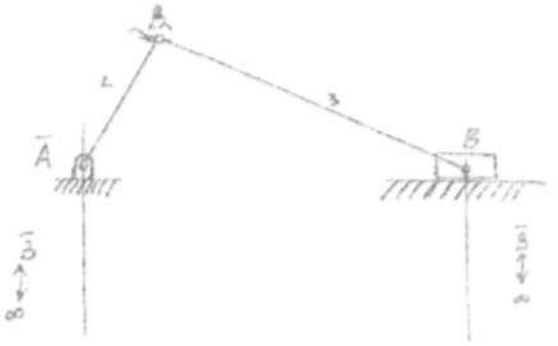
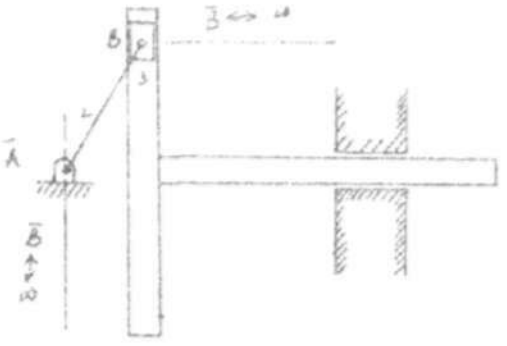


fig. 1·03: Kinematically identical mechanisms.

fig. 1·04: Equivalent mechanisms with respect link 3.







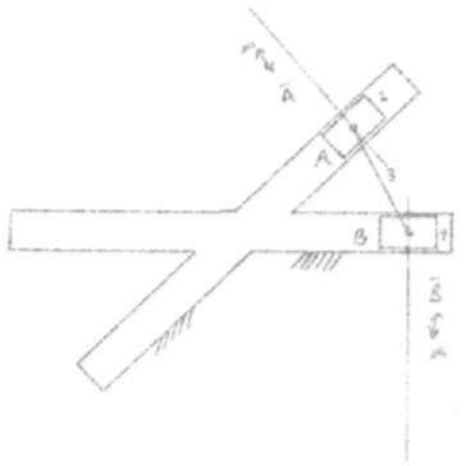
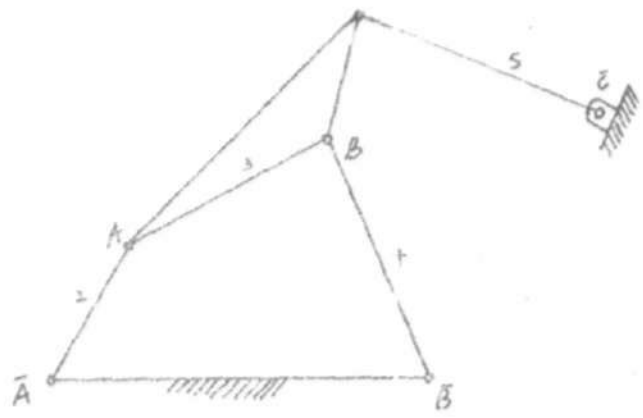


fig. 1·05 : Equivalent mechanisms with respect to links 2 & 4.

fig. 1·06 : 4-bar mechanis... with one or ... infinitely long link



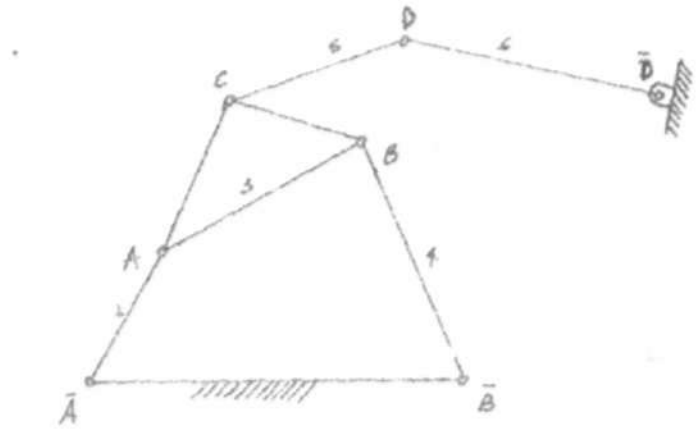


fig. 1.07: Mechanisms with larger number of links in which the 4-bar mechanism features.

1.1. Application of the 4-bar mechanism:

The 4-bar mechanism has its uses in many machines; some of which are to convert or transform motion; to guide; to generate some desired path or function; or maybe to give some mechanical advantage.

(a) Fork-lift truck:

In this application, the mechanism serves to guide the forks upwards and back at a slight angle, in an approximate straight line.

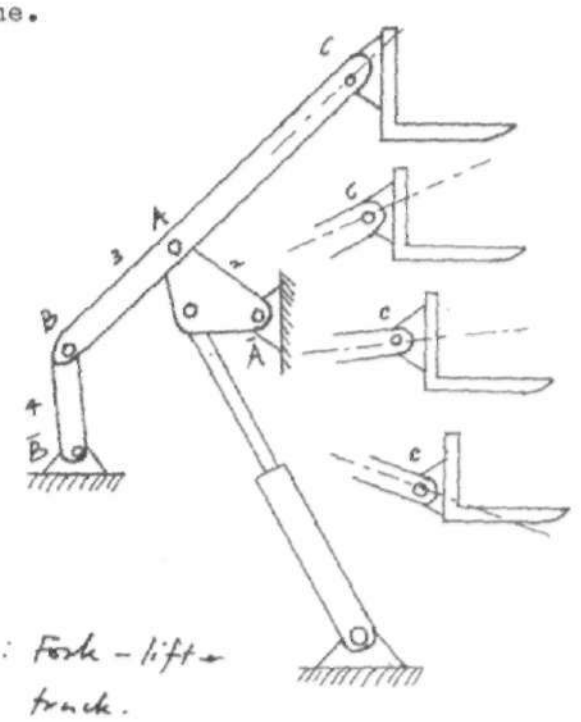


fig. 1.08 : Fork-lift-truck.

the 4-bar mechanism such that coupler point C would have the desired approximate straight motion over the necessary range of motion.

(b) Jib crane:

The double rocker mechanism is, besides other applications, utilized in slewing cranes. A horizontal draft is required for the load attached at coupler point C and a portion of the D-curve described by point C is used.

(c) Drive of a grinding wheel, small lathe or sewing machine:

A crank rocker mechanism is utilized in the treadle mechanisms for sewing machines, grinding wheels and light lathes.

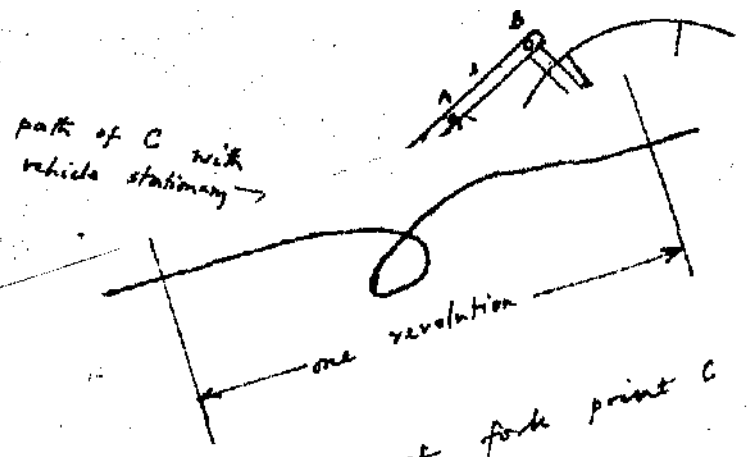


fig 1.12: actual path of fork point C with vehicle in motion.

...se, is a 6-bar linkage, ...
...rts of the human body.

Hay harvester:

A crank rocker mechanism is again used in the hay harvester shown in fig. 1.11. Link 3 supports a flexibly suspended fork.

The end of the fork C describes an oval-shaped curve. The actual path of the fork point, with the vehicle in motion, is similar to an extended cycloidal curve (fig. 1.12)

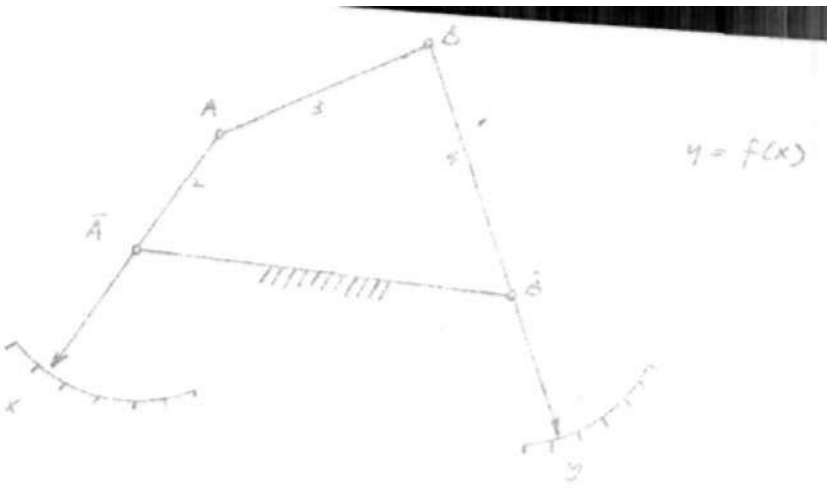


Fig 1·13: Computing linkage

of the output ...

crank displacements.

Usually, the designed 4-bar mechanism is not used to deduce the relationships between the variable x and y; it is, however, used in performing some mechanical function.

(f) Intermittent feed for film bands:

With distinct selection of relative lengths of a 4-bar mechanism, D-shaped coupler curves are obtained. The approximate straight line portion is used, in the present instance, to provide

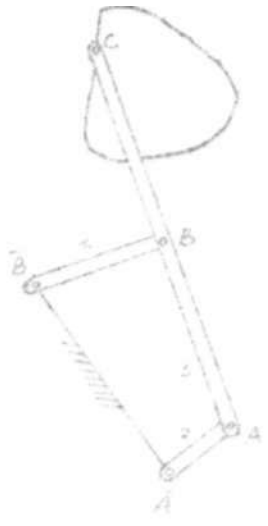


fig 1.9 : Intermittent feed for film bands.

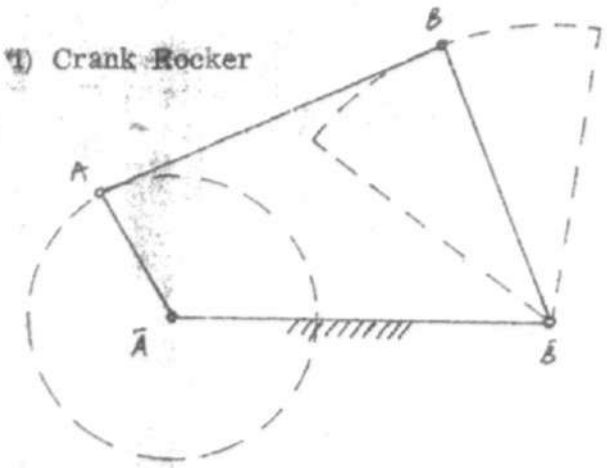
CHAPTER TWO

CLASSIFICATION OF 4-BAR MECHANISMS

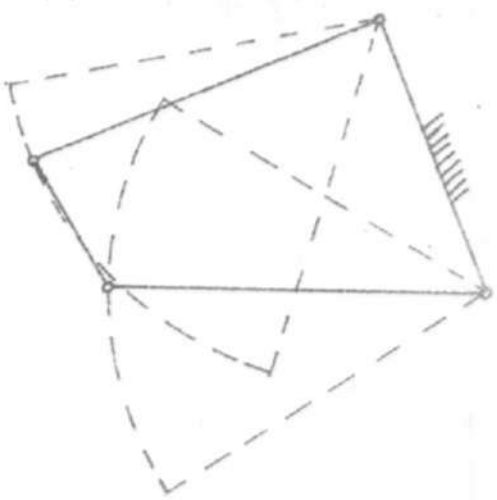
There are three types of 4-bar mechanisms, namely: the crank-rocker, the double crank (drag link), and the double rocker (double lever) mechanisms.

A mechanism and its inversions may be capable of forming any of the three types, depending on the fixed link of the linkage.

(i) Crank-Rocker



(ii) Double-Rocker



(iii) Crank-Rocker

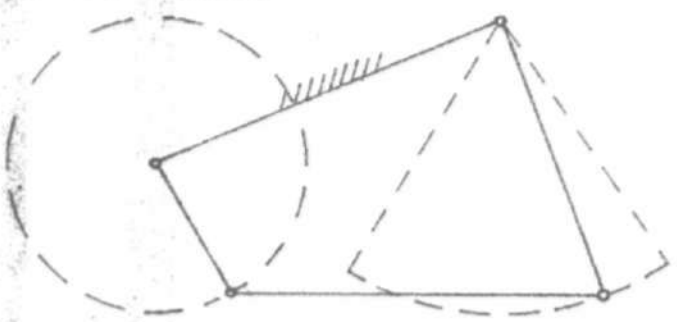


Fig. 2.01a:
Mechanism $\bar{A}AB\bar{B}$ with its inversions.

(iv) Double-Crank

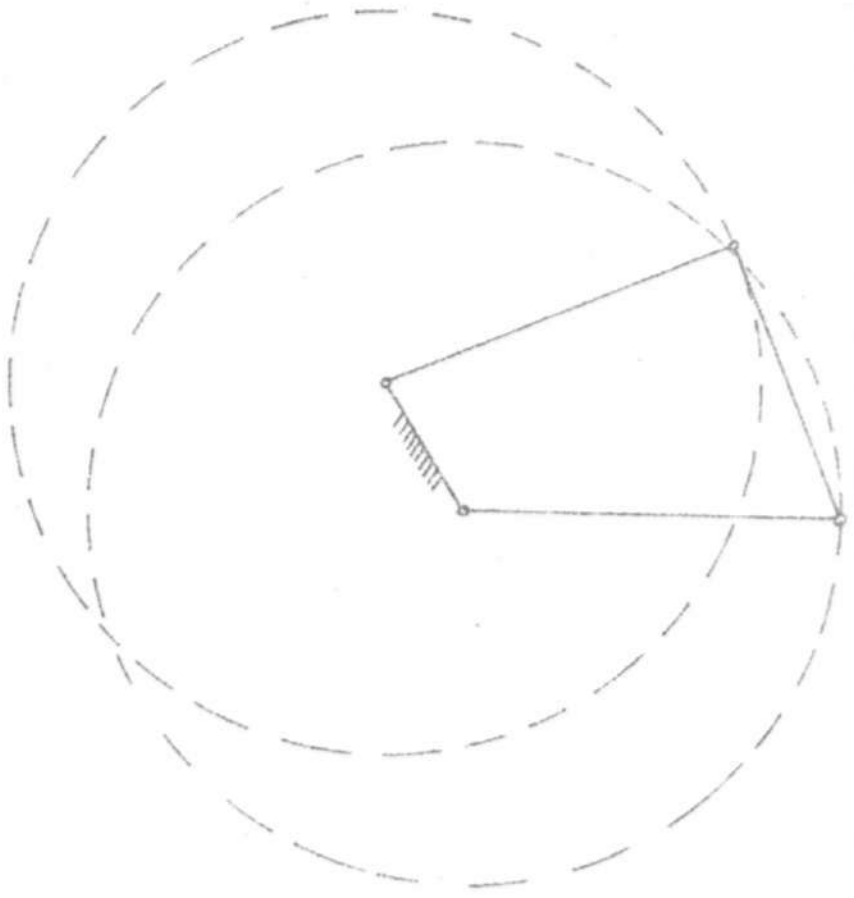
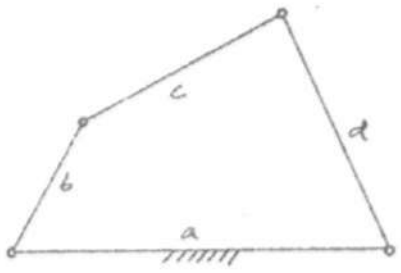


Fig. 2.01b:
Mechanism $\bar{A}AB\bar{B}$ with its inversions.

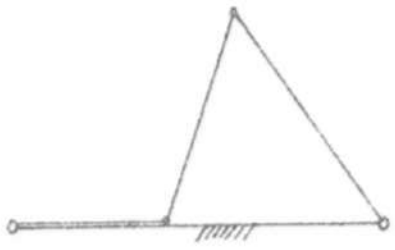
To classify 4-bar mechanism, various possible limit positions are considered and the properties grouped under certain general 4-bar configurations, as shown (Fig. 2.03).

I



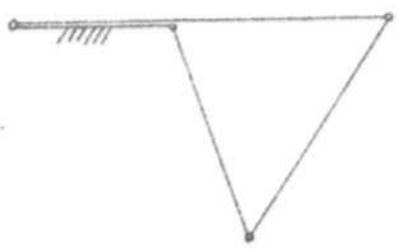
reference mechanism:
a — length of fixed link
abcd — clockwise rotation.

II



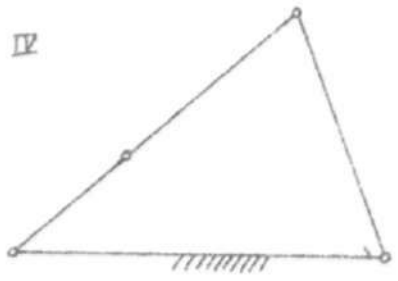
(i) $c < d + (a-b)$
$\Rightarrow b+c < d+a$
(ii) $d < c + (a-b)$
$\Rightarrow b+d < a+c$
(iii) $(a-b) < c+d$
$\Rightarrow a < b+c+d$
(iv) $b < a$

III



(i) $c < d + (b-a)$
$\Rightarrow a+c < b+d$
(ii) $d < c + (b-a)$
$\Rightarrow a+d < b+c$
(iii) $(b-a) < c+d$
$\Rightarrow b < c+d+a$
(iv) $a < b$

IV

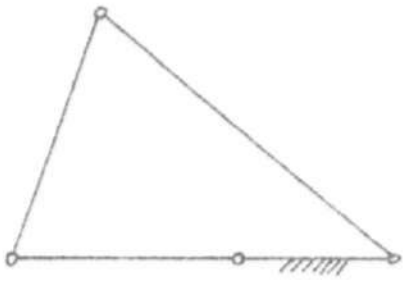


(i) $b+c < a+d$

(ii) $d < a+b+c$

(iii) $a < b+c+d$

V

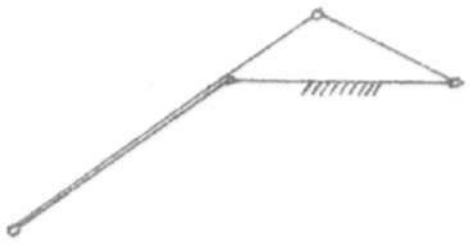


(i) $a+b < c+d$

(ii) $c < d+a+b$

(iii) $d < a+b+c$

VI



(i) $(c-b) < d+a$

$\Rightarrow c < d+a+b$

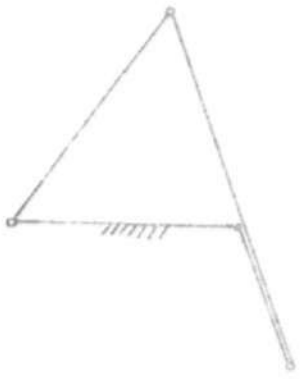
(ii) $d < (c-b)+a$

$\Rightarrow b+d < a+c$

(iii) $a < d+(c-b)$

$\Rightarrow a+b < c+d$

VII



(i) $(c-d) < a+b$

$\Rightarrow c < d+a+b$

(ii) $b < a+(c-d)$

$\Rightarrow b+d < a+c$

(iii) $a < b+(c-d)$

$\Rightarrow a+d < b+c$

VIII



(i) $(b-c) < a+d$

$\Rightarrow b < c+d+a$

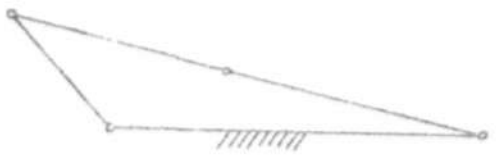
(ii) $a < (b-c)+d$

$\Rightarrow a+c < b+d$

(iii) $d < a+(b-c)$

$\Rightarrow c+d < a+b$

IX

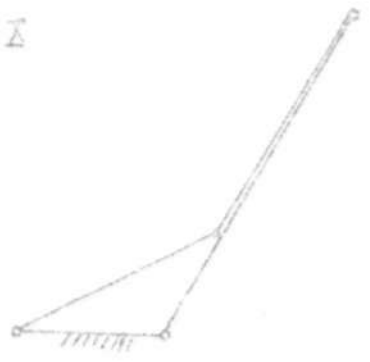


(i) $c+d < a+b$

(ii) $a < b+c+d$

(iii) $b < c+d+a$

I



(i) $(d-c) < a+b$

$\Rightarrow d < a+b+c$

(ii) $a < b+(d-c)$

$\Rightarrow a+c < b+d$

(iii) $b < a+(d-c)$

$\Rightarrow b+c < a+d$

XI

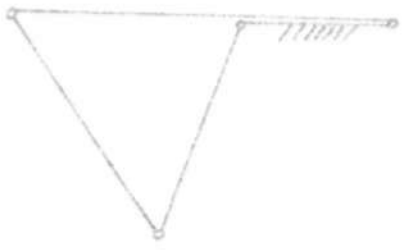


(i) $a+d < b+c$

(ii) $b < a+c+d$

(iii) $c < d+a+b$

XII



(i) $d < a+b+c$

(ii) $b < c+(d-a)$

$\Rightarrow a+b < c+d$

(iii) $c < b+(d-a)$

$\Rightarrow a+c < b+d$

(iv) $a < d$

fig 2.02: Limit positions in various 4-bar mechanisms.

The limit positions defined in the various situations depicted in Fig. 16 can be shown on a graph of d against c with a(b) constant for varying values of b(a). It makes for better clarity to have two sets of graphs: one for the case of $b < a$ and $a = 1$; this means that the other link lengths b, c, d are relative to the fixed link length a. The other is for the case $a < b$ and $b = 1$; now the lengths c, d, a, are relative to b.

The full classification of the general configurations possible within the possible link dimensions are shown in Fig. 2.03.

2.1. Conditions For Crank-Rocker Mechanisms:

For a 4-bar mechanism, the crank must be capable of turning at the limit positions; when the angular displacement with the fixed link is either 0° or 180°. This is shown in diagrams II, III, V, XI, XII of Fig. 16.

b-Crank:

When the crank is that whose length is denoted by b, the the conditions of diagrams II, IV, should generally be satisfied. These can be stated as follows:

(i) $a + b < c + d$

(ii) $a - b > /c - d/$ ----------(20)

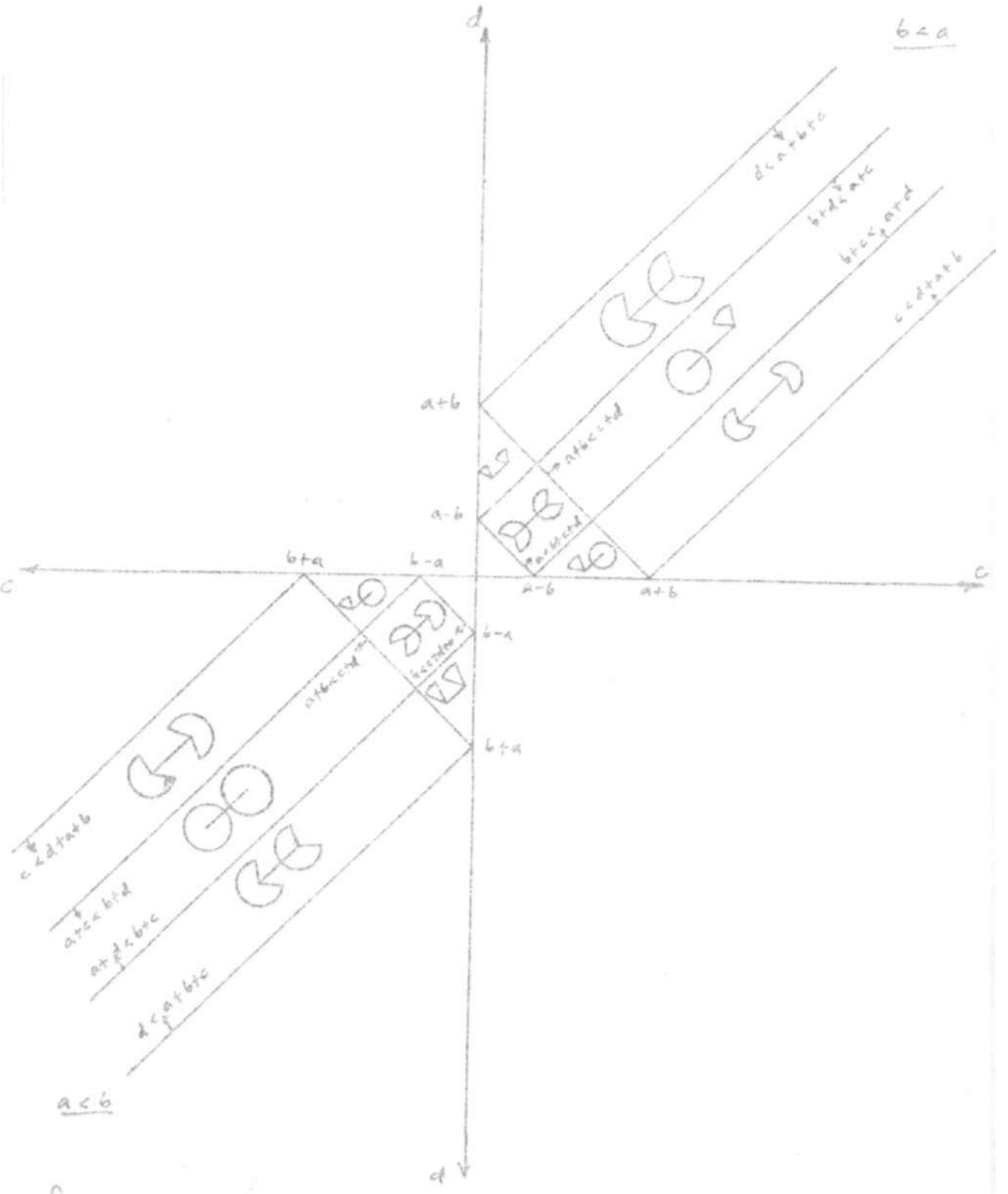


fig. 2·03. classification chart for 4-bar mechanisms.

d-Crank:

With the link designated with length d as crank, the conditions are:

(i) $a + d \leq b + c$

(ii) $a - d \geq |b - c|$ ----------(202)

2.2. Conditions For Double-Crank Mechanisms:

In double-crank (drag link) mechanisms, the two cranks of length b and d rotate. The links of lengths b and d must then be capable of turning at the limit positions where their angular displacements with the fixed link is either 0° or 180°. The conditions are:

(i) $b - a \geq |d - c|$

(ii) $d - a \geq |b - c|$ ----------(203)

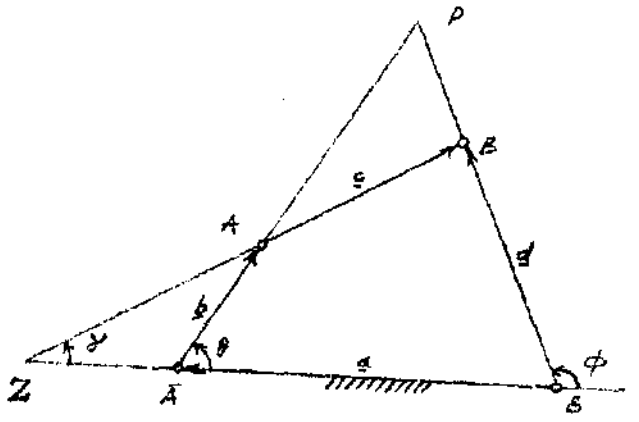
CHAPTER THREE

VELOCITY AND ACCELERATION

In high speed mechanisms, the inertia forces due to the accelerations of the moving parts may be very high. Inertia forces cause a change in the relationship between the input and output forces and also create additional stresses in the members of the mechanism.

3.1. Relative Velocity:

The relative velocity of one point to another is the differencebetween absolute velocities of the two points.



$\underline{a} = ae^{i\alpha}$
$\underline{b} = be^{i\theta}$
$\underline{c} = ce^{i\gamma}$
$\underline{d} = de^{i\phi}$

fig. 3.01.

Considering the 4-bar mechanism in Fig. 3.01, and taking $\bar{A}$ as the zero point;

$$\underline{r}_a = \underline{b} \; ; \; \underline{r}_b = \underline{b} \rightarrow \underline{c} = -\underline{a} \rightarrow \underline{d} \qquad \text{--------(301)}$$

Differentiating equation (301) with respect to time t,

$$\frac{d\underline{r}_a}{dt} = \frac{d\underline{b}}{dt} = i \frac{d\theta}{dt} \underline{b}$$

$$\therefore \; \underline{v}_a = \frac{d\underline{r}_a}{dt} = i\omega_2 \underline{b} \qquad \text{----------(302)}$$

Similarly;

$$\underline{v}_b = \frac{d\underline{r}_b}{dt} = \underline{i}\omega_2 x\underline{b} \rightarrow \underline{i}\omega_3 x\underline{c} \qquad \text{----------(303)}$$

or

$$\underline{v}_b = \frac{d\underline{r}_b}{dt} = \underline{i}\omega_4 x\underline{d} \qquad \text{----------(304)}$$

where $\frac{d\theta}{dt} = \omega_2$

$$\frac{d\gamma}{dt} = \omega_3$$

$$\frac{d\phi}{dt} = \omega_4$$

Equation (302) shows that the velocity of point A is perpendicular to the link $\bar{A}A$ and in the direction of the angular velocity ω_2. Also equation (304) shows that the velocity of point B is perpendicular to the link $\bar{B}B$ and in the direction of the angular velocity ω_4.

The relative velocity of point B to A $\underline{v}_{ba}$ is given by

$\underline{v}_{ba} = \underline{v}_b - \underline{v}_a$.

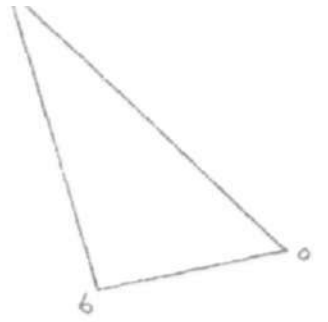


Fig. 3.02: vector velocity diagram of 4-bar mechanism in fig. (3.01)

Equation (305) shows that ... int A is perpendicular to the link AB.

The vector velocity diagram of the 4-bar linkage $\bar{A}A\bar{B}\bar{B}$ of Fig. (3.01) is shown in Fig. (3.02) with ω_2 anticlockwise.

$$\angle aob = \phi - \theta$$

$$\angle oab = \theta - \gamma$$

It can then be seen that triangles PAB (Fig. 3.01) and oab (Fig. 3.02) are similar. Point o on Fig. 3.02 is a point of zero velocity and so point P is also the velocity pole of the given

configuration of the 4-bar mechanism $\bar{A}A\bar{B}\bar{B}$ for the coupler AB. The coupler AB then rotates about point P for the given configuration. As the crank rotates, the position of the velocity pole 'P' changes. Point P then acts as an instantaneous centre of rotation for the coupler AB with reference to the fixed link $\bar{A}\bar{B}$.

3.2. Instantaneous Centre:

An instantaneous centre is two coincident points in two different bodies (and hence can be referred to as a point). There are several definitions for the instantaneous centre and it can be defined as a point common to two bodies, having the same linear velocity in each, about which the bodies rotate relative to each other , permanently or momentarily.

3.3. Kennedy's Theorem:

Kennedy's theorem states that any three bodies moving relatively to one another have three positions of instantaneous centre, all of which lie in a straight line.

In a 4-bar mechanism, there are four links, 1, 2, 3, 4. There are three possible combinations of selecting three bodies from the four: 123, 134, 234. Therefore there are three lines, each containing three instantaneous centres (Fig. 3.03).



There are then six instantaneous centres of rotation in a 4-bar mechanism. The four pivots $\bar{A}$, A, B, $\bar{B}$ constitute the permanent instantaneous centres of rotation and the location of the instantaneous centres of links 3 relative to 1, P, and links 2 relative to 4, Z, depends on the angular displacement of the driving crank.

3.4. Velocities By Instantaneous Centre Method:

The instantaneous centre provides a means of determining velocity relationships of mechanisms without drawing the velocity diagram.

The velocity of a point on a body is proportional to its distance from the instantaneous of the body relative to a fixed link. In the 4-bar mechanism of Fig. (3.01), the fixed link is represented by 1 and so all instantaneous centres connected with the fixed link 1 can be used to determine velocities relative to the fixed link. Such instantaneous centres are $\bar{A}$, $\bar{B}$ and P. $\bar{A}$ is the centre of rotation of link 2 relative to the fixed link and $\bar{B}$ is the centre of rotation relative to the fixed link; P is the instantaneous centre of rotation of link 3 (coupler AB) relative to the fixed link.

Considering the mechanism shown in Fig. 3.04,

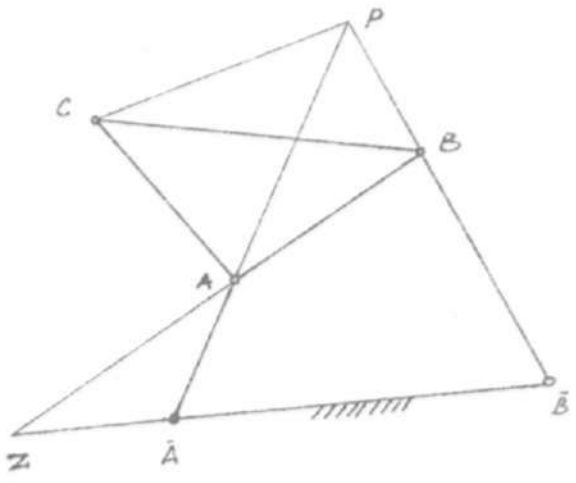
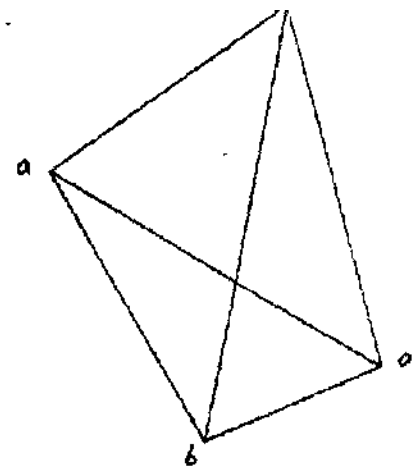


fig. 3.04.



$\Delta ABC \sim \Delta abc$

fig. 3.05 : velocity diagram for 4-bar mechanism of fig. 3.04.

$$\underline{v}_o = \underline{v}_b + \underline{v}_{ob} \qquad \text{----------(307)}$$

Equations (306) and (307) enable us to get at the velocity of point C on the vector velocity diagrams for a given angular velocity of the driving crank (Fig. 3.05).

The velocity of point C is given on the velocity diagram by oc. The problem can be solved more easily by instantaneous centre method. For the given configuration of the 4-bar mechanism, the instantaneous centre of link 3 relative to the fixed link is located at point P (Fig. 3.04).

$$\frac{v_a}{PA} = \frac{v_b}{PB} = \frac{v_c}{PC} = \omega_3 \text{ ----------(308)}$$

where ω_3 is the angular velocity of link 3.

It can be shown that triangles ABC (Fig. 3.04) and abc (Fig.3.05) are similar.

3.5. Relative Acceleration:

The relative acceleration of one point to the other is the difference in their absolute acceleration values. The absolute accelerations have to be taken to a common frame of reference.

Considering the 4-bar mechanism of Fig. 3.01, the relative accelerations of points A and B relative to the fixed link are the absolute accelerations with respect to the fixed link.

$$\underline{a}_a = \frac{d\underline{v}_a}{dt} = \frac{d^2\underline{r}_a}{dt^2} \text{ ----------(309)}$$

From equation (302), $\underline{v}_a = \frac{d\underline{r}_a}{dt} = \frac{d\underline{b}}{dt} = \underline{i}\ \frac{d\theta}{dt} \times \underline{b}$

$$\therefore \underline{a}_a = i\ \frac{d^2\theta}{dt^2} \times \underline{b} \rightarrow i^2\left(\frac{d\theta}{dt}\right)^2 \underline{b}$$

$$= \underline{i}\ \alpha_2 \times \underline{b} \rightarrow \omega_2^2 \underline{b} \text{ ----------(310)}$$

(where $\alpha_2 = \frac{d^2\theta}{dt^2}$)

Similarly, $\underline{a}_b = \frac{d\underline{v}_b}{dt} = \frac{d^2\underline{r}_b}{dt^2}$ ----------(311)

From equation (304), $\underline{v}_b = \underline{i}\ \frac{d\phi}{dt} \times \underline{d}$

Differentiating with respect to time, t:

$$\underline{a}_b = i\ \frac{d^2\phi}{dt^2} \times \underline{d} \rightarrowtail i^2\left(\frac{d\phi}{dt}\right)^2 \underline{d}$$

$$= \underline{i}\ \alpha_4 \times \underline{d} \longrightarrow \omega_4^2\ \underline{d} \quad \text{----------(312)}$$

$$\left(\alpha_4 = \frac{d^2\phi}{dt^2}\right)$$

From equation (303), $\underline{v}_b = \underline{i}\ \frac{d\theta}{dt} \times \underline{b} \rightarrowtail i\ \frac{d\beta}{dt}\ \underline{c}$

$$\therefore\ \underline{a}_b = (\underline{i}\ \alpha_2 \times \underline{b} \longrightarrow \omega_2^2\ \underline{b}) \rightarrowtail (\underline{i}\alpha_3 \times \underline{c} \longrightarrow \omega_3^2\ \underline{c}) \text{--(313)}$$

$$\left(\alpha_3 = \frac{d^2\gamma}{dt^2}\right)$$

Equation (310) shows that the acceleration of point A has two components: one tangential and in the direction of the angular acceleration α_2 and the other along the link towards the centre of rotation.

Equation (312) also shows that point B has two components of acceleration, the tangential and normal components.

The relative acceleration of point B to point A, $\underline{a}_{ba}$ is given by;

$$\underline{a}_{ba} = \underline{a}_b \rightarrow \underline{a}_a \quad \text{----------(314)}$$

Substituting for values of $\underline{a}_b$ and $\underline{a}_a$ in equations (3.13) and (3.10) respectively, equation (3.14) becomes;

$$\underline{a}_{ba} = (\underline{i}\alpha_3 \longrightarrow \omega_3^2)\underline{c} \quad \text{----------(3.15)}$$

Therefore the relative acceleration of point B to point A, $\underline{a}_{ba}$, also has two components of acceleration; the tangential and the normal.

$$\underline{a} = \underline{a}^t \twoheadrightarrow \underline{a}^n \quad \text{----------(3.16)}$$

The acceleration of a point is the sum of the tangential and normal components of acceleration of the point.

$$\underline{a}^t = \underline{i}\alpha \times \underline{l} \quad \text{----------(3.17)}$$

The tangential component of acceleration, $\underline{a}^t$, is the product of the angular acceleration and the length of the link, l, and it is in the direction of the angular acceleration.

$$\underline{a}^n = -\omega^2\underline{l} \quad \text{----------(3.18)}$$

The normal or centripetal component of acceleration, $\underline{a}^n$, is the product of the square of the angular velocity of the link and length of the link l, directed towards the centre of rotation.

From equation (3.14),

$$\underline{a}_b = \underline{a}_a \twoheadrightarrow \underline{a}_{ba} \quad \text{----------(3.19)}$$

$$= \underline{a}_a^t \twoheadrightarrow \underline{a}_a^n \twoheadrightarrow \underline{a}_{ba}^t \twoheadrightarrow \underline{a}_{ba}^n \quad \text{----------(3.20)}$$

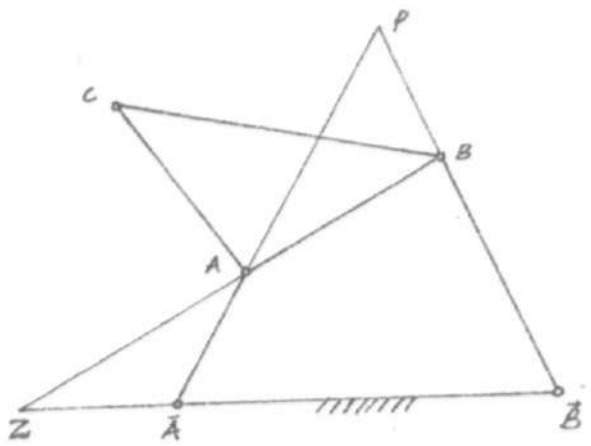
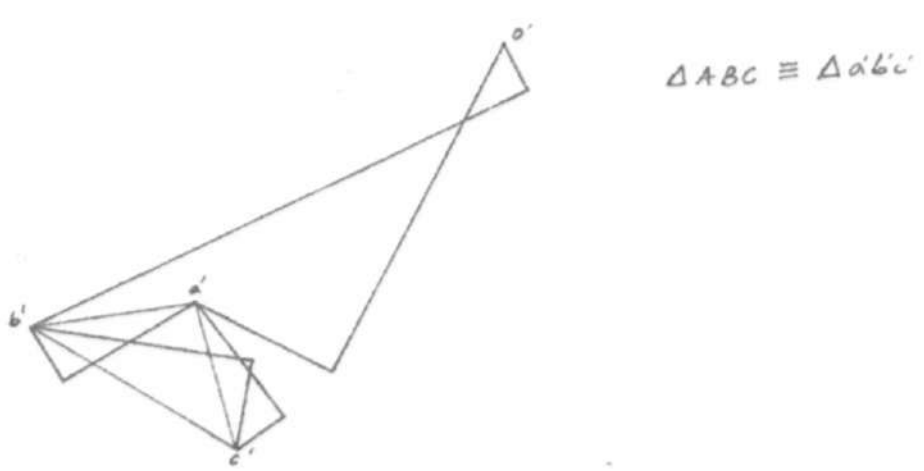


fig. 3.04: redrawn.



$\Delta ABC \equiv \Delta a'b'c'$

fig. 3.06: vector acceleration diagram for mechanism of fig. 3.04.

For a given mechanism configuration, the directions of all the components of acceleration for any required point is apparent. The magnitude of the normal components of accelerations depends on the angular velocity values. To draw a vector acceleration diagram it is usual to get the velocity values either by drawing the vector velocity diagram or using the instantaneous centre method.

The acceleration diagram of the 4-bar mechanism of Fig. (3.04) is shown in Fig.(3.06). The acceleration of point C of Fig. (3.04) is given by the following equations:

$$\underline{a}_c = \underline{a}_a^t + \underline{a}_a^n + \underline{a}_{ca}^t + \underline{a}_{ca}^n \quad \text{----------(321)}$$

$$\underline{a}_c = \underline{a}_b^t + \underline{a}_b^n + \underline{a}_{cb}^t + \underline{a}_{cb}^n \quad \text{----------(322)}$$

The magnitudes of $\underline{a}_a^n$, $\underline{a}_b^n$, $\underline{a}_{ca}^n$, $\underline{a}_{cb}^n$, can be got from the velocity diagram of Fig. (3.05).

For different values of angular acceleration of the driving crank, the shapes of the acceleration diagrams will vary. In the case of the vector velocity diagram, for a given position of a 4-bar mechanism, the shapes of the velocity diagrams will be similar, the size depending on the angular velocity of the driving crank.

From equations (316) to (322);

$$\underline{a}_a = \underline{a}_a^t + \underline{a}_a^n$$

$$\underline{a}_b = \underline{a}_b^t \rightarrow\!\!\!\!\rightarrow \underline{a}_b^n$$

$$= \underline{a}_a^t \rightarrow\!\!\!\!\rightarrow \underline{a}_a^n \rightarrow\!\!\!\!\rightarrow \underline{a}_{ba}^t + \underline{a}_{ba}^n$$

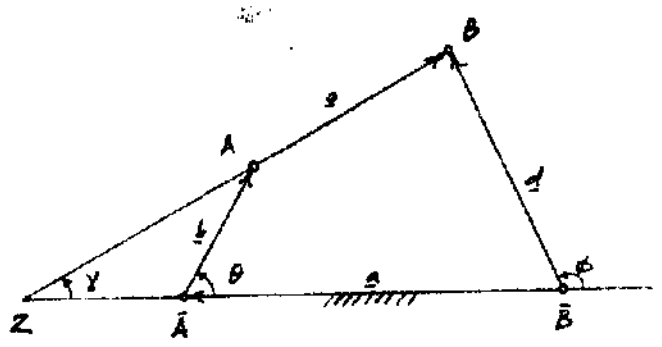
$$\underline{a}_c = \underline{a}_a \rightarrow\!\!\!\!\rightarrow \underline{a}_{ca}$$

$$= \underline{a}_a^t \rightarrow\!\!\!\!\rightarrow \underline{a}_a^n \rightarrow\!\!\!\!\rightarrow \underline{a}_{ca}^t \rightarrow\!\!\!\!\rightarrow \underline{a}_{ca}^n$$

$$\underline{a}_c = \underline{a}_b \rightarrow\!\!\!\!\rightarrow \underline{a}_{cb}$$

$$= \underline{a}_b^t \rightarrow\!\!\!\!\rightarrow \underline{a}_b^n \rightarrow\!\!\!\!\rightarrow \underline{a}_{cb}^t \rightarrow\!\!\!\!\rightarrow \underline{a}_{cb}^n$$

The tangential components of acceleration are all perpendicular to the link and the normal are directed along the link towards the centre of rotation.



$\underline{a} = ae^{i\pi}$

$\underline{b} = be^{i\theta}$

$\underline{c} = ce^{i\gamma}$

$\underline{d} = de^{i\phi}$

Fig. 3·07

...be derived by the use of complex numbers.

The 4-bar mechanism $\bar{A}AB\bar{B}$ is described by the vector and complex equations (323) and (324)

$$\underline{a} + \underline{b} + \underline{c} + \underline{d} = 0 \quad \text{----------(323)}$$

$$-a + be^{i\theta} + ce^{i\gamma} - de^{i\phi} = 0 \quad \text{----------(324)}$$

With angular velocities

$$\omega_2 = \frac{d\theta}{dt} = \dot{\theta};\ \omega_3 = \frac{d\gamma}{dt} = \dot{\gamma};\ \omega_4 = \frac{d\phi}{dt} = \dot{\phi}$$

and angular accelerations

$$\alpha_2 = \dot{\omega}_2 = \ddot{\theta};\ \alpha_3 = \dot{\omega}_3 = \ddot{\gamma};\ \alpha_4 = \dot{\omega}_4 = \ddot{\phi}$$

Differentiating equation (324(with respect to time, t,

$$i\,\dot{\theta}be^{i\theta} + i\dot{\gamma}ce^{i} - i\dot{\phi}de^{i\phi} = 0 \quad \text{----------(325)}$$

From equation (325)

$$\omega_2 b(\cos\theta + i\sin\theta) + \omega_3 c(\cos\gamma + i\sin\gamma) - \omega_4 d(\cos\phi + i\sin\phi) = 0$$

--------(326)

$$\therefore \quad \omega_3 c\cos\gamma = \omega_4 d\cos\phi - \omega_2 b\cos\theta \text{ -------(327)}$$

$$\omega_3 c\sin\gamma = \omega_4 d\sin\phi - \omega_2 b\sin\theta \text{ -------(328)}$$

Dividing equation (327) by equation (328);

$$\frac{\cos\gamma}{\sin\gamma} = \frac{\omega_4 d\cos\phi - \omega_2 b\cos\theta}{\omega_4 d\sin\phi - \omega_2 b\sin\theta}$$

$$\therefore \quad \frac{\omega_4}{\omega_2} = \frac{b\sin(\theta-\gamma)}{d\sin(\phi-\gamma)} \text{ ----------(329)}$$

From equation (326)

$$\omega_4 d\cos\phi = \omega_2 b\cos\theta + \omega_3 c\cos\gamma \text{ -----(330)}$$

$$\omega_4 d\sin\phi = \omega_2 b\sin\theta + \omega_3 c\sin\gamma \text{ -----(331)}$$

From equations (330) and (331);

$$\frac{\omega_3}{\omega_2} = \frac{b\sin(\theta-\phi)}{c\sin(\phi-\gamma)} \text{ ----------(332)}$$

For the relationship between the crank angular velocity and acceleration and the complex and follower angular acceleration, equation (325) is differentiated with respect to time, t, to give equation (333).

$$(i\ddot{\theta} + i^2\dot{\theta}^2)\underline{b} + (i\ddot{\gamma} + i^2\dot{\gamma}^2)\underline{c} - (i\ddot{\phi} + i^2\dot{\phi}^2)\underline{d} = 0 \quad \text{---------(333)}$$

or

$$(i\alpha_2 - \omega_2^2)be^{i\theta} + (i\alpha_3 - \omega_3^2)ce^{i\gamma} - (i\alpha_4 - \omega_4^2)de^{i\theta} = 0 \quad \text{---(334)}$$

Equation (334) gives rise to equations (335) and (336).

$$\alpha_2 b \sin\theta + \omega_2^2 b\cos\theta + \alpha_3 c \sin\gamma + \omega_3^2 c\cos\gamma - \alpha_4 d \sin\phi - \omega_4^2 d\cos\phi = 0 \quad \text{----------(335)}$$

$$\alpha_2 b \cos\theta - \omega_2^2 b\sin\theta + \alpha_3 c\cos\gamma - \omega_3^2 c\sin\gamma - \alpha_4 d\cos\phi - \omega_4^2 d\sin\phi = 0 \quad \text{----------(336)}$$

From equation (335);

$$\alpha_3 c \sin\gamma = \alpha_4 d \sin\phi + \omega_4^2 d\cos\phi - \alpha_2 b\sin\theta - \omega_2^2 b\cos\theta - \omega_3^2 c\cos\gamma \quad \text{----------(337)}$$

From equation (336);

$$\alpha_3 c\cos\gamma = \alpha_4 d\cos\phi - \omega_4^2 d\sin\phi - \alpha_2 b\cos\theta + \omega_2^2 b\sin\theta + \omega_3^2 c\sin\gamma \quad \text{----------(338)}$$

Solving equations (337) and (338);

$$\alpha_4 = \frac{b}{d}\left(\alpha_2 \frac{\sin(\theta-\gamma)}{\sin(\phi-\gamma)} - \omega_2^2 \frac{\cos\theta-\gamma}{\sin(\phi-\gamma)}\right) - \frac{c}{d}\omega_3^2 \frac{1}{\sin(\phi-\gamma)} + \omega_4^2 \cot(\phi-\gamma) \quad \text{----------(339)}$$

Substituting for ω_3^2 and ω_4^2 in equation (339):

$$\alpha_4 = \frac{b}{d}\left(\alpha_2 \frac{\sin(\theta-\gamma)}{\sin(\phi-\gamma)} - \frac{\omega_2^2 \cos(\theta-\gamma)}{\sin(\phi-\gamma)}\right)$$

$$- \frac{b^2 c}{d^3}\omega_2^2 \frac{\sin^2(\phi-\theta)}{\sin^3(\phi-\gamma)} + \frac{b^2}{d^2}\omega_2^2 \frac{\cos(\phi-\gamma)\sin^2(\theta-\gamma)}{\sin^3(\phi-\gamma)} \quad \text{---(340)}$$

For the value of the angular acceleration of the coupler, α_3, equations (335) and (336) are written as;

$$\alpha_4 d \sin\phi = \alpha_2 b \sin\theta + \omega_2^2 b \cos\theta + \alpha_3 c \sin\gamma + \omega_3^2 c \cos\gamma$$
$$- \omega_4^2 d \cos\phi \quad \text{----------(341)}$$

$$\alpha_4 d \cos\phi = \alpha_2 b \cos\theta - \omega_2^2 b \sin\theta + \alpha_3 c \cos\gamma - \omega_3^2 c \sin\gamma$$
$$- \omega_4^2 d \sin\phi \quad \text{----------(342)}$$

Solving equations (341) and (342);

$$\alpha_3 = \frac{b}{c}\left(\alpha_2 \frac{\sin(\theta-\phi)}{\sin(\phi-\gamma)} + \omega_2^2 \frac{\cos(\phi-\theta)}{\sin(\phi-\gamma)}\right)$$
$$+ \omega_3^2 \frac{\cos(\phi-\gamma)}{\sin(\phi-\gamma)} - \omega_4^2 \frac{d}{c\sin(\phi-\gamma)} \quad \text{----------(343)}$$

Substituting for ω_3^2 and ω_4^2 in equation (343);

$$\alpha_3 = \frac{b}{c}\left(\alpha_2 \frac{\sin(\theta-\phi)}{\sin(\phi-\gamma)} + \omega_2^2 \frac{\cos(\phi-\theta)}{\sin(\phi-\gamma)}\right)$$
$$+ \frac{b^2}{c^2}\omega_2^2 \frac{\cos(\phi-\gamma)\sin^2(\phi-\theta)}{\sin^3(\phi-\gamma)} - \omega_2^2 \frac{b^2 d}{c^3}\frac{\sin^2(\theta-\gamma)}{\sin^3(\phi-\gamma)} \quad \text{----(344)}$$

From the derived values of ω_3, ω_4, α_3, α_4, it can be said that to keep down inertia forces in 4-bar mechanisms, the angle between the coupler and the follower $(\phi - \delta)$ - which is termed the transmission angle, μ - should be kept as near 90° as possible. For most designs, $/90 - \mu/_{max}$ is less than 50°.

The relationship between the angular velocities of the links of a 4-bar mechanism can be derived in some other ways. One of such is by the use of the instantaneous centres.

Considering the 4-bar mechanism of Fig. 3.08;

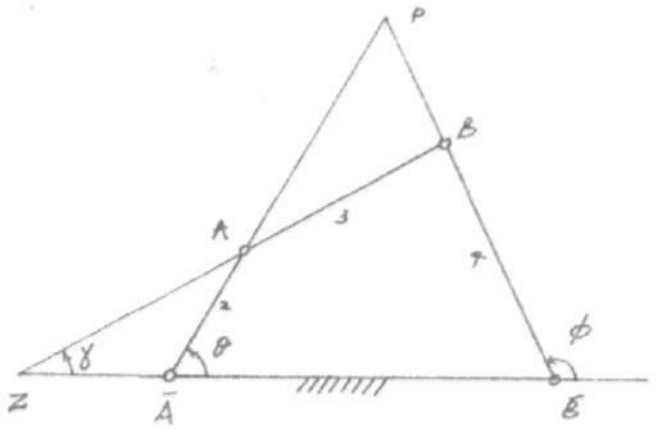


fig. 3.08.

the velocity of point A is given by

$$\underline{v}_a = \underline{\omega}_2 \times \underline{\bar{A}A} = \underline{\omega}_2 \times \underline{b} \text{ ----------(345)}$$

Similarly the velocity of point B is given by

$$\underline{v}_b = \underline{\omega}_4 \times \underline{BB} = \underline{\omega}_4 \times \underline{d} \quad \text{----------(346)}$$

Since point P is the instantaneous centre of rotation for the link 3 to the fixed link for the given configuration; then

$$\frac{\underline{v}_a}{PA} = \frac{\underline{v}_b}{PB} = \omega_3 \quad \text{----------(347)}$$

Substituting equations (345) and (346) in equation (27);

$$\frac{\omega_2 b}{PA} = \frac{\omega_4 d}{PB} = -\omega_3$$

$$\therefore \frac{\omega_4}{\omega_2} = \frac{b}{d} \times \frac{PB}{PA} \quad \text{----------(348)}$$

But $\frac{PA}{\sin(\phi - \gamma)} = \frac{PB}{\sin(\theta - \gamma)}$

$$\therefore \frac{\omega_4}{\omega_2} = \frac{b \sin(\theta - \gamma)}{d \sin(\phi - \gamma)} \quad \text{----------(349)}$$

From equation (348);

$$\frac{\omega_3}{\omega_2} = -\frac{b}{PA} \quad \text{----------(350)}$$

But $\frac{PA}{\sin(\phi - \gamma)} = \frac{AB}{\sin(\phi - \theta)} = \frac{c}{\sin(\phi - \theta)}$

$$\therefore \frac{\omega_3}{\omega_2} = -\frac{b \sin(\phi - \theta)}{c \sin(\phi - \gamma)} \quad \text{----------(351)}$$

Considering the instantaneous centre located at Z,

$$\therefore \quad \frac{\omega_4}{\omega_2} = \frac{Z\bar{A}}{Z\bar{B}} \qquad \text{----------(352)}$$

$$= \frac{Z\bar{A}}{Z\bar{A} + \bar{A}\bar{B}} = \frac{Z\bar{A}}{Z\bar{A} + a}$$

From Fig. 3.08;

$$\frac{Z\bar{A}}{\sin(\theta - \gamma)} = \frac{\bar{A}A}{\sin \gamma} = \frac{b}{\sin \gamma} \qquad \text{----------(353)}$$

Also $$\frac{Z\bar{B}}{\sin(\phi - \gamma)} = \frac{\bar{B}B}{\sin \gamma} = \frac{d}{\sin \gamma} \qquad \text{----------(354)}$$

Substituting equations (353) and (354) in (352);

$$\frac{\omega_4}{\omega_2} = \frac{b \sin(\theta - \gamma)}{d \sin(\phi - \gamma)} \qquad \text{----------(355)}$$

From equation (352) it can be seen that $\frac{\omega_4}{\omega_2}$ is a maximum or a minimum when $Z\bar{A}$ is a minimum.

Since point Z is the instantaneous centre for links 2 and 4, it then has the same velocity in both links.

$$\underline{v}_z = \omega_2 . Z\bar{A} = \omega_4 . Z\bar{B} \qquad \text{----------(356)}$$

3.7. Prescribed Velocities And Accelerations:

By the use of complex numbers, it is possible to find a 4-bar mechanism for prescribed values of velocities and accelerations of the various links.

Considering the 4-bar mechanism of Fig. (3.09);

$$\underline{a} \mapsto \underline{b} \mapsto \underline{c} \rightarrow \underline{d} = 0 \qquad \text{----------(357)}$$

With angular velocities given by;

$$\omega_2 = \frac{d\theta}{dt} = \dot{\theta} \; ; \; \omega_3 = \frac{d\gamma}{dt} = \dot{\gamma} \; ; \; \omega_4 = \frac{d\phi}{dt} = \dot{\phi}$$

and angular accelerations given by;

$$\alpha_2 = \dot{\omega}_2 = \ddot{\theta} \; ; \; \alpha_3 = \dot{\omega}_3 = \ddot{\gamma} \; ; \; \alpha_4 = \dot{\omega}_4 = \ddot{\phi}$$

Equation (357) can be written as

$$-a + be^{i\theta} + ce^{i\gamma} - de^{i\phi} = 0 \qquad \text{----------(358)}$$

Differentiating equation (358) with respect to time, t:

$$0 + i\dot{\theta}be^{i\theta} + i\dot{\gamma}ce^{i\gamma} - i\dot{\phi}de^{i\phi} = 0 \qquad \text{----------(359)}$$

Also differentiating equation (359) with respect to time, t:

$$0 + (i\ddot{\theta} + i^2\dot{\theta}^2)be^{i\theta} + (i\ddot{\gamma} + i^2\dot{\gamma}^2)ce^{i\gamma} - (i\ddot{\phi} + i^2\dot{\gamma}^2)de^{i\phi} = 0 \quad --(360)$$

Equations (358) to (360) can be written as in equations (i) to (iii);

$$-1 \cdot \underline{a} + 1 \cdot \underline{b} + 1 \cdot \underline{c} - 1 \cdot \underline{d} = 0 \quad \text{----------(i)}$$

$$0 \cdot \underline{a} + \omega_2 \cdot \underline{b} + \omega_3 \cdot \underline{c} - \omega_4 \underline{d} = 0 \quad \text{----------(ii)}$$

$$0 \cdot \underline{a} + (\alpha_2 + i\omega_2^2) \cdot \underline{b} + (\alpha_3 + i\omega_3^2)\underline{c} - (\alpha_4 + i\omega_4^2)\underline{d} = 0 \quad --(iii)$$

The set of equations has the following solutions:

$$\underline{b} = \left[(\omega_3\alpha_4 - \omega_4\alpha_3) + i\omega_3\omega_4(\omega_4 - \omega_3)\right] / D$$

$$\underline{c} = \left[(\omega_4\alpha_2 - \omega_2\alpha_4) + i\omega_2\omega_4(\omega_2 - \omega_4)\right] / D$$

$$\underline{d} = \left[(\omega_3\alpha_2 - \omega_2\alpha_3) + i\omega_2\omega_3(\omega_2 - \omega_3)\right] / D$$

$$\underline{a} = \underline{d} - \underline{b} - \underline{c} \quad \text{----------(361)}$$

where D, the denominator is given by

$$D = \left[(\omega_4\alpha_3 - \omega_3\alpha_4) + (\omega_2\alpha_3 + \omega_3\alpha_2) - (\omega_2\alpha_4 + \omega_4\alpha_2)\right]$$
$$+ i\left\{(\omega_3 - \omega_4)\left[\omega_3\omega_4 + \omega_2(\omega_3 + \omega_4) + \omega_2^2\right]\right\}$$

Usually the denominator, D, is often omitted and set equal to unity. This does not affect the relative lengths of the 4-bar mechanism derived for a given set of angular velocity and acceleration values.

Ex(3.01) A 4-bar mechanism is to be designed where the following velocity and acceleration values apply in one of its phases.

$$\omega_2 = 10\ s^{-1};\ \omega_3 = 2\ s^{-1};\ \omega_4 = 6\ s^{-1}$$

$$\alpha_2 = 0;\ \alpha_3 = 10\ s^{-2};\ \alpha_4 = 0$$

Solution: Substituting the given values in equation 41 and taking D = 1;

$$\underline{b} = -60 + 48\,i$$

$$\underline{c} = 240\,i$$

$$\underline{d} = -100 + 160\,i$$

$$\underline{a} = -40 - 128\,i$$

The derived 4-bar mechanism is shown in Fig. (3.10)

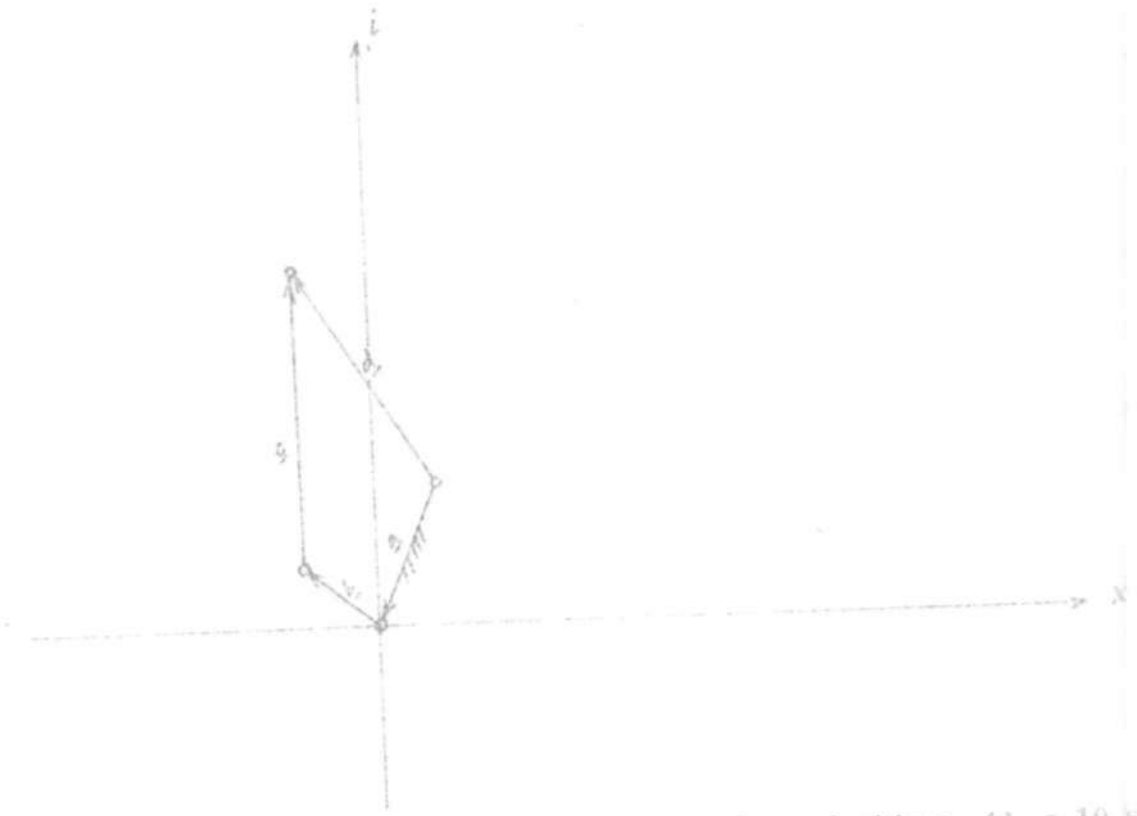


Fig. 3.10: 4-bar linkage with prescribed angular velocities: $\omega_2 = 10$ s^{-1}, $\omega_3 = 2$ s^{-1}, $\omega_4 = 6$ s^{-1}, and angular accelerations: $\alpha_2 = 0$, $\alpha_3 = 10$ s^{-2}, $\alpha_4 = 0$; assuming D is unity.

If the value for the denominator were taken into account, then

$$D = 160 - 792\,i$$

The values of $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$, then become

$$\underline{b} \triangleq -.073 - .061\,i$$

$$\underline{c} \triangleq -.2911 + .0588\,i$$

$$\underline{d} \triangleq -.218 - .0821\,i$$

$$\underline{a} \triangleq -.1461 - .0799\,i$$

The derived 4-bar mechanism is shown in Fig. (3.11). It can be seen that the denominator does not affect the link length ratios; as such, both mechanisms have the same motion characteristics.

Ex(3.02) Derive a 4-bar mechanism that will have an approximate speed ratio of cranks, $\frac{\omega_4}{\omega_2} = -1$, for a portion of its crank rotation.

Solution: For approximate constant speed ratio of cranks,

$$\alpha_2 = \alpha_4 = 0$$

Assuming $\omega_2 = -\omega_4 = 2\ s^{-1}$.

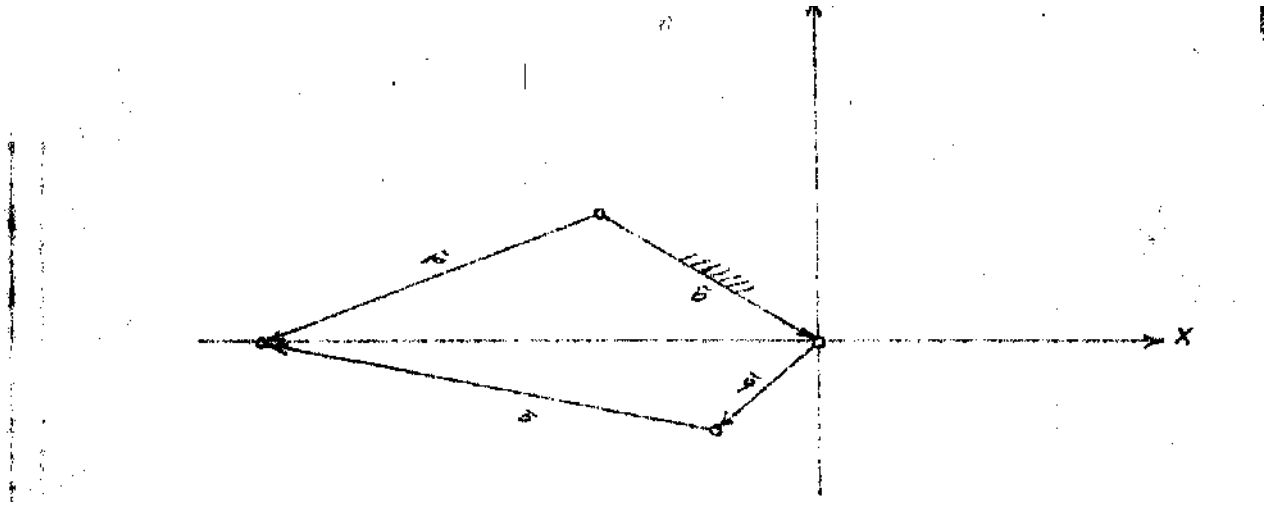


fig 3.11: 4 - bar linkage with prescribed angular velocities and accelerations:

$\omega_2 = 10\,s^{-1}$, $\omega_3 = 2\,s^{-1}$, $\omega_4 = 6\,s^{-1}$

$\alpha_2 = 0$, $\alpha_3 = 10\,s^{-2}$, $\alpha_4 = 0$

From equation (41); assuming D = 1

$$\underline{c} = -16\,i$$

$$\underline{b} = 2\alpha_3 + 2\omega_3(2 + \omega_3)i$$

$$\underline{d} = -2\alpha_3 + 2\omega_3(2 - \omega_3)i$$

$$\underline{a} = \underline{d} - \underline{b} - \underline{c} = -4\alpha_3 - 4\omega_3^2 i + 16\,i$$

It is then possible to assign arbitrary values to ω_3 and α_3.

Taking $\omega_3 = 0$ and $\alpha_3 = 4\ s^{-2}$

then

$\underline{b} = 8$

$\underline{c} = -16\,i$

$\underline{d} = -8$

$\underline{a} = -16 + 16\,i$

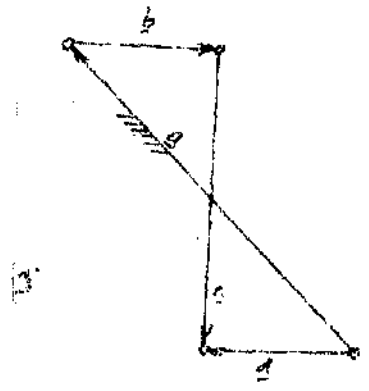


Fig. 3.12

The derived 4-bar mechanism is shown in Fig. (3.12)

3.8. Velocities And Accelerations Of Points On The Coupler:

Given a point C on the coupler of a 4-bar mechanism $\bar{A}AB\bar{B}$ (Fig. 3.13) whose position is defined relative to a specific frame of reference, the velocity and acceleration values can be found for the particular configuration.

Let $\lambda = \gamma + \alpha$

then $\frac{d\lambda}{dt} = \dot{\lambda} = \frac{d\gamma}{dt} = \dot{\gamma} = \omega_3$ (α = constant)

$$\frac{d^2\lambda}{dt^2} = \ddot{\lambda} = \ddot{\gamma} = \alpha_3$$

From Fig. 3.13, $x = b \cos\theta + m \cos\lambda$ ----------(362)

$y = b \sin\theta + m \sin\lambda$ ----------(363)

Differentiating equations (362) and (363) with respect to time, t:

$$\frac{dx}{dt} = \dot{x} = -b\omega_2 \sin\theta - m\omega_3 \sin\lambda \quad \text{--------(364)}$$

$$\frac{d^2x}{dt^2} = \ddot{x} = -b(\omega_2^2 \cos\theta + \alpha_2 \sin\theta) - m(\omega_3^2 \cos\lambda + \alpha_3 \sin\lambda) \text{---(365)}$$

$$\frac{dy}{dt} = \dot{y} = b\omega_2 \cos\theta + m\omega_3 \cos\lambda \quad \text{----------(366)}$$

$$\frac{d^2y}{dt^2} = \ddot{y} = -b(\omega_2^2 \sin\theta - \alpha_2 \cos\theta) - m(\omega_3^2 \sin\lambda - \alpha_3 \cos\lambda) \text{---(367)}$$

The velocity of point C on the coupler is given by

$$\underset{\sim}{v}_c = (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} \quad \text{----------(368)}$$

and the acceleration, $\underset{\sim}{a}_c$, is given by

$$\underset{\sim}{a}_c = (\ddot{x}^2 + \ddot{y}^2)^{\frac{1}{2}} \quad \text{----------(369)}$$

Substituting in equation (368);

$$(\underset{\sim}{v}_c)^2 = (b\omega_2)^2 + (m\omega_3)^2 + 2\,mb\omega_2\omega_3 \cos(\theta - \lambda) \quad \text{-----(370)}$$

Ex(3.03) Given a 4-bar mechanism of Fig. 3.04, find the velocity and acceleration values of point C. Assume $\omega_2 = 10\ s^{-1}$ and $\alpha_2 = 40\ s^{-2}$.

From Fig. (304);

$\theta = 60^\circ$; $\delta = 29.3^\circ$; $\phi = 113.3^\circ$

b = 4 ; o = 6 ; d = 7

m = 5.6; $\alpha = 96^\circ$

From equation (351);

$$\frac{\omega_3}{\omega_2} = -\frac{b}{c}\frac{\sin(\phi - \theta)}{\sin(\phi - \delta)}$$

$$\therefore \quad \omega_3 = -10.\frac{4}{6}.\frac{\sin 53.3^\circ}{\sin 84^\circ} = -5.375 \text{ s}^{-1}$$

$$\lambda = -\delta + \alpha = 125.3^\circ$$

Substituting in equations (364) to (367);

$$\dot{x}_c = -10.08 \text{ s}^{-1}$$

$$\dot{y}_c = 37.40 \text{ s}^{-1}$$

$$\therefore \quad \underline{v}_c = (\dot{x}_c^2 + \dot{y}_c^2)^{\frac{1}{2}} = 38.73 \text{ s}^{-1}$$

This is shown on the redrawn Fig. (3.04).

Substituting in equation (344);

$$\alpha_3 = 8 \text{ s}^{-2}$$

Substituting now in equations (365) and (367);

$$\ddot{x}_c = -281.63 \text{ s}^{-2}$$

$$\ddot{y}_c = -424.31 \text{ s}^{-2}$$

$$\therefore \quad /\underline{a}_c/^2 = (\ddot{x}_c^2 + \ddot{y}_c^2)^{\frac{1}{2}} = 509.27 \text{ s}^{-2}$$

The acceleration is shown on the redrawn Fig. 3.04.

CHAPTER FOUR

FORCES IN 4-BAR MECHANISMS

A mechanism designed to a kinematic specification may not operate successfully if the forces in the links are greater than can be readily coped with.

The principle of statics states that when a body is in a state of equilibrium under the action of various forces, the force vectors form a closed polygon, the lines of action of the forces pass through a single point.

In the 4-bar mechanism of Fig. 4·01, a given force Q_2 acts on link 2 and the magnitude of a balancing force Q_4 on link 4 with the given direction is to be found. Force can be transmitted through the coupler, link 3, in an axial direction only and the forces are the two ends of the coupler, are equal in magnitude and opposite in direction.

For equilibrium of link 2, the forces acting on the link pass through a point; this is the intersection of Q_2 and Q_{32}, the axial force in the coupler. The force acting through the fixed joint, Q_{12}, passes through the intersection of Q_2 and Q_{32}. The polygon of forces due to Q_2, Q_{12} and Q_{32} must also close, for equilibrium to be achieved.

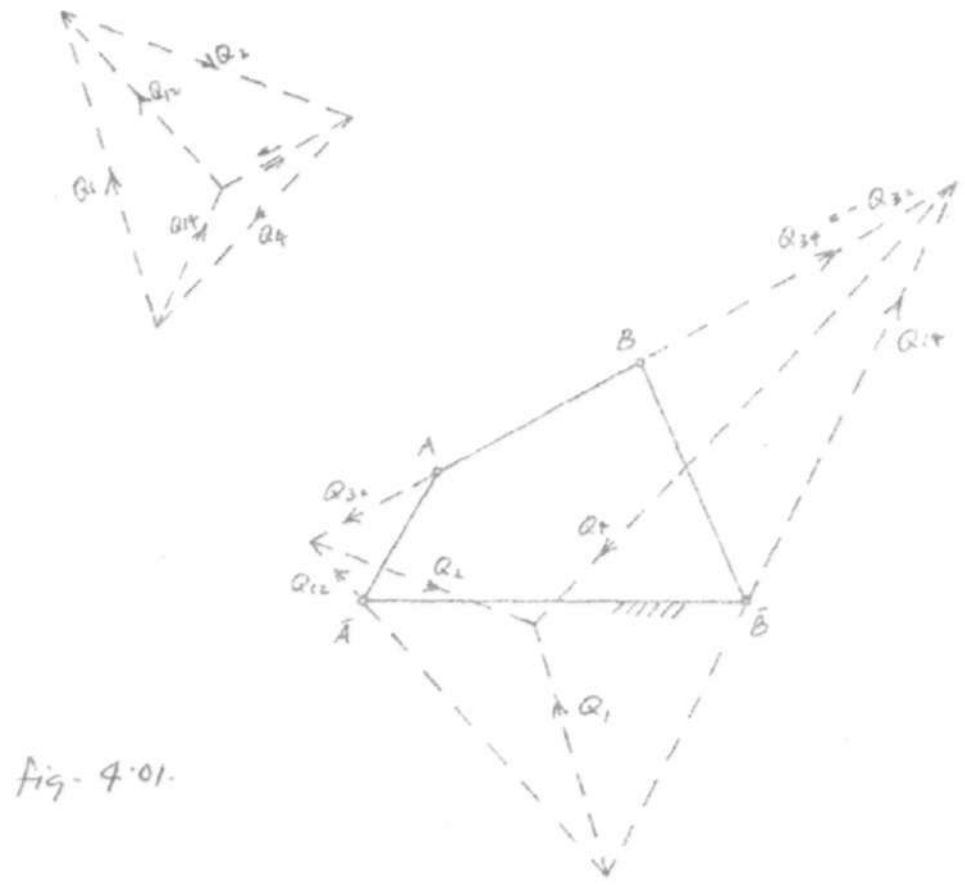


Fig- 4·01.

Similarly, for the equilibrium of link 4; Q_4, Q_{34}, Q_{14} form a closed triangle and intersect at a single point. The axial forces Q_{34} and Q_{32} are related by the equation; $Q_{34} = - Q_{32}$ ------(401)

For the equilibrium of the 4-bar mechanism, forces Q_1, Q_2, Q_4, balance. The force exerted by the frame, Q_1, is the vector sum of Q_{12} and Q_{14}. $Q_1 = Q_{12} + Q_{14}$ ----------(402)

4.1. Torque Ratio:

Neglecting frictional losses, power input at the input crank should be equal to power output at the output crank.

$$T_i \omega_i = T_o \omega_o \quad \text{----------(403)}$$

For the 4-bar linkage of Fig. (4.01),

$$\omega_i = \omega_2 \text{ and } \omega_o = \omega_4$$

$$\therefore T_i = T_2 \text{ and } T_o = T_4$$

(T - torque).

$$\therefore \frac{T_2}{T_4} = \frac{\omega_4}{\omega_2} = \frac{Z\bar{A}}{Z\bar{B}} \quad \text{----------(404)}$$

4.2. Inertia Forces:

The forces due to the inertia effects cannot be determined until the mass distribution in the mechanism is known. The total load on a link is due to the combined effects of static and dynamic forces.

The resultant inertia force, R, in a link is the product of the mass of the link and the acceleration of the centre of gravity of the link, $\underline{a}_g$; and it acts parallel to the acceleration.

$$R = m \underline{a}_g \quad \text{----------(405)}$$

where R is the inertia force

m is the mass of the link

and $\underline{a}_g$, the acceleration of the centre of gravity.

The torque due to inertia, T, is given by

$$T = I\alpha \qquad \text{----------(406)}$$

where I is the polar mass moment of inertia,

and α is the angular acceleration of the link.

The relationship between the torque, T, and the inertia force, R, is given by

$$T = Re \qquad \text{----------(407)}$$

where e is the shortest distance from the centre of gravity to the line of action of the inertia force, R.

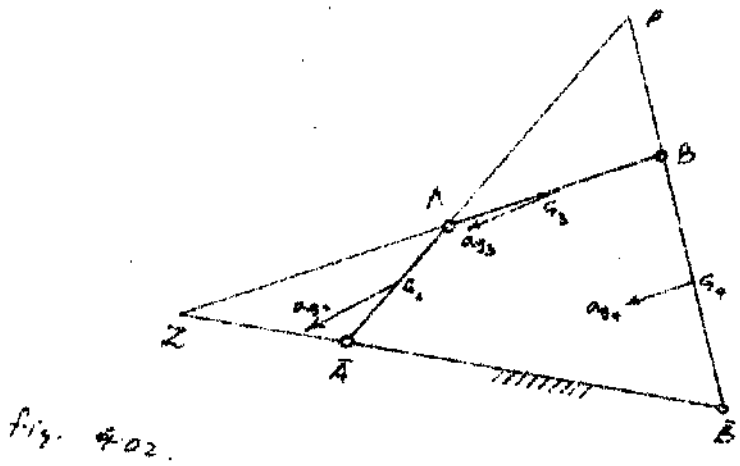


fig. 4.02.

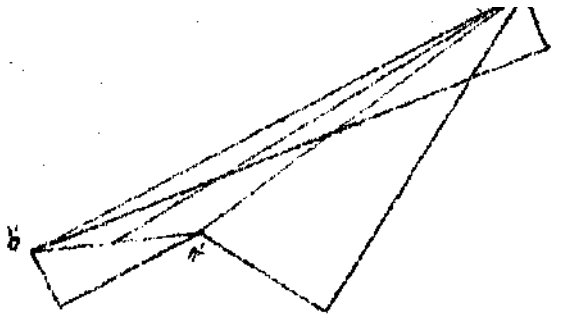


fig. 4·03: Acceleration diagram of mechanism in fig. 4·02.

The driving torque, Q_i, in the driving link to overcome inertia effects can be evaluated by using the principle of virtual work. With a virtual rotation $\delta\theta_i$ of the driving link

$$Q_i\,\delta\theta_i - \sum m\,\underline{a}_g \cdot \delta\underline{s} - \sum I\alpha\,\delta\lambda = 0 \quad \text{------(408)}$$

where $\delta\underline{s}$ is the virtual displacement of the centre of gravity

and $\delta\lambda$ is the virtual rotation of a member.

If the virtual displacements take place in a time δt,

$$Q_i\,\frac{d\theta_i}{dt} - \sum m\,\underline{a}_g \cdot \frac{d\underline{s}}{dt} - \sum I\alpha\,\frac{d\lambda}{dt} = 0 \quad \text{----------(409)}$$

when δt tends to zero.

Assuming $\frac{d\theta_i}{dt} = \omega_i$

$$Q_i\omega_i - \sum m \underline{a}_g \cdot \underline{v}_g - \sum I\alpha\omega = 0 \quad \text{----------(4.10)}$$

where $\underline{v}_g$ is the velocity of the centre of gravity

and ω is the angular velocity of a member.

Considering the 4-bar mechanism of Fig. (4.02), for the driving link, link 2; from equation (4.10);

$$Q_{i2}\cdot\omega_2 - m_2\alpha_2(\bar{A}G_2)^2\omega_2 - I_2\omega_2\alpha_2 = 0$$

$$\therefore \quad Q_{i2} = m_2\alpha_2(\bar{A}G_2)^2 + I_2\alpha_2$$

$$= \left[m_2(\bar{A}G_2)^2 + I_2\right]\alpha_2 \quad \text{----------(4.11)}$$

This is shown in Fig. (4.04).

The inertia torque due to link 3, Q_{i3}, is given by

$$\omega_2\cdot Q_{i3} - m_3 \underline{a}_{g3} \cdot \underline{v}_{g3} - I_3\alpha_3\omega_3$$

$$\therefore \quad Q_{i3} = \frac{1}{\omega_2}\left[m_3 \underline{a}^t_{g3}\, \underline{v}_{g3} + I_3\alpha_3\omega_3\right] \quad \text{----------(4.12)}$$

This is illustrated in Fig. (4.06).

The inertia torque due to link 4, Q_{i4}, is given by

$$Q_{i4}\cdot\omega_2 - m_4\alpha_4(\bar{B}G_4)^2\omega_4 - I_4\alpha_4\omega_4$$

$$\therefore \quad Q_{i4} = \frac{\omega_4}{\omega_2}\alpha_4\left[m_4(\bar{B}G_4)^2 + I_4\right] \quad \text{----------(4.13)}$$

The net torque needed to drive the mechanism, Q_i, is given by

$$Q_i = Q_{i2} + Q_{i3} + Q_{i4} \quad \text{----------(4.14)}$$

Assuming that for the 4-bar mechanism of Fig.(4.02), there is a uniform mass/unit length value of m_x, the Q_{i2}, Q_{i3}, Q_{i4} can be calculated.

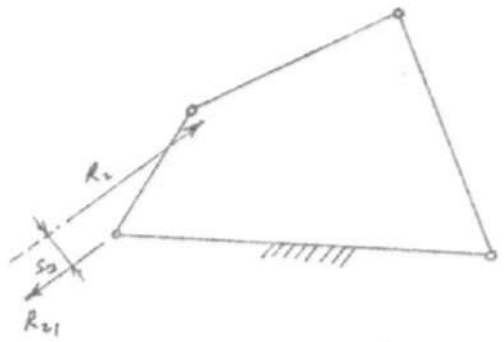


fig. 4.04.

$$Q_{i1} = R_2 \cdot S_2$$

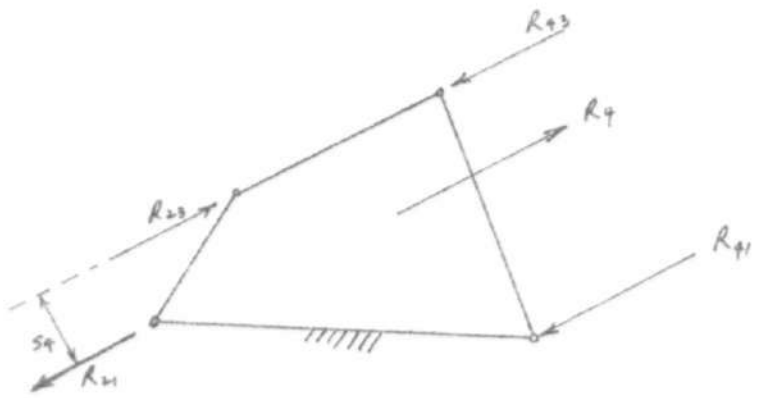


fig. 4.05.

$$R_{23} = -R_{43}$$

$$Q_{i4} = R_{23} \cdot S_4$$

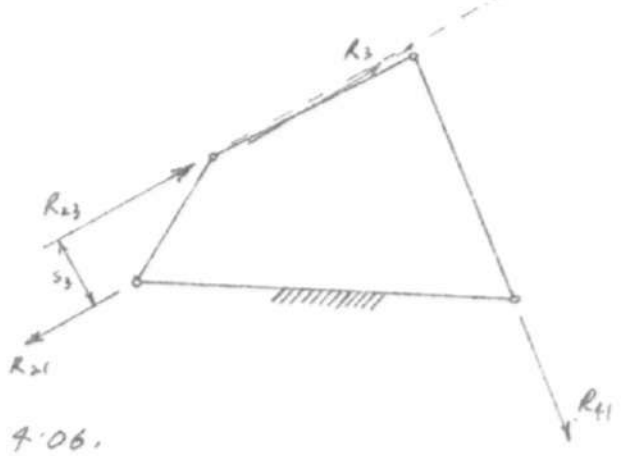


fig. 4.06.

$$Q_{i3} = R_{23} \cdot S_3$$

$I = (m_x l).\frac{l^2}{12}$ where l is the length of link.

The torque necessary to drive link 2, Q_{i2}, is then given by, (substituting in equation);

$$Q_{i2} = \left[m_x.\bar{A}A(\bar{A}G_2)^2 + Im_x\bar{A}A \cdot \frac{(\bar{A}A)^2}{12}\right]\alpha_2$$

$$= 40\left[4\, m_x x\, 2^2 + m_x x\, 4 \times \frac{4^2}{12}\right]$$

$$= 853.3\, m_x$$

Similarly the torque necessary to drive link 3, Q_{i3}, is given by;

$$Q_{i3} = 6\, m_x\left[27.6 \times 352 + \frac{6^2}{12} \times 10.67 \times 5.5\right]$$

$$= 5930\, m_x$$

The value of Q_{i4}, the torque required to drive link 4 of the mechanism in Fig. 4.02, is given by;

$$Q_{i4} = 0.3 \times 83.43\left[m_x x\, 7 \times 12.5 + m_x x\, 7 \times \frac{7^2}{12}\right]$$

$$= 2905.4\, m_x$$

The values for $\omega_2, \omega_3, \omega_4, \alpha_2, \alpha_3, \alpha_4$, and the lengths of the link members are got from Figs. 4.02 and 4.03.

$$\therefore \quad Q_i = Q_{i2} + Q_{i3} + Q_{i4} = 7688.7\, m_x$$

CHAPTER FIVE

COUPLER CURVES

The paths traced by points belonging to the coupler, link 3, of a 4-bar mechanism are termed coupler curves. These curves are of sixth order with varying shapes.

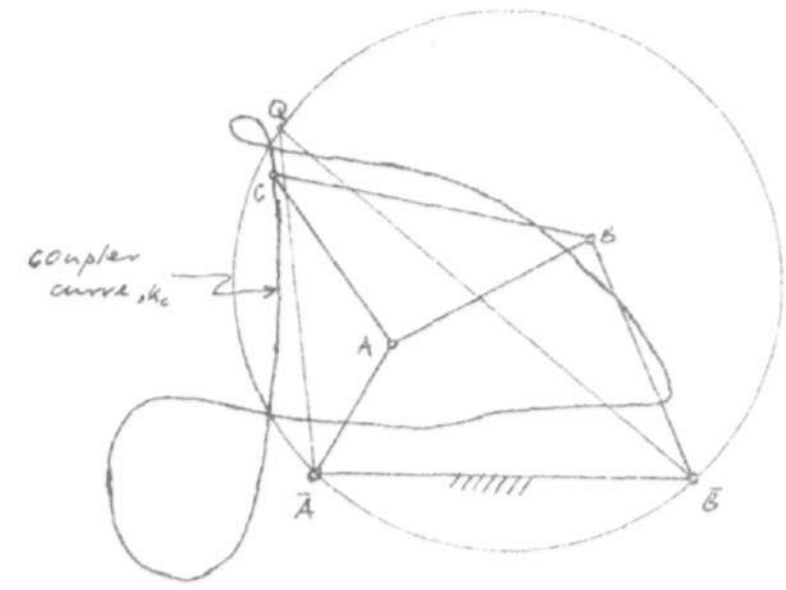


fig. 5.01: Coupler curve, k_c, of point C of 4-bar mechanism $\bar{A}AB\bar{B}$.

5.1. Coupler Curve Equations:

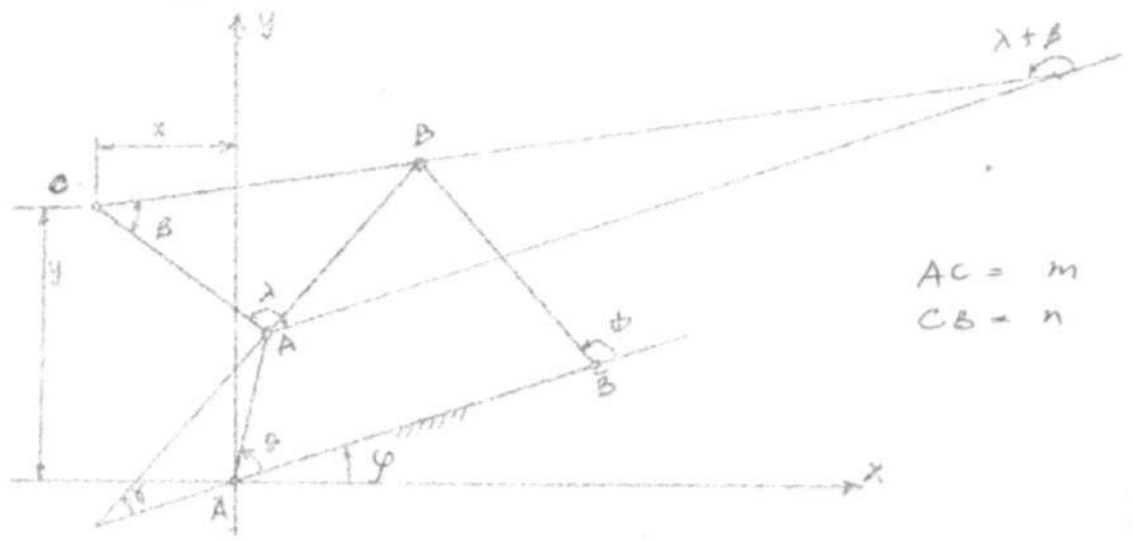


fig 5.02.

From Fig. (5.02),

$$x_A = x - m\cos(\lambda+\varphi);\ y_A = y - m\sin(\lambda+\varphi) \text{ --------(501)}$$

$$x_B = x - n\cos(\lambda+\varphi+\beta);\ y_B = y - n\sin(\lambda+\varphi+\beta) \text{ --(502)}$$

Also,

$$x_A^2 + y_A^2 - b^2 = 0;\ (x_B - a\cos\varphi)^2 + (y_B - a\sin\varphi)^2 - d^2 = 0 \text{ ---(503)}$$

From equations (501), (502) (503);

(i) $$x^2 + y^2 - 2mx\cos(\lambda+\varphi) - 2my\sin(\lambda+\varphi) + m^2 - b^2 = 0 \text{ -----(504)}$$

(ii) $x^2 + y^2 - 2\,n\,x\cos(\lambda+\varphi+\beta) - 2\,n\,y\sin(\lambda+\varphi+\beta)$

$- 2\,a\,x\cos\varphi - 2\,a\,y\sin\varphi + 2\,a\,n\cos\varphi\cos(\lambda+\varphi+\beta)$

$+ 2\,a\,n\sin\varphi\sin(\lambda+\varphi+\beta) + n^2 + a^2 - d^2 = 0$ -------(505)

Equations (504) and (505) can be written as

$$2\,m\left[x\cos(\lambda+\varphi) + y\sin(\lambda+\varphi)\right] = x^2 + y^2 + m^2 - b^2 \quad \text{-----(506)}$$

$$2\left[n\,x\cos(\lambda+\varphi+\beta) + n\,y\sin(\lambda+\varphi+\beta) + a\,x\cos\varphi\right.$$

$$\left. + a\,y\sin\varphi + an\cos(\lambda+\beta)\right] = x^2 + y^2 + n^2 + a^2 - d^2 \quad \text{----(507)}$$

From equation (506);

$$\cos\lambda\left[2\,m\,x\cos\varphi + 2\,m\,y\sin\varphi\right] + \sin\left[-2\,mx\sin\varphi\right.$$

$$\left. + 2\,m\,y\cos\varphi\right] = x^2 + y^2 + m^2 - b^2 \quad \text{----------(508)}$$

From equation (507);

$$\cos\lambda\left[2\,n\,x\cos(\varphi+\beta) + 2\,n\,y\sin(\varphi+\beta) + e\,a\,n\cos\beta\right]$$

$$+ \sin\lambda\left[-2\,n\sin(\varphi+\beta) + 2\,n\,y\cos(\varphi+\beta) - 2\,a\,n\sin\beta\right]$$

$$= x^2 + y^2 + n^2 + a^2 - d^2 - a\,x\cos\varphi - a\,n\sin\varphi \quad \text{-----(509)}$$

From equation (501);

$$\cos\lambda = \frac{(x\cos\varphi + y\sin\varphi) - b\cos(\theta)}{m} \quad \text{----------(510)}$$

$$\sin\lambda = \frac{(y\cos\varphi - x\sin\varphi) - b\sin(\theta)}{m} \quad \text{----------(511)}$$

since $x_A = b\cos(\theta + \varphi)$; $y_A = b\sin(\theta + \varphi)$.

Substituting equations (510) and (511) in equation (509);

$$\frac{2n}{m}\Big[(x\cos\varphi + y\sin\varphi) - b\cos\theta\Big]\Big[x\cos(\varphi+\beta) + y\sin(\varphi+\beta) + a\cos\beta\Big] + \frac{2n}{m}\Big[y\cos(\varphi+\beta) - x\sin(\varphi+\beta) - a\sin\beta\Big]\Big[y\cos\varphi - x\sin\varphi) - b\sin\theta\Big]$$

$$= x^2 + y^2 + n^2 + a^2 - d^2 - a\,x\cos\varphi - a\,y\sin\varphi \text{ ------(512)}$$

Equation (512) can be written as

$$x^2\Big[1 - \frac{2n}{m}\cos\beta\Big] + y^2\Big[1 - \frac{2n}{m}\cos\beta\Big] + \frac{2n\,x}{m}\Big[\Big\{b\sin(\theta+\beta) - a\sin\beta\Big\}\sin\varphi + \Big\{b\cos(\theta+\beta) + a\cos\beta - \frac{m\,a}{n}\Big\}\cos\varphi\Big] + \frac{2n\,y}{m}\Big[\Big\{\frac{m\,a}{n} - b\cos(\theta+\beta) - a\cos\beta\Big\}\sin\varphi + \Big\{b\sin(\theta+\beta) - a\sin\beta\Big\}\cos\varphi\Big] + a^2 + n^2 - d^2 - \frac{2n\,a\,b}{m}\cos(\theta-\beta) = 0 \text{ ---(513)}$$

For a particular 4-bar mechanism configuration, x, y depend on the value of θ.

Solving equations (508) and (509) for $\cos\lambda$ and $\sin\lambda$;

$$\cos\lambda = \frac{(m\,y\cos\varphi - m\,x\sin\varphi)\left[x^2 + y^2 + n^2 + a^2 - d^2 - a\,x\cos\varphi - a\,y\sin\varphi\right]}{2\,m\,n\left[x^2\sin\beta + y^2\sin\beta + a\,x\sin(\beta-\varphi) + a\,y\cos(\beta-\varphi)\right]} + \frac{\left[n\,x\sin(\varphi+\beta) - n\,y\cos(\varphi+\beta)\right]}{2\,m\,n\left[(x^2\sin\beta + y^2\sin\beta + a\,n\sin\beta\right]\left[x^2 + y^2 + m^2 - b^2\right]}{+ a\,x\sin(\beta-\varphi) + a\,y\cos(\beta-\varphi)\right]} \quad \text{---------(514)}$$

$$\sin\lambda = \frac{\left[n\,x\cos(\varphi+\beta) + n\,y\sin(\varphi+\beta) + a\,n\cos\beta\right]}{2\,m\,n\left[x^2\sin\beta + y^2\sin\beta + a\,x\sin(\beta-\varphi)\right]} \left[x^2 + y^2 + m^2 - b^2\right] - \left[mx\cos\varphi + m\,y\sin\varphi\right]\left[x^2 + y^2 + n^2 + a^2 - d^2 - a\,x\cos\varphi - a\,y\sin\varphi\right]}{+ a\,y\cos(\beta-\varphi)\right] \quad 2\,m\,n\left[x^2\sin\beta + y^2\sin\beta + a\,x\sin(\beta-\varphi) + a\,y\cos(\beta-\varphi)\right]} \quad \text{---------(515)}$$

but $\sin^2\lambda + \cos^2\lambda = 1$

$$\therefore K_1^2 + K_2^2 = K_3^2 \quad \text{---------(516)}$$

where

$$K_1 = (m\,y\cos\varphi - m\,x\sin\varphi)\left[x^2 + y^2 + n^2 + a^2 - d^2 - a\,x\cos\varphi - a\,y\sin\varphi\right]$$

$$+\left[n\,x\sin(\varphi+\beta) - n\,y\cos(\varphi+\beta) + a\,n\sin\beta\right]\left[x^2 + y^2 + m^2 - b^2\right]$$

$$K_2 = \left[n\,x\cos(\varphi+\beta) + n\,y\sin(\varphi+\beta) + a\,n\cos\beta\right]\left[x^2 + y^2 + m^2 - b^2\right]$$

$$-\left[m\,x\cos\varphi + my\sin\varphi\right]\left[x^2 + y^2 + n^2 + a^2 - d^2 - a\,x\cos\varphi - a\,y\sin\varphi\right]$$

$$K_3 = 2\,m\,n\left[x^2\sin\alpha + y^2\sin\beta + a\,x\sin(\beta-\varphi) + a\,y\cos(\beta-\varphi)\right]$$

where $\varphi = 0$; equation (5.16) becomes

$$n^2\left[(x-a)^2 + y^2\right](x^2 + y^2 + m^2 - b^2)^2 - 2\,m\,n\left[(x^2 + y^2 - a\,x)\cos\beta + a\,y\sin\beta\right] + \left[x^2 + y^2 + m^2 - b^2\right](x-a)^2 + y^2 + n^2 - d^2\Big]$$

$$+ m^2(x^2 + y^2)\left[(x-a)^2 + y^2 + n^2 - d^2\right]^2 - 4\,m^2\left[(x^2 + y^2 - a\,x)\sin\beta - a\,y\cos\beta\right]^2 = 0 \quad \text{----------(5.17)}$$

It can be seen that the coupler curve equation is of the sixth order.

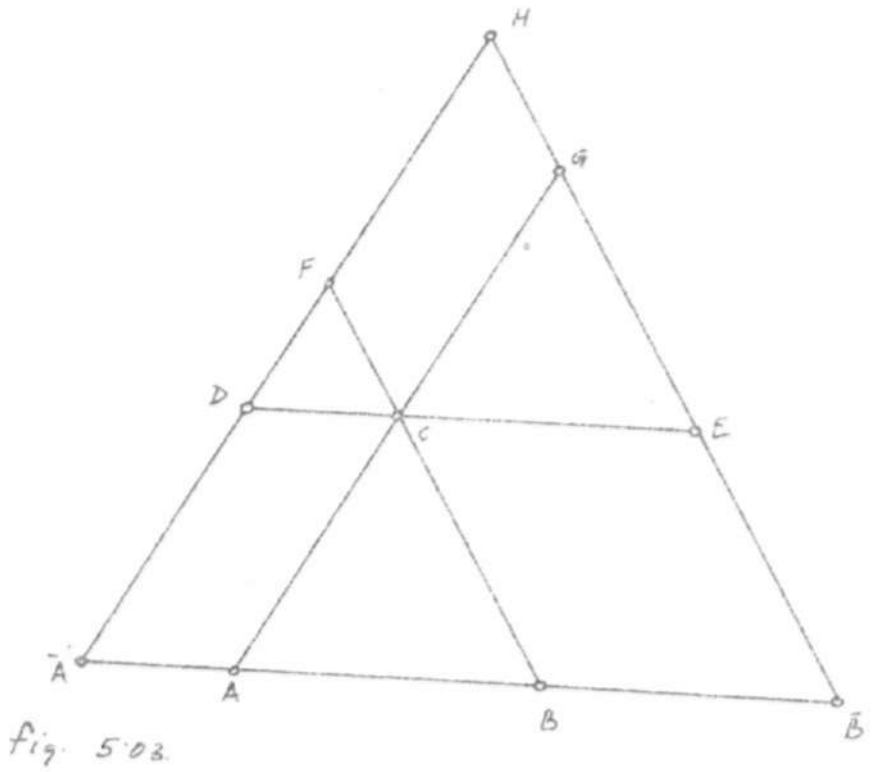


fig. 5·03.

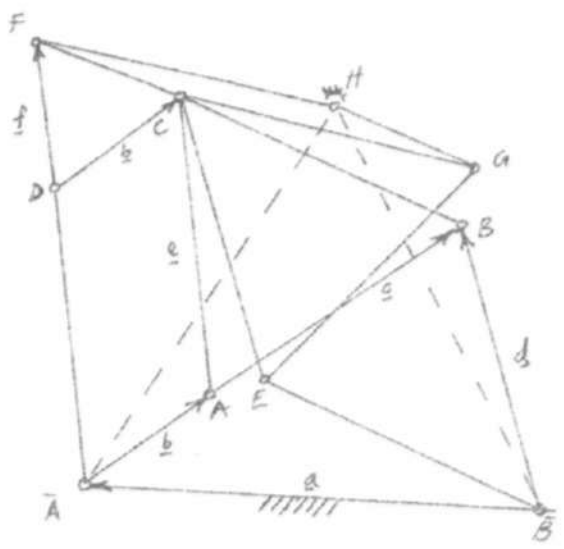


fig. 5·04.

5.2. Robert's Theorem:

In Fig. (5.03), triangles $\bar{A}\bar{B}H$, ABC, DCE, CEG are all similar. The triangles remain similar even when the links are displaced relative to one another. The same linkage is shown in Fig. (5.04) with relative displacements between the links.

If the linkage is fixed at points $\bar{A}$, $\bar{B}$ and H, there occurs three different 4-bar mechanisms joined at a common coupler point C. The 4-bar mechanisms designated by $\bar{A}AB\bar{B}$, $\bar{A}DFH$, $\bar{B}EGH$, then trace a common coupler point curve at coupler point C.

From Fig. (5.04),

$$\underline{e} = (\underline{c} + \underline{\Omega} \times \underline{c})k \qquad \text{---------- (5.18)}$$

where $\underline{c} = \underline{AB}$

$\underline{e} = \underline{AC}$

k = constant

$\underline{\Omega}$ = a vector normal to plane of mechanism.

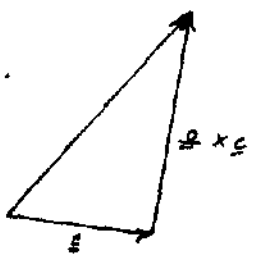


fig. 5.05.

Similarly, $\underline{DF} = \underline{f} = (b + \Omega_4 \times \underline{b})k$ ----------(519)

where $\underline{b} = \bar{A}\bar{A}$

and $\underline{CG} = \underline{g} = -(\underline{d} + \Omega_2 \times \underline{d})k$ ----------(520)

where $\underline{d} = \bar{B}\bar{B}$

$$\bar{A}\bar{H} = \bar{A}\underline{D} + \underline{DF} + \underline{FH}$$

$$= \underline{e} + \underline{f} + \underline{g}$$

$$= k\left[(\underline{b} + \underline{c} - \underline{d}) + \Omega_1 \times (\underline{b} + \underline{c} - \underline{d})\right]$$

$$= -k\left[\underline{a} + \Omega_2 \times \underline{a}\right]$$

$$= \left[\underline{a} + \underline{b} + \underline{c} - \underline{d} = 0\right] \quad \text{----------(521)}$$

Since $\bar{A}\bar{B}$ is a fixed vector, $\bar{A}\underline{H}$ is also a fixed vector. H is then a fixed point since $\bar{A}$ is a fixed point. It can also be seen that triangle $\bar{A}\bar{B}H$ is similar to triangle ABC.

Since H is a fixed point, ADFH, $\bar{A}A\bar{B}\bar{B}$, and $\bar{B}\bar{B}GH$ can then act as different 4-bar mechanisms and the three mechanisms have a common coupler point C that describes a coupler curve for the three 4-bar mechanisms. The three mechanisms that describe the same coupler path are termed cognates.

5.3. Cognate Linkages:

It has been shown that a given 4-bar mechanism has two other 4-bar mechanisms 'cognate' to it. The coupler curve of a 4-bar mechanism can also be traced by a particular 5-bar mechanism with a certain restriction imposed on it. This means that, in all, there are six, 4- and 5-bar mechanisms, cognate linkages.

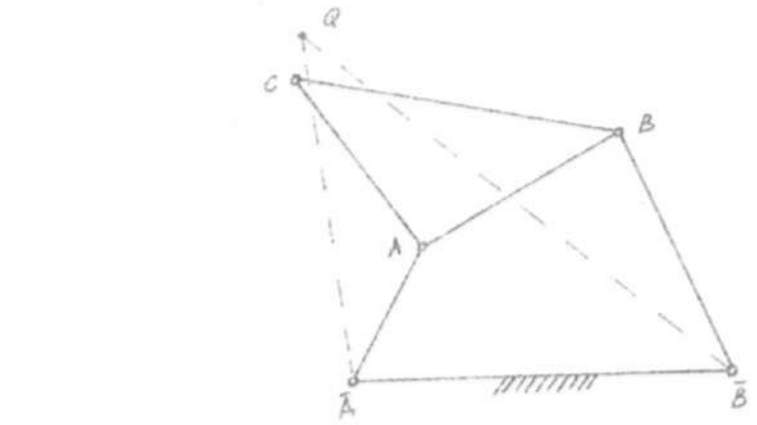
Given a 4-bar mechanism, the two other cognate 4-bar mechanisms can be found and the 5-bar cognate mechanisms corresponding to each of the three 4-bar mechanisms can also be found.

Cognate 4-bar mechanisms: Given the 4-bar mechanism $\bar{A}AB\bar{B}$, with coupler point C, of Fig. (42 a), the cognate 4-bar mechanism can be found as follows (Fig. 5.06):

Triangle $\bar{A}\bar{B}Q$ is made similar to triangle ABC.

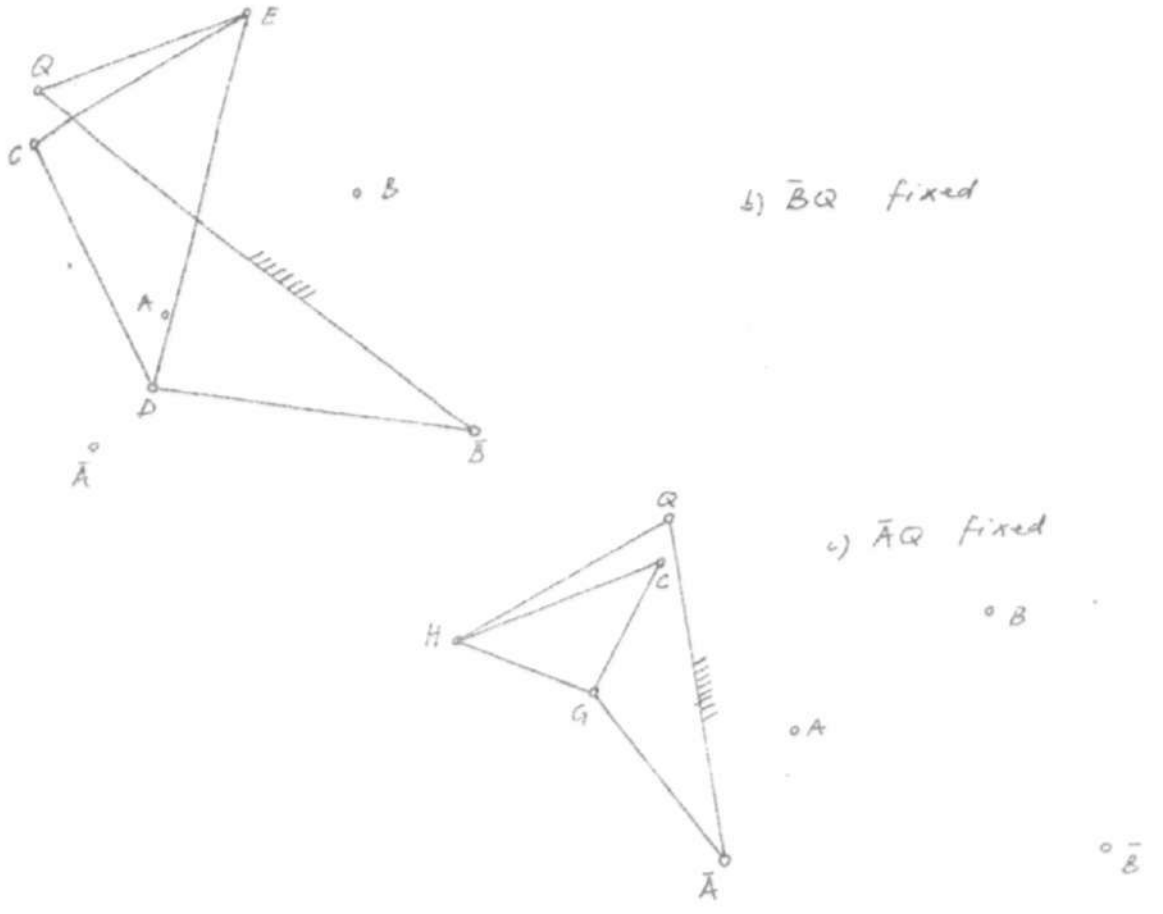
(i) For the cognate linkage with BQ fixed, triangle DCE is made similar to triangle $\bar{B}\bar{A}Q$, with $\bar{B}BCD$ being a parallelogram. QE and BD are the two cranks in the cognate 4-bar mechanism $\bar{B}DEQ$, coupler point C, with $\bar{B}Q$ fixed.

(ii) For the cognate linkage with $\bar{A}Q$ fixed, triangle GCH is made similar to triangle $\bar{A}\bar{B}Q$, with $\bar{A}ACG$ being a parallelogram. $\bar{A}G$ and QH are the two cranks in the 4-bar mechanism $\bar{A}GHQ$, coupler point C, with $\bar{A}Q$ fixed.



$\Delta ABC \equiv \bar{A}\bar{B}Q$

a) $\bar{A}\bar{B}$ fixed



b) $\bar{B}Q$ fixed

c) $\bar{A}Q$ fixed

fig 5.06: Cognate 4-bar mechanisms.

Cognate 5-bar mechanisms: In the cognate 5-bar mechanisms, the two cranks are driven in the same direction at the same speed. This can be achieved by gearing or by use of a parallelogram linkage to connect the two cranks, as shown in Fig. (5.07).

Considering the 4-bar mechanism in Fig. (5.06 a) with $\bar{A}\bar{B}$ fixed, $\bar{A}ACM$ and $\bar{B}BCN$ are two parallelograms. $\bar{A}MCN\bar{B}$ is the 5-bar mechanism cognate to the 4-bar mechanism with the two of the 5-bar mechanisms $\bar{A}M$, $\bar{B}N$ driven in the same direction and at the same angular speed. $\bar{A}FN\bar{B}$ is the parallelogram to ensure same angular velocity of cranks 5 and 8.

Point C is the common coupler point.

The same procedure can be followed to derive the 5-bar mechanisms cognate to the two other cognate 4-bar mechanisms in Fig. (5.06).

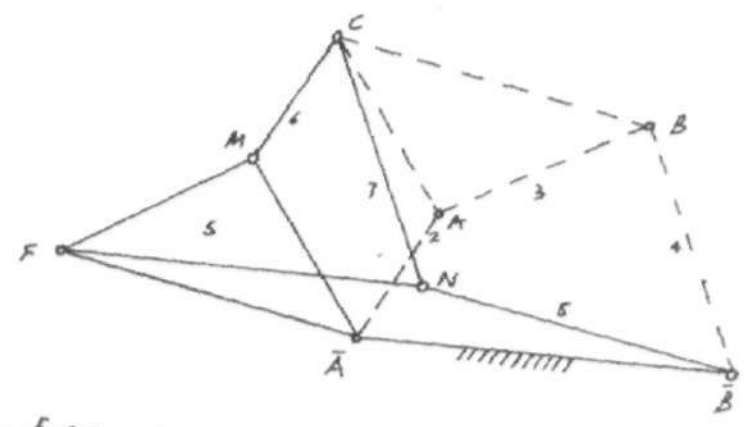


fig. 5.07: 5-bar mechanism cognate to $\bar{A}ABB\bar{B}$. $\bar{A}FN\bar{B}$ is a parallelogram to ensure same angular velocity of links 5 and 8.

5.4. Double Points Of A Coupler Curve:

The double points of a coupler curve are points on the curve crossed by the coupler point twice. Any double points of a coupler curve lies on the circle through vertices of triangle $\bar{A}\bar{B}Q$ which is similar to the coupler triangle ABC (Fig. 5.04).

Double points for given crank positions: Double points of coupler curves are of interest; they enable the coupler point to come back to the same point after some crank angular displacement.

Given two pairs of crank positions $\bar{A}A_1$, $\bar{A}A_2$ and $\bar{B}B_1$, $\bar{B}B_2$ to correspond to the same double point on the coupler curve of point C; the double point, C^1, is the intersection of the perpendicular bisectors of A_1A_2 and B_1B_2 (Fig. 5.08).

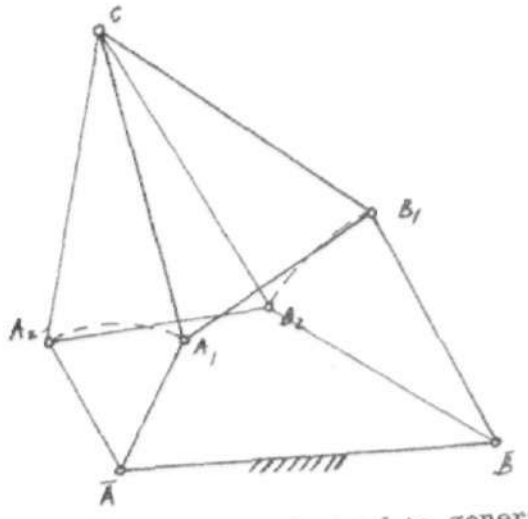


Fig. 5.08: 4-bar mechanism derived to generate a double point for two given crank positions.

5.5. Coupler Curves With Cusps:

The coupler point coinciding with the instantaneous centre of link 3, coupler, relative to the fixed link, describes a coupler curve with a cusp (an end, pointed, with an indefinable tangent) or, also called, a return point (Fig. 5.09).

Considering the 4-bar mechanism in Fig. (10), the coupler point C describes a coupler point with two cusps coincides the instantaneous centres at phases $\bar{A}A_1B_1\bar{B}$, $\bar{A}A_2B_2\bar{B}$ and A_1, A_2 are the intersections of the crank circle, radius $\bar{A}A$, and the circle through points $\bar{A},\bar{B}$, A_1 (Fig. 5.10).

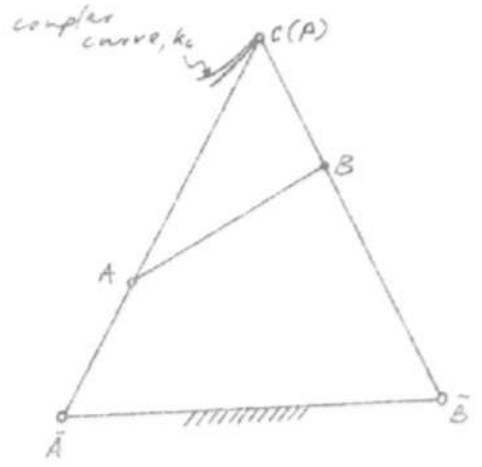


fig. 5:09.

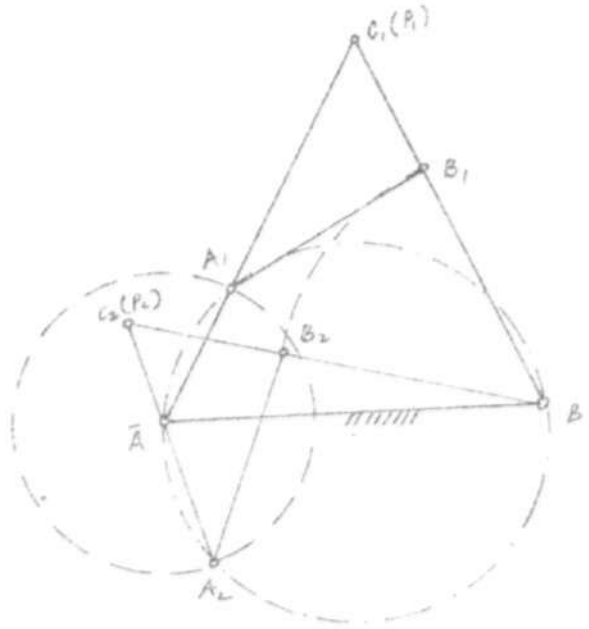


Fig. 5:10: 4 - bar mechanism to generate coupler curve with two cusps, at $C_1(P_1)$ and $C_2(P_2)$

CHAPTER SIX

FINITE DISPLACEMENTS

It is important in the synthesis of 4-bar mechanisms to be able to derive suitable 4-bar linkages that will satisfy various link displacement requirements. The methods to be discussed involve graphical layout and construction. The mechanisms derived are accurate for the positions considered.

6.1. Poles:

The pole concept is rather useful in the synthesis of linkage mechanisms. A plane can move from one position to another by pure rotation about a fixed point. The fixed point is termed the pole for the plane for moving by pure rotation from the initial position to the final position. If any of the initial and final positions changes, the position of the pole also changes. For infinitesimal plane displacement, the pole coincides with the velocity pole.

6.2. Two Link Positions:

The geometric properties of two finitely near positions of a link is of great value in the synthesis of crank mechanisms.

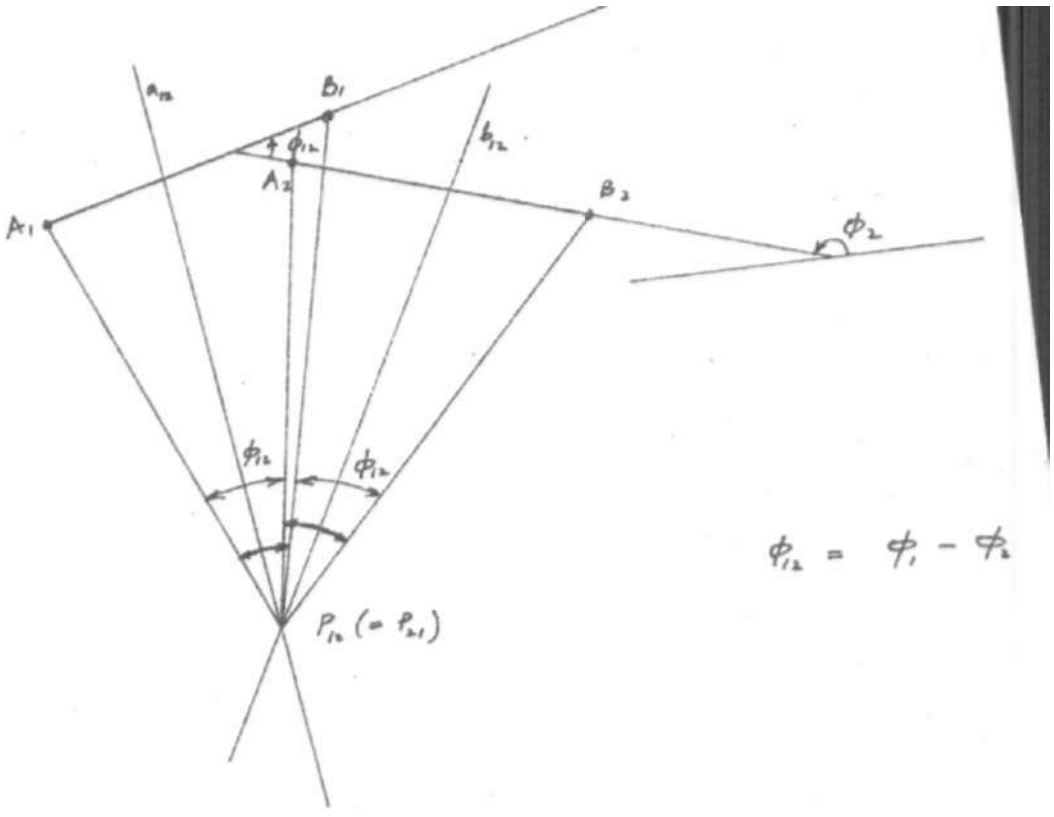


fig. 6.01.

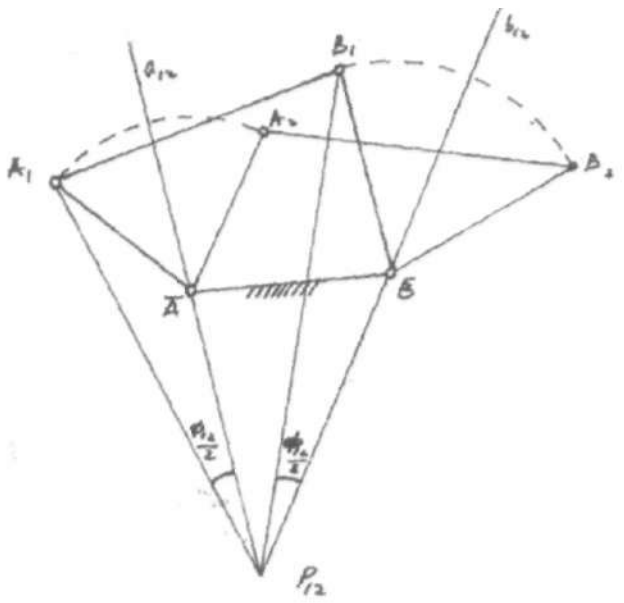
..... of $a_1 a_2$, a_{12},

and of $B_1 B_2$, b_{12} (Fig. 6.01).

For pure rotation, triangles $A_1P_{12}B_1$ and $A_2P_{12}B_2$ are similar, therefore, angles $A_1P_{12}B_1$ and $A_2P_{12}B_2$ are equal. Consequently, angles $A_1P_{12}A_2$ and $B_1P_{12}B_2$ are also equal. The angle of rotation, ϕ_{12}, from position 1 to position 2 is the difference in the inclinations of the links at positions 1 and 2.

The simplest way of moving link AB from position 1 to position 2 is by pivoting the link at the pole, P_{12}.

If AB represents the coupler of a 4-bar mechanism, A_1 can be made to move to A_2 by having the fixed end, $\bar{A}$, of the crank $\bar{A}A$ on the perpendicular bisector of A_1A_2, a_{12}. Similarly B_1 can be moved to B_2 by fixing $\bar{B}$ on the perpendicular bisector of B_1B_2, b_{12}. Consequently, there are infinite numbers of possible 4-bar mechanism to move coupler AB from position 1 to position 2 (Fig. 6.02).



$\angle A_1 P_{12} A = \angle B_1 P_{12} B = \frac{\phi_{12}}{2}$

fig. 6.02: 4-bar mechanism for two link positions.

6.3. Three Link Positions:

Pole Triangle: Given three finitely separated link positions C_1D_1, C_2D_2, C_3D_3 as in Fig. (6.03), the pole triangle $P_{12}P_{23}P_{13}$ may be constructed. P_{12} is the intersection of the perpendicular bisectors of C_1C_2, c_{12}, and D_1D_2, d_{12}. P_{23} is the intersection of c_{23} and d_{23} and P_{13} is the intersection of c_{13} and d_{13}.

The images of C_1 about $P_{12}P_{13}$, C_2 about $P_{12}P_{23}$ and C_3 about $P_{23}P_{13}$ coincide at C' (Fig. 6.03).

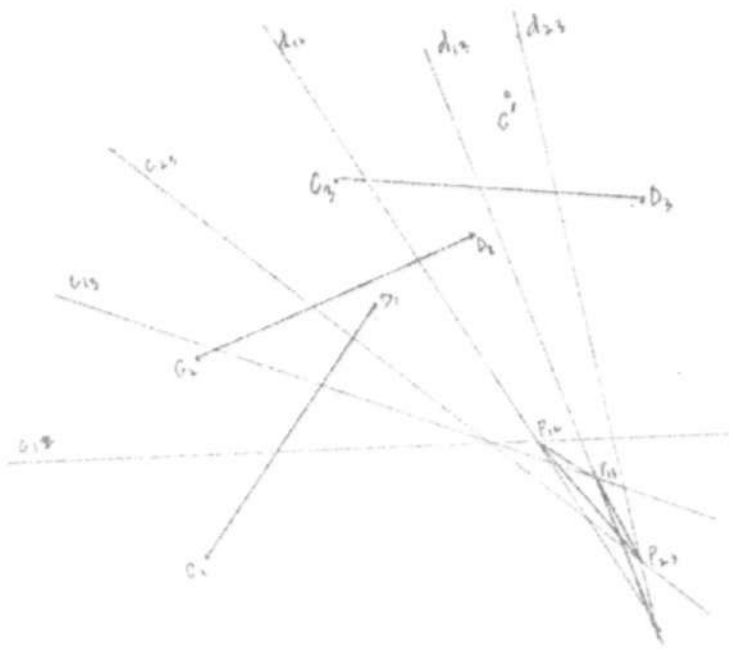
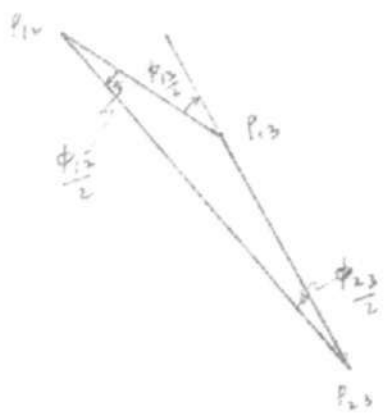


fig. 6.03: Three link positions with pole triangle $P_{12}P_{23}P_{13}$



$$\frac{\phi_{13}}{2} = \frac{\phi_{12}}{2} + \frac{\phi_{23}}{2} \qquad (603)$$

fig. 6.04: Pole triangle in fig. 6.03 enlarged.

$P_{12}P_{23}$, $P_{23}P_{13}$ respectively coinciding at D^1.

Three finitely near link positions: Given three link positions as in Fig. (6.05); link CD is the coupler of a 4-bar mechanism whose fixed pivots coincide with the intersections of c_{12}, c_{23}, c_{13} and d_{12}, d_{23}, d_{13} (Fig. 6.05). If the 4-bar mechanism derived is not so desirable, some other pair of points along the coupler length can be chosen as the moving pivots A, B. Then different sets of perpendicular bisectors will have to be drawn and new fixed pivot position located.

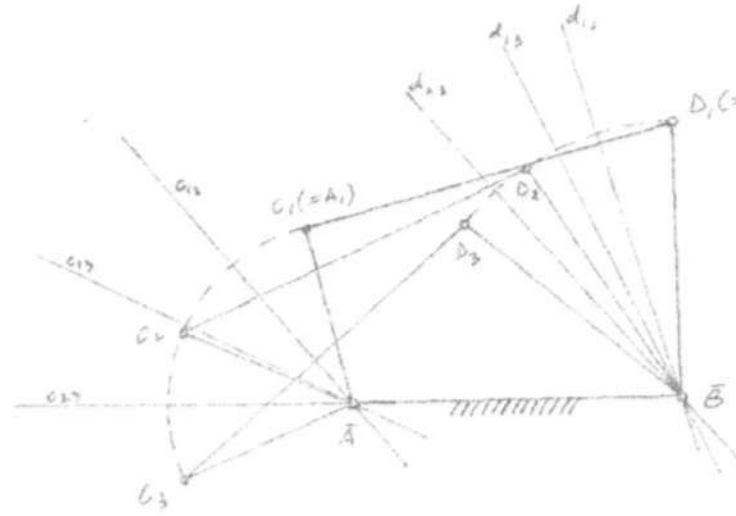


fig. 6.05 : 4-bar mechanism derived for the finitely near link positions.

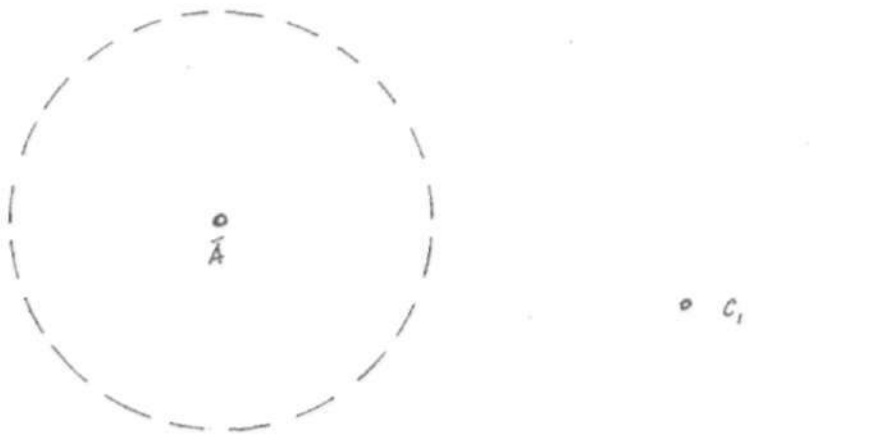


fig. 6.06 :

Ex(6.01):

Given three point positions, C_1, C_2, C_3; crank radius $\bar{A}A$ and a fixed pivot position $\bar{A}$ as in Fig. (6.06), a suitable 4-bar mechanism is to be derived to satisfy the given conditions.

Solution:

Choosing a convenient length AC, points A_1, A_2, A_3 are located. Poles P_{12}, P_{23}, P_{13} are then determined. A suitable position is then chosen for the other fixed pivot B.

Poles P_{12}, P_{23} are used to locate point B_2. From equation (602);

$$\angle A_2 P_{12} \bar{A} = \angle B_2 P_{12} \bar{B} = \phi_{12}/2$$

Also

$$\angle A_2 P_{23} \bar{A} = 180^\circ - \angle B_2 P_{23} \bar{B} = \phi_{23}/2.$$

The derived 4-bar mechanism is shown in Fig. (6.07). There exists infinite number of solutions depending on the location of the fixed pivot $\bar{B}$.

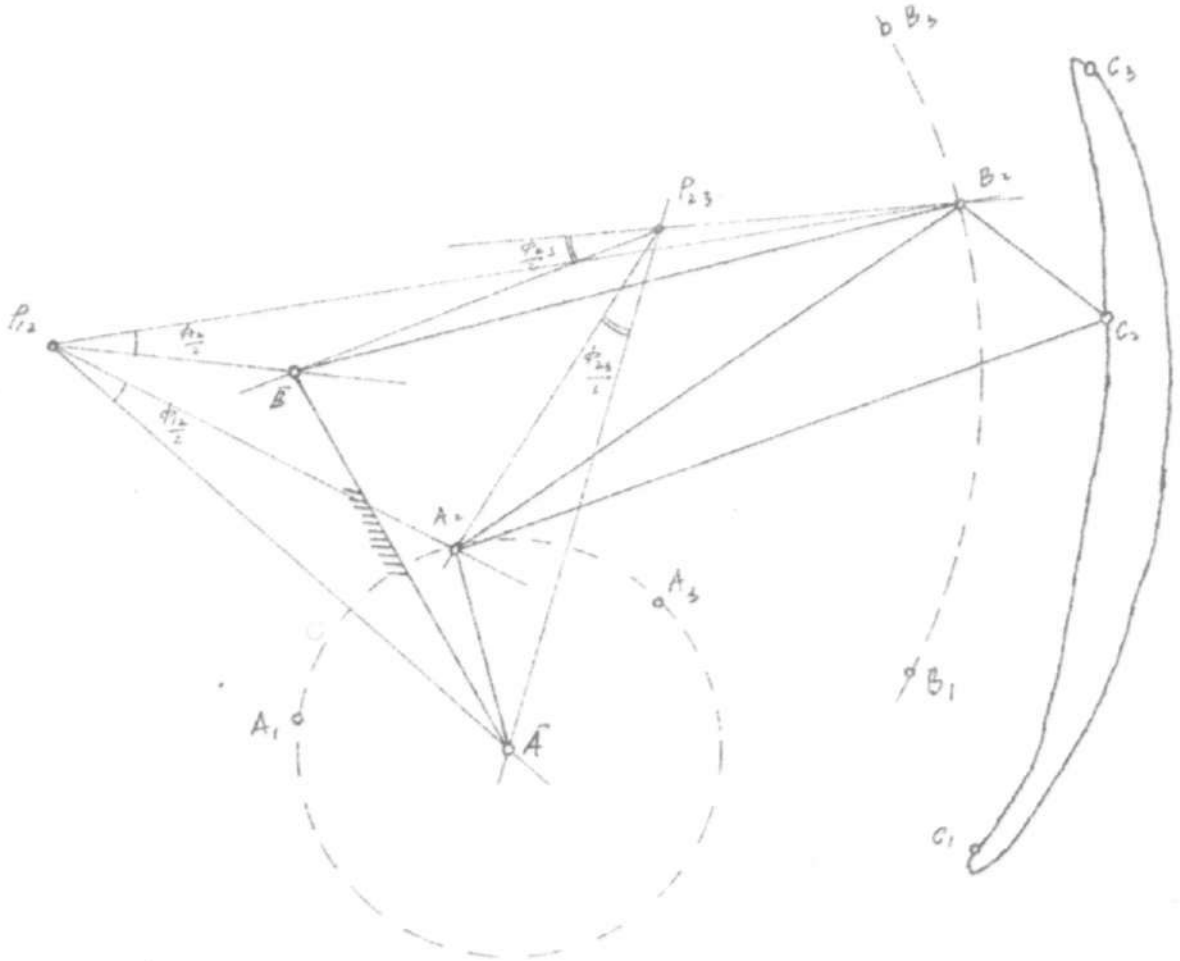


fig. 6.07: 4-bar mechanism designed for three given point-positions C_1, C_2, C_3.

Ex(6.02):

Approximate Straight-Line Motion:

A slider crank mechanism can be used to develop a 4-bar mechanism whose coupler point has an approximately straight path for limited range of motion.

As shown in Fig. (6.08), points A_1, A_2, A_3, are marked off on the circle of radius $\bar{A}A$ and A_2, A_3 are made symmetrical about the normal to $\bar{A}C$ through $\bar{A}$; this is to ensure that pole P_{23} will be at infinity. Points C_1, C_2, C_3, are marked out on the path of the slider of the assumed slider crank motion. Poles P_{12}, P_{13}, are then found.

A convenient point is made the fixed pivot $\bar{B}$. Point B_1 is found from the following relationships:

From equation (602);

$$\angle A_1 P_{12} \bar{A} = \angle B_1 P_{12} \bar{B} = \phi_{12}/2$$

$$\angle A_1 P_{13} \bar{A} = \angle B_1 P_{13} \bar{B} = \phi_{13}/2.$$

The derived 4-bar mechanism is shown in Fig. (6.09)

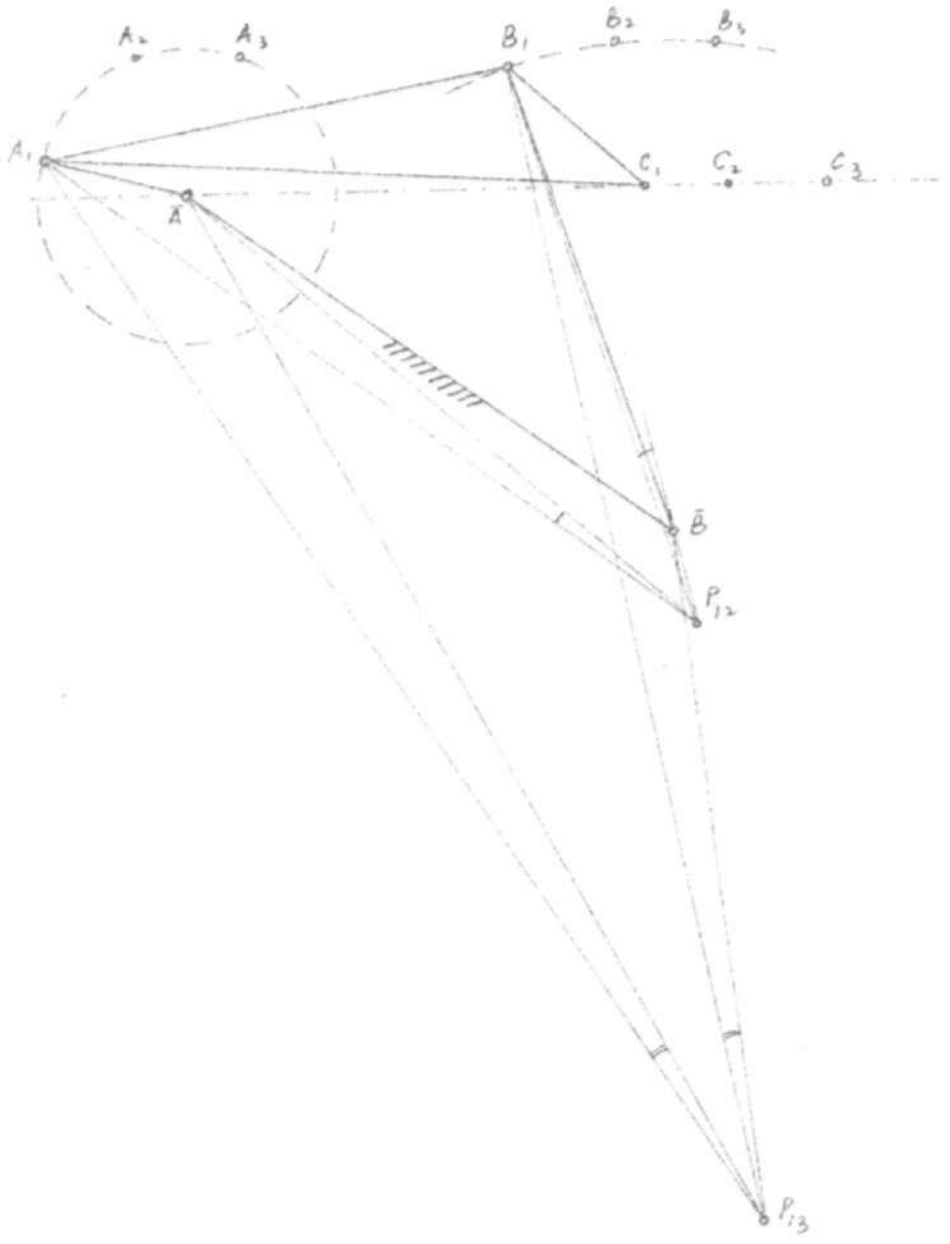


fig. 6.08: Approximate straight line motion of coupler point C, derived from the motion of a slider crank [R. Beyer]

6.4. Four Link Positions:

Four finitely near link positions give rise to six poles: P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}.

Opposite pole quadilaterals: Opposite poles are two poles whose subscripts do not contain any common numeral. P_{12} and P_{34} are a pair of opposite poles. Two pairs of opposite poles make up an opposite pole quadilateral. The adjacent poles have subscripts with a common numeral. There are three possible opposite pole quadilaterals for four finitely near link positions; these are $P_{12}\,P_{14}\,P_{34}\,P_{23}$, $P_{12}\,P_{13}\,P_{34}\,P_{24}$, and $P_{14}\,P_{13}\,P_{23}\,P_{24}$ (Fig. 6.09).

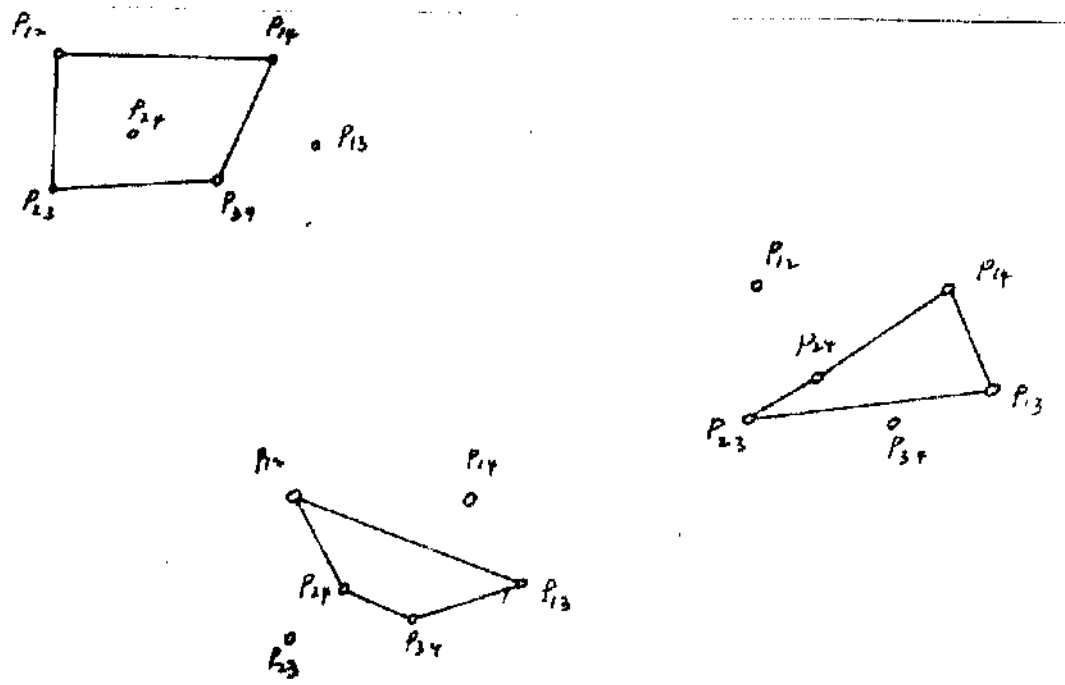


Fig. 6.09: Opposite pole quadilaterals for four link positions.

Centre Curve:

The centre curve is the locus of all possible locations of the fixed pivots of a 4-bar mechanism, $\bar{A}$, $\bar{B}$.

To plot the centre curve, the poles are found and an opposite pole quadilateral is chosen, say $P_{12}\ P_{24}\ P_{34}\ P_{13}$. The intersection of $P_{12}\ P_{24}$, and $P_{13}\ P_{34}$, $\widetilde{\pi}_{14}$ and the intersection of $P_{12}\ P_{13}$ and $P_{24}\ P_{34}$, $\widetilde{\pi}_{23}$, are also found.

Considering Fig. 6.10, K_1 is the circle through points $P_{13}\ P_{34}\ \widetilde{\pi}_{23}$. q_1 is the perpendicular bisector of $P_{12}\ P_{13}$ and q_2 is the perpendicular bisector of $P_{24}\ P_{34}$. A line is drawn from a point on K_1 through P_{13} to meet q_1 at Q_1 and another from the same point on K_1 through P_{34} to meet q_2 at Q_2. The locus of the intersections of the circles drawn about Q_1 with radius $Q_1\ P_{13}$ and the circles about Q_2 with radius $Q_2\ P_{34}$ is the centre curve. It passes through all the poles and the pole pair intersections $\widetilde{\pi}$.

There are other methods of drawing the centre curve. One of such is to choose the opposite sides of an opposite pole quadilateral, say $P_{12}\ P_{23}$ and $P_{14}\ P_{34}$ (or $P_{12}\ P_{14}$ and $P_{23}\ P_{24}$)

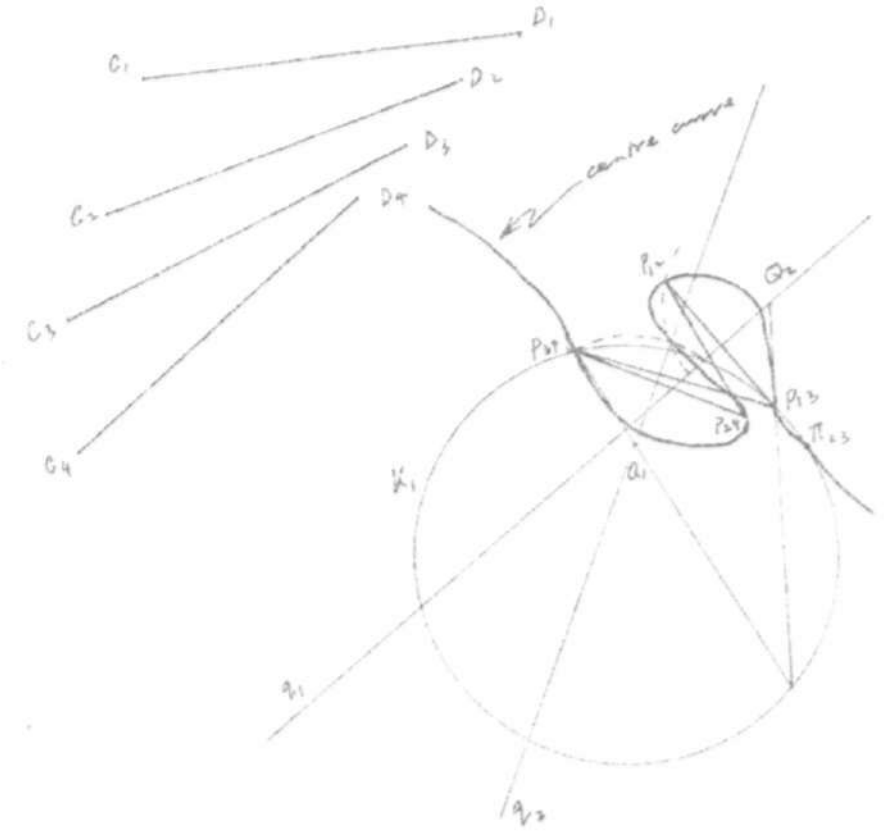


fig. 6·10: Centre curve for four link positions.

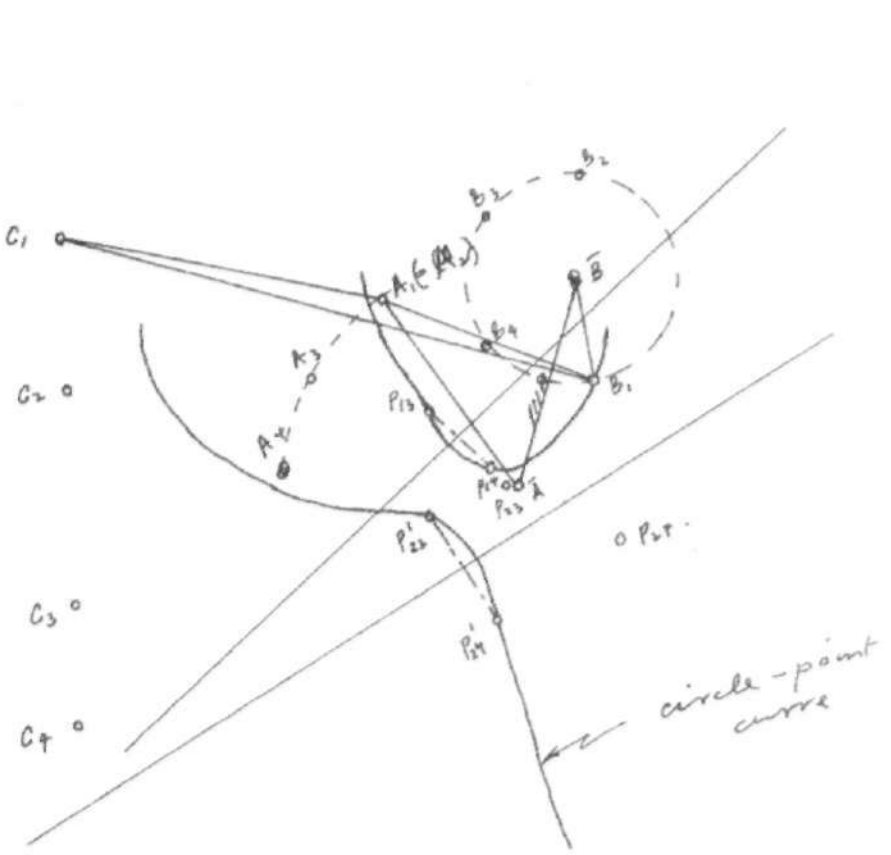


fig. 6·12: 4-bar mechanisms derived for four given points on a straight line.

With $P_{12}\,P_{23}$ as a chord of a circle of arbitrary radius, r_2, is drawn. With $P_{14}\,P_{24}$ as a chord, another circle whose radius,

Circle Point Curve:

The circle point curve is the locus of all possible locations of the moving pivots, A_1 B, at a particular position.

To construct a circle point curve, the poles are found. Considering the example in Fig. 6.12, the circle point curve is constructed for position 1. The circle point curve then gives the location (locus) of A_1 and B_1, the moving pivots of the 4-bar mechanism whose coupler lies at the given positions.

An opposite pole quadilateral is chosen, say P_{13} P_{14} P_{24} P_{23}. $P_{24}^{\,1}$ and $P_{23}^{\,1}$ are determined. $P_{24}^{\,1}$ is the image of P_{24} about P_{12} P_{14} and $P_{23}^{\,1}$ is the image of P_{23} about P_{12} P_{13}.

The opposite pole quadilateral about position 1, P_{13} P_{14} $P_{24}^{\,1}$ $P_{23}^{\,1}$, is then used to plot the circle point curve.

A circle of radius r_{34} is drawn with P_{13} P_{14} as chord and another circle of radius $r_{34}^{\,1}$ is drawn with $P_{23}^{\,1}$ $P_{24}^{\,1}$ as chord.

$$\frac{r_{34}}{r_{34}^{\,1}} = \frac{P_{13}\,P_{14}}{P_{23}^{\,1}\,P_{24}^{\,1}}$$

The circle point curve is the locus of the intersections of the two circles.

A suitable 4-bar mechanism can be derived by selecting various points for the moving pivots A_1, B, and then using equation (602) to locate the fixed pivots $\bar{A}_1$ $\bar{B}$, and then choosing the best of the derived mechanisms.

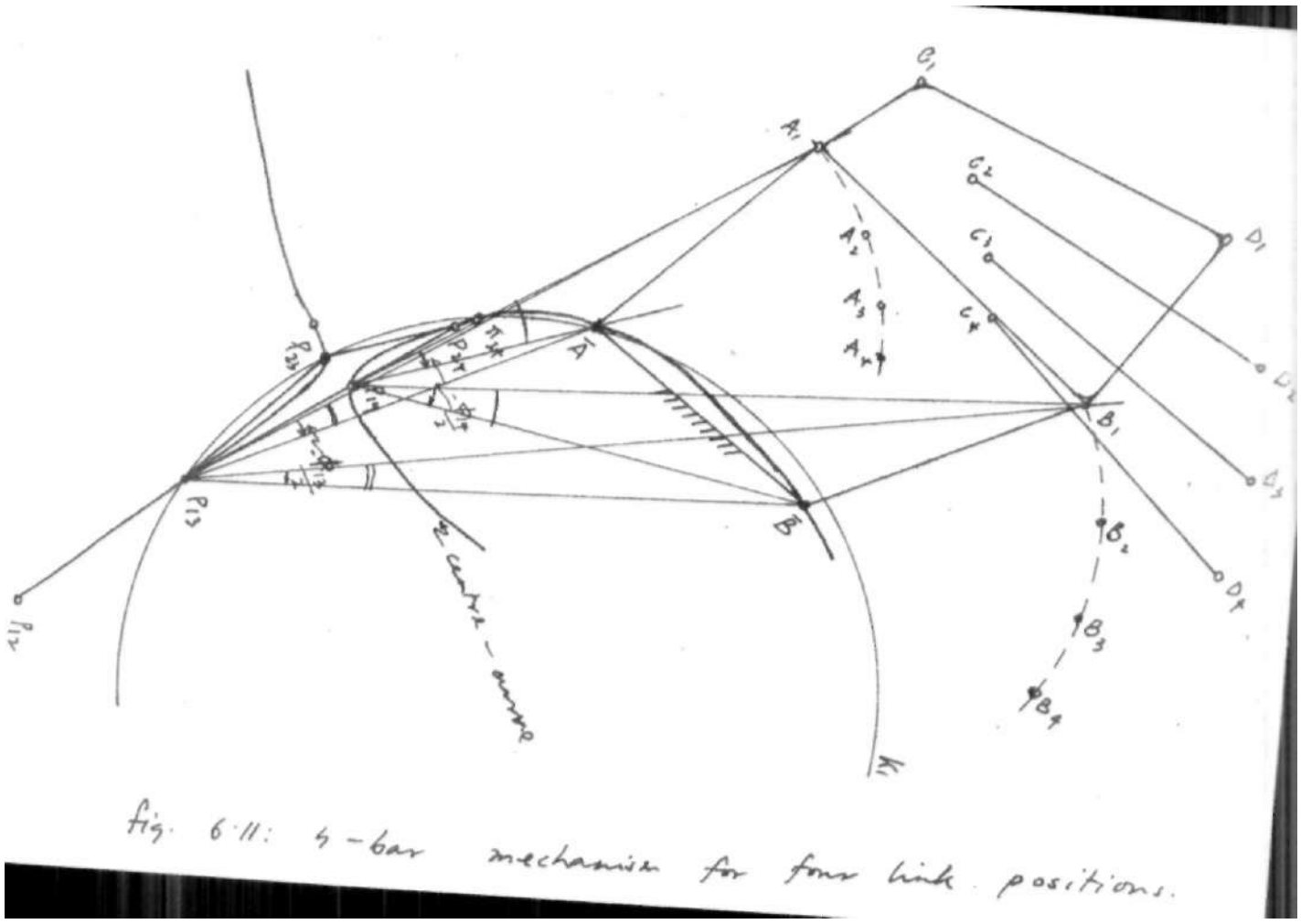


fig. 6.11: 4-bar mechanism for four link positions.

6.5. Five Link Positions:

It may also be possible to derive a 4-bar mechanism to have coupler positions correspond to five given link positions. It is necessary that the fixed pivots of the 4-bar mechanism correspond to the intersections of centre curves for a chosen pair of four link positions.

Five groups of four finitely near link positions are possible from five link positions. These are 1234, 1235, 1245, 1345, 2345. An example is shown in Fig. 6.13.

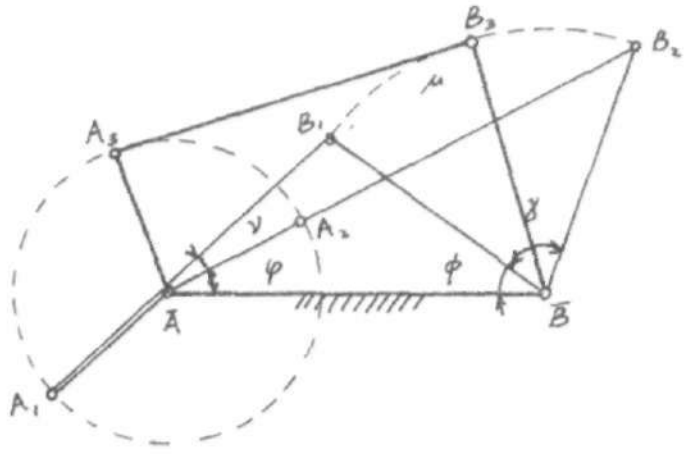
CHAPTER SEVEN

POSITION COORDINATION

Position coordination plays an important role in 4-bar mechanism design. It is usually necessary to coordinate crank angular displacements for particular motion of a 4-bar mechanism.

7.1. Crank-Rocker Mechanism:

A crank-rocker 4-bar mechanism is particularly useful for many purposes. It is useful for converting rotary motion to rocking (oscillatory) motion or vice-versa. It can be used for quick return motion; and some other functions.



$a + b < c + d$
$a - b < |c - d|$

fig. 7.01 : 4 - bar crank - rocker mechanism.

Referring to the crank-rocker mechanism of Fig. 7.01, as the crank $\bar{A}A$ rotates, the rocker $\bar{B}B$ oscillates through angle ψ.

When the rocker swings from position $\bar{B}B_1$ to $\bar{B}B_2$, the crank has an angular displacement of $(180 - v)^\circ$ anticlockwise and the return motion takes the crank $\bar{A}A$ through an angular displacement of $(180 + v)^\circ$. For constant angular velocity of the crank, it is seen that the time for the forward and reverse strokes are unequal. A 4-bar mechanism then acts as a quick return mechanism. For constant angular velocity of crank $\bar{A}A$, the time ratio, T.R., of the swings is given by;

$$T.R. = \frac{180 + v}{180 - v} \qquad \text{----------} \quad (701)$$

Considering the crank rocker mechanisms of Fig. , at the limit positions of the rocker, the following conditions hold:

$$d^2 = a^2 + (b + c)^2 - 2a(b+ c)\cos\psi \qquad \text{----------} \quad (702)$$

$$d^2 = a^2 + (c - b)^2 - 2a(c - b)\cos(v + \psi) \qquad \text{--------} \quad (703)$$

$$(c - b)^2 = a^2 + d^2 - 2ad\cos\phi \qquad \text{----------} \quad (704)$$

$$(c + b)^2 = a^2 + d^2 - 2ad\cos(\gamma + \phi) \qquad \text{----------} \quad (705)$$

Also,

$$\frac{c - b}{\sin\phi} = \frac{d}{\sin(v + \psi)} \qquad \text{----------} \quad (706)$$

$$\frac{c + b}{\sin(\gamma + \phi)} = \frac{d}{\sin\psi} \qquad \text{----------} \quad (707)$$

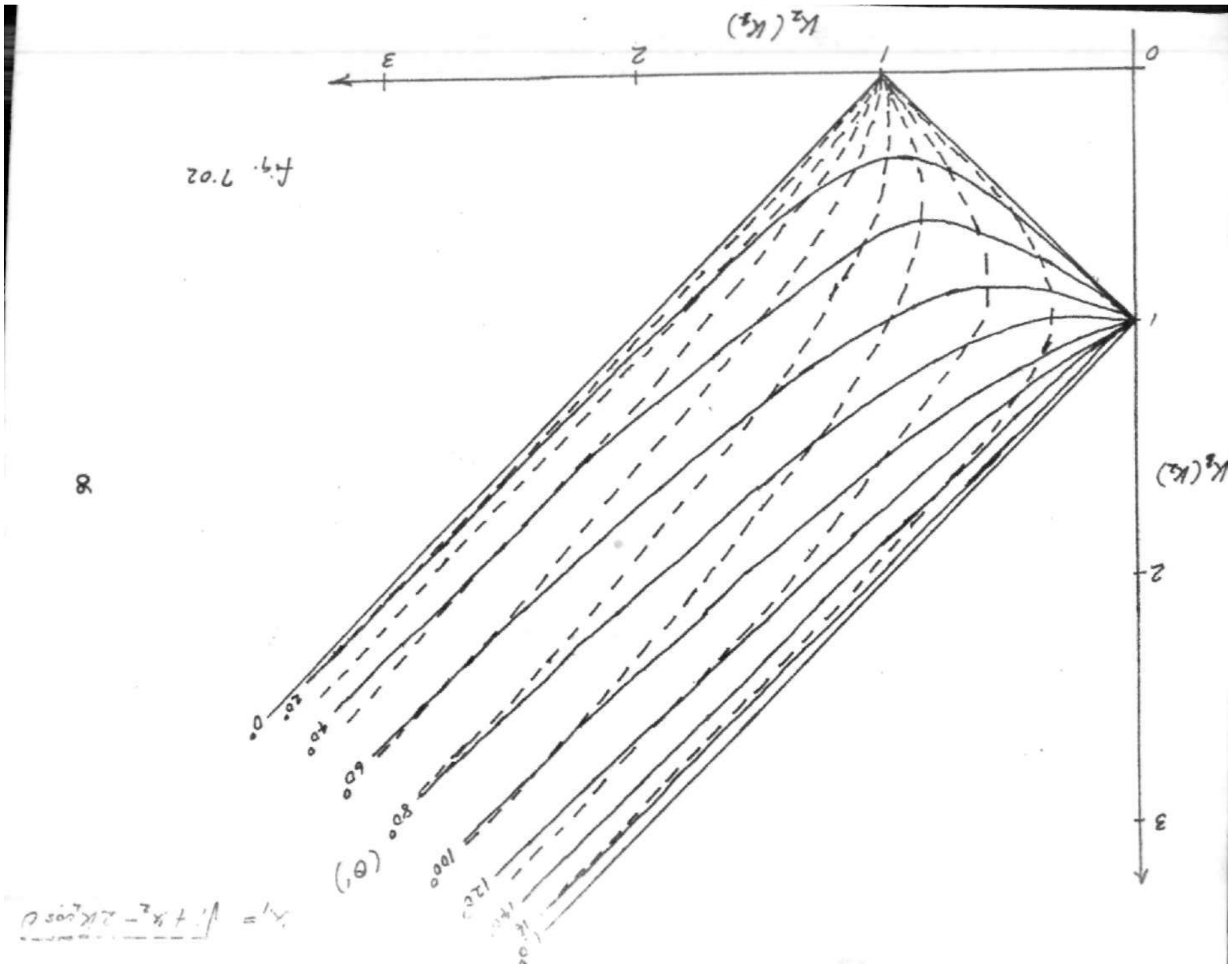


Fig. 7.02

$\lambda_1 = \sqrt{1 + \chi_z - 2\chi_z \cos\theta}$

Dividing equations (7.02) to (7.05) by a^2, they become;

$$\left(\frac{d}{a}\right)^2 = 1 + \left[\left(\frac{b}{a}\right) + \left(\frac{c}{a}\right)\right]^2 - 2\left(\frac{b}{a} + \frac{c}{a}\right)\cos\gamma \quad \text{------(708)}$$

$$\left(\frac{d}{a}\right)^2 = 1 + \left(\frac{c}{a} - \frac{b}{a}\right)^2 - 2\left(\frac{c}{a} - \frac{b}{a}\right)\cos(v + \psi) \quad \text{------(709)}$$

$$\left(\frac{c}{a} - \frac{b}{a}\right)^2 = 1 + \left(\frac{d}{a}\right)^2 - 2\frac{d}{a}\cos\phi \quad \text{----------(710)}$$

$$\left(\frac{c}{a} + \frac{b}{a}\right)^2 = 1 + \left(\frac{d}{a}\right)^2 - 2\frac{d}{a}\cos(\gamma + \phi) \quad \text{----------(711)}$$

The four equations are similar and can be written as

$$K_1^2 = 1 + K_2^2 - 2K_2\cos\theta^1 \quad \text{----------(712)}$$

There are now three variables K_1, K_2, θ^1. Plotting graphs of K_1 against K_2 for varying values of θ^1 can be used as an easy way to solving angular displacement problems in 4-bar mechanisms (Fig. 7.02).

The problems can also be solved graphically. For example, designing for given values of v and γ as shown in Fig. (7.03).

The value of the crank length, b, of the link $\bar{A}A$ is found from the relationship;

$$c + b = \bar{A}B_2 \quad \text{----------(713)}$$

$$c - b = \bar{A}B_1 \quad \text{----------(714)}$$

There are infinite numbers of possible designs of 4-bar mechanism for the given conditions. The locus of possible positions for $\bar{A}$ is the circle through points B_1, B_2, $\bar{A}$.

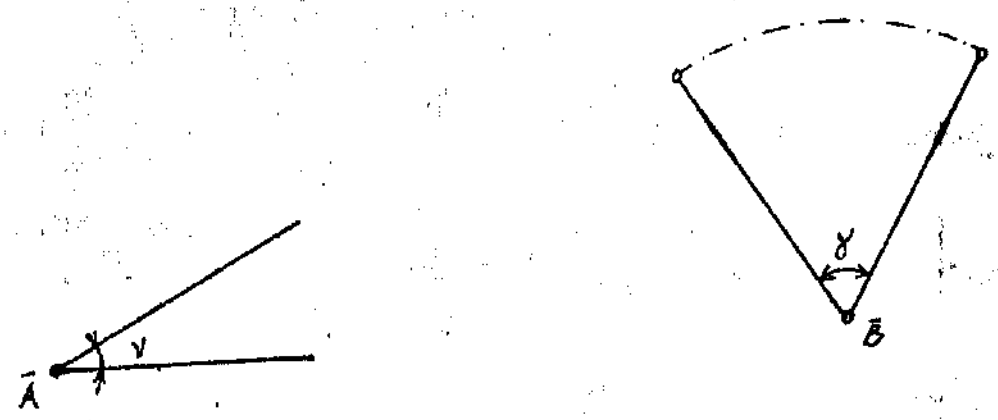


fig. 7.03 : Given values for ν and γ.

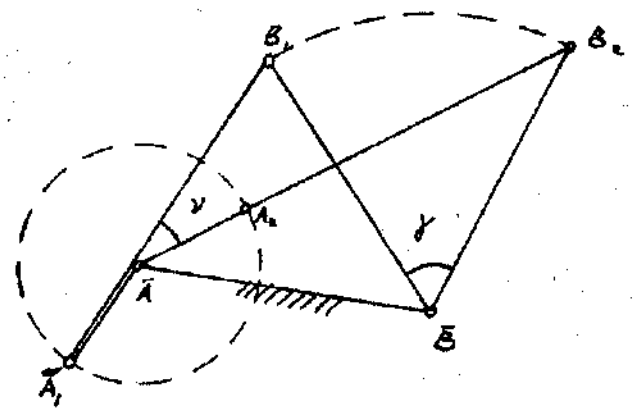


fig. 7.04 : 4-bar mechanism for the given values of ν and γ.

7.2. Double-Rocker Mechanisms:

In double-rocker 4-bar mechanisms, the input and output links, links 2 and 4, do not prescribe complete circular motion but rock or oscillate about their fixed pivots, $\bar{A}$, $\bar{B}$. Double-rocker mechanisms do not act as quick return mechanisms as the time ratio (T.R.) is equal to unity.

Output Angles: In double rocker (double lever) mechanisms, the input angle as well as the output angle varies. There are five major types of double-rocker mechanisms, considering the variations in input and output angles, as can be seen from the classification graph of Fig. (2.03).

Case I: As shown in Fig. (7.05), the input and output links do not cross the fixed link line.

From the classification graph, it can be seen that the conditions for double-rocker of the case being considered can be written as follows:

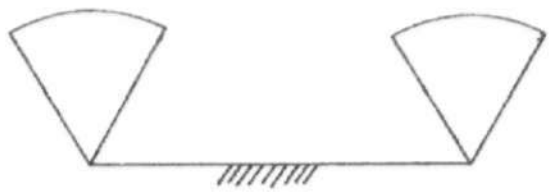
$$(i)\quad a + c < b + d \qquad \text{----------(7.15)}$$

$$(ii)\quad c + d < a + b$$

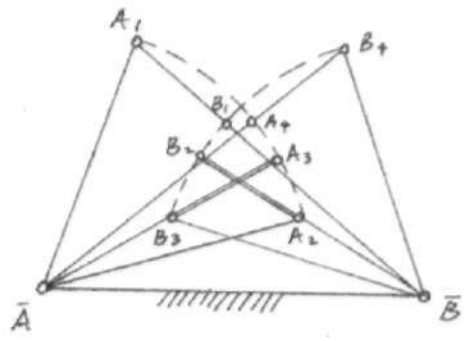
Considering the example of the type I double-rocker mechanism shown in Fig. (7.06), the following relations hold:

$$(c + d)^2 = a^2 + b^2 - 2ab \cos (\phi_1 + \gamma_1) \qquad \text{----------(7.16)}$$

$$(d - c)^2 = a^2 + b^2 - 2ab \cos \phi_1 \qquad \text{----------(7.17)}$$



7·05 : 4- bar double-rocker mechanism considered in case I.



7·06 : 4 - bar mechanism of type considered in case I.

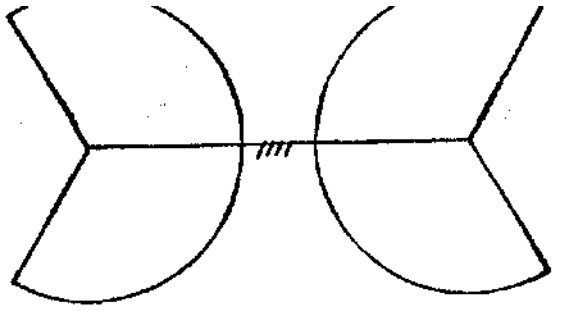


fig. 7.07: 4-bar double-rocker mechanism considered in case II.

line on the outside but do so on the fixed link itself.

From the classification graph of Fig. (2.03), the conditions for 4-bar double-rocker mechanisms of the type shown in Fig. (7.07) are as follows:

(i) $c + d < a + b$

$a - b > |c - d|$ ----------(720)

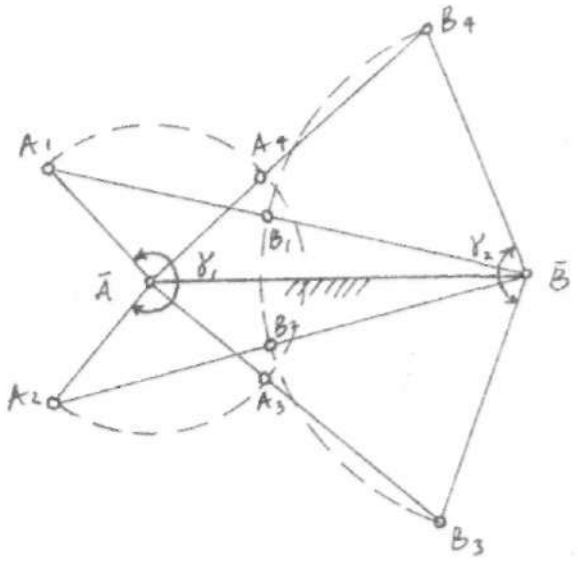


fig. 7.08 : 4-bar mechanism of type considered in case II.

Considering the mechanism of Fig. (7.08), the following relations hold:

$$(c + d)^2 = a^2 + b^2 - 2ab \cos \frac{\gamma_1}{2} \quad \text{----------(721)}$$

$$(b + c)^2 = a^2 + d^2 - 2ad \cos \frac{\gamma_2}{2} \quad \text{----------(722)}$$

Case III: 4-bar double-rocker mechanisms considered under Case III have of the input and output links, the input link crossing the fixed line on the inside and the output link crossing the fixed link line

on the outside; as shown in Fig. (7.09).

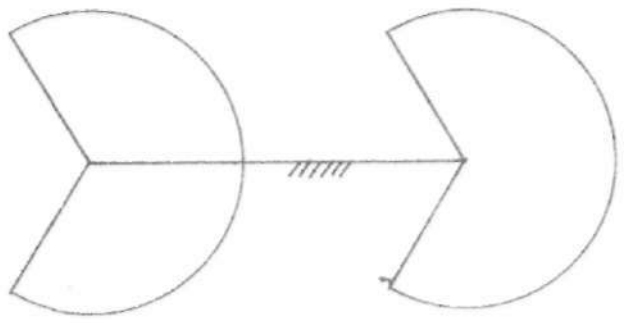


Fig. 7.09: 4-bar double-rocker mechanism considered in case III.

From the classification graph, conditions for 4-bar double-rocker mechanisms of the type considered in Case III are as follows:

(i) $c + d < a + b$

(ii) $b - a > /c - d/$ ----------(723)

Considering the mechanism of Fig. (7.10) in the limit positions, the following hold:

$$(c + d)^2 = b^2 + a^2 - 2ab \cos \frac{\gamma_1}{2} \quad \text{----------(724)}$$

$$(b - c)^2 = d^2 + a^2 - 2ad \cos (180 - \frac{\delta_2}{2}) \quad \text{----------(725)}$$

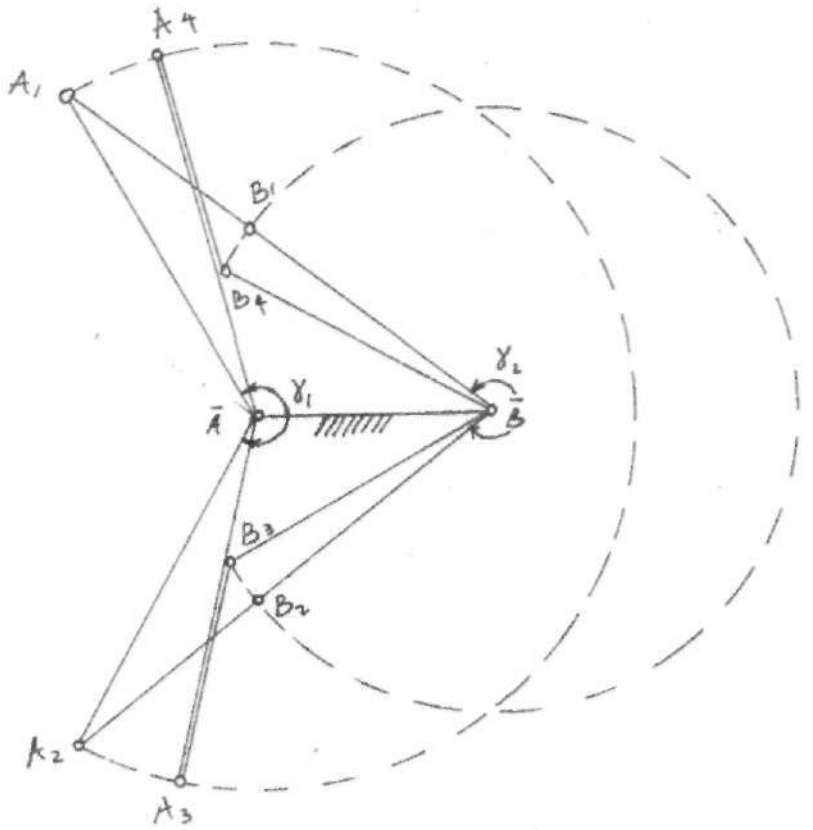


Fig. 7.10 : 4-bar mechanism of type considered in case III.

Case IV: The 4-bar double-rocker mechanisms considered under Case IV, as shown in Fig. (7.11), have input and output links that cross the fixed link line only on the outside.

From the classification graph, the conditions for the mechanisms considered under Case IV are as follows:

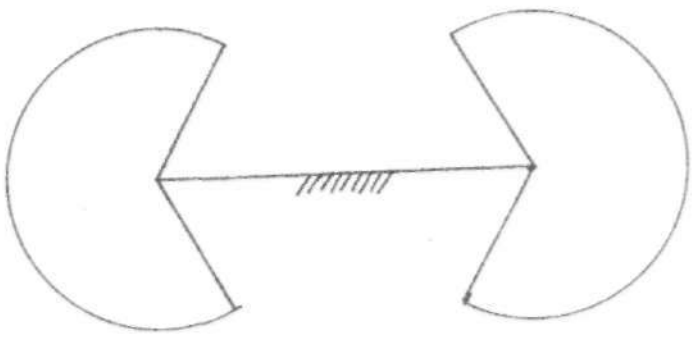


fig. 7·11 : 4-bar double-rocker mechanism considered in case IV.

(i) $|a - b| < c - d$

(ii) $a + b < c + d$ ----------(726)

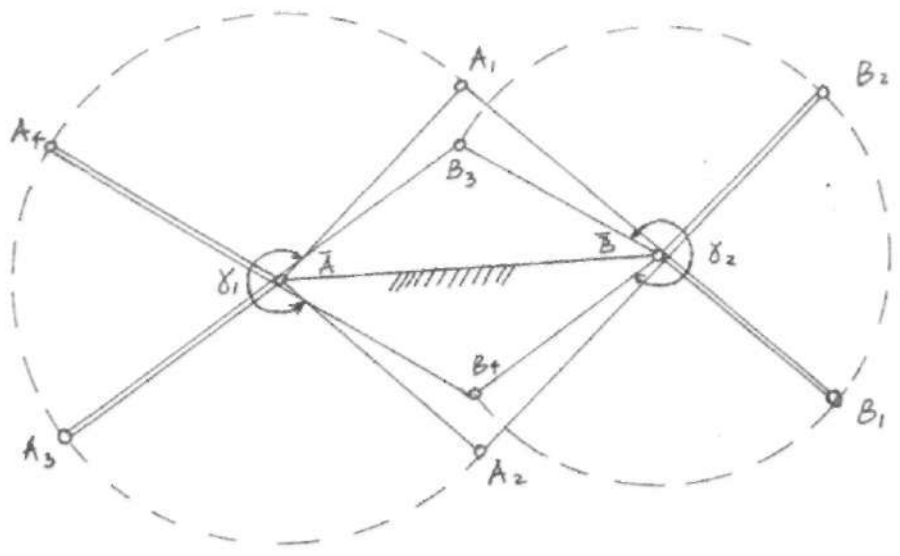


fig. 7·12 : 4-bar mechanism of type considered in case IV.

Considering the mechanism in Fig. (7.12), the following relations hold:

$$(c - d)^2 = a^2 + b^2 - 2ab \cos\left(180 - \frac{\gamma_1}{2}\right) \text{ ----------(727)}$$

$$(c - b)^2 = a^2 + d^2 - 2ad \cos\left(180 - \frac{\gamma_2}{2}\right) \text{ ----------(728)}$$

Case V: The 4-bar double-rocker mechanisms considered under Case V, as shown in Fig. (7.13), have input links that cross the fixed link lines only on the outside and output links that cross the fixed link lines only on the inside.

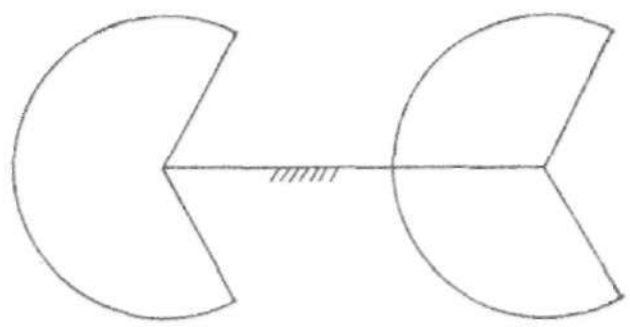
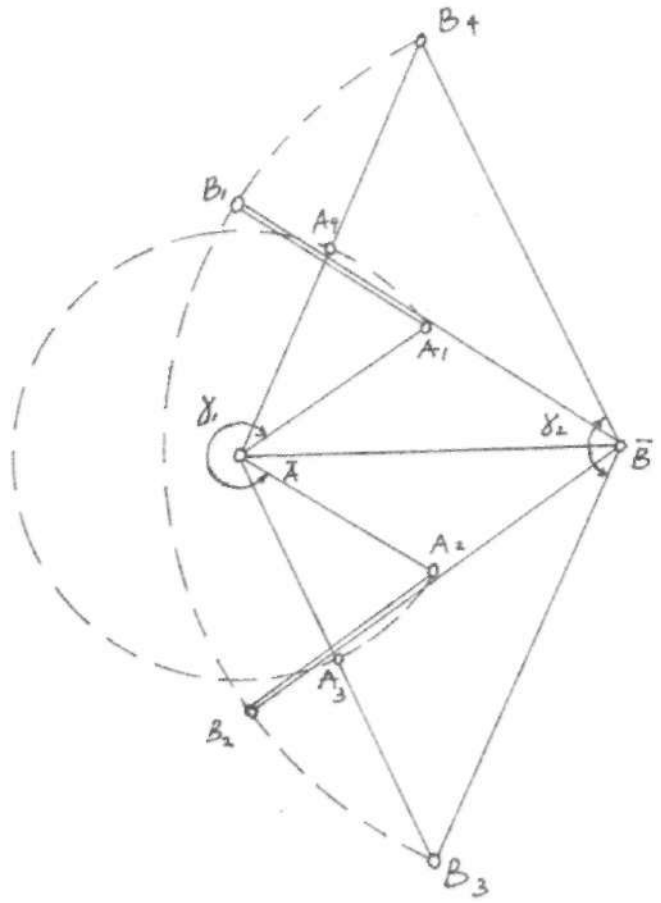


fig. 7.13 : 4-bar double-rocker mechanism considered in case V

From the classification graph of Fig. (2.03), the conditions for the 4-bar double-rocker mechanisms considered in Case V can be written as follows:

(i) $a + b < c + d$

(ii) $|a - b| < d - c$ ----------(729)



7.14 : 4 - bar mechanism of type considered in case V.

From the mechanism in Fig. (7.14), it can be developed that;

$$(d - c)^2 = a^2 + b^2 - 2ab \cos\left(180 - \frac{\gamma_1}{2}\right) \text{ ----------(730)}$$

$$(b + c)^2 = a^2 + d^2 - 2ad \cos\frac{\gamma_2}{2} \text{ ----------(731)}$$

The general equation for output-input angles relationships to link lengths in a 4-bar double-rocker mechanism can be written as before in equation (712); dividing equations (716) to (729) by a^2:

$$K_1^2 = 1 + K_2^2 - 2K_2 \cos\theta^1 \text{ ----------(712)}$$

Therefore the graphs of Fig. (7.02) can be used to analyse problems on input-output angles in double-rocker mechanisms.

7.3. Transmission Angle:

The relationship between velocity and acceleration of links 3 and 4 to angular velocity value of link 2, shows the importance of the angle between the driven link and the coupler - the transmission angle (Fig. 7.15).

The transmission angle, μ, should be kept as close to 90° as possible so as to avoid jamming.

Considering the crank-rocker mechanism in Fig. (7.15),

$$b^2 + a^2 - 2ab \cos\theta = c^2 + d^2 - 2cd \cos\mu \text{ ---------(732)}$$

Differentiating with respect to angle θ, equation (732) becomes

$$2ab \sin\theta = 2cd \sin\mu \frac{d\mu}{d\theta} \text{ ----------(733)}$$

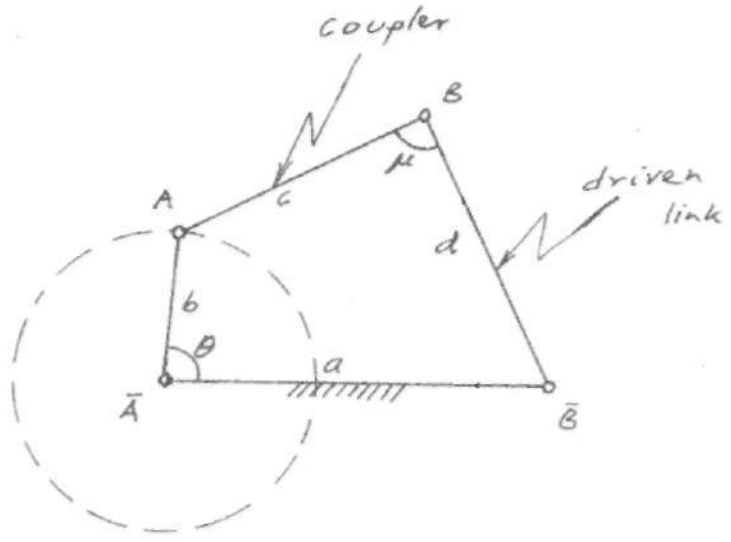


fig. 7.15 : 4 - bar mechanism showing the transmissi
angle — μ.

For turning values, $\frac{d\mu}{d\theta} = 0$

$\therefore \quad \sin\theta = 0$

$\therefore \quad \theta = 0^{\circ}$ or

$\theta = 180^{\circ}$

Differentiating equation (733) with respect to θ

$$2ab\cos\theta = 2cd\cos\mu\left(\frac{d\mu}{d\theta}\right)^2 + 2cd\sin\mu\frac{d^2\mu}{d\theta^2} \quad \text{--------(734}$$

When $\frac{d\mu}{d\theta} = 0$ and $\theta = 0^{\circ}$

$$\frac{d^2\mu}{d\theta^2} = \frac{2ab}{2ad\sin\mu} > 0 \quad \text{----------(735)}$$

When $\frac{d\mu}{d\theta} = 0$ and $\theta = 180^{\circ}$

$$\frac{d^2\mu}{d\theta^2} = -\frac{2ab}{2cd\sin\mu} < 0 \qquad \text{----------(736)}$$

Equation (735) shows that the transmission angle, μ, is minimum, μ_{min}, when $\theta = 0^{\circ}$ and equation (736) shows that the transmission angle is maximum, μ_{max}, when the crank angle, θ, is equal to 180°. These are shown on Fig. (7.16) for the mechanism shown in Fig. (7.15).

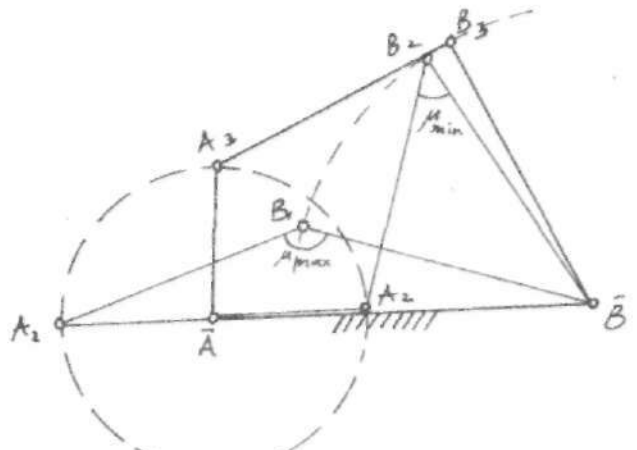


Fig. 7.16: 4-bar mechanism showing minimum and maximum values of the transmission angle.

Considering Fig. (7.16), the following expressions hold:

$$(a - b)^2 = c^2 + d^2 - 2cd \cos \mu_{min} \text{ ----------(737)}$$

$$(a + b)^2 = c^2 + d^2 - 2cd \cos \mu_{max} \text{ ----------(738)}$$

Equations (737) and (738) are similar to the types got and the general graphs drawn in Fig. (7.02) can be used in some respect to analyse problems on transmission angle designs.

For low speed cranks, it is generally desired that $/90 - \mu/_{max}$ should not be more than 50°. This means that $140^\circ \geq \mu \geq 40^\circ$. It follows that for extreme values,

$$\mu_{min} = 40^\circ$$

and $$\mu_{max} = 140^\circ$$

Substituting in equations (737) and (738),

$$(a - b)^2 = c^2 + d^2 - 2cd \cos 40^\circ \text{ ----------(739)}$$

$$(a + b)^2 = c^2 + d^2 - 2cd \cos 140^\circ \text{ ----------(740)}$$

For crank rocker mechanisms, $b < a$: the driving link length is less than the length of the fixed link. Equations (739) and (740) can then be written as

$$(1 - \frac{b}{a})^2 = (\frac{c}{a})^2 + (\frac{d}{a})^2 - 2\frac{c}{a}.\frac{d}{a} \cos 40^\circ \text{ ---------(741)}$$

$$(1 + \frac{b}{a})^2 = (\frac{c}{a})^2 + (\frac{d}{a})^2 - 2\frac{c}{a}.\frac{d}{a} \cos 140^\circ \text{ ---------(742)}$$

If the fixed link length, a, is made equal to unity, then graphs of $d^*(\frac{d}{a})$, against $c^*(\frac{c}{a})$, can be plotted for values of $b^*(\frac{b}{a})$ varying from 0 to 1 (Fig. 7.17).

For double-crank (drag-link) mechanisms, a < b: the driving link length is more than the length of the fixed link. Equations (739) and (740) can be written as;

$$(1 - \frac{a}{b})^2 = (\frac{c}{b})^2 + (\frac{d}{b})^2 - 2\frac{c}{b}.\frac{d}{b}\cos 40^\circ \text{ ----------(743)}$$

$$(1 + \frac{a}{b})^2 = (\frac{c}{b})^2 + (\frac{d}{b})^2 - 2\frac{c}{b}.\frac{d}{b}\cos 140^\circ \text{ ----------(744)}$$

Graphs of $d^*(\frac{d}{b})$, against $c^*(\frac{c}{b})$, can then be plotted for varying values of $a^*(\frac{a}{b})$, ranging from 0 to 1 (Fig. 7.17).

It will be seen that the graphs got in both considerations are the same as shown in Fig. (7.17).

From the graphs in Fig. (7.17), it is advisable that the ratio of the driving crank length to the fixed link length, $\frac{b}{a}$, be no more than 0.4 for low speed cranks, to prevent jamming. It is also easily seen that for good transmission angles, the coupler length, c, and the driven crank length, d, should be about equal.

$$\frac{b}{a} \leqslant 0.4; \quad c \simeq d \gtrsim 0.8a \text{ ----------(745)}$$

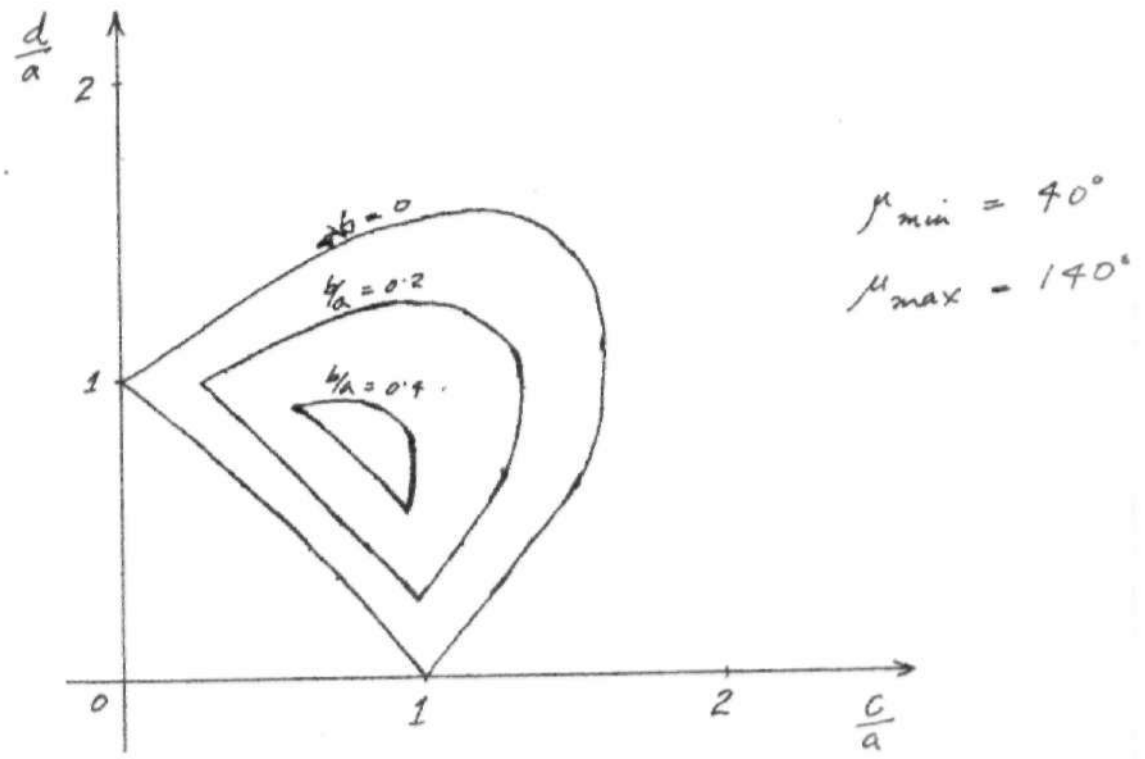


fig. 7·17 : Graphs showing possible link length ratios for $|90 - \mu| \leq 50°$.

Ex:(7.01) Design a 4-bar crank-rocker mechanism having an output angle, γ, of 100° and whose minimum angle between the output link and the fixed link is 40°, and the ratio of the input link length, b, to the fixed link length, a, be 0.4.

Solution: Referring to the notations given in the crank-rocker mechanism of Fig. (7.01),

$$\gamma = 100^{\circ}$$

$$\phi = 40^{\circ}$$

$$\frac{b}{a} = 0.4$$

The design problem can be solved in a number of ways; some of which are shown in methods (i), (ii).

(i) From equation (710),

$$\left(\frac{c}{a} - \frac{b}{a}\right)^2 = 1 + \left(\frac{d}{a}\right)^2 - 2\frac{d}{a}\cos\phi$$

$$\therefore \quad \left(\frac{c}{a} - 0.4\right)^2 = 1 + \left(\frac{d}{a}\right)^2 - 2\frac{d}{a}\cos 40^{\circ} \text{ ----------(746)}$$

Substitutimg in equation (711),

$$\left(\frac{c}{a} + \frac{b}{a}\right)^2 = 1 + \left(\frac{d}{a}\right)^2 - 2\frac{d}{a}\cos(\phi + \gamma)$$

$$\left(\frac{c}{a} + 0.4\right)^2 = 1 + \left(\frac{d}{a}\right)^2 + 2\frac{d}{a}\cos 40^{\circ} \text{ ----------(747)}$$

Solving equations (746) and (747) simultaneously,

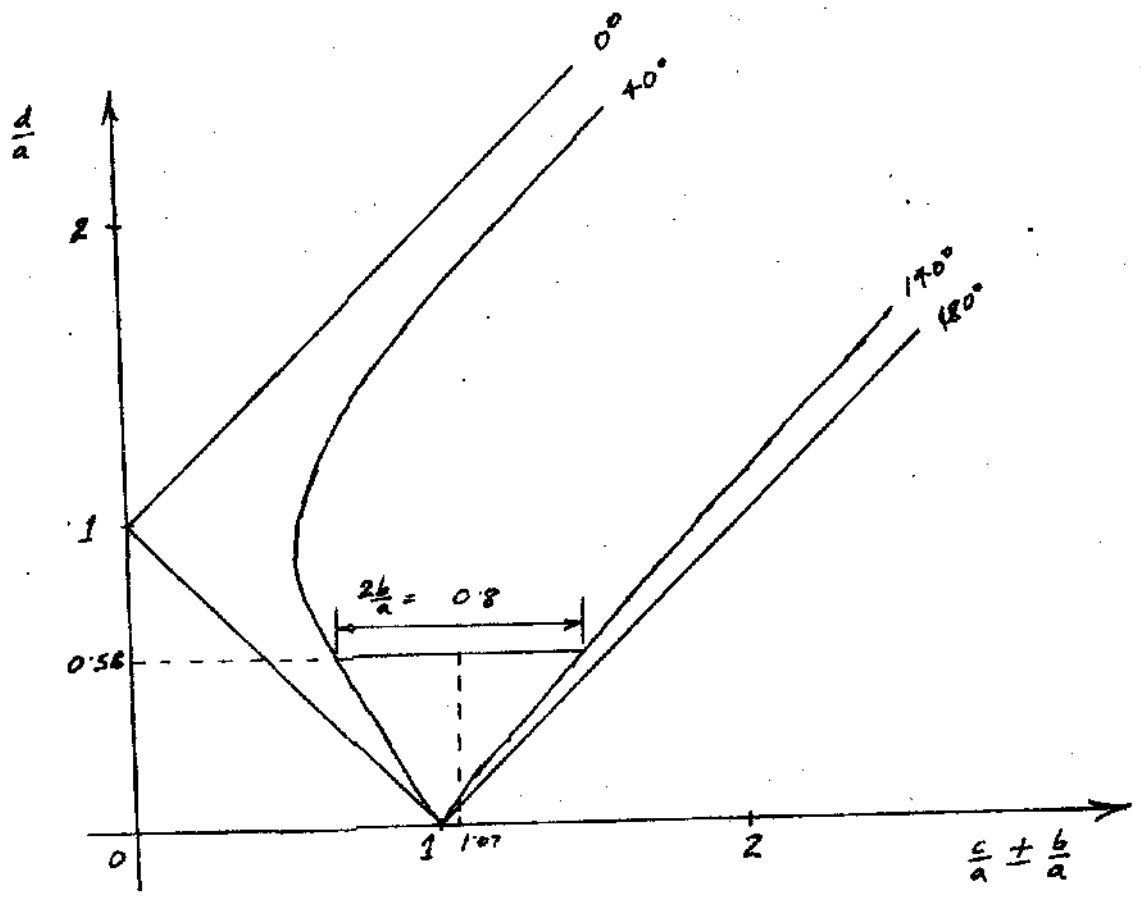
$$\frac{c}{a} = 1.075$$

$$\frac{d}{a} = 0.561$$

(ii) Checking the graphs of Fig. (7.02) and noting that for the two equations;

$K_2 \equiv c^* - b^*$; $K_1 \equiv d^*$, $\theta^1 = \phi$ for equation (7.10)

$K_2 \equiv c^* + b^*$; $K_1 \equiv d^*$, $\theta^1 = (\phi + \delta)$ for equation (7.11).



g. 7.18: Using the displament curves for 4 - bar mechanism design.

As shown in Fig. (7.18), when the horizontal displacement between the two curves, the one for $\theta^1 = 40^\circ$ and the one for $\theta^1 = 140^\circ$, is $2\frac{b}{a}$, and in the present case, equals 0.8, then the position determines the relative length $\frac{d}{a}$. The relative length $\frac{c}{a}$ is determined as the midpoint between the two curves, at the indicated position, on the abscissa.

Methods (i) and (ii) are found to yield the same results. The designed 4-bar mechanism is shown in Fig. (7.19).

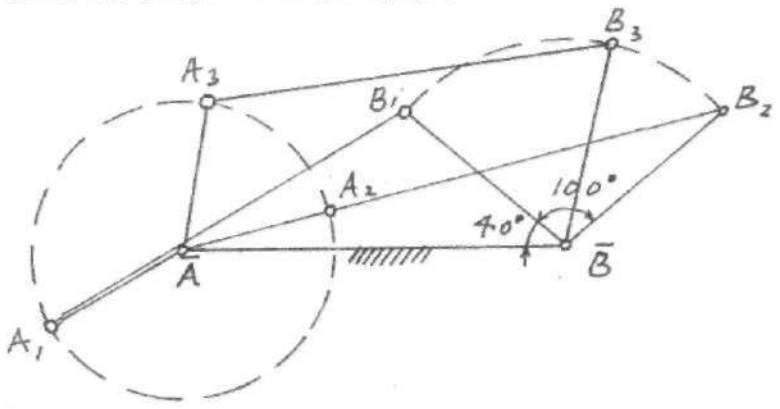


fig. 7.19 : The derived 4-bar mechanism for Ex. 7.01.

7.4. Two-Position Design:

The 4-bar mechanism is a useful way in converting one form of crank angular displacement into another. Given specific input and output angles, an infinite number of 4-bar mechanisms can be derived.

For example, with input and output angles as shown in Fig.(720), suitable positions are selected for the fixed pivots, $\bar{A}$, $\bar{B}$. Choosing a convenient length for the driving link, $\bar{A}A$, points A_1, A_2 are marked such that angle $A_1\ \bar{A}A_2$ corresponds to the input angle.

A_2 is rotated about $\bar{B}$, in the opposite sense, through an angle equal to the output angle to locate point A_2^1. The perpendicular bisector of $A_1A_2^1$ is the locus of possible location of point B_1 (Fig. 7.21.

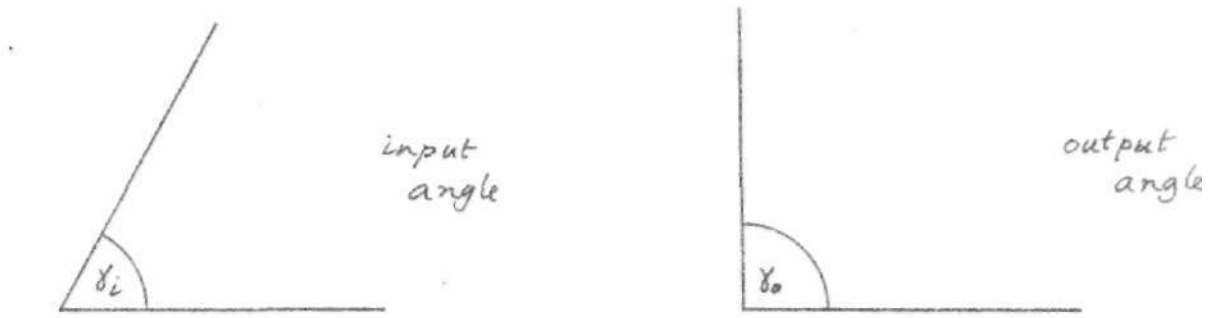


Fig. 7.20: Input and output angles for two-position design of 4-bar mechanism (example).

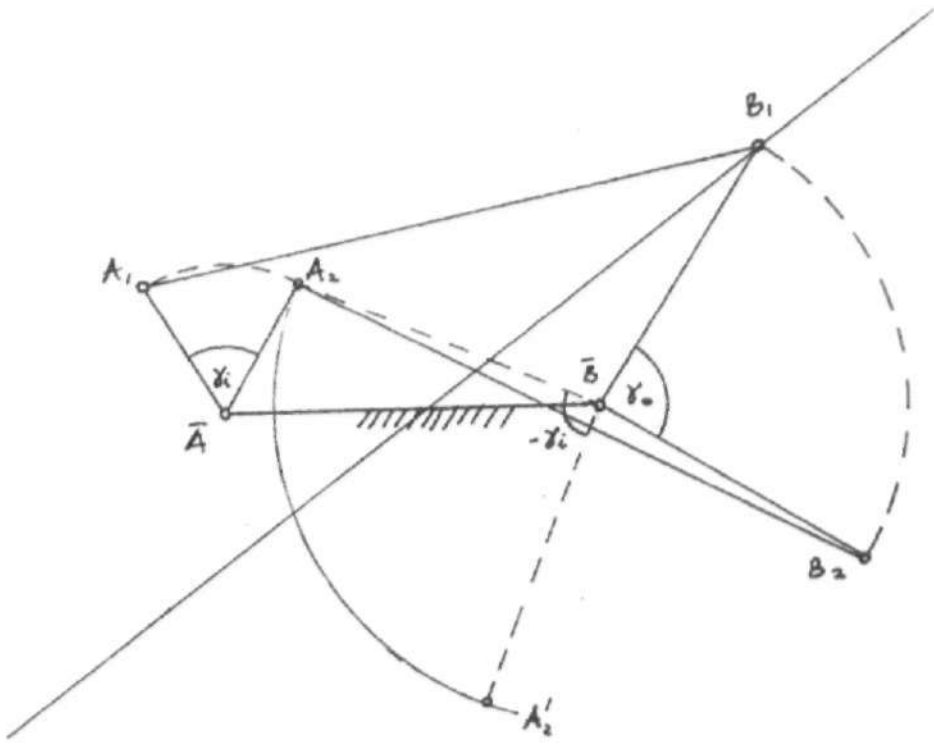
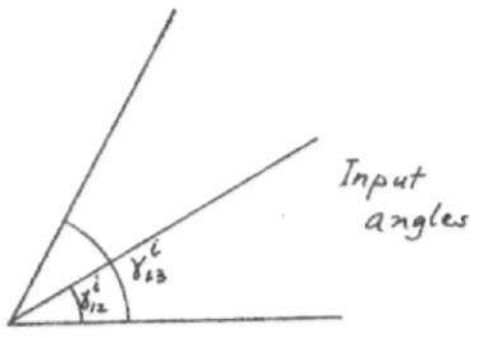


fig. 7.21 : Two-position design of a 4-bar mechanism.



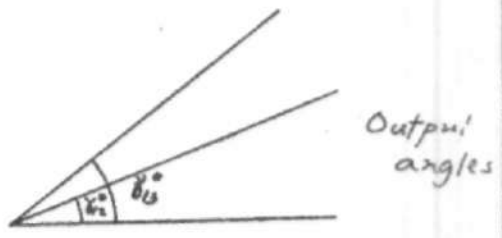


fig. 7.22 : Input and output angles for three position coordination (example).

7.5. Three-Position Coordination:

There are also various methods for deriving a suitable 4-bar mechanism for three angular position coordination, given the fixed link and also the driving crank lengths.

For example, with input and output angular link displacements as shown in Fig. (7.22), two methods are used to get at the required 4-bar mechanism. There still exists an infinite number of possible 4-bar mechanism as the initial position of the driving link can be chosen arbitrarily.

Method (i): Points R_{12}, subtending an angle $\frac{\gamma^{i}_{12}}{2}$ with the fixed link at $\bar{A}$ and an angle $\frac{\gamma^{o}_{12}}{2}$ with the fixed link at $\bar{B}$, and R_{13}, subtending an angle $\frac{\gamma'_{13}}{2}$ with the fixed link at $\bar{A}$ and also subtending an angle $\frac{\gamma^{o}_{13}}{2}$ with the fixed link at $\bar{B}$, are located (Fig. 7.23).

A convenient position is chosen for the driving link at state 1. The angle subtended by the fixed link, $\bar{A}\bar{B}$, at R_{12} is equal to the angle subtended by the coupler at state 1, A_1B_1. The same holds for the angle subtended at R_{13}. Point B_1 is located using the two stated conditions.

Method (ii): Points A_2^1 and A_3^1 are located. Point A_2^1 is point A_2 rotated about point $\bar{B}$ through an angle γ^{o}_{12} but in the opposite sense. Point A_3^1 is point A_3 rotated about point $\bar{B}$ through an angle γ^{o}_{13} but in the opposite sense. Point B_1 is the centre of the circle through points A_1, A_2^1, A_3^1 (Fig. 7.24).

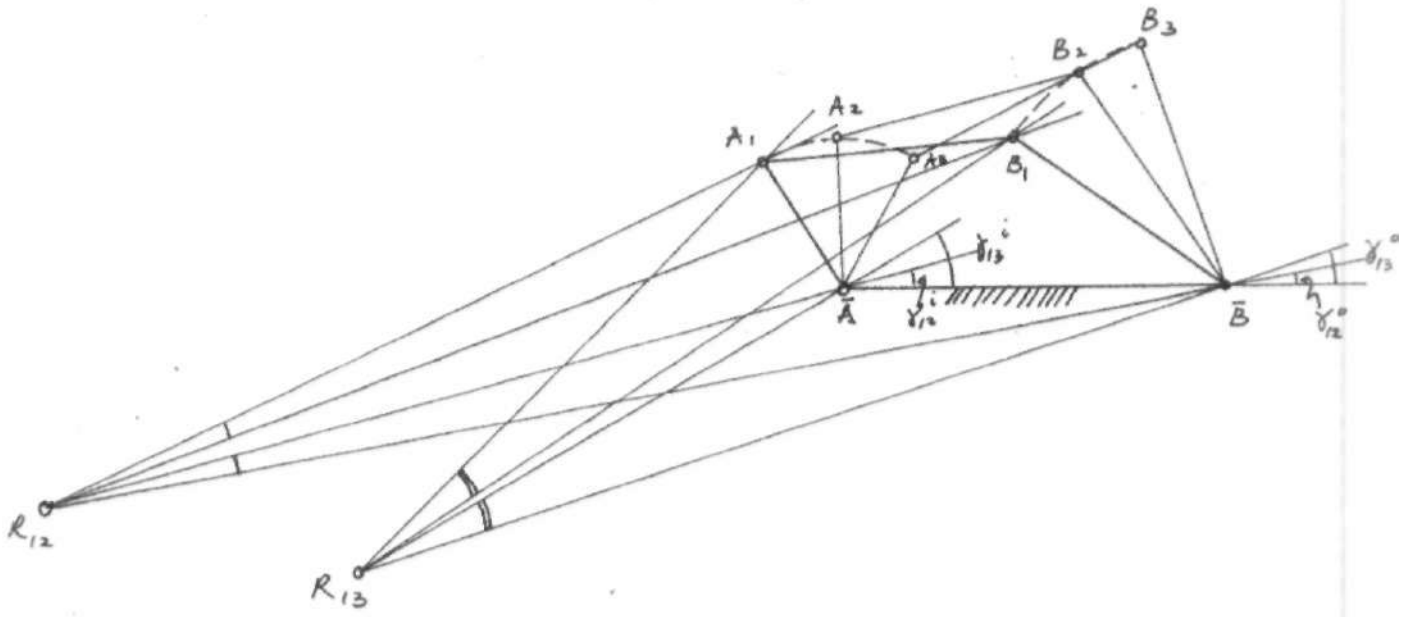


fig. 7·23: 4-bar mechanism for three-position coordination derived using method (i).

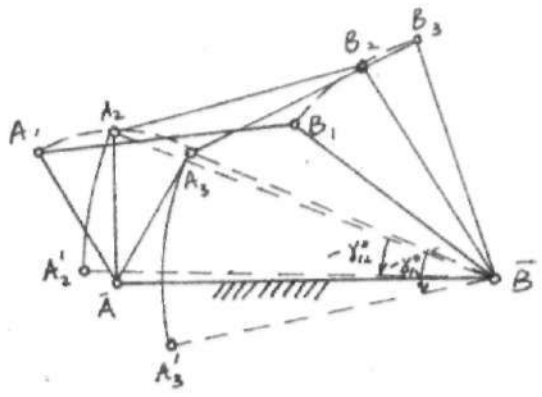


fig. 7·24: 4-bar mechanism for three-position coordination by method (ii).

The same problem can be solved by the approach given under coordination of crank motion (Ex. 9.01).

7.6. Four-Position Coordination:

For four position coordination of input and output links of a 4-bar mechanism, the fixed link can be determined as a starting point.

For example, given the input and output angular displacements as shown in Fig. (7.25), a suitable 4-bar mechanism can be derived in the following way.

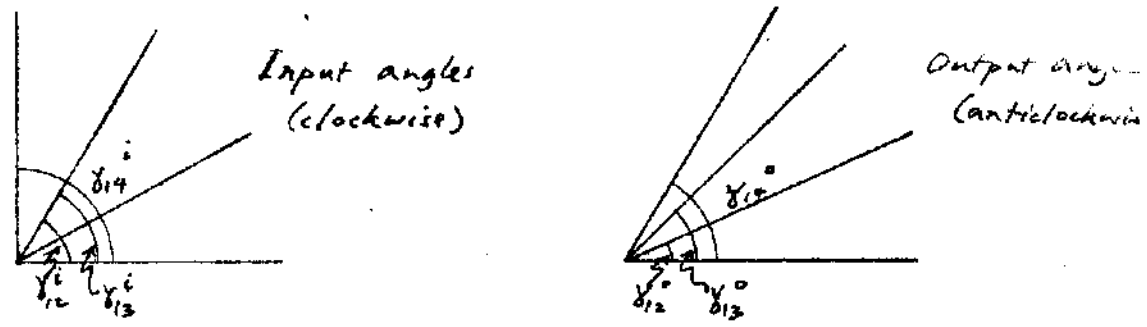


fig. 7.25 : Input and output angular displacements for four-position coordination (example).

Point $\bar{B}$ is then rotated about point $\bar{A}$ through angles γ^i_{12}, γ^i_{13}, γ^i_{14}, in the opposite sense, to locate points $B_2{}^1$, $B_3{}^1$, $B_4{}^1$. Some link $\bar{BC}$ is assumed to be connected at $\bar{B}$. When point $\bar{B}$ is rotated about point $\bar{A}$, link $\bar{BC}$ also experiences some angular displacement.

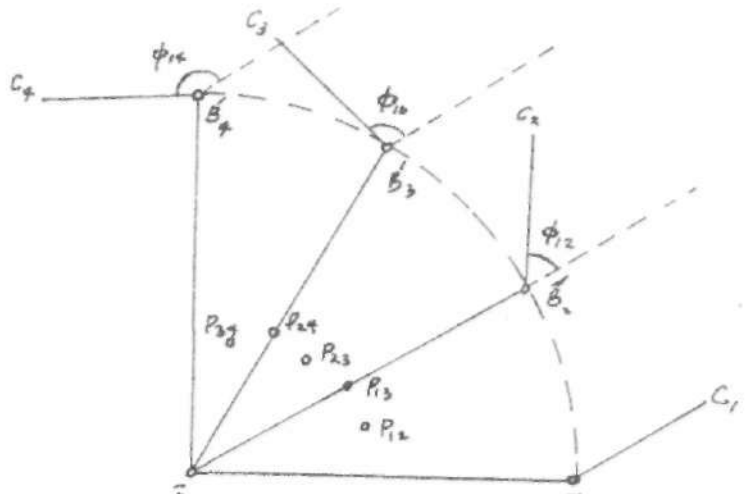
The angular displacements of the link at the three other positions to the initial position are given by;

$$\phi_{12} = \gamma^{i}_{12} - \gamma^{o}_{12}$$

$$\phi_{13} = \gamma^{i}_{13} - \gamma^{o}_{13}$$

$$\phi_{14} = \gamma^{i}_{14} - \gamma^{o}_{14}$$

as shown in Fig. (7.26)

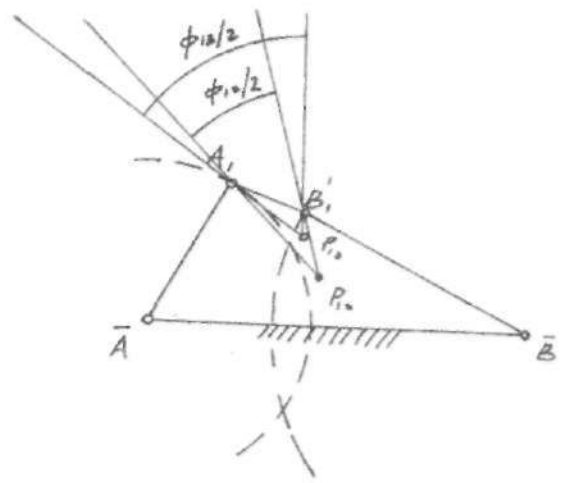


7.26: Inverting the mechanism for four-position design.

The poles of the assumed link $\bar{B}C$ in the four positions are then found and some are shown in Fig. (7.26).

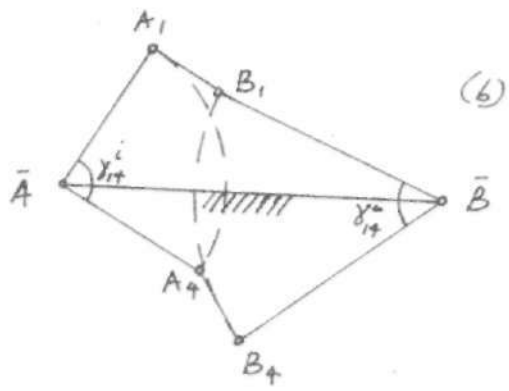
The circle point curve is then plotted and a suitable point on it selected for point position A_1.

The curve passes through the poles, so any convenient pole can be chosen for the location of a moving pivot at a particular state. In Fig. (7.27), pole P_{24} is chosen as point A_1. Poles P_{12} and P_{13} are used to locate point B_1. Angle $A_1P_{12}B_1$ equals $\delta_{12}/2$ and angle $A_1P_{13}B_1$ equals $\delta_{13}/2$.



(a)

Fig. 7.27 Four-position design for 4-bar mechanism.



(b)

CHAPTER EIGHT

POINT POSITION REDUCTION

The essence of point position reduction is in selecting points of a given number of positions where two or more will coincide. The objective is to have the available point positions reduced to three, and the three points corresponding to a greater number of points.

8.1. *4-Bar Mechanism For Four Point Positions:*

Given four point positions C_1, C_2, C_3, C_4, as in Fig.(8.01); the four points can be made to lie on the coupler curve of a 4-bar mechanism.

Pivot point $\bar{B}$ is assumed at any convenient position on c_{14}, the perpendicular bisector of C_1 and C_4. Triangle $C_1A_1\bar{B}$ is made congruent to triangle $C_4A_4\bar{B}$. Either of points A_1 and A_4 can be chosen arbitrarily.

Pivot A is chosen on a convenient point on the perpendicular bisector of A_1A_4, a_{14}.

Triangle $C_1A_1\bar{B}_2$ is made congruent to triangle $C_2A_2\bar{B}$. Triangle $C_1A_1\bar{B}_3$ is also made congruent to triangle $C_3A_3\bar{B}$. B_1 is the centre of the circle passing through $\bar{B}$, $\bar{B}_2$, $\bar{B}_3$.

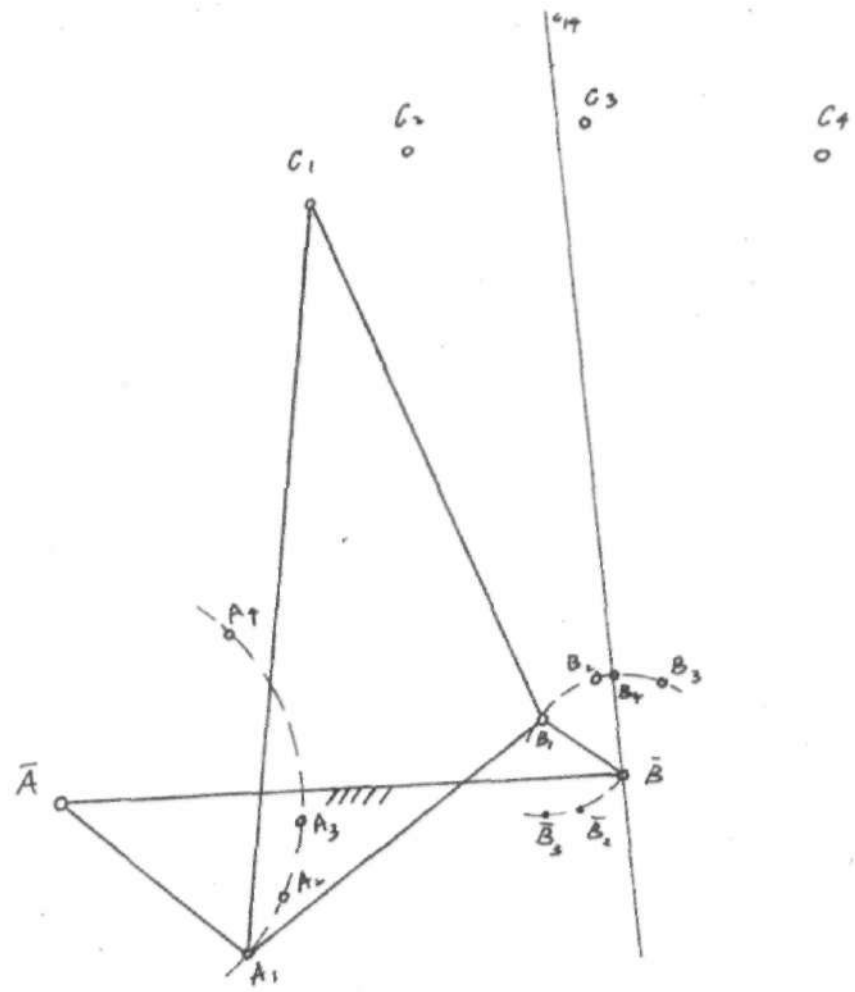


fig. 8·01: 4 - bar mechanism derived for four point - positions

8.2. 4-Bar Mechanism For Five Point Positions:

The point position reduction method can also be used to derive a suitable 4-bar mechanism to have a given set of five point positions on its coupler curve.

For example, with five point positions C_1, C_2, C_3, C_4, C_5, as in Fig. (8.02), fixed pivot point $\bar{B}$ is chosen as the intersection of two perpendicular bisectors of two distinct pairs of points, say C_1C_5 and C_2C_3.

With a convenient coupler length AC, triangle $A_1\bar{B}C_1$ is made congruent to triangle $A_4\bar{B}C_4$ and also, triangle $A_2\bar{B}C_2$ is made congruent to triangle $A_3\bar{B}C_3$.

Pivot point $\bar{A}$ is the intersection of A_1A_2 and A_3A_5 (Fig. 8.02).

To locate B_1, triangle $C_1A_1\bar{B}_2$ is made congruent to triangle $C_2A_2\bar{B}$ and triangle $C_1A_1\bar{B}_4$ is made congruent to triangle $C_4A_4\bar{B}$. B_1 is the centre of the circle that passes through points $\bar{B}$, $\bar{B}_2$, $\bar{B}_4$. It will be noticed that $\bar{B}_1$, $\bar{B}_5$ coincide at $\bar{B}$ and $\bar{B}_2$, $\bar{B}_3$ coincide at $\bar{B}_2$.

The derived 4-bar mechanism is shown in Fig. (8.02).

8.3. 4-Bar Mechanism For Five Point Positions With A Sixth On A Given Curve:

It is possible to derive a 4-bar mechanism whose coupler point will pass through five given positions on a curve with a sixth point chosen for a better approximation of the curve.

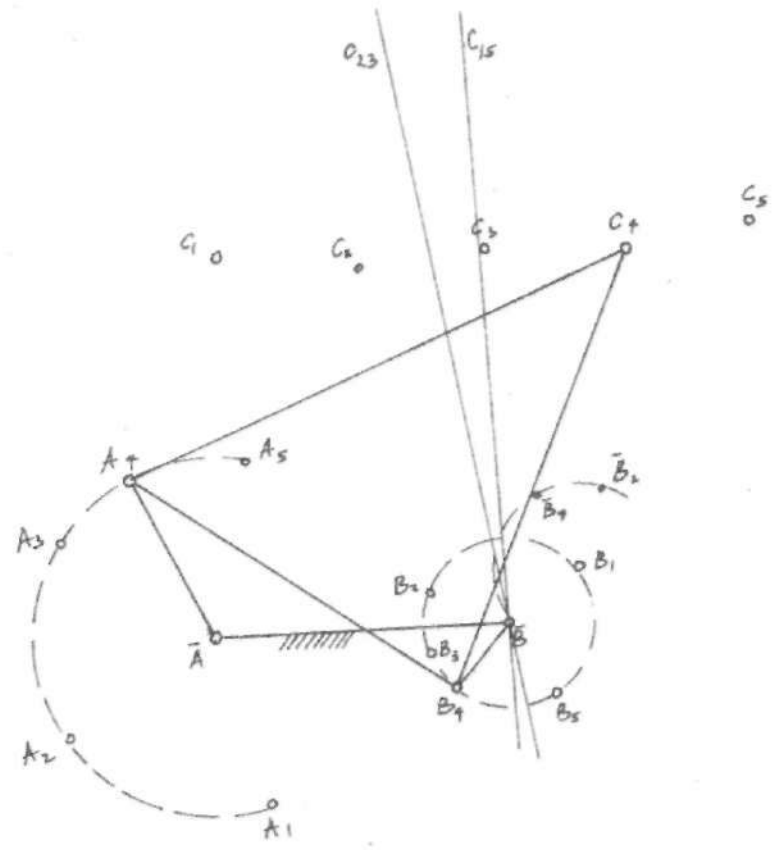


fig. 8·02 : 4-bar mechanism derived for five point-positions.

For example, given the curve of Figure (8.03) and the coupler point C of a 4-bar mechanism to pass accurately through points C_1, C_2, C_3, C_4, C_5, a sixth point C_6 will be chosen on the curve for better approximation.

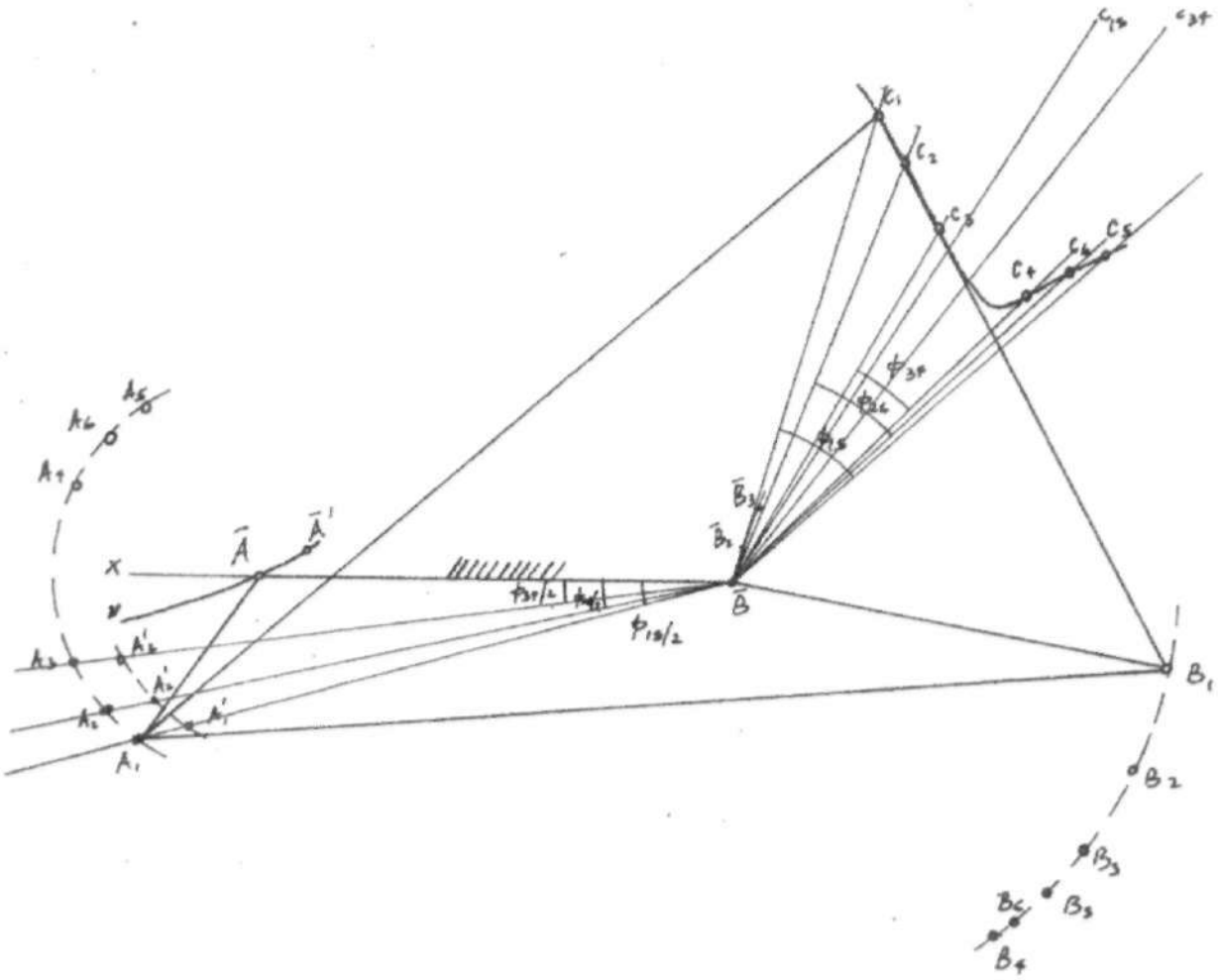


Fig. 8.03 : 6-bar mechanism for 5 points with a sixth on a given curve

Point $\bar{B}$ is chosen as the intersection of the perpendicular bisectors of the distinct pairs of points, say C_1C_5 and C_3C_4. With radius $\bar{B}C_2$, an arc is described about $\bar{B}$ to cut the given curve at C_6, different from C_2. This is the sixth point position on the curve.

Choosing an arbitrary coupler length AC, positions A_1^1, A_2^1, A_3^1 are marked with angles $A_1^1\bar{B}X$, $A_2^1\bar{B}X$, $A_3^1\bar{B}X$ being equal to half the angles $C_1\bar{B}C_5$, $C_3\bar{B}C_4$, $C_2\bar{B}C_6$, respectively. $\bar{A}^1$ is the centre of the circle through points A_1^1, A_2^1, A_3^1. This procedure is repeated for varying lengths of coupler length AC and a curve, v, is got which is the locus of $\bar{A}^1$. Point $\bar{A}$ is the intersection of the curve v and the line $\bar{B}X$ (Fig.8.03).

If the derived mechanism is found unsuitable, another design can be done for a different fixed link line.

8.4. 4-Bar Mechanism To Generate A Double Point:

(i) At Beginning And End: For the mechanism considered (Fig. 8.04), the double point is at the beginning and end of the coupler curve; therefore the coupler curve is closed at coupler point $C_1 (= C_5)$. Fixed pivot $\bar{B}$ is the intersection of two distinct pairs of points, say C_1, C_4 and C_2, C_3.

Choosing a convenient coupler length AC and point position A_1, triangle $A_4\bar{B}C_4$ is made congruent to triangle $A_1\bar{B}C_1$. Triangles $A_2\bar{B}C_2$ and $A_3\bar{B}C_3$ are also made congruent to each other.

Fixed pivot $\bar{A}$ is the intersection of perpendicular bisectors of A_1A_2 and A_3A_4. Point A_5 is then marked out.

Triangle $A_1C_1\bar{B}_2$ is made congruent to triangle $A_2C_2\bar{B}$ and also triangle $A_1C_1\bar{B}_5$ is made congruent to triangle $A_5C_5\bar{B}$. B_1 is the centre of the circle through points $\bar{B}$, $\bar{B}_2$, $\bar{B}_5$ (Fig. 8.04).

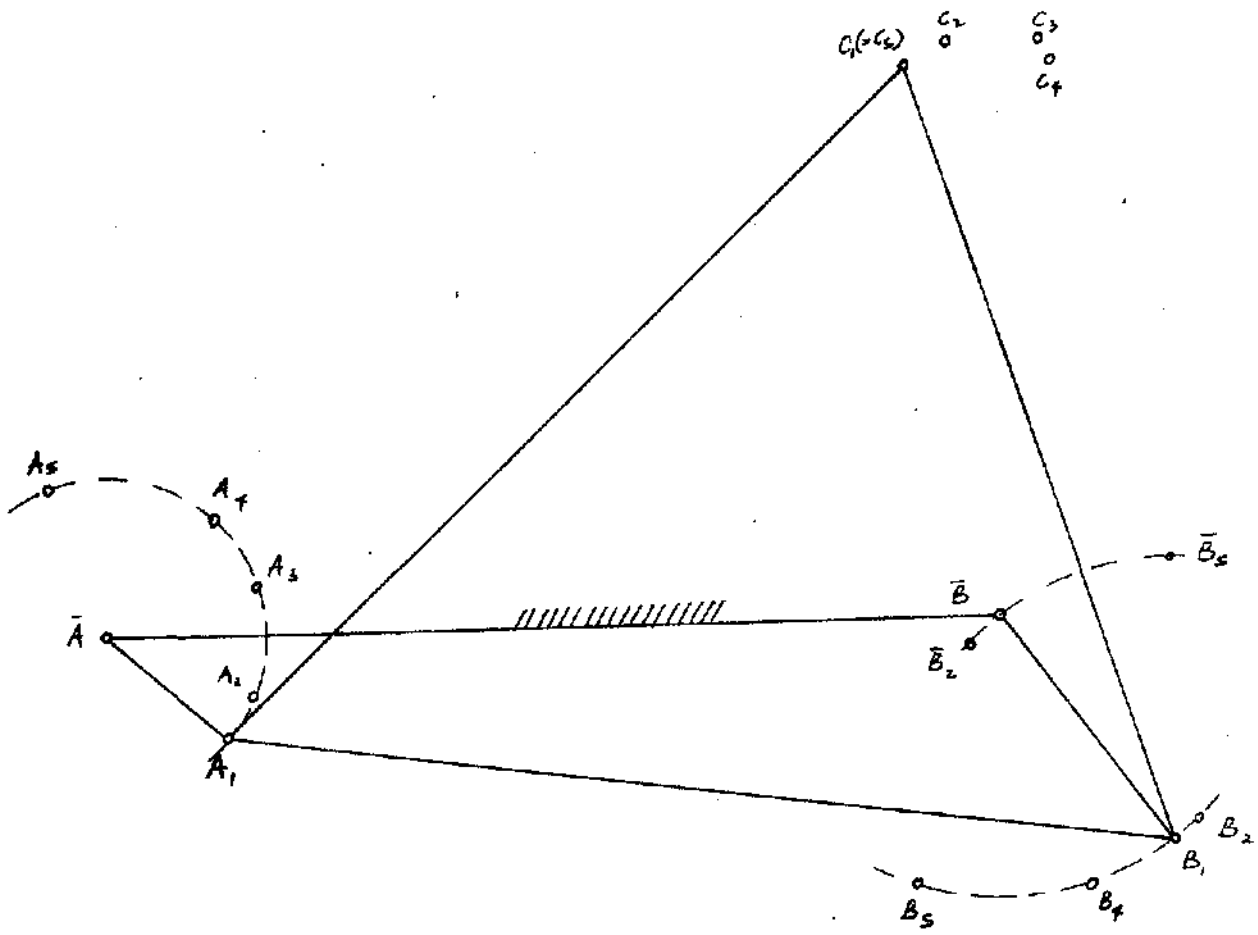


fig. 8.04 : 4-bar mechanism designed to generate a double point at $C_1(\equiv C_5)$.

(ii) At Interior Of Coupler Curve: It is also possible to derive a 4-bar mechanism (or several) that has a double point on the interior of its coupler point curve.

For example, in Fig. (8.05), a coupler C of a 4-bar mechanism is to run through C_1, C_2, C_3, C_4, C_5, with a double point at point $C_2(= C_3)$.

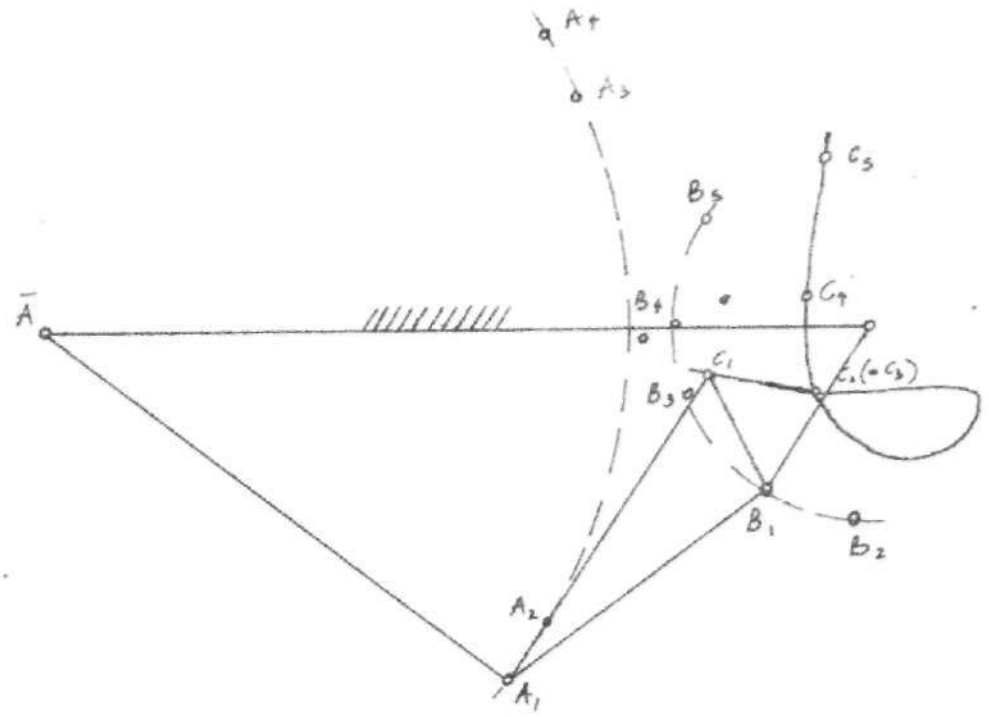


fig. 8·05 : 4-bar mechanism designed to generate a double-point at $C_2 (= C_3)$.

Two distinct point pairs, say C_1; C_5 and C_2; C_4, are chosen and their perpendicular bisector c_{15} and c_{24} intersect at the fixed pivot point, $\bar{B}$.

Choosing a convenient coupler length AC and a point position A_1, triangle $A_5\bar{B}C_5$ is made congruent to triangle $A_1\bar{B}C_1$. Choosing a point A_2 on the radius length C_2A, triangle $A_4\bar{B}C_4$ is also made congruent to triangle $A_2\bar{B}C_2$. Point A_3 is then located, different from point A_2, on the arc of radius $\bar{A}A$; where $\bar{A}$ is the intersection of the perpendicular bisectors of A_1A_2 and A_4A_5.

Triangle $A_1C_1\bar{B}_2$ is made congruent to triangle $A_2C_2\bar{B}$ and triangle $A_1C_1\bar{B}_3$ is made congruent to triangle $A_3C_3\bar{B}$.

B_1 is the centre of the circle through points $\bar{B}(=\bar{B}_1=\bar{B}_5)$, $\bar{B}_2(=B_4)$, $\bar{B}_5$.

8.5. 4-Bar Mechanism For Four Point Positions With B_1 as A Rotation Point:

Using the point position reduction method, a moving pivot, B_1, say, can be assumed as pole of one or more point positions pairs.

For example, given coupler point positions C_1, C_2, C_3, C_4, as in Fig. (8.06), moving pivot $B_1(=B_4)$ is chosen on any convenient point of the perpendicular bisector or C_1C_4, c_{14}.

Choosing a convenient coupler length AB, and point position A_1, triangle $A_4B_4C_4$ is made congruent to triangle $A_1B_1C_1$. Points A_2, A_3 are then marked out.

Triangles $C_2B_2A_2$, $C_3B_3A_3$, $C_4B_4A_4$ are all made congruent to triangle $C_1B_1A_1$. $\bar{B}$ is the centre of the circle through points

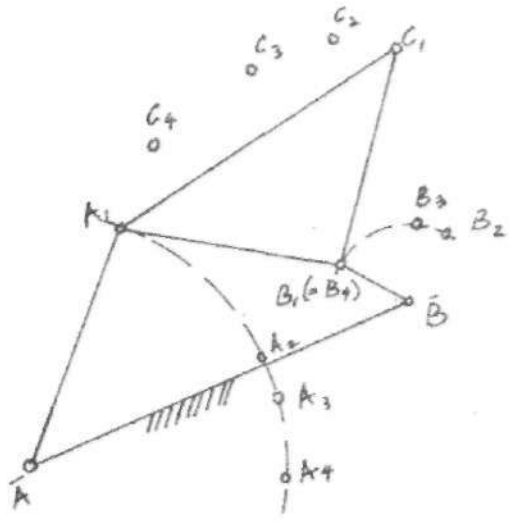


fig. 8·06 : 4-bar mechanism derived for four point-positions; with B_1 as a rotation point.

8.6. Correlation Of Point Positions And Angles:

Mechanism design problems involving correlation of point positions and angles can also be simplified by the use of point position reduction.

(i) Four Point Positions And Three Crank Angles:

As in Fig. (8.07), given four point positions C_1, C_2, C_3, C_4, to be traversed in adjacent angular crank position displacements, θ_{12}, θ_{23}, θ_{34}, the 4-bar mechanism is derived using point position reduction method as follows:

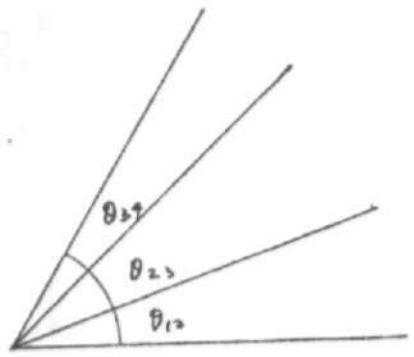
Fixed pivot $\bar{A}$ is chosen as for the perpendicular bisector of any two point positions, say, C_1, C_2, such that angle $C_1\bar{A}C_2$ equals θ_{12}.

Coupler point positions C_2, C_3, C_4, are rotated about fixed pivot $\bar{A}$ through angles $-\theta_{12}$, $-\theta_{13}$, $-\theta_{14}$, respectively to yield point positions C_2^1, C_3^1, C_4^1, respectively. C_2^1 is found to coincide with C_1. A_1 is the centre of the circle through points $C_1(=C_2^1)$, C_3^1, C_4^1. Points A_2, A_3, A_4, are the determined.

$B_3(=B_4)$ is assumed on any convenient point on the perpendicular bisector of C_3C_4, c_{34}.

Triangles $A_1C_1B_1$, $A_2C_2B_2$, are made congruent to triangle $A_3C_3B_3$. Triangles $A_3C_3B_3$ and $A_4C_4B_4$ are already congruent to one another as points B_3 and B_4 coincide as point position reduction (Fig. 8.07).

$\bar{B}$ is the centre of the circle through points B_1, B_2, $B_3(=B_4)$.



Crank angles

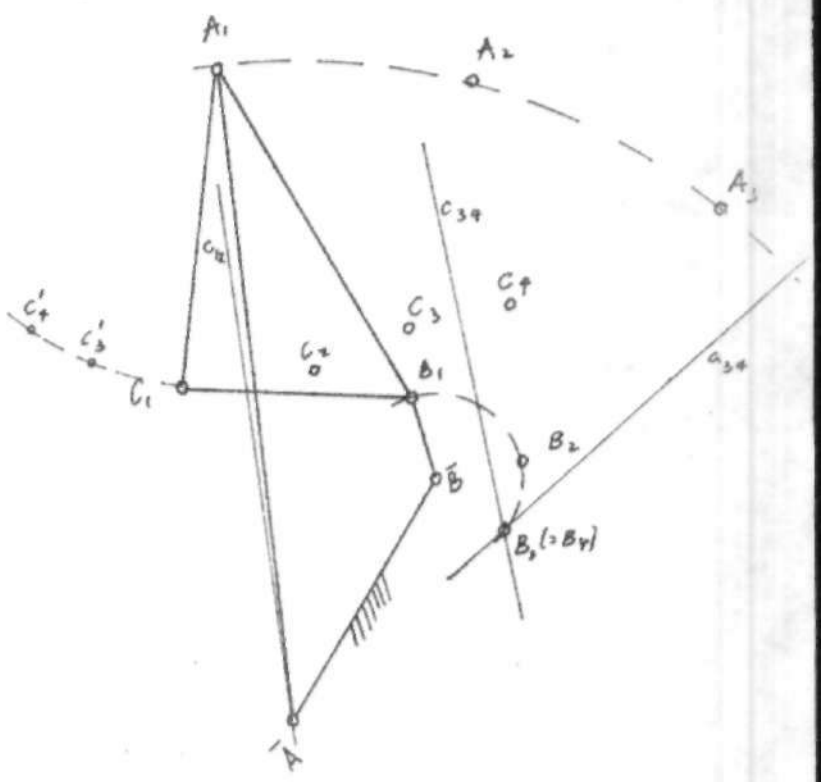


fig. 8·07: Correlation of four point-positions and three crank angles.

8.7. Correlation Of Coupler Positions:

It has been stated that the circle point curve represents the geometric locus of all movable pivot points and the centre curve represents the locus of all fixed pivots. These curves are known to pass through the relevant poles of the mechanism under consideration. Therefore, the pivots can then be located on the poles, minimizing construction work, to derive suitable 4-bar mechanisms for given coupler link positions. Three methods are shown here for deriving suitable 4-bar mechanisms, if possible, for four given coupler positions.

(i) Fixed Pivots As Poles: Given four coupler positions, C_1D_1, C_2D_2, C_3D_3, C_4D_4, as in Fig. (8.08), the procedure for deriving a 4-bar mechanism whose coupler coincides with the given positions, is as follows:

Fixed pivot $\bar{A}$ is chosen to coincide with the pole of two coupler positions; P_{12}, say. The other fixed pivot, $\bar{B}$, is also chosen to coincide with the pole of another pair of coupler positions, P_{34}, say.

Triangles $C_1D_1\bar{A}_2$, $C_2D_2\bar{A}$, are made congruent and triangles $C_1D_1\bar{A}_3$, $C_3D_3\bar{A}$, are also made congruent. Triangle $C_1D_1\bar{A}_4$ is also made congruent to triangle $C_4D_4\bar{A}$. A_1 is the centre of the circle through points $\bar{A}(=\bar{A}_2)$, $\bar{A}_3$, $\bar{A}_4$.

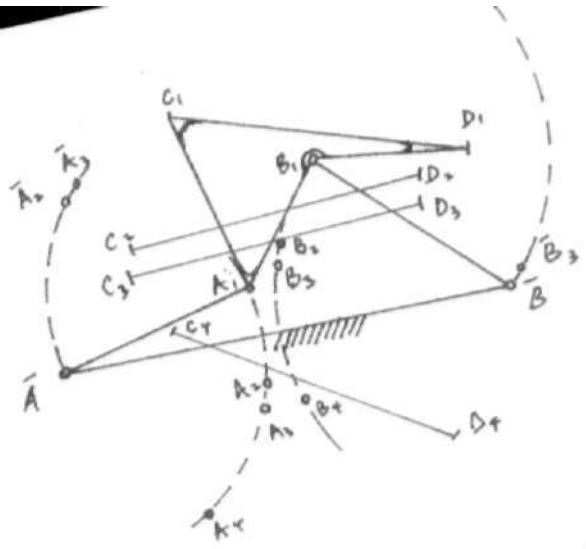


fig. 8.08: 4-bar mechanism for four coupler positions with the fixed pivots as poles. ($\bar{A} = P_{34}$, $\bar{B} = P_{12}$).

Similarly, triangles $C_1D_1\bar{B}_2$, $C_2D_2\bar{B}$, are made congruent; triangles $C_1D_1\bar{B}_3$, $C_3D_3\bar{B}$, are made congruent; and triangles $C_1D_1\bar{B}_4$, $C_4D_4\bar{B}$, are made congruent. B_1 is the centre of the circle theough point $\bar{B}$, $\bar{B}_2$, $\bar{B}_3(=\bar{B}_4)$.

(ii) One Moving Pivot And One Fixed Pivot As Poles:

A similar problem is solved as in the preceding case by making one moving pivot A and a fixed pivot $\bar{B}$ coincide with the poles of the pairs of coupler positions, resulting in point position reduction.

As in Fig. (8.09), $A_1(=A_2)$ is the pole P_{12} and $\bar{B}$ coincides with P_{34}.

$\Delta C_1D_1A_1 \equiv \Delta C_2D_2A_2 \equiv \Delta C_3D_3A_3 \equiv \Delta C_4D_4A_4$. A is the centre of the circle through points $A_1(=A_2)$, A_3, A_4.

Similarly, $\Delta C_1D_1\bar{B}_2 \equiv \Delta C_2D_2\bar{B}$;

$\Delta C_1D_1\bar{B}_3 \equiv \Delta C_3D_3\bar{B}$;

$\Delta C_1D_1\bar{B}_4 \equiv \Delta C_4D_4\bar{B}$.

B_1 is the centre of the circle through points $\bar{B}$, $\bar{B}_2$, $\bar{B}_3(=\bar{B}_4)$

(iii) <u>Moving Pivots As Poles</u>: A similar problem as in the two preceding examples is here solved by having the two moving pivot positions A_1, B_3 coincide with different poles of distinct pairs of coupler positions.

$A_1(=A_2)$ is made to coincide with the pole of coupler link positions 1 and 2, P_{12}; and $B_3(=B_4)$ lie on P_{34}.

$\Delta C_1D_1A_1 \equiv \Delta C_2D_2A_2 \equiv \Delta C_3D_3A_3 \equiv \Delta C_4D_4A_4$. $\bar{A}$ is then the centre of the circle through points $A_1(=A_2)$, A_3, A_4.

Similarly, $\Delta C_1D_1B_1 \equiv \Delta C_2D_2B_2 \equiv \Delta C_3D_3B_3 \equiv \Delta C_4D_4B_4$. $\bar{B}$ is the crntre of the circle through points B_1, B_2, $B_3(=B_4)$

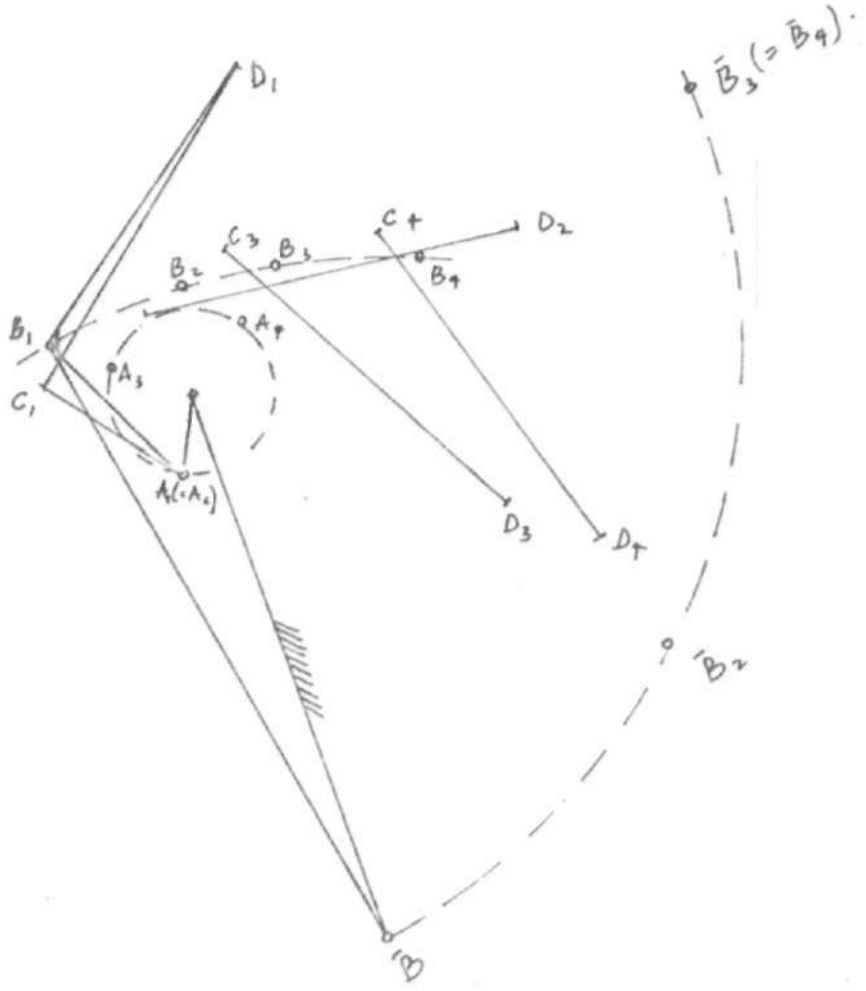


fig. 8.09: 4-bar mechanism for four coupler positions; P_{12} as moving pole, $A_1(=A_2)$, P_{34} as fixed pole $\bar{B}$.

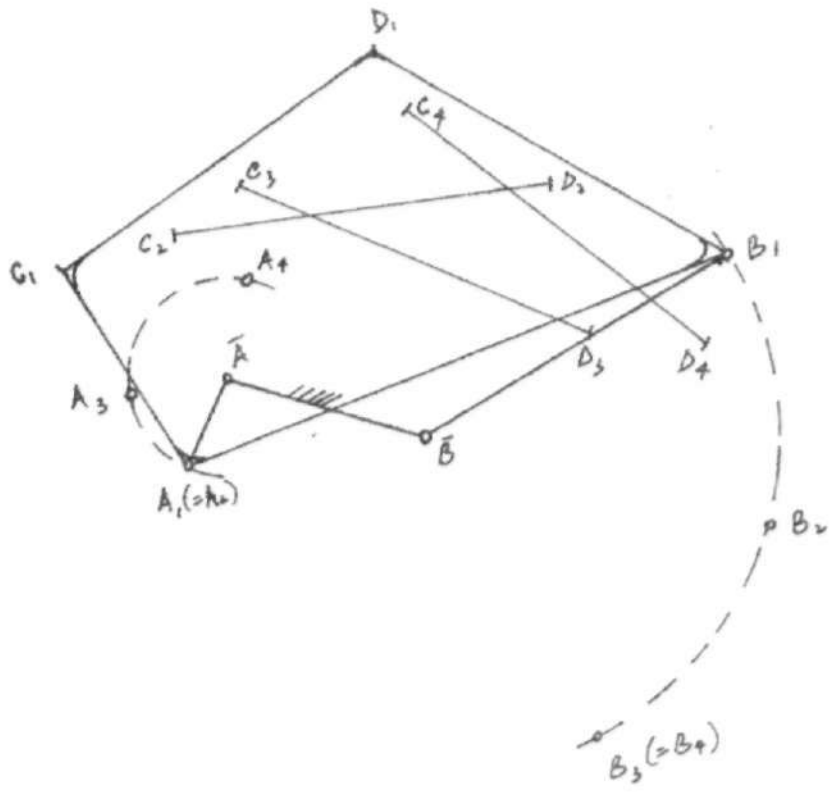


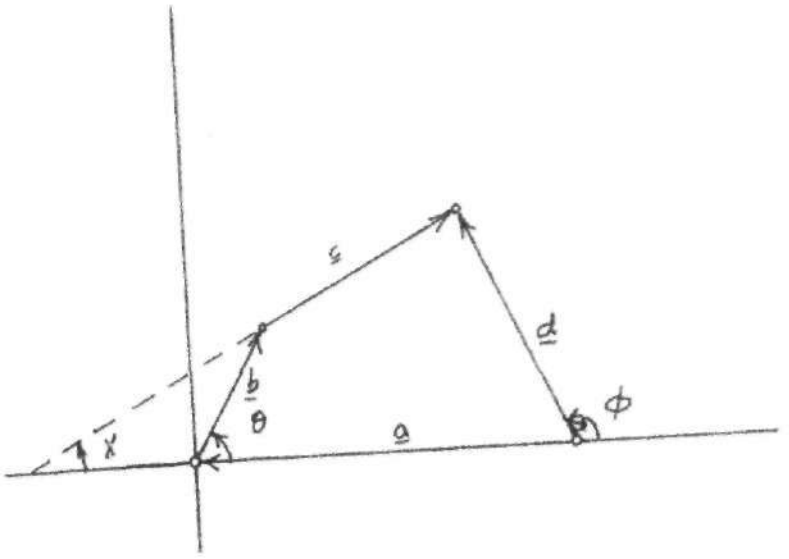
fig. 8.10 : 4-bar mechanism derived for four coupler positions; moving pivots as poles.

CHAPTER NINE

ANALYTICAL CO-ORDINATION OF CRANK MOTION

9.1. Frendenstein's Equation:

Considering the 4-bar mechanism of Fig. (9.01), the following equations can be deduced by the use of 'vectors' and



$\underline{a} = ae^{i\pi}$
$\underline{b} = be^{i\theta}$
$\underline{c} = ce^{i\gamma}$
$\underline{d} = de^{i\phi}$

fig 9.01 : Notations used for deriving the Frendenstein's equation

$$\underline{b} + \underline{c} - \underline{d} + \underline{a} = 0 \text{ ----------} (901)$$

Equation (901) can be written as

$$be^{i\phi} + ce^{i\delta} - de^{i\phi} - a = 0 \quad \text{----------(902)}$$

Equation (902) yields equations (903) and (904)

$$b\cos\theta + c\cos\delta - d\cos\phi - a = 0 \quad \text{----------(903)}$$

$$b\sin\theta + c\sin\delta - d\sin\phi = 0 \quad \text{----------(904)}$$

Eliminating δ from equations (903) and (904),

$$c^2 = a^2 + b^2 + d^2 - 2bd\cos(\theta - \phi) - 2ab\cos\theta + 2ad\cos\phi \quad \text{----------(905)}$$

$$\therefore \quad R_1\cos\phi - R_2\cos\theta + R_3 = \cos(\theta - \phi) \quad \text{----------(906)}$$

where $R_1 = \frac{a}{b}$, $R_2 = \frac{a}{d}$, $R_3 = (a^2 + b^2 + d^2 - c^2)/2bd$

Equation (906) is the Frendenstein equation.

Ex.9.01: Three Coordinated Crank Positions:

Design a 4-bar mechanism such that crank and follower angles will be coordinated as follows:

θ	60	90	120	degrees
ϕ	120	130	150	

Solution: $R_1\cos\phi - R_2\cos\theta + R_3 = \cos(\theta - \phi)$ ----------(906)

Substituting in equation (906);

$$R_1 \cos 120 - R_2 \cos 60 + R_3 = \cos(-60) \quad \text{------(a)}$$

$$R_1 \cos 130 - R_2 \cos 90 + R_3 = \cos(-40) \quad \text{------(b)}$$

$$R_1 \cos 150 - R_2 \cos 120 + R_3 = \cos(-30) \quad \text{------(c)}$$

Solving equations (a) to (c) simultaneously;

$R_1 = 2.5$

$R_2 = 1.25$

$R_3 = 2.31$

Assuming $a = 1$, then

$b = 0.4$

$c = 0.53$

$d = 0.8$

The derived 4-bar mechanism is shown in Fig. (9.02).

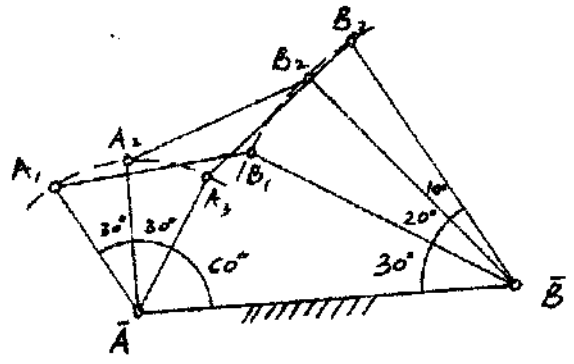


fig. 9.02: 4-bar mechanism derived for three coordinated crank positions.

9.2. 4-Bar Function Generators:

The graphical treatment of 4-bar function generators has been explained under position coordination and more so under point position reduction. Analytically, as by graphical methods, it is possible to generate a function with six precision points but after four precision points, the problem becomes arduous.

For a three-point precision analytical solution, the starting angles, the ranges of rotation of the cranks, the spacing of the precision points, and the directions of rotation can be chosen arbitrarily. For a 4-bar function generator designed analytically for a four-point precision accuracy, all the variables stated for a three-point precision solutions may not be arbitrarily chosen as a suitable solution may not exist.

It is advisable to limit the ranges of the angular displacements of the cranks to 120° and for higher (than three) precision points, the cranks be made to rotate in opposite directions.

Chebiechew spacing of precision points is a good starting point in the design of 4-bar function generators. For a three-point precision design the precision points are spaced as shown in Fig. (9.03); for a four point precision point in Fig. (9.04); and for a fiv-point precision in Fig. (9.05).

They may be written as follows:

For a 3-point spacing;

$$X_1 = X_o + 0.067\Delta x$$
$$X_2 = X_o + 0.5\Delta x$$
$$X_3 = X_o + 0.933\Delta x$$

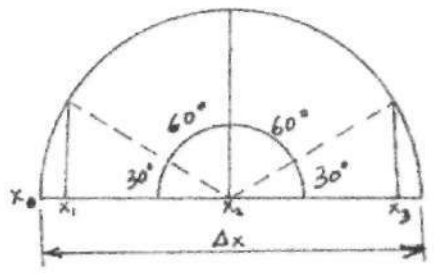


fig. 9.03 : Chebicher three-point spacing.

For a 4-point spacing;

$$X_1 = X_o + 0.038\,\Delta x$$
$$X_2 = X_o + 0.309\,\Delta x$$
$$X_3 = X_o + 0.692\,\Delta x$$
$$X_4 = X_o + 0.962\,\Delta x$$

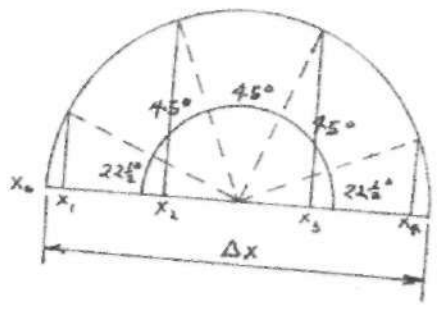


fig. 9.04 : Chebicher four-point spacing.

For a 5-point spacing;

$$X_1 = X_o + 0.024\,\Delta x$$

$$X_2 = X_o + 0.206\,\Delta x$$

$$X_3 = X_o + 0.500\,\Delta x$$

$$X_4 = X_o + 0.794\,\Delta x$$

$$X_5 = X_o + 0.976\,\Delta x$$

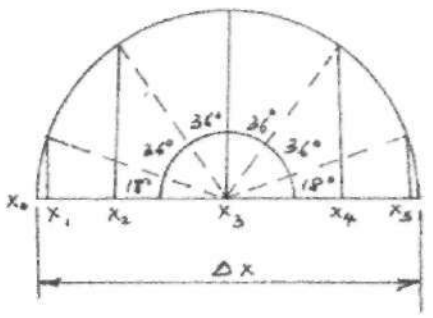


fig. 9·05 : Chebicher five – point spacing.

9.3. Function-Generator With Optimum Sensitivity:

For a function, $y = f(x)$, the theoretical sensitivity, s, is defined as

$$y = f(x) \quad X_o \leqslant X \leqslant X_o + \Delta x$$

$$s = \frac{dy}{dx} = f^1(x)$$

$$= \frac{dy/dt}{dx/dt} \qquad \text{----------(907)}$$

Let X_1, X_2, X_3, be the three precision points and y_1, y_2, y_3, the corresponding values of y.

$$\frac{X - X_1}{\Delta x} = \frac{\theta - \theta_1}{\Delta \theta} \qquad \text{----------(908)}$$

$$\frac{y - y_1}{\Delta y} = \frac{\phi - \phi_1}{\Delta \phi} \qquad \text{----------(909)}$$

where $\Delta\theta$ is the range of the input crank and $\Delta\phi$ is the range of the output crank.

From equation (907),

$$s = \frac{\frac{d\phi}{dt}}{\frac{d\theta}{dt}} \cdot \frac{\frac{\Delta y}{\Delta \phi}}{\frac{\Delta x}{\Delta \theta}} = \frac{\omega_4}{\omega_2} \cdot q \qquad \text{----------(910)}$$

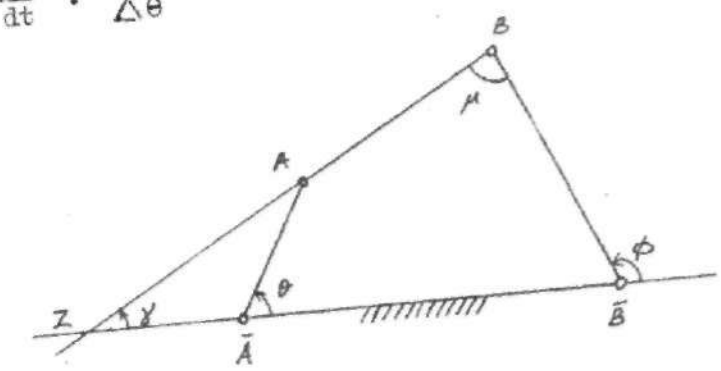


fig. 9.06: 4-bar mechanism.

where $q = \frac{\Delta y}{\Delta \phi} / \frac{\Delta x}{\Delta \theta} = \frac{Ky}{Kx}$

From equation (910)

$$\frac{\omega_4}{\omega_2} = \frac{b}{d} \frac{\sin(\theta - \not{\alpha})}{\sin(\phi - \not{\delta})} = \frac{s}{q}$$

$$\therefore \quad \sin\delta\left[\cos\phi - \frac{bq}{sd}\cos\theta\right] = \cos\delta\left[\sin\phi - \frac{bq}{sd}\sin\theta\right] \text{ --(911)}$$

$$\text{then} \qquad \tan\delta = \frac{\sin\phi - \frac{bq}{sd}\sin\theta}{\cos\phi - \frac{bq}{sd}\cos\theta} \text{ ----------(912)}$$

$$\text{but} \qquad \cos\delta = \frac{1}{(1 + \tan^2\delta)^{\frac{1}{2}}}$$

and letting $k_1 = \frac{b}{d}$

$$\cos\delta = \frac{\cos\phi - k_1\frac{q}{s}\cos\theta}{\left[1 + \frac{k_1^2 q^2}{s^2} - \frac{2k_1 q}{s}\cos(\theta - \phi)\right]^{\frac{1}{2}}} \text{ --------(913)}$$

From Fig. (9.06); $\delta = \phi - \mu$, where μ is the transmission angle.

Also from Fig. (9.06);

$$b\cos\theta + c\cos\gamma - d\cos\phi - a = 0 \text{ --------(914)}$$

$$\therefore \quad k_1\cos\theta + k_2\cos\delta - k_3 = \cos\phi \text{ ----------(915)}$$

where $k_2 = \frac{c}{d}$ and $k_3 = \frac{a}{d}$

Equation (915) can then be written for the three precision points

$$k_1\cos\theta_1 + k_2\cos\delta_1 - k_3 = \cos\phi_1 \text{ --------(915a)}$$

$$k_1 \cos(\theta_1 + \Delta\theta_1) + k_2 \cos \gamma_2 - k_3 = \cos \phi_2 \quad \text{----(915b)}$$

$$k_1 \cos(\theta + \Delta\theta_2) + k_2 \cos \gamma_3 - k_3 = \cos \phi_3 \quad \text{----(915c)}$$

where $\Delta\theta_1 = \theta_2 - \theta_1$

and $\Delta\theta_2 = \theta_3 - \theta_2$

For optimum design, the second precision point should correspond to $\mu_2 = 90$ degrees.

$$\therefore \tan \gamma_2 = \tan (\phi_2 - 90^\circ)$$

$$= \frac{\sin \phi_2 - k_1 \frac{q}{s_2} \sin (\theta_1 + \Delta\theta_1)}{\cos \phi_2 - k_1 \frac{q}{s_2} \cos(\theta_1 + \Delta\theta_1)} \quad \text{----------(916)}$$

$$= p$$

$$\therefore k_1 = \frac{\frac{s_2}{q}(p \cos \phi_2 - \sin \phi_2}{\left[p \cos(\theta_1 + \Delta\theta_1) - \sin(\theta_1 + \Delta\theta_1)\right]} \quad \text{--------(917)}$$

Eliminating k_2 and k_3 from equations (915a) to (916),

$$k_1 \left[(\cos \gamma_2 - \cos \gamma_3) \left\{ \cos(\theta_1 + \Delta\theta_1) - \cos \theta_1 \right\} - (\cos \gamma_2 - \cos \gamma_1) \times \left\{ \cos(\theta_1 + \Delta\theta_1) - \cos(\theta_1 + \Delta\theta_2) \right\} \right]$$

$$= (\cos \gamma_2 - \cos \gamma_3)(\cos \phi_2 - \cos \phi_1) - (\cos \gamma_2 - \cos \gamma_1)$$

$$(\cos \phi_2 - \cos \phi_3) \quad \text{----------(918)}$$

ϕ and $\Delta\phi$ can be chosen arbitrarily and also $\Delta\theta$. k_1 is evaluated from equation (917) and substituted into equation (918). θ_1 then becomes the only unknown. Equation (918) is then solved to yield the value for θ_1.

* This treatment was originally presented by A. C. Rao.

9.4. Coordination Of Crank Angular Displacement Derivatives:

The relation hip between the input and output angular displacements, θ and ϕ, is given by;

$$R_1 \cos\phi - R_2 \cos\theta + R_3 = \cos(\theta - \phi) \quad \text{-------(906)}$$

Differentiating with respect to θ;

1st differentiation:

$$- R_1 \sin\phi \frac{d\phi}{d\theta} + R_2 \sin\theta = - \sin(\theta - \phi) * \left[1 - \frac{d\phi}{d\theta}\right] \quad \text{----(919)}$$

2nd differentiation:

$$- R_1 \cos\phi . \left(\frac{d\phi}{d\theta}\right)^2 - R_1 \sin\phi \frac{d^2\phi}{d\theta^2} + R_2 \cos\theta$$

$$= - \cos(\theta - \phi) * \left[1 - \frac{d\phi}{d\theta}\right]^2 + \frac{d^2\phi}{d\theta^2} \sin(\theta - \phi) \quad \text{----------(920)}$$

3rd differentiation:

$$R_1 \sin\phi \left[\left(\frac{d\phi}{d\theta}\right)^3 - \frac{d^3\phi}{d\theta^3}\right] + R_1 \cos\phi \left[- 3 \frac{d\phi}{d\theta} \cdot \frac{d^2\phi}{d\theta^2}\right] - R_2 \sin\theta$$

$$= \sin(\theta - \phi) * \left[\left(1 - \frac{d\phi}{d\theta}\right)^3 + \frac{d^3\phi}{d\theta^3}\right] + \cos(\theta - \phi) * \left[3 \frac{d^2\phi}{d\theta^2}\left(1 - \frac{d\phi}{d\theta}\right)\right] \quad \text{---(921)}$$

4th differentiation:

$$R_1 \sin\phi\left[6\left(\frac{d\phi}{d\theta}\right)^2 \frac{d^2\phi}{d\theta^2} - \frac{d^4\phi}{d\theta^4}\right] + R_1\cos\phi\left[\left(\frac{d\phi}{d\theta}\right)^4 - 4\frac{d\phi}{d\theta}\frac{d^3\phi}{d\theta^3} - 3\left(\frac{d^2\phi}{d\theta^2}\right)^2\right] - R_2\cos\theta = \sin(\theta-\phi)\left[\frac{d^4\phi}{d\theta^4} - 6\frac{d^2\phi}{d\theta^2}\left(1-\frac{d\phi}{d\theta}\right)^2\right] + \cos(\theta-\phi)\left[\left(1-\frac{d\phi}{d\theta}\right)^4 + 4\frac{d^3\phi}{d\theta^3}\left(1-\frac{d\phi}{d\theta}\right) - 3\left(\frac{d^2\phi}{d\theta^2}\right)^2\right] \text{ -----(922)}$$

Substituting $n_i = \frac{d^i\phi}{d\theta^i}\left[n_1 = \frac{d\phi}{d\theta},\ n_2 = \frac{d^2\phi}{d\theta^2},\ \ldots\right]$ in equations (919) to (922); and having a general equation of the form written in equation (906) as follows;

$$R_1(p_i \sin\phi + q_i \cos\phi) + R_2(r_i \sin\theta + s_i \cos\theta) = t_i \sin(\theta-\phi) + u_i \cos(\theta-\phi) \text{ ----------(923)}$$

the values of p_i, q_i, r_i, s_i, t_i, u_i, are shown in the table.

i	p_i	q_i	r_i	s_i	t_i	u_i
1	$-n_1$	0	1	0	$-(1-n_1)$	0
2	$-n_2$	$-n_1^2$	0	1	n_2	$-(1-n_1)^2$
3	$-n_1^3+n_3$	$3n_1n_2$	1	0	$-(1-n_1)^3-n_3$	$-3n_2(1-n_1)$
4	$n_4-6n_1^2n_2$	$4n_1n_3+3n_2^2-n_1$	0	1	$6n_2(1-n_1)^2-n_4$	$-(1-n_1)^4+3n_2^2-4n_3(1-n_1)$

From equation (923), with i = 1

$$R_2 = \frac{t_1 \sin(\theta - \phi) + u_2 \cos(\theta - \phi)}{\sin \theta} - \frac{R_1 (p_1 \sin \phi + q_1 \cos \phi}{\sin \theta} \quad \text{---(924)}$$

Substituting in equation (923), when i = 2

$$R_1 \left[p_2 \sin \phi + q_2 \cos \phi - \cot \theta (p_1 \sin \phi + q_1 \cos \phi) \right]$$

$$= t_2 \sin(\theta - \phi) + u_2 \cos(\theta - \phi) - \cot \theta \left[t_1 \sin(\theta - \phi) + u_1 \cos(\theta - \phi) \right] \text{---(925)}$$

Substituting in equation (923), when i = 3

$$R_1 \left[p_3 \sin \phi + q_3 \cos \phi - (p_1 \sin \phi + q_1 \cos \phi) \right]$$

$$= (t_3 - t_1) \sin(\theta - \phi) + (u_3 - u_1) \cos(\theta - \phi) \quad \text{----------(926)}$$

Dividing equation (925) by equation (926), and rearranging;

$$\lambda_1 \tan^2 \theta + \lambda_2 \tan \theta + \lambda_3 = 0 \quad \text{----------(927)}$$

where
$$\lambda_1 = Q_2(P_3 - P_1) - P_2(Q_3 - Q_1)$$
$$\lambda_2 = P_1(Q_3 - R_2) - P_2(R_3 - R_1) + P_3(R_2 - Q_1)$$
$$\lambda_3 = P_1(R_3 - R_1) - R_1(P_3 - P_1)$$

and
$$P_i = p_i \sin \phi + q_i \cos \phi$$
$$Q_i = t_i \cos \phi + u_i \sin \phi$$
$$R_i = -t_i \sin \phi + u_i \cos \phi$$

Ex(9.02): Design a 4-bar mechanism for approximate angular velocity ratio of - 1.

Solution: For a velocity ratio of - 1,

$$n_1 = -1, \quad n_2 = 0, \quad n_3 = 0.$$

i	p_i	q_i	t_i	u_i
1	1	0	- 2	0
2	0	- 1	0	- 4
3	1	0	- 8	0

Various 4-bar mechanisms for varying values of angle ϕ are derived and the best chosen.

For example, for an angle $\phi = 135^\circ$

$$P_1 = 1, \quad P_2 = 1, \quad P_3 = 1$$
$$Q_1 = 2, \quad Q_2 = -4, \quad Q_3 = 8$$
$$R_1 = 2, \quad R_2 = 4, \quad R_3 = 8$$

and
$$\lambda_1 = -4(0) - 1(6) = -6$$
$$\lambda_2 = 1(4) - 1(6) + 1(2) = 0$$
$$\lambda_3 = 1(6) - 2(0) = 6$$

Substituting in equation (927),

$$\therefore \quad -6 \tan^2\theta + 6 = 0$$
$$\tan^2\theta = 1$$
$$\theta = 45^\circ, 135^\circ, 225^\circ, 315^\circ.$$

For $\theta = 135^\circ$, substituting in equations, (906),(923),

$$R_1 \cos 135 - R_2 \cos 135 - R_3 = \cos 0 \quad \text{----------(a)}$$

$$R_1(.707) + R_2 \sin 135^\circ = -2 \sin 0 + 0 \cos 0 \quad \text{-------(b)}$$

$$R_1(.707) + R_2 \cos 135^\circ = 0 \sin 0 - 4 \cos 0 \quad \text{--------(c)}$$

$$R_1(.707) + R_2 \sin 135^\circ = -8 \sin 0 + 0 \cos 0 \quad \text{-------(d)}$$

These become

$$-.707 R_1 + .707 R_2 + R_3 = 1$$

$$.707 R_1 + .707 R_2 = 0$$

$$.707 R_1 - .707 R_2 = -4$$

$$.707 R_1 + .707 R_2 = 0$$

Solving the equations;

$$-R_1 = +R_2 = 2\sqrt{2}$$

$$R_3 = -3$$

Assuming the fixed link length, a = 100 cm

$$b = 25\sqrt{2}, \quad d = 25\sqrt{2}, \quad c = 50\sqrt{2}$$

The derived mechanism is shown in Fig. (9.07).

It should be remembered that the optimum mechanism, in the present instance, is derived after a number of trials.

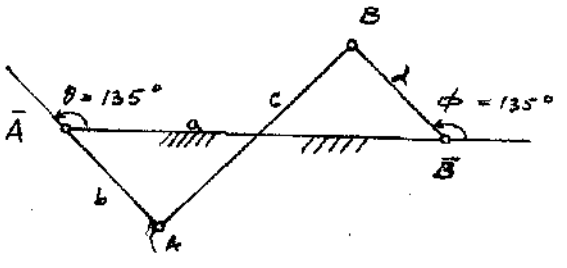


fig. 9.07: 4-bar mechanism with velocity ratio of cranks being approx. -1.

9.5. Alternate Form Of Results For Fourth Order Approximation:

Optimum design of 4-bar mechanism to generate a given function is usually found from a number of trials, involving repeated calculations for various values of output angle ϕ. It is more convenient to use an alternate form of the equation given below; equation (933).

$$\lambda_1 \tan^2\theta + \lambda_2 \tan\theta + \lambda_3 = 0 \quad \text{----------(923)}$$

From equation (923),

$$\lambda_1 = (t_2\cos\phi + u_2\sin\phi)\left[(p_3 - p_1)\sin\phi + (q_3 - q_1)\cos\phi\right]$$

$$- (p_2\sin\phi + q_2\cos\phi)\left[(t_3 - t_1)\cos\phi + (u_3 - u_1)\sin\phi\right] \text{---(928)}$$

Dividing equation (928) by $\cos^2\phi$

$$\frac{\lambda_1}{\cos^2\phi} = (t_2 + u_2 \tan\phi)\left[(p_3 - p_1)\tan\phi + (q_3 - q_1)\right]$$

$$- (p_2 \tan\phi + q_2)\left[(t_3 - t_1) + (u_3 - u_1)\tan\phi\right]$$

$$= \tan^2\phi\left[u_2(p_3 - p_1) - p_2(u_3 - u_1)\right] + \tan\phi\left[t_2(p_3 - p_1)\right.$$

$$\left. - p_2(t_3 - t_1) - q_2(u_3 - u_1)\right] + \left[t_2(q_3 - q_1) - q_2(t_3 - t_1)\right] \text{---(929)}$$

Similarly;

$$\frac{\lambda_2}{\cos^2\phi} = \tan^2\phi\left[p_1(t_2 + u_3) + p_2(t_3 - t_1) - p_3(t_2 + u_2)\right]$$

$$+ \tan\phi\left[p_1(t_3 - u_2) + q_1(t_2 + u_3) + p_2(u_1 - u_3) + q_2(t_3 - t_1)\right.$$

$$\left. + p_3(u_2 - t_1) - q_3(t_2 + u_1)\right]$$

$$+\left[q_1(t_3 - u_2) + q_2(u_1 - u_3) + q_3(t_1 + u_2)\right] \text{ ----------(930)}$$

Also similarly,

$$\frac{\lambda_3}{\cos^2\phi} = \tan^2\phi\left[p_1(t_1 - t_3) + t_1(p_3 - p_1)\right] + \tan\phi\left[p_1(u_3 - u_1)\right.$$

$$\left. + q_1(t_1 - t_3) + t_1(q_3 - q_1) + u_1(p_1 - p_3)\right]$$

$$+\left[q_1(u_3 - u_1) + u_1(q_1 - q_3)\right] \text{ ----------(931)}$$

The alternate form of equation (923) can then be written as;

$$\lambda_1^{\,1} \tan^2\theta + \lambda_2^{\,1} \tan\theta + \lambda_3^{\,1} = 0 \quad \text{----------(932)}$$

where $\lambda_i^{\,1} = j_i + k_i \tan\phi + l_1 \tan^2\phi$

$$j_1 = t_2(q_3 - q_1) - q_2(t_3 - t_1) = t_2q_3 - q_2t_3 + t_1q_2$$
$$j_2 = q_1(t_3 - u_2) + q_2(u_1 - u_3) + q_3(u_2 - t_1) = 0$$
$$j_3 = q_1(u_3 - u_1) + u_1(q_1 - q_3) = 0$$

$$k_1 = t_2(p_3 - p_1) + u_2(q_3 - q_1) + p_2(t_1 - t_3) + q_2(u_1 - u_3) = 0$$
$$k_2 = p_1(t_3 - u_2) + q_1(t_3 + u_3) + p_2(u_1 - u_3) + q_2(t_3 - t_1) + p_3(u_2 - t_1) - q_3(t_2 - u_1)$$
$$k_3 = p_1(u_3 - u_1) + q_1(t_1 - t_3) + t_1(q_3 - q_1) + u_1(p_1 - p_3) = 0$$

$$l_1 = u_2(p_3 - p_1) + p_2(u_1 - u_3) = u_2(p_3 - p_1) - p_2u_3$$
$$l_2 = p_1(t_2 + u_3) + p_2(t_3 - t_1) - p_3(t_2 + u_2) = 0$$
$$l_3 = p_1(t_1 - t_3) + t_1(p_3 - p_1) = t_1p_3 - p_1t_3$$

Substituting for j_i, k_i, l_i, in equation (923), the alternate form of equation (923) becomes;

$$(j_1 + l_1 \tan^2\phi) \tan^2\theta + (k_2 \tan\phi) \tan\theta + l_3 \tan^2\phi = 0 \quad \text{--(933)}$$

where $j_1 = t_2q_3 - q_2t_3 + t_1q_2$

$l_1 = u_2p_3 - u_2p_1 - u_3p_2$

$l_3 = t_1p_3 - p_1t_3$; $k_2 = l_1 - l_3 = j_1$.

Ex(9.03): Design a 4-bar mechanism to mechanize the function $y = x^2$ between the limits $x = 1$ and $x = 6$.

Solution: Choosing input crank range, $\Delta\theta$, to be 60° and output crank range, $\Delta\phi$, to be 70°, then

$$K_x = \frac{\Delta x}{\Delta\theta} = \frac{6 - 1}{60} = 0.833 \text{ units/deg.}$$

$$K_y = \frac{\Delta y}{\Delta\phi} = \frac{36 - 1}{-70} = -0.5 \text{ unit/deg.}$$

where $x_i = x_o + K_x(\theta_i - \theta_o)$

and $y_i = y_o + K_y(\phi_i - \phi_o)$

$$n_1 = \frac{d\phi}{d\theta} = \frac{K_x}{K_y} \cdot \frac{dy}{dx} = \frac{K_x}{K_y}(2x)$$

$$n_2 = \frac{d^2\phi}{d\theta^2} = \frac{K_x^2}{K_y} \cdot \frac{d^2y}{dx^2} = \frac{K_x^2}{K_2}(2)$$

$$n_3 = \frac{d^3\phi}{d\theta^3} = \frac{K_x^3}{K_y} \cdot \frac{d^3y}{dx^3}$$

$$= 0$$

at $x = 4$,

$$n_1 = -1.333$$

$$n_2 = -0.028$$

$$n_3 = 0.$$

The table of values for p_i, q_i, t_i, u_i, then becomes;

i	p_i	q_i	t_i	u_i
1	1.333	0	0.333	0
2	0.028	- 1.777	- 0.028	- .111
3	2.369	0.112	- 0.037	- .028

Stating equation (933);

$$(j_1 + l_1 \tan^2\phi)\tan^2\theta + (k_2 \tan\phi)\tan\theta + l_3 \tan^2\phi = 0$$

and substituting for j_1, l_1, l_3, k_2,

$$j_1 = -.660626$$

$$l_1 = -.114212$$

$$l_3 = .838198$$

$$k_2 = -.291784$$

for $\phi = +45^\circ$, equation (933) becomes

$$-.775 \tan^2\theta - .292 \tan\theta + .838 = 0 \quad \text{---------(934)}$$

Solving equation (934),

$$\theta = 40^\circ, 129^\circ, 220^\circ, 309^\circ.$$

For one of the solutions, $\theta = 40^\circ$,

$$R_1 \cos 45^\circ - R_2 \cos 40^\circ + R_3 = \cos(-5^\circ) \text{ ----------(a)}$$

$$1.333\, R_1 \sin 45^\circ + R_2 \sin 40^\circ = 0.333 \sin(-5^\circ) \text{ ------(b)}$$

$$-1.749\, R_1 \sin 45^\circ + R_2 \cos 40^\circ = +.028 \sin 5^\circ - .111 \cos 5^\circ \text{--(c)}$$

Solving equations (a) to (c) for R_1, R_2, R_3,

$$R_1 = 0.668$$

$$R_2 = -0.9$$

$$R_3 = -.169$$

Assuming $a = 10$ cm

$$b = \frac{a}{R_1} = 15.0 \text{ cm}$$

$$d = \frac{a}{R_2} = -11.1 \text{ cm}$$

$$c = 19.83 \text{ cm}$$

The derived 4-bar mechanism is shown in Fig. (9.08).

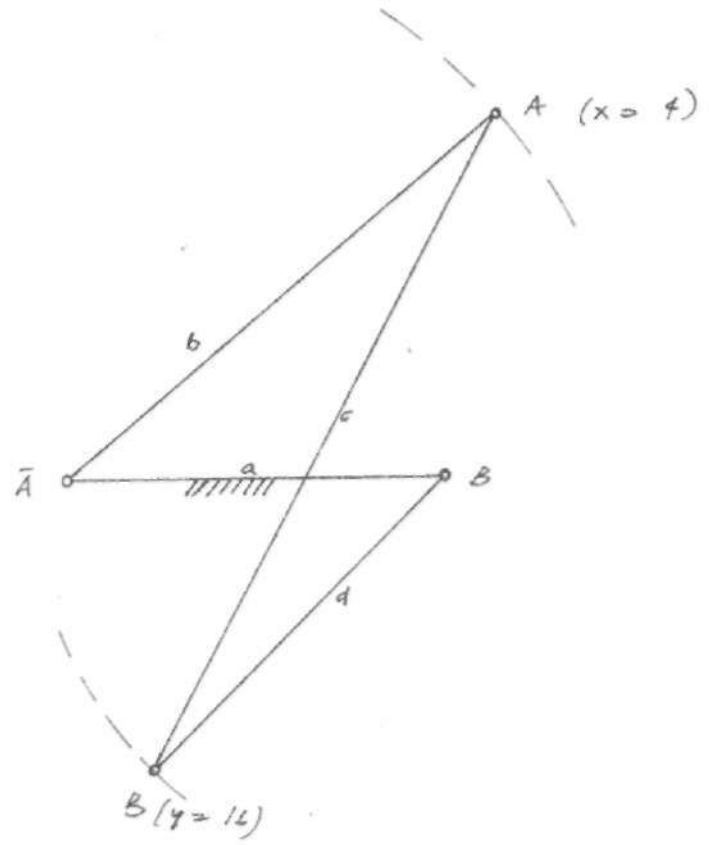


fig. 9·08: 4-bar function-generator for $y = x^2$. $(1 < x < 5)$.

The derived 4-bar mechanism in the present instance does not go through the given ranges of input and output angles. Another value of precision point can then be selected.

9.6. 4-Point Approximation Of Coordinated Crank Displacements:

A 4-bar mechanism designed to generate a given function can be made to have four precision points. It is also possible to have five or six precision points; but at these stages, the calculations become rather cumbersome. It is also theoretically possible to have seven precision points but this necessitates having arbitrary scales for the relationships between the angular displacements of the cranks and the function to be generated.

Writing the frendenstein's equation for four precision points corresponding to four values of θ and ϕ;

$$R_1 \cos \phi_o - R_2 \cos \theta_o + R_3 = \cos(\theta_o - \phi_o) \text{ ---------(a)}$$

$$R_1 \cos \phi_1 - R_2 \cos \theta_1 + R_3 = \cos(\theta_1 - \phi_1) \text{ ---------(b)}$$

$$R_1 \cos \phi_2 - R_2 \cos \theta_2 + R_3 = \cos(\theta_2 - \phi_2) \text{ ---------(c)}$$

$$R_1 \cos \phi_3 - R_2 \cos \theta_3 + R_3 = \cos(\theta_3 - \phi_2) \text{ ---------(d)}$$

Putting $\theta_i = \theta_o + \alpha_i$ and $\phi_i = \phi_o + \beta_i$

with $\gamma_i = \alpha_i - \beta_i$, then equations (a) to (d) can be written as in equation (935)

$$R_1 \cos(\phi_o + \beta_i) - R_2 \cos(\theta_o + \alpha_i) + R_3 = \cos(\theta_o - \phi_o + \gamma_i) \text{ ---(935)}$$

for i = 0, 1, 2, 3.

Eliminating R_1, R_2, R_3, from equation (935),

$$\Omega_i \tan^2 \theta_o + \Omega_2 \tan \theta_o + \Omega_3 = 0 \text{ ----------(936)}$$

where Ω_1, Ω_2, Ω_3, are expressed in terms of ϕ_o.

Rewriting equations (a) to (d),

$$R_1 \cos \phi_1 - R_2 \overset{x}{0} + R_3 = \sin \phi_1 \quad \text{----------(a1)}$$

$$R_1 \cos(\phi_1 + 15) - R_2.(0.5) + R_3 = \cos(45 - \phi_1) \quad \text{------(b1)}$$

$$R_1 \cos(\phi_1 + 45) \; 0 \; R_2.\frac{\sqrt{3}}{2} + R_3 = \cos(\phi_1 + 15) \quad \text{-------(c1)}$$

$$-R_1 \sin \phi_1 - R_2.1 + R_3 = -\sin \phi_1 \quad \text{----------(d1)}$$

Subtracting equation (a1) from equations (b1) to (d1),

$$R_1\left[\cos(\phi_1 + 15) - \cos \phi_1\right] - R_2(0.5) = \cos(45 - \phi_1) - \text{Sin } \phi_1 \text{ --(a2)}$$

$$R_1\left[\cos(\phi_1 + 45) - \cos \phi_1\right] - R_2.(\tfrac{\sqrt{3}}{2}) = \cos(\phi_1 + 15) - \sin \phi_1 \text{ --(b2)}$$

$$-R_1\left[\sin \phi_1 + \cos \phi_1\right] - R_2(1) = -2 \sin \phi_1 \quad \text{----------(c2)}$$

Eliminating R_1 and R_2 in equations (a2) to (c2),

$$\tan^2 \phi_1(0.004) - \tan \phi_1(0.149) + 0.103 = 0 \quad \text{----------(937)}$$

Solving equation (937),

$$\phi_1 = 35^\circ, 88.4^\circ, 215^\circ, 268.4^\circ.$$

Choosing $\phi_1 = 215^\circ$, equations (a) to (d) become;

$$R_1 \cos 215^\circ + R_3 = \sin 215^\circ \quad \text{----------(a3)}$$

$$R_1 \cos 230^\circ - 0.5\, R_2 + R_3 = \cos 170^\circ \quad \text{----------(b3)}$$

$$R_1 \cos 260^\circ - \tfrac{\sqrt{3}}{2} R_2 + R_3 = \cos 230^\circ \quad \text{----------(c3)}$$

$$+R_1 \sin 35^\circ - R_2 + R_3 = \sin 35^\circ \quad \text{----------(d3)}$$

Ex (9.04): Design a 4-bar mechanism to generate the function $y = -\frac{x}{8}(x + 2)$ in the range $0 < x < 6$. Both cranks are to rotate 90° for this range of x.

Solution:

The Chebichev four-point spacing will be advisable for good approximation of function to be generated; but for easier computation, the precision points chosen now are x = 0, 2, 4, 6.

For $\theta_1 = 90^\circ$

i	x_i	y_i	θ_i	ϕ_i
1	0	0	90°	ϕ_1
2	2	- 1	60°	$\phi_1 + 15^\circ$
3	4	- 3	30°	$\phi_1 + 45^\circ$
4	6	- 6	0°	$\phi_1 + 90^\circ$

The design is considered for opposite rotation of the cranks.

Writing Freudenstein's equation for the four precision points;

$$R_1 \cos \phi_1 - R_2 \cos 90 + R_3 = \cos(90 - \phi_1) \text{ ----------(a)}$$

$$R_1 \cos(\phi_1 + 15) - R_2 \cos 60 + R_3 = \cos(60 - \phi_1 - 15) \text{--(b)}$$

$$R_1 \cos(\phi_1 + 45) - R_2 \cos 30 + R_3 = \cos(30 - \phi_1 - 45) \text{--(c)}$$

$$R_1 \cos(\phi_1 + 90) - R_2 \cos 0 + R_3 = \cos(0 - \phi_1 - 90) \text{--(d)}$$

Solving equations (a3),(b3), and (d3) for R_1, R_2, R_3;

$$-.82\,R_1 + R_3 = -.574 \quad \text{----------(a4)}$$

$$-.643\,R_1 - 0.5\,R_2 + R_3 = -.985 \quad \text{----------(b4)}$$

$$.574\,R_1 - R_2 + R_3 = .574 \quad \text{----------(c4)}$$

Solving equations (a4) to (c4),

$$R_1 = 2.0$$

$$R_2 = 1.64$$

$$R_3 = 1.066$$

If a = 10 cm,

then b = 5 cm, d = 6.1 cm and c = 9.86 cm.

The derived 4-bar mechanism is shown in Fig. (9.09)

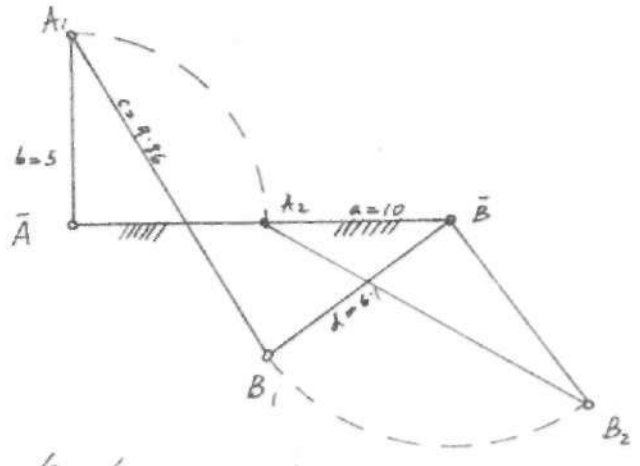


fig. 9.09 : 4-bar mechanism designed to generate the function $y = -\frac{x}{8}(x+2)$ with four precision points.

9.7. 5-Point Approximation:

Five point approximation can be carried out for the generation of a given function by considering two separate sets of four precision points. The frendenstein equation is written for the five precision points as follows;

$$R_1\cos(\phi_0+\beta_i)-R_2\cos(\theta_0+\alpha_i)+R_3=\cos(\theta_0-\phi_0+\gamma_i) \quad \text{-----(938)}$$

for i = 0, 1, 2, 3, 4.

There are four ways of selecting a set of four from the five equations. Two sets are solved; eliminating R_1, R_2, R_3, from each set and the two resulting equations solved for θ_0 and ϕ_0.

For a given function,

$$y = f(x), \quad x_i \leq x \leq x_f \quad \text{----------(939)}$$

where x_i and x_f are the initial and final values of x; the ranges for the input and output cranks, $\Delta\theta$ and $\Delta\phi$ respectively, are selected. The ranges should be kept not more than 90 degrees in order to achieve a good approximation. The cranks could be made to turn in the same direction or in opposite direction, whichever is seen to give a better approximation. The Chebichev five-point precision spacing is advised.

CHAPTER TEN

PATH CURVATURE

10.1. Poles And Centrodes:

As shown when discussing two-link position theory, a lamina may be moved from one given position to another by rotating about a fixed point, the pole. When the displacement of a link is infinitesimal, the pole coincides with instantaneous centres of rotation.

Considering the 4-bar mechanism shown in Fig. (10.01), instantaneous centres I_{13} and I_{24} change their locations with angular displacement of the cranks. Consequently the coupler, link 3, will trace a path of instantaneous centres relative to the fixed link. The locus of the locations of the instantaneous centre I_{13} is the fixed (polode) centrode of link 3 relative to link 1. The moving centrode (polode) is the locus of the instantaneous centres, I_{31}, of link 1 relative to 3; in this case link 3 is considered the fixed link.

Similarly, the moving and fixed polodes of link 2 relative to link 4 can be found. The motion of one link to another can be described as the rolling of the moving polode to the fixed polode of the respective links.

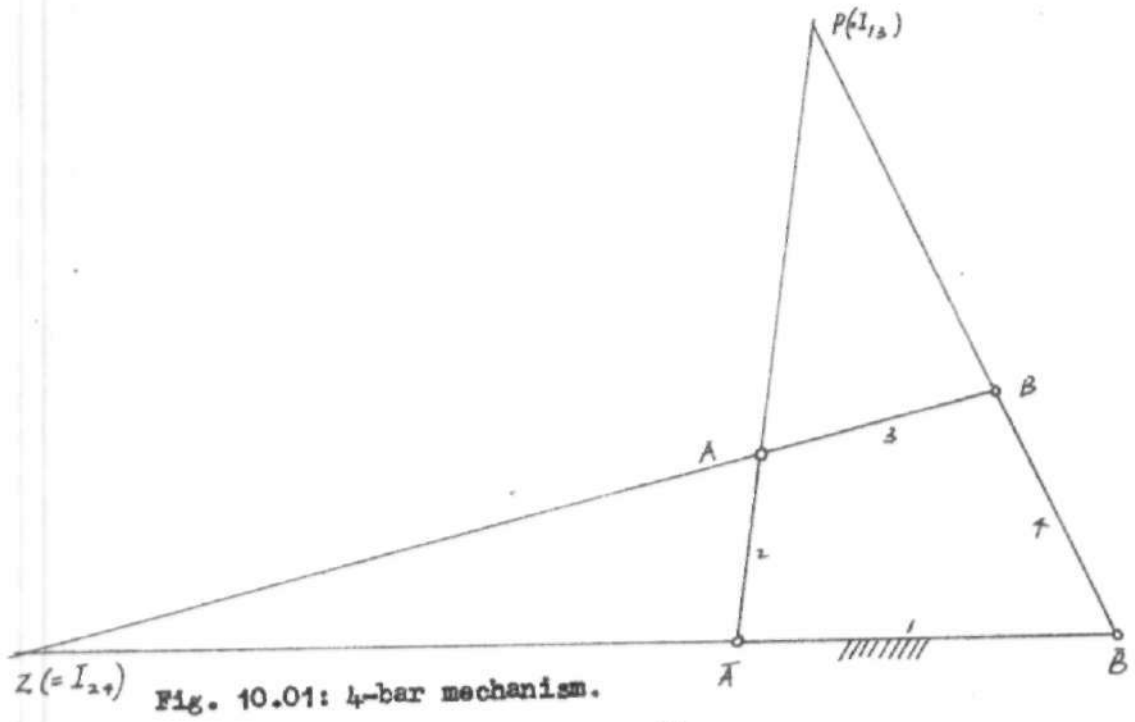


Fig. 10.01: 4-bar mechanism.

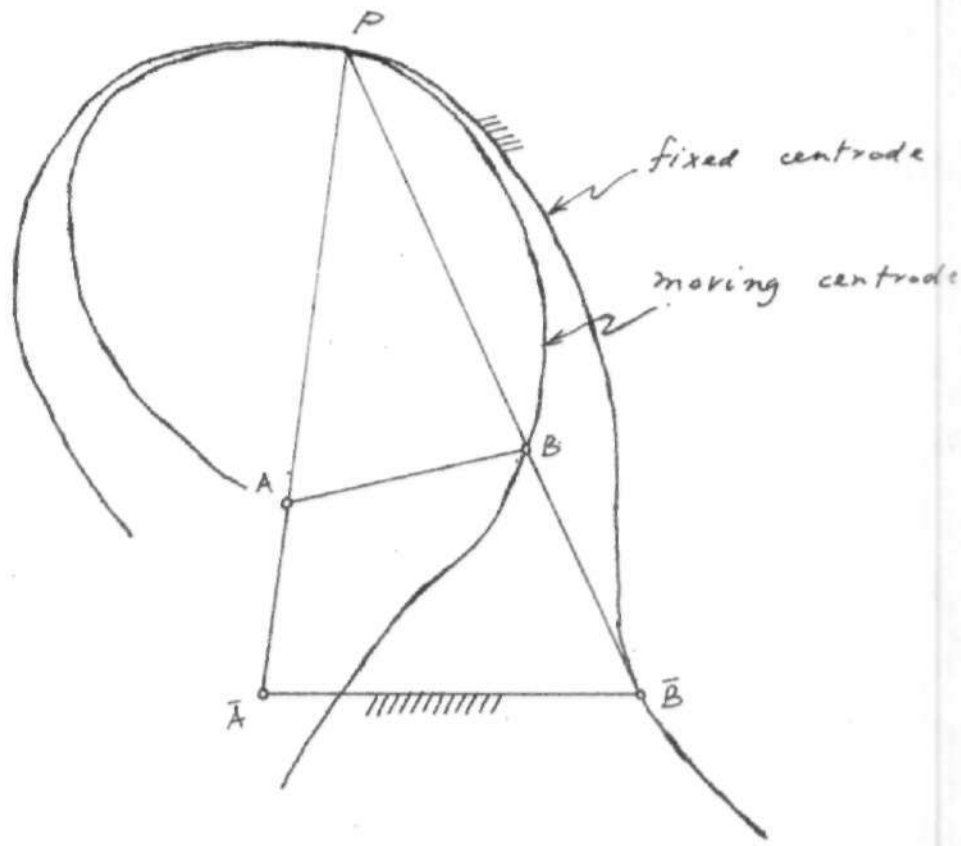


Fig. 10.02: Fixed and moving centredes of coupler motion relative to the fixed link.

The relative motion of two rigid bodies is equivalent to the rolling of two curves, the centrodes (or polodes). The point of contact between the centrodes is the instantaneous centre (pole) (Fig. 10.03).

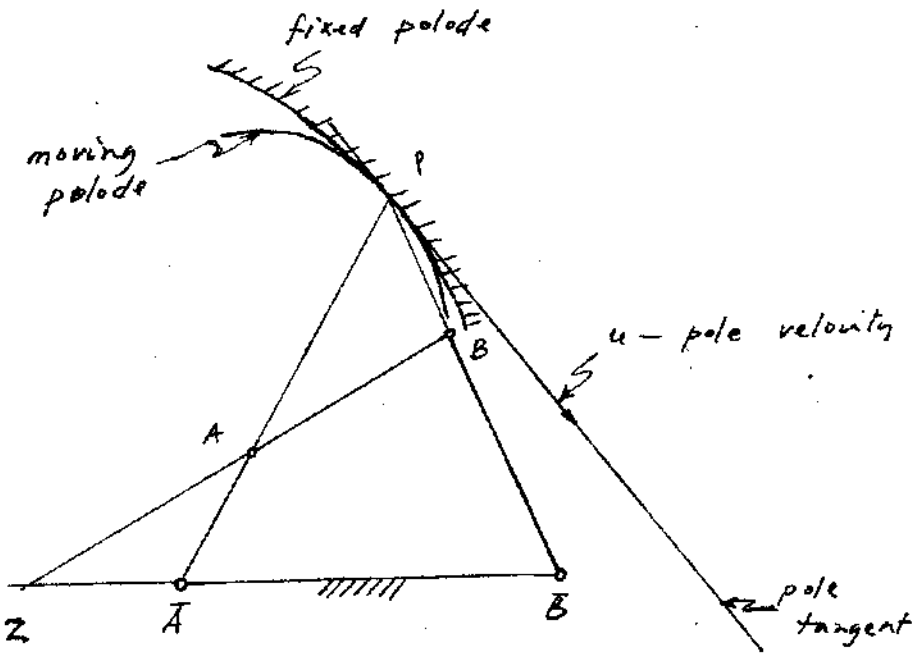


Fig. 10.03.

The pole tangent is the common tangent to the fixed and moving polodes at the pole, P (Fig. 10.03). The pole velocity is the instantaneous velocity at which the pole shifts position.

10.02. Equations For The Centrodes (Polodes):

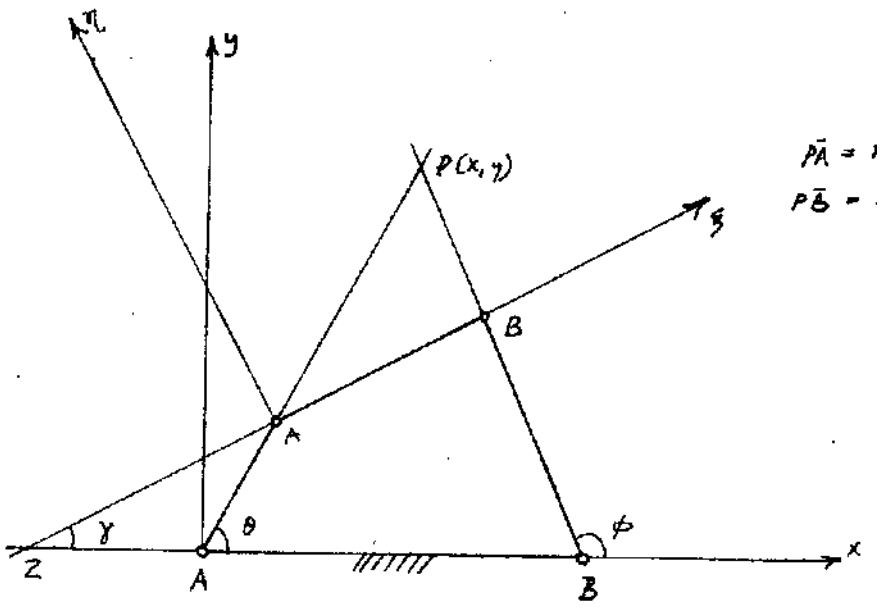


fig. 10.04.

(a) The equation of the fixed centrode (polode) can be derived, using the parameters of Fig. (10.04), as follows;

$$x^2 + y^2 = r_1^2 \text{ ----------(1001)}$$

and $$(x - a)^2 + y^2 = r_2^2 \text{ ----------(1002)}$$

$$/AB/^2 = c^2 = (x_b - x_a)^2 + (y_b - y_a)^2 \text{ ------(1003)}$$

$$x_a = \frac{b}{r_1} x \; ; \; y_a = \frac{b}{r_1} y$$

$$x_b = a\left(\frac{r_2 - d}{r_2}\right) + \frac{d}{r_2} x \; ; \; y_b = \frac{d}{r_2} y \text{ ----------(1003a)}$$

Substituting in equation (1003),

$$\left[a\left(\frac{r_2 - d}{r_2}\right) + x\left(\frac{d}{r_2} - \frac{b}{r_1}\right)\right]^2 + y\left[\frac{d}{r_2} - \frac{b}{r_1}\right]^2 = c^2 \quad \text{-----(1004)}$$

$$\therefore \quad \frac{a^2(r_2 - d)^2}{r_2^2} + 2ax\left(\frac{r_2 - d}{r_2}\right)\left(\frac{dr_1 - br_2}{r_1 r_2}\right) + \frac{x^2(dr_1 - br_2)^2}{r_1^2 r_2^2}$$

$$+ \frac{y^2(dr_1 - br_2)^2}{r_1^2 r_2^2} = c^2 \quad \text{----------(1004a)}$$

From equations (1001) and (1002),

$$x = \frac{r_1^2 - r_2^2 + a^2}{2a}$$

$$\text{and} \quad y_2 = r_1^2 - \frac{(r_1^2 - r_2^2 + a^2)^2}{4a^2} \quad \text{----------(1005)}$$

Substituting for r_1 and r_2 in equation (1004) gives the equation of the fixed polode in cartesian form.

$$\frac{a^2\left[\left\{(x-a)^2 + y^2\right\}^{\frac{1}{2}} - d\right]^2}{(x-a)^2 + y^2} + \frac{2ax\left[\left\{(x-a)^2 + y^2\right\}^{\frac{1}{2}} - d\right]}{\left\{(x-a)^2 + y^2\right\}^{\frac{1}{2}}}$$

$$\times \frac{\left[d\left\{x^2 + y^2\right\}^{\frac{1}{2}} - b\left\{(x-a)^2 + y^2\right\}^{\frac{1}{2}}\right]}{\left\{x^2 + y^2\right\}^{\frac{1}{2}}\left\{(x-a)^2 + y^2\right\}^{\frac{1}{2}}}$$

$$+ \frac{x^2\left[d\left\{x^2 + y^2\right\}^{\frac{1}{2}} - b\left\{(x-a)^2 + y^2\right\}^{\frac{1}{2}}\right]^2}{(x^2 + y^2)\left[(x-a)^2 + y^2\right]}$$

$$+ \frac{y^2\left[d\left\{x^2+y^2\right\}^{\frac{1}{2}} - b\left\{(x-a)^2+y^2\right\}^{\frac{1}{2}}\right]^2}{(x^2+y^2)\left[(x-a)^2+y^2\right]} = c^2 \quad \text{-----(1006)}$$

Equation (1006) can be written as

$$(x^2+y^2)\left[d\left\{x^2+y^2\right\}^{\frac{1}{2}} - b\left\{(x-a)^2+y^2\right\}^{\frac{1}{2}}\right]^2$$

$$+ 2ax(x^2+y^2)^{\frac{1}{2}}\left[\left\{(x-a)^2+y^2\right\}^{\frac{1}{2}} - d\right]\left[d\left\{x^2+y^2\right\}^{\frac{1}{2}}\right.$$

$$\left. - b\left\{(x-a)^2+y^2\right\}^{\frac{1}{2}}\right] + a^2\left[\left\{(x-a)^2+y^2\right\}^{\frac{1}{2}} - d\right]^2$$

$$(x^2+y^2) - c^2(x^2+y^2)\left[(x-a)^2+y^2\right] = 0 \quad \text{--------(1007)}$$

Equation for the fixed polode can be written in another form by substituting for x and y as in equation (1005), in equation (1004).

$$\frac{a^2(r_2-d)^2}{r_2^2} + (r_1^2 - r_2^2 + a^2)\left(\frac{r_2-d}{r_2}\right)\left(\frac{dr_1-br_2}{r_1r_2}\right)$$

$$+ \frac{(r_1^2-r_2^2+a^2)^2}{4a^2}\,\frac{(dr_1-br_2)^2}{r_1^2r_2^2} + \left[r_1^2 - \frac{(r_1^2-r_2^2+a^2)^2}{4a^2}\right] x$$

$$\frac{(dr_1-br_2)^2}{r_1^2r_2^2} = c^2 \quad \text{----------(1008)}$$

Equation (1008) then becomes

$$a^2 r_1^2 (r_2 - d)^2 + r_1 (r_1^2 - r_2^2 + a^2)(r_2 - d)(dr_1 - br_2)$$

$$+ r_1^2 (dr_1 - br_2)^2 = c^2 r_1^2 r_2^2 \quad \text{----------(1008a)}$$

(b) For the moving polode, using the parameters of Fig.(10.04), the equation can be derived as follows;

$$\eta^2 + \xi^2 = (r_1 - b)^2 \quad \text{----------(1009)}$$

$$(\xi - c)^2 + \eta^2 = (r_2 - d)^2 \quad \text{----------(1010)}$$

but $(\xi_{\bar{b}} - \xi_{\bar{a}})^2 + (\eta_{\bar{b}} - \eta_{\bar{a}})^2 = a^2$ ----------(1011)

and $\xi_{\bar{a}} = \left(\frac{-b}{r_1 - b}\right)\xi$, $\xi_{\bar{b}} = \xi\left(\frac{d}{r_2 - d}\right) + c\left(\frac{r_2}{r_2 - d}\right)$

$$\eta_{\bar{a}} = \left(\frac{-b}{r_1 - b}\right)\eta \ , \ \eta_{\bar{b}} = \left(\frac{-d}{r_2 - d}\right)\eta \quad \text{----------(1011a)}$$

Substituting in equation (1011),

$$\left[c\left(\frac{r_2}{r_2 - d}\right) + \xi\left\{\left(\frac{d}{r_2 - d}\right) + \left(\frac{b}{r_1 - b}\right)\right\}\right]^2 + \eta^2\left[\left(\frac{d}{r_2 - d}\right) - \left(\frac{b}{r_1 - b}\right)\right]^2$$

$$= a^2 \quad \text{----------(1012)}$$

The equation can be written in terms of ξ and η by substituting for r_1 and r_2. ξ and η can then be substituted for x and y.

From equations (1009) and (1010),

$$2\xi c - c^2 = (r_1 - b)^2 - r_2 - d)^2$$

$$\therefore \quad 2\xi c = (r_1 - b)^2 - (r_2 - d)^2 + c^2$$

$$\xi = \frac{(r_1 - b)^2 - (r_2 - d)^2 + c^2}{2c}$$

$$\eta^2 = (r_1 - b)^2 - \xi^2 \qquad \text{---------- (1013)}$$

$$= (r_1 - b)^2 - \frac{\left[(r_1 - b)^2 - (r_2 - d)^2 + c^2\right]^2}{4c^2}$$

Substituting in equation (1012);

$$\frac{c^2 r_2^2}{(r_2 - d)^2} + \frac{r_2}{(r_2 - d)}\left[\frac{d}{(r_2 - d)} + \frac{b}{(r_1 - b)}\right]\left[(r_1-b)^2-(r_2-d)^2+c^2\right]$$

$$+ (r_1 - b)^2\left[\frac{d}{(r_2 - d)} - \frac{b}{(r_1 - b)}\right]^2 = a^2 \qquad \text{---------- (1014)}$$

This is the equation for the moving polode in terms of r_1 and r_2.

10.03. Bobillier Theorem:

The angle between one of the cranks and the pole tangent is equal to the angle between the 'collineation axis', PZ, and the second crank (Fig. 10.05).

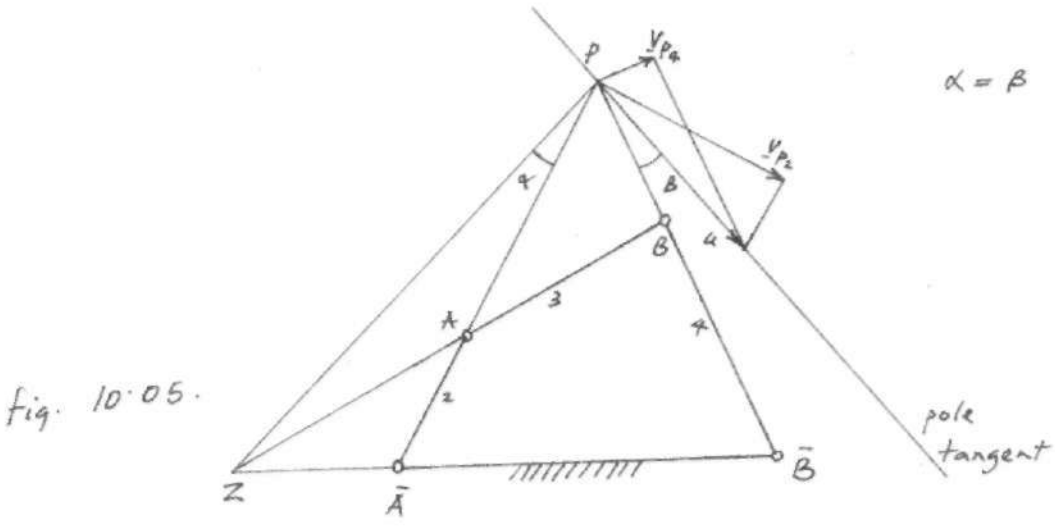


Fig. 10·05.

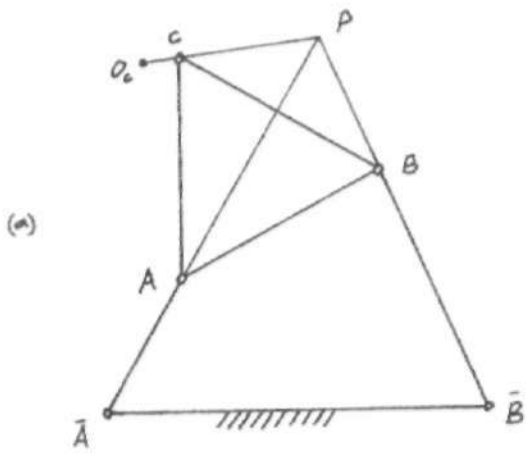
The pole velocity, u, can be found by the construction illustrated in Fig. (10.06).

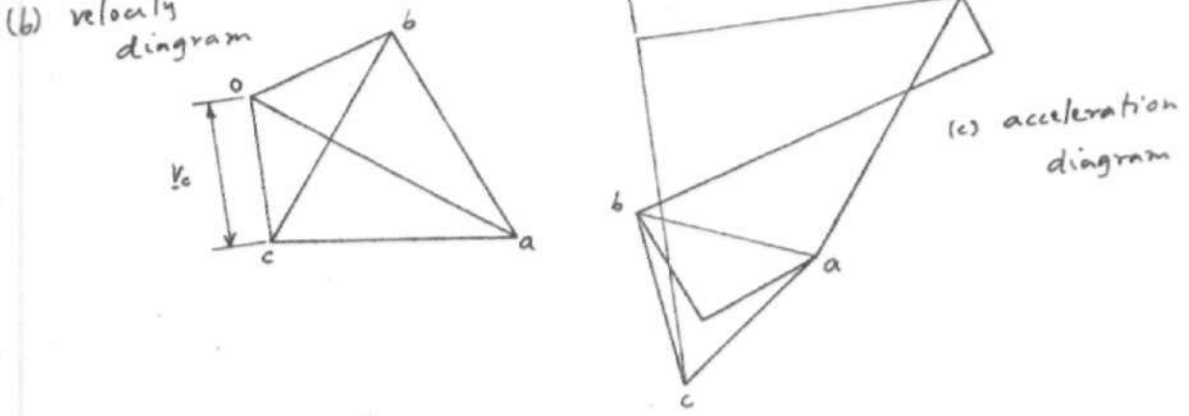
10.4. Radius Of Curvature:

The radius of curvature, p, of the path of a point can be determined from the results of a velocity and acceleration analysis, through the relations;

$$p = \frac{\underline{V}^2}{\underline{a}^n} \qquad \text{----------(1015)}$$

where $\underline{V}$ is the velocity and $\underline{a}^n$ is the normal component of acceleration of the point considered.





$$a_c^t = \frac{V_c^2}{a_c^n}$$

fig. 10·06.

For points on the coupler of a 4-bar mechanism, the point, c, its centre of rotation, O_c, and the pole, all lie in the same line (Fig. 10.06).

10.5. Acceleration Of Point At Pole:

The point of a moving body coincident with the pole has instantaneous acceleration equal to - $u\omega^2$, where u is the pole velocity and ω is the angular velocity of the moving body.

This can be proved by considering the rolling of the polodes. Replacing the polodes with their osculatory circles with centres O and C, the value of the instantaneous acceleration of a point at the pole is readily shown. An 'osculatory circle' or 'circle of curvature' is the circle with centre at the instantaneous centre for the point and passing through the point in consideration. The osculatory circles replacing the polodes are shown in Fig. (10.07).

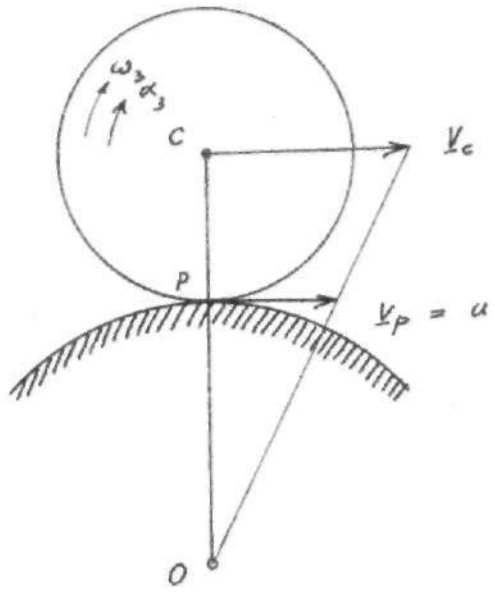


fig. 10·07.

The acceleration of the point at pole, P_2, is given by:

$$\underline{a}_{P_3} = \underline{a}_c^{\,n} \mapsto \underline{a}_{P_{32}}^{\,n} \qquad \text{----------(1016)}$$

$$\text{but } \underline{a}_c^{\,n} = \underline{V}_c^{\,2} / OC \qquad \text{----------(1017)}$$

$$\text{and } \underline{a}_{P_3 c}^{\,n} = (\underline{V}_{P_3 c})^2 / CP = \underline{V}_c^{\,2} / CP$$

$$\therefore \underline{a}_{P_3} = \underline{V}_c^{\,2} \left(\frac{1}{OC} + \frac{1}{CP}\right)$$

$$= + \underline{V}_c \frac{OP}{OC} \cdot \underline{V}_c / CP \qquad \text{----------(1018)}$$

$$= - \underline{V}_P \cdot \omega_3 \qquad \text{----------(1019)}$$

$$= - u \omega_3 \qquad \text{----------(1019a)}$$

Comparing the acceleration polygon for $\underline{a}_{P_3}$ with the polygon for determining the pole velocity u, (Fig. 10.08), it can be seen that the two are similar.

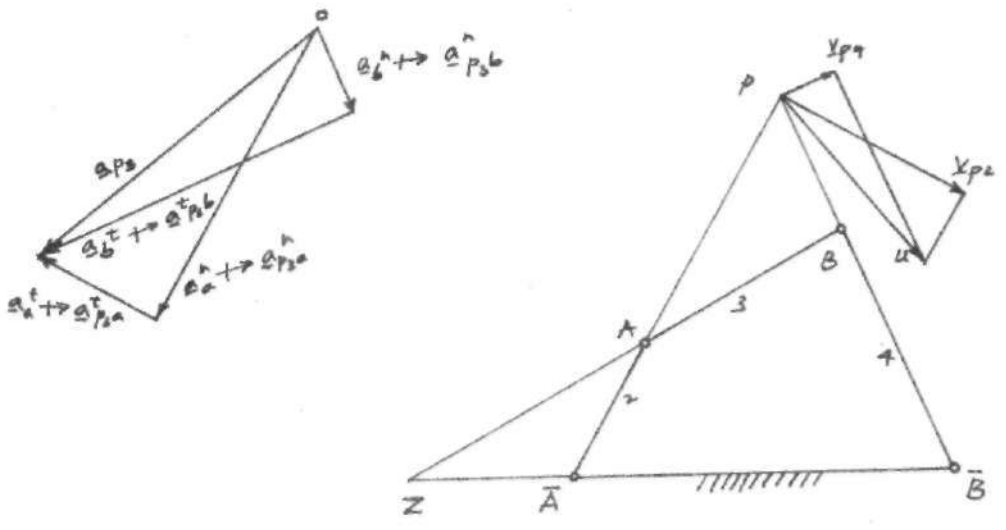


fig. 10.08.

10.6. Acceleration Of Coupler Points In Terms Of Acceleration Of Point At Pole:

The acceleration of any point, C, on the coupler can be written as:

$$\underline{a}_c = -u\omega_3 \rightarrow PC.\omega_3^2 \rightarrow PC.\alpha_3 \quad \text{----------(1020)}$$

where u - pole velocity

ω_3 - angular velocity of the coupler

α_3 - angular acceleration of the coupler

PC - distance from pole to point C.

The normal and tangential components of the acceleration of point C are then given by;

$$\underline{a}_c^{\,n} = -u\omega_3 \cos\gamma_c \rightarrow (PC)\omega_3^2 \quad \text{----------(1021)}$$

$$\underline{a}_c^{\,t} = -u\omega_3 \sin\gamma_c \rightarrow (PC)\alpha_3 \quad \text{----------(1022)}$$

where angle γ_c is the angle between the pole normal and PC (Fig. 10.09).

10.7. The Inflection Circle:

Let J_c be any point having zero normal component of acceleration, then from equation (1021),

$$0 = -u\omega_3 \cos\gamma_c \rightarrow (PJ_c)\omega_3^2 \quad \text{----------(1023)}$$

$$\therefore PJ_c = \frac{u}{\omega_3}\cos\gamma_c \quad \text{----------(1024)}$$

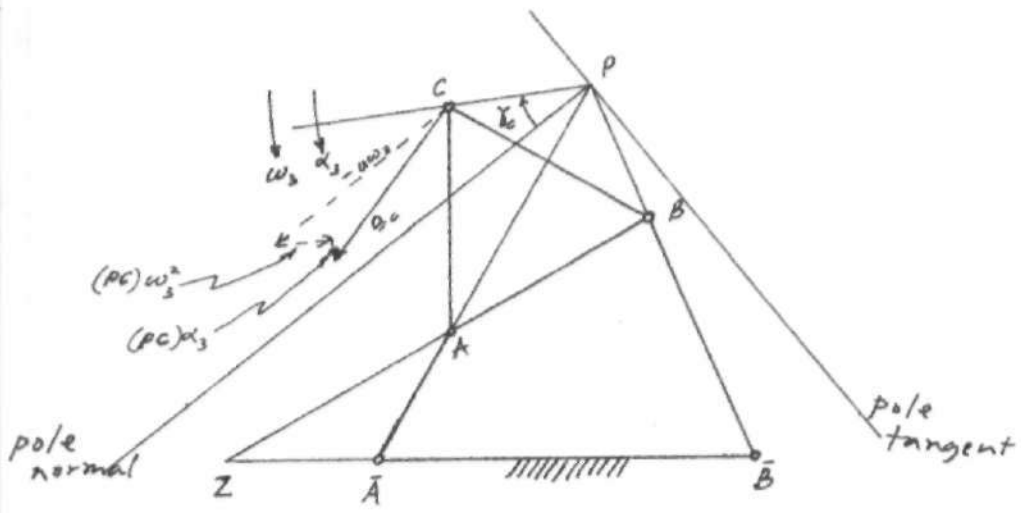


fig. 10·09.

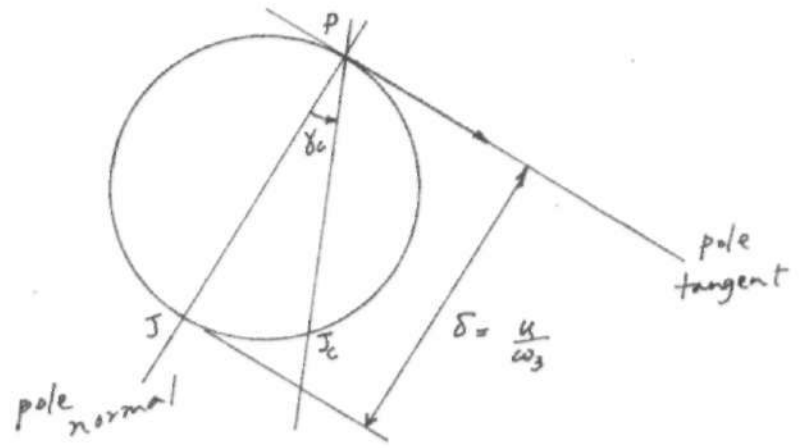


fig. 10·10 : The inflection circle.

Equation (1024) is the equation of a circle, diameter, $\mathcal{J}$, $= \frac{u}{\omega_3} (= PJ)$, termed the inflection circle.

From Fig. (10.10), point U, the 'inflection pole' has a velocity equal to that of the pole velocity, u.

10.8. Points Having Zero Tangential Acceleration:

Let K_c be any point having zero tangential acceleration, then from equation (1022),

$$0 = -u\omega_3 \sin\gamma_c \leftrightarrow (PK_c)\alpha_3 \text{ ----------(1025)}$$

$$\therefore PK_c = \frac{u\omega_3}{\alpha_3} \sin\gamma = PJ\left(\frac{\omega_3^2}{\alpha_3}\right)\sin\gamma \text{ -------(1026)}$$

Equation (1026) is the equation of the circle, diameter $\frac{u\omega_3}{\alpha_3} = PJ\left(\frac{\omega_3^2}{\alpha_3}\right)$, that is the locus of points with zero tangential component of acceleration (Fig. 10.11). The circle is termed the 'Bresse circle'.

The acceleration centre, J_o, is the point with zero acceleration. This is the intersection of the inflection and Bresse circles. The acceleration of any point, C, on the coupler is proportional to its distance from the acceleration centre, J_o.

$$\underline{a}_c = CJ_o(\omega_3^2 \leftrightarrow \alpha_3) \text{ ----------(1027)}$$

The velocity centre, P, also falls on both the inflection and Bresse circles but it has an acceleration, $-u\omega_3$, and this is because point P is at the cusp of its path with the tangent to its path being undefined.

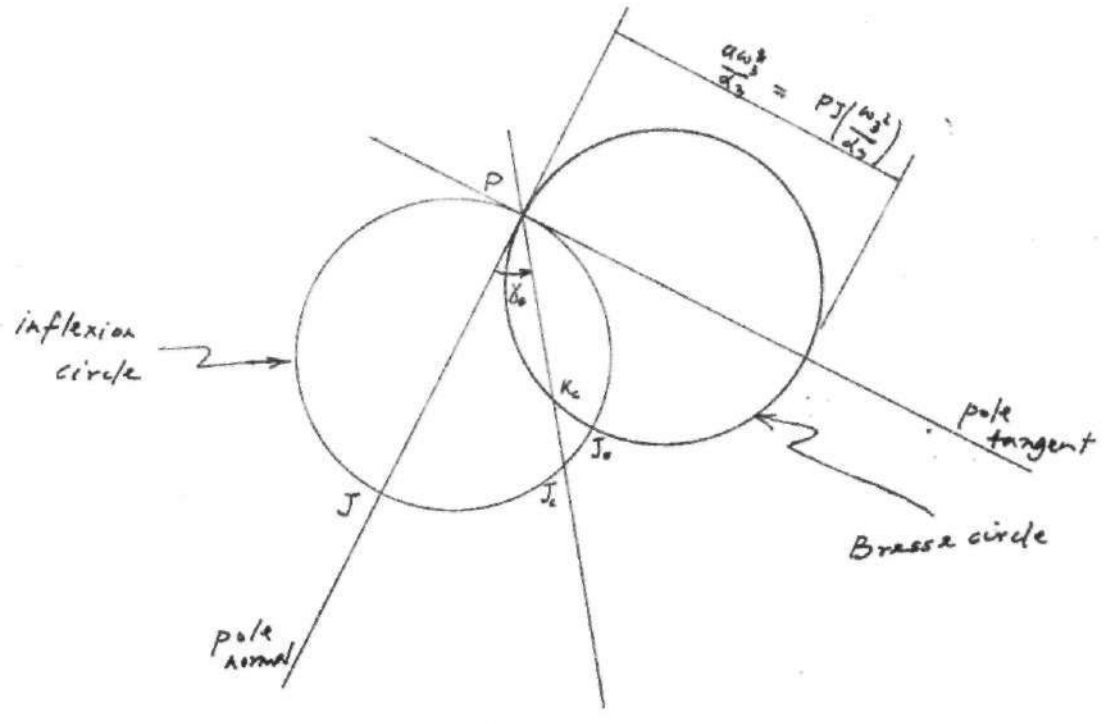


fig. 10·11 : Bresse circle with the inflexion circle, showing the acceleration centre, J_o.

10.9. Euler-Savary Equation:

Let O_c be the centre of curvature of point C on the coupler, then from equation (1015),

$$O_cC = \frac{V_c^2}{a_c^n}$$

$$\therefore O_cC = \frac{(PC)^2\omega_3^2}{-u\omega_3\cos\gamma_o + (PC)\omega_3^2}$$

$$= \frac{(PC)^2}{-\frac{u}{\omega_3}\cos\gamma_c + PC}$$

$$= \frac{(PC)^2}{-PJ_c + PC}$$

$$\therefore \quad O_cC = \frac{(PC)^2}{J_cC} \qquad \text{----------(1028)}$$

Equation (1028) is one form of the Euler-Savary equation.

$$O_cC = -PO_c + PC \qquad \text{----------(1029)}$$

$$= \frac{(PC)^2}{-PJ_c + PC} \qquad \text{----------(1030)}$$

$$\therefore \quad PO_c.PJ_c - PC.PJ_c - PO_c.PC = 0 \qquad \text{----------(1030a)}$$

Equation (1030) gives another form of the Euler-Savary equation as

$$\frac{1}{PC} - \frac{1}{PO_c} = \frac{1}{PJ_c} \qquad \text{----------(1031)}$$

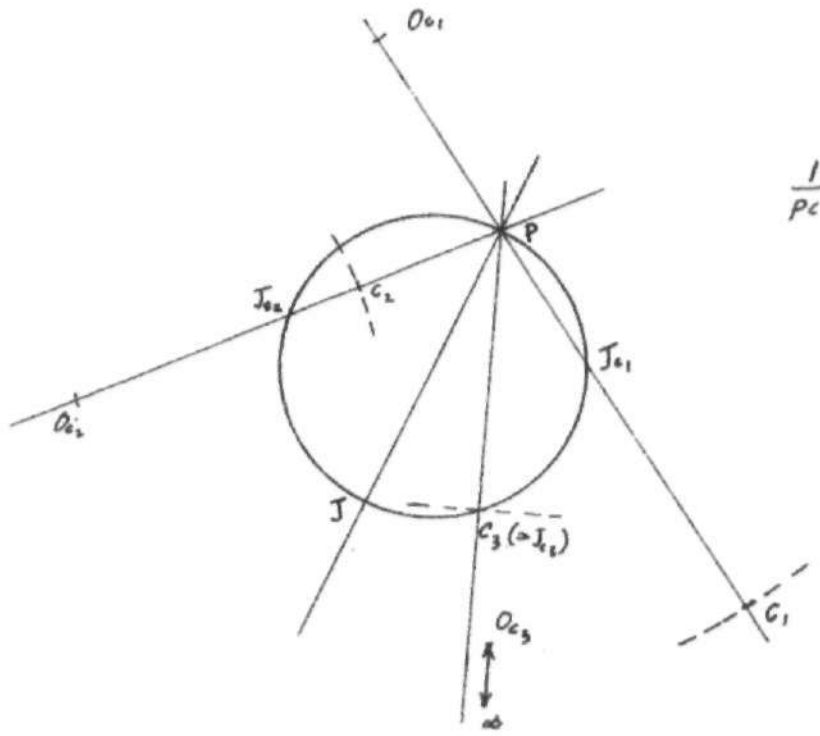
Ex.(10.01):

Determine the inflection circles for the given 4-bar mechanism (Fig. 10.13).

Solution: For point A on the coupler,

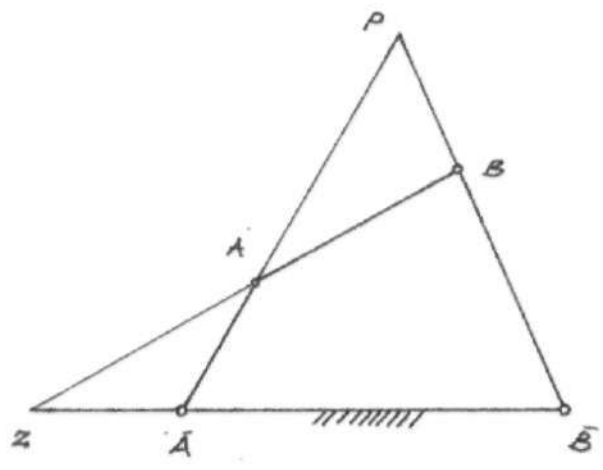
$$\frac{1}{PJ_a} = \frac{1}{PA} - \frac{1}{PO_a} = \frac{1}{PA} - \frac{1}{P\bar{A}}; \quad \left[O_a \equiv \bar{A}\right]$$

$$= \frac{1}{7.5} - \frac{1}{11.5}; \quad \therefore \quad PJ_a = 21.56$$



$$\frac{1}{Pc} - \frac{1}{PO_c} = \bar{P}.$$

fig. 10·12 : Location of centres of curvature using the Euler – Savary equation.



$\bar{A}A$ = 4·0
$\bar{B}B$ = 7·0
PA = 7·5
PB = 3·8
AB = 6·0
$\bar{A}\bar{B}$ = 10·0

fig. 10·13 .

For point B on the coupler,

$$\frac{1}{PJ_b} = \frac{1}{PB} - \frac{1}{PO_b}$$

$$\therefore \frac{1}{PJ_b} = \frac{1}{PB} - \frac{1}{P\bar{B}} ; \quad \left[O_b \equiv \bar{B} \right]$$

Substituting;

$$\frac{1}{PJ_b} = \frac{1}{3.8} - \frac{1}{10.8}$$

$$\therefore PJ_b = 5.86$$

The inflection circle can then be drawn through points P, J_a, J_b.

10.10. Graphical Solutions Of The Euler-Savary Equation:

There are a number of ways of proving the Euler-Savary equation and there also are a number of ways used in solving the equation. The most common of these constructions are the Hartmann and Bobillier constructions.

(a) Hartmann Construction; Knowing the pole velocity u and the velocity of a point, C, on the coupler, the centre of curvature, O_c, of the point is derived. This is a kinematic construction since it involves velocity vector as well as distances.

The construction is shown in Fig. (10.15). From Fig. (10.15),

$$\frac{O_c C}{O_c P} = \frac{V_c}{u \cos \gamma_c} = \frac{(PC)\omega_3}{u \cos \gamma_c} \qquad \text{----------(1032)}$$

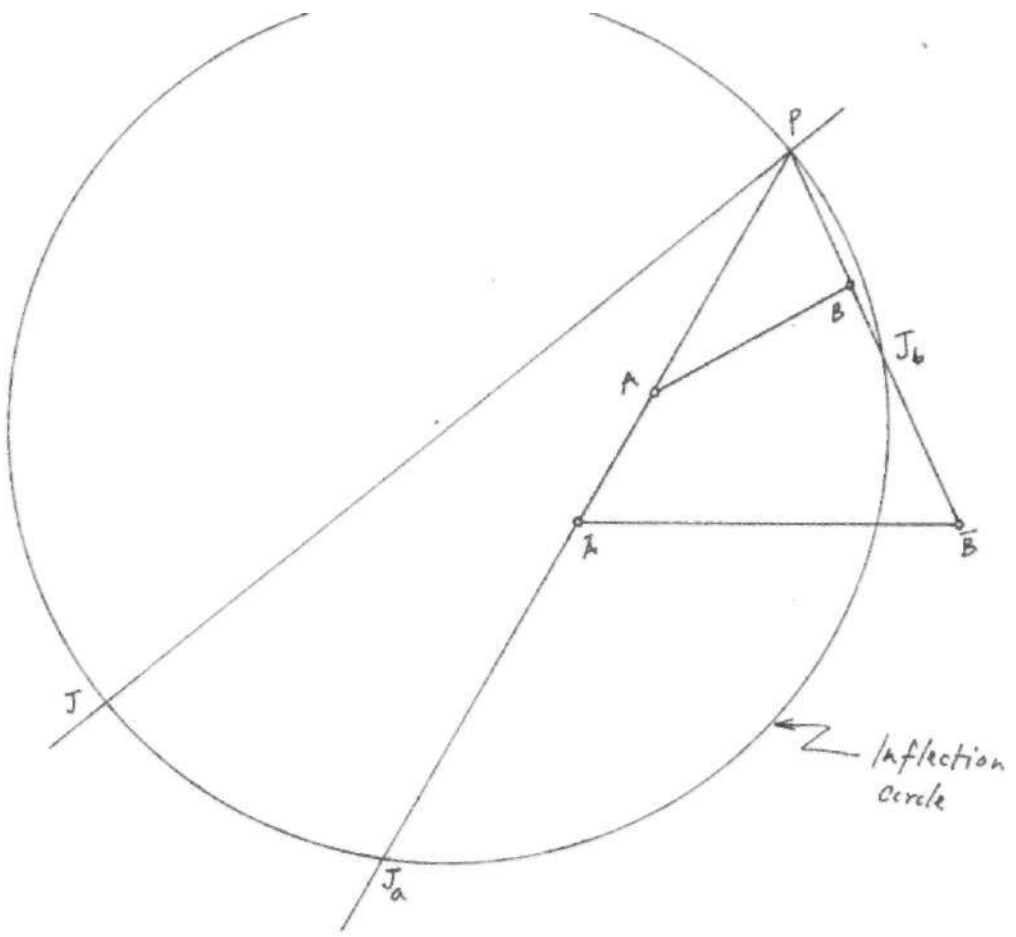


fig. 10·14 : Inflection circle of the 4 - bar mechanism of fig. (10·13).

$$PJ = \frac{u}{\omega_3}, \quad \therefore \quad \omega_3 = \frac{u}{PJ}$$

Equation (1032) then becomes;

$$\frac{O_c C}{O_c P} = \frac{PC}{PJ \cos \gamma_c}, \quad \therefore \quad \frac{O_c C}{O_c P} = \frac{PC}{PJ_c}$$

$$\therefore \quad \frac{O_c P + PC}{O_c P} = \frac{PC}{PJ_c} \qquad \left[PO_c = - O_c P \right]$$

$$\therefore \quad \frac{1}{PC} - \frac{1}{PO_c} = \frac{1}{PJ_c}$$

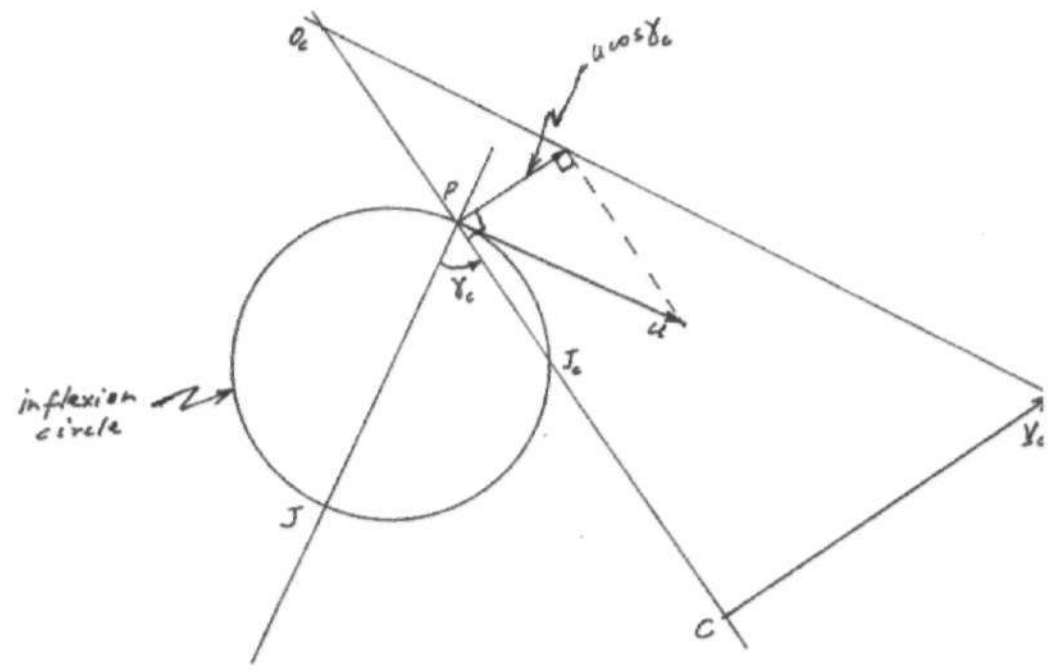


fig. 10·15: The Hartmann construction.

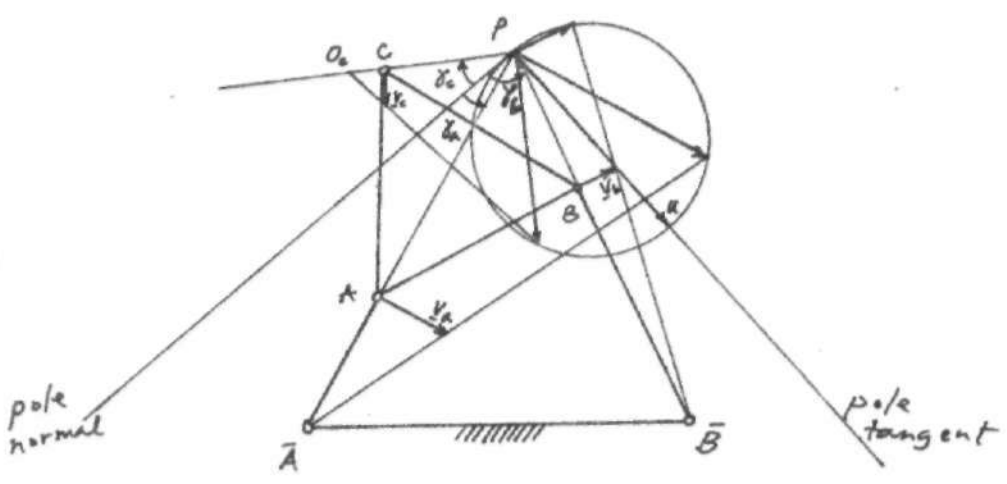


fig. 10·16: Hartmann construction for locating the centre of curvature of a point, C, on the coupler of a 4-bar mechanism.

(b) Bobillier Constructions:

(i) For a given 4-bar mechanism, the inflection circle can be drawn by the use of Bobillier construction as in Fig. (10.17).

Pole P is located and so is instantaneous centre I_{24} located at point Z. PD, parallel to $\bar{A}\bar{B}$, meets ZB at D. Line DJ_aJ_b, parallel to the collineation axis, PZ, meets $P\bar{A}$ at J_a and $P\bar{B}$ at J_b.

The inflection circle is drawn through points P, J_a, J_b.

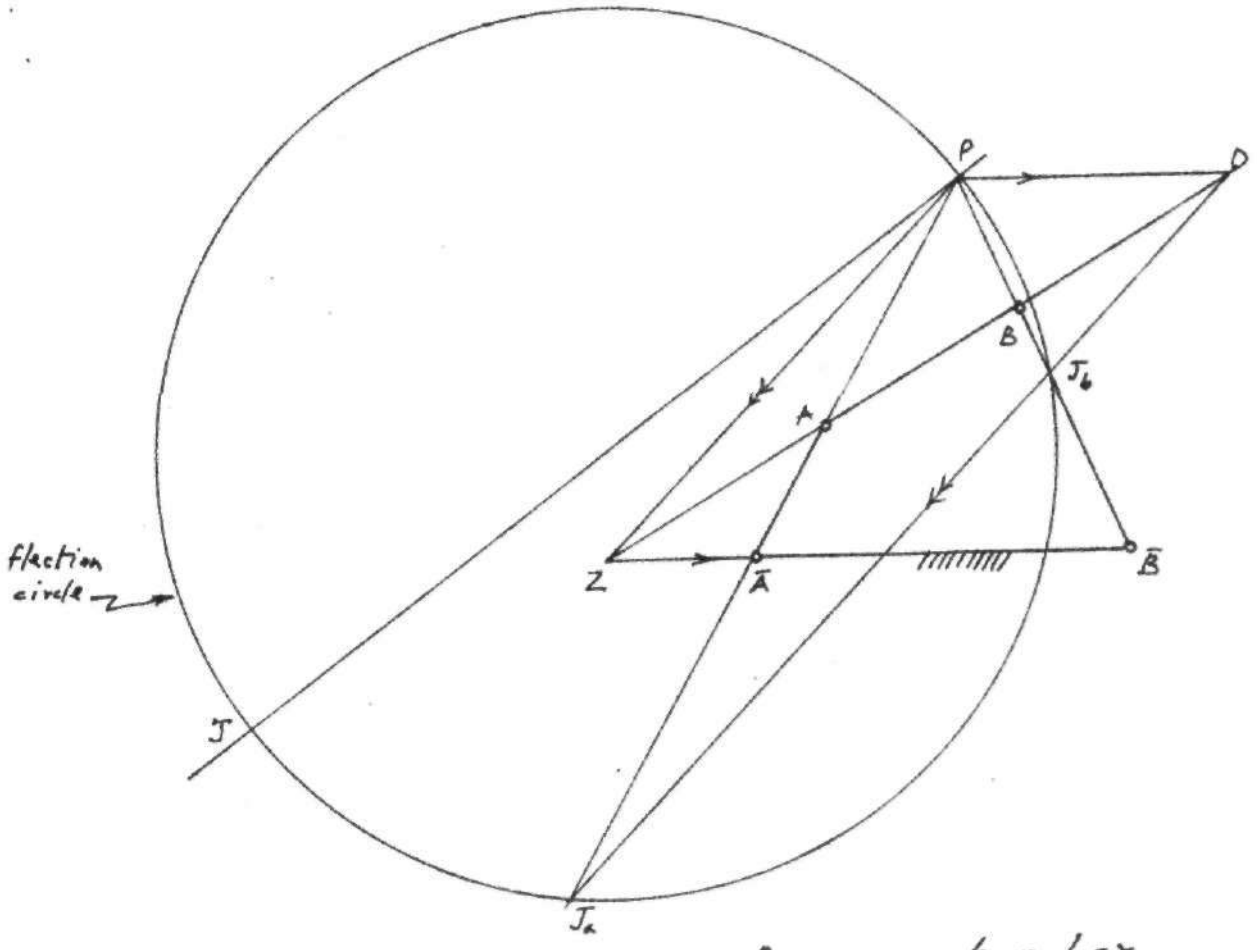


fig. 10.17: Inflection circle for a 4-bar mechanism by Bobillier construction.

(ii) This second Bobillier construction is used to locate the centre of curvature of path for a point on the coupler, when the inflection circle is known.

Considering the example in Fig. (10·18), J_c is located and PD is made parallel to J_cJ and PD meets line CJ at D. DO_c, parallel to JP, meets CP at O_c (Fig. 10.18).

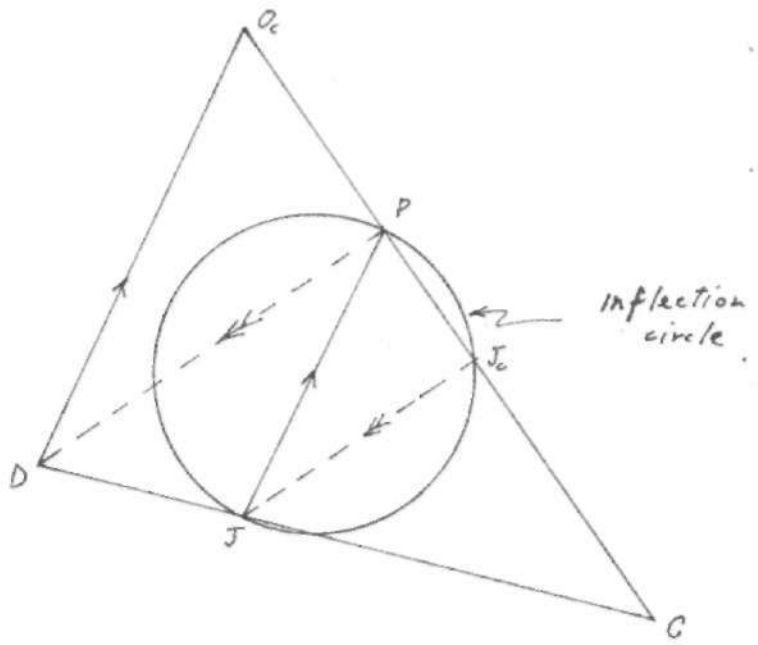


fig. 10·18: Location of centre of curvature by Bobillier construction.

Proof: Triangle CDO_c is similar to triangle CJP

$$\therefore \quad \frac{O_cC}{PC} = \frac{O_cD}{PJ}$$

but $O_cD = O_cP/\cos\gamma_c$

$$\therefore \frac{O_cC}{PC} = \frac{O_cP}{PJ\cos\gamma} = \frac{O_cP}{PJ_c}$$

$$\frac{O_cP + PC}{PC} = \frac{O_cP}{PJ_c}$$

$$\therefore \frac{1}{PC} - \frac{1}{PO_c} = \frac{1}{PJ_c}.$$

10.11. Polode Curvature:

The radius of curvature of the fixed polode, R_f, can be determined from the pole velocity, u, and the corresponding pole acceleration, $\underline{a}_p$.

$$R_f = u^2/a_p^n \text{ ----------(1033)}$$

Considering the 4-bar mechanism of Fig. (10.19), the acceleration of a point at the pole, $\underline{a}_p$, may be found by relating it to the acceleration values of two points on the coupler, A and B; assuming α_3 is zero.

$$\underline{a}_p = \underline{a}_{p_2} \rightarrow \underline{a}_{pp_2} = \underline{a}_{p_2} \rightarrow {}_{pp_2}\omega_2 \rightarrow \underline{a}^t_{pp_2}$$

but $\underline{a}_{p_2} = \underline{a}_a \left(\frac{\bar{AP}}{\bar{AA}}\right)$ and $\underline{a}_a = -u\omega_3 \rightarrow \underline{a}^n_{ap_3}$

$$\therefore \underline{a}_p = -u\omega_3 \cdot \frac{\bar{P}}{\bar{AA}} \rightarrow {}_{pp_2}\omega_2$$

$$\rightarrow \left[\underline{a}^n_{ap_3} \cdot \frac{\bar{AP}}{\bar{AA}} + a^t_{pp_2}\right] \text{ ----------(1034)}$$

Similarly;

$$\underline{a}_p = \underline{a}_{p_4} \rightarrow \underline{a}_{pp_4} = -u\omega_3 \cdot \frac{\overline{BP}}{\overline{BB}} \rightarrow 2\underline{v}_{pp_2}\omega_4$$

$$\rightarrow \left[\underline{a}^{n}_{bp_3} \cdot \frac{\overline{BP}}{\overline{BB}} + a^{t}_{pp_4}\right] \qquad \text{----------(1035)}$$

The pole velocity, u, and the components of velocity, $\underline{v}_{pp_2}$ and $\underline{V}_{pp_1}$, are found as illustrated in Fig. (10.19).

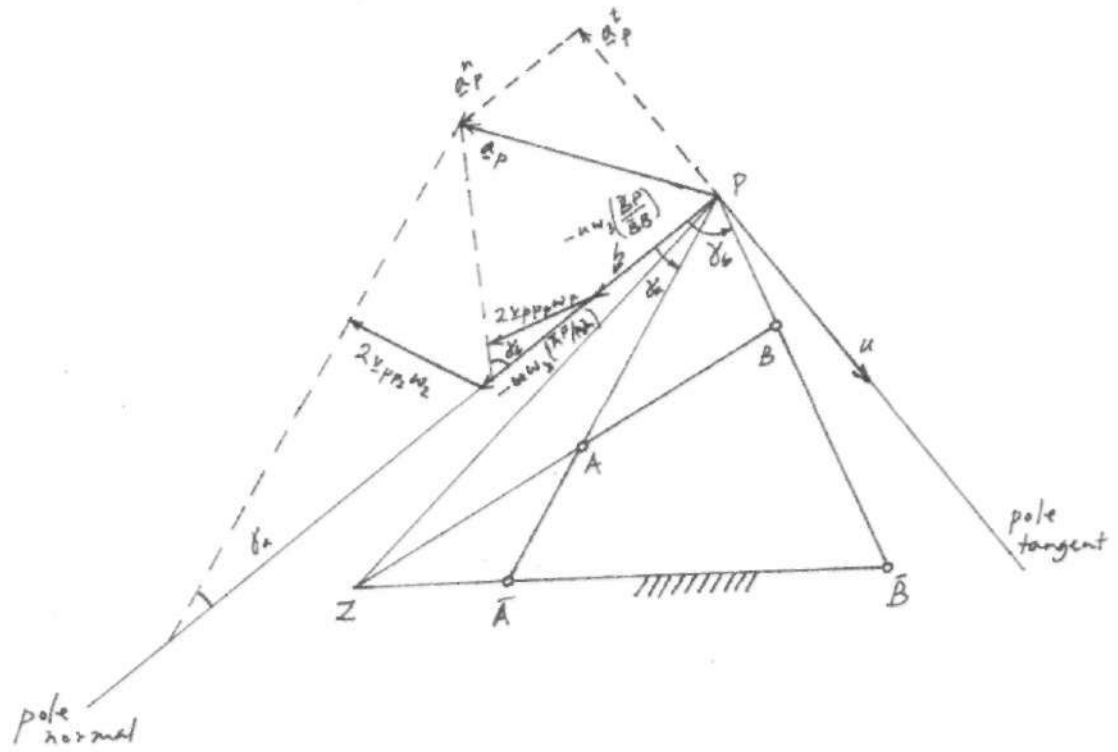


fig. 10·19.

The simultaneous solution of equations (1034) and (1035) is shown in Fig. (10.19). The component in bracket in equation (1034)

is parallel to link $\bar{A}A$ and the component in bracket in equation (1035) is parallel to link $\bar{B}B$.

From Fig. (10.19);

$$\underline{a}_p^n = -u\omega_3 \frac{\bar{A}P}{\bar{A}A} + \frac{2v_{-pp_2}\omega_2}{\sin\gamma_a} - \left[-u\omega_3 \frac{\bar{A}P}{\bar{A}A} + \frac{2v_{-pp_2}\omega_2}{\sin\gamma_a} + u\omega_3 \frac{\bar{B}P}{\bar{B}B} - \frac{2v_{-pp_4}\omega_4}{\sin\gamma_b}\right] \frac{\sin\gamma_b \cos\gamma_a}{\sin(\gamma_b - \gamma_a)} \quad \text{-------(1036)}$$

also,

$$\underline{a}_p^n = -u\ 3\frac{\bar{B}P}{\bar{B}B} + \frac{2v_{-pp_4}\omega_4}{\sin\gamma_b} - \left[-u\omega_3 \frac{\bar{A}P}{\bar{A}A} + \frac{2v_{-pp_2}\omega_2}{\sin\gamma_c} + u\ 3\frac{\bar{B}P}{\bar{B}B} - \frac{2v_{-pp_4}\omega_4}{\sin\gamma_b}\right] \frac{\sin\gamma_a \cos\gamma_b}{\sin(\gamma_b - \gamma_a)} \quad \text{---------(1037)}$$

but

$$\underline{v}_{pp_2} = u \sin\gamma_a;$$

$$\underline{v}_{pp_4} = u \sin\gamma_b;$$

$$\omega_2 = \omega_3 \frac{PA}{\bar{A}A};$$

and $$\omega_4 = \omega_3 \frac{PB}{\bar{B}B}$$

Adding equations (1036) and (1037) and substituting for $\underline{v}_{pp_2}$, $\underline{v}_{pp_4}$, ω_2, ω_4;

$$\frac{2\underline{a}_p^n}{u\ 3} = \left(-\frac{\bar{A}P + 2PA}{\bar{A}A}\right)\left[1 - \frac{\sin\gamma_b \cos\gamma_a}{\sin(\gamma_b - \gamma_a)} - \frac{\sin\gamma_a \cos\gamma_b}{\sin(\gamma_b - \gamma_a)}\right] + \left(-\frac{\bar{B}P + 2PB}{\bar{B}B}\right)\left[1 + \frac{\sin\gamma_b \cos\gamma_a}{\sin(\gamma_b - \gamma_a)} + \frac{\sin\gamma_a \cos\gamma_b}{\sin(\gamma_b - \gamma_a)}\right] \text{---(1038)}$$

but $$\frac{a_p^n}{u\omega_3} = \frac{a_p^n}{u^2} \cdot \frac{u}{\omega_3} = \frac{PJ}{R_f}.$$

Therefore,

$$\frac{2(PJ)}{R_f} = \left[3\frac{PA}{AA} - 1\right](1 - K) + \left[3\frac{PB}{BB} - 1\right](1 + K)$$

where $$K = \frac{\sin(\gamma_b + \gamma_a)}{\sin(\gamma_b - \gamma_a)} \qquad \text{----------(1039)}$$

Equation (1039) gives an expression for determining the radius of curvature of the fixed polode, R_f, at the point of contact of the polodes.

The radius of curvature of the moving polode, R_m, can be determined by the use of the Euler-Savory equation.

$$\frac{1}{R_m} - \frac{1}{R_f} = \frac{1}{PJ}$$

$$\therefore \frac{1}{R_m} = \frac{1}{PJ} + \frac{1}{R_f}$$

$$\therefore \frac{2PJ}{3R_m} = \frac{PA}{AA}(1 - K) + \frac{PB}{BB}(1 + K) \qquad \text{----------(1040)}$$

where $$K = \frac{\sin(\gamma_b + \gamma_a)}{\sin(\gamma_b - \gamma_a)}$$

There are other suitable expressions for the radii of curvature of the fixed and moving polodes. Some are shown in equations (1041) to (1053).

Referring to the notations of Fig. (10.20);

$$\frac{1}{R_f} = \frac{3}{\tan\gamma_a - \tan\gamma_b}\left(\frac{\sin\gamma_a}{P\bar{A}} - \frac{\sin\gamma_b}{P\bar{B}}\right) + \frac{1}{PJ} \qquad \text{------(1041)}$$

and $$\frac{1}{R_m} = \frac{3}{\tan\gamma_a - \tan\gamma_b}\left(\frac{\sin\gamma_a}{PA} - \frac{\sin\gamma_b}{PB}\right) - \frac{1}{PJ} \qquad \text{------(1042)}$$

From Fig. (10.20),

$$\frac{PZ}{\sin\theta} = \frac{P\bar{A}}{\sin(\phi - \theta)} \qquad \text{----------(1043)}$$

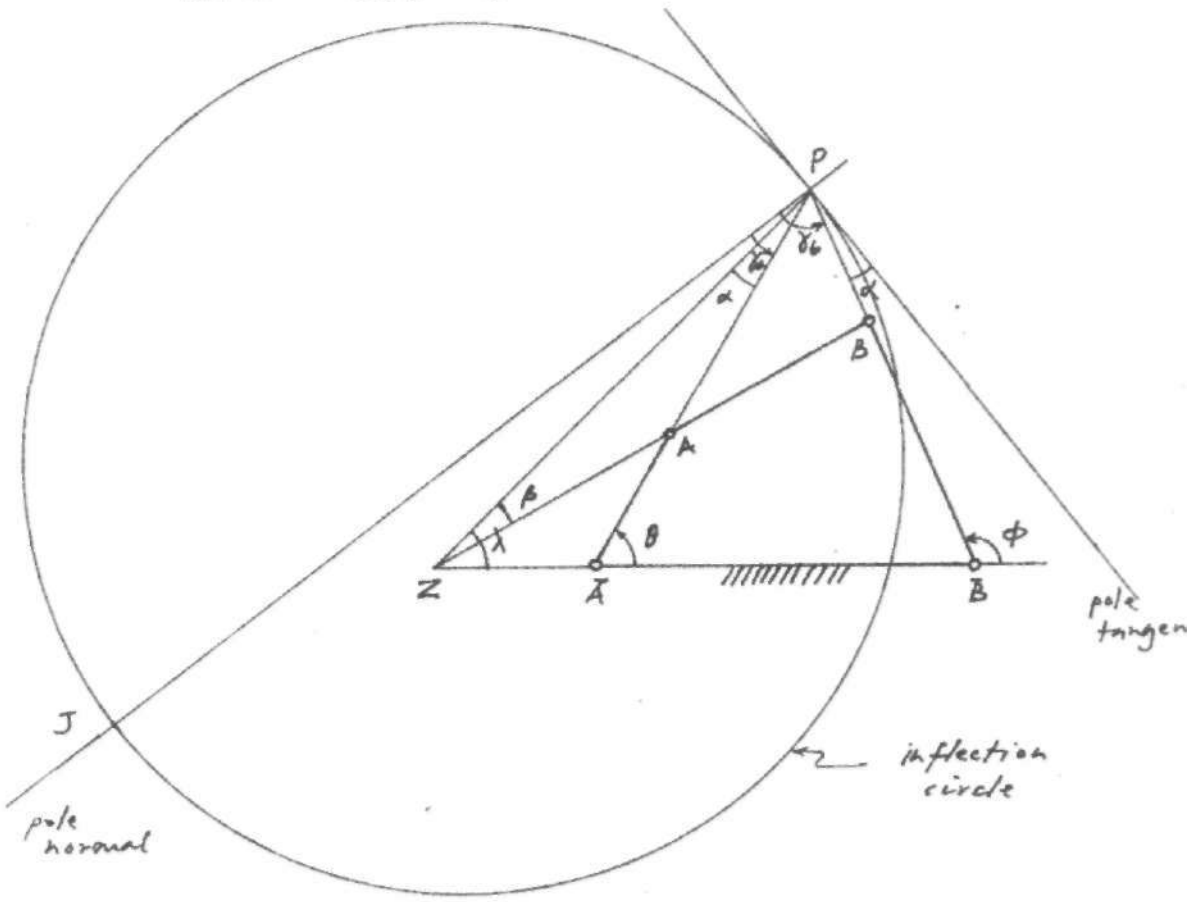


fig. 10.20:

and $\lambda + \alpha = \theta$

$\gamma_b + \alpha = 90^\circ$

$\therefore \theta = 90 - (\gamma_b - \lambda)$

Equation (1043) can then be written as

$$PZ = \frac{P\bar{A}\cos(\gamma_b - \lambda)}{\sin(\phi - \theta)} \qquad \text{----------(1044)}$$

Similarly, $$PZ = \frac{PA\cos(\gamma_b - \beta)}{\sin\beta} \qquad \text{----------(1045)}$$

$$= \frac{P\bar{B}\cos(\gamma_a - \lambda)}{\sin(\phi - \theta)} \qquad \text{----------(1046)}$$

$$= \frac{PB\cos(\gamma_a - \beta)}{\sin\beta} \qquad \text{----------(1047)}$$

Writing the Euler-Savary equations for coupler points A and B,

$$\frac{1}{PJ\cos\gamma_a} = \frac{1}{PA} - \frac{1}{P\bar{A}} \qquad \text{----------(1048)}$$

and $$\frac{1}{PJ\cos\gamma_b} = \frac{1}{PB} - \frac{1}{P\bar{B}} \qquad \text{----------(1049)}$$

Substituting equations (1044) to (1047) in equations (1047) and (1048);

$$PJ = \frac{PZ}{\cos\gamma_a \cos\gamma_b(\cot\lambda - \cot\beta)} \qquad \text{----------(1050)}$$

Equations (1041) and (1042) can then be written as

$$\frac{1}{R_f} = \frac{1}{PJ}\left[\frac{2\tan\beta + \tan\lambda}{\tan\lambda - \tan\beta}\right] \qquad \text{----------(1051)}$$

and $$\frac{1}{R_m} = \frac{1}{PJ}\left[\frac{\tan\beta + 2\tan\lambda}{\tan\lambda - \tan\beta}\right] \qquad \text{----------(1052)}$$

Furthermore, they can be written as

$$R_f = \frac{PJ}{m-1} \quad \text{and} \quad R_m = \frac{PJ}{m}$$

where $$m = \frac{2q+1}{q-1} \qquad \text{----------(1053)}$$

and $$q = \frac{\tan\lambda}{\tan\beta}$$

Ex.(10.02): Design a 4-bar mechanism for specified radii of curvature, as shown in Fig. (10.21). The cylinder is to be constrained to make approximate rolling contact with the flat fixed polode.

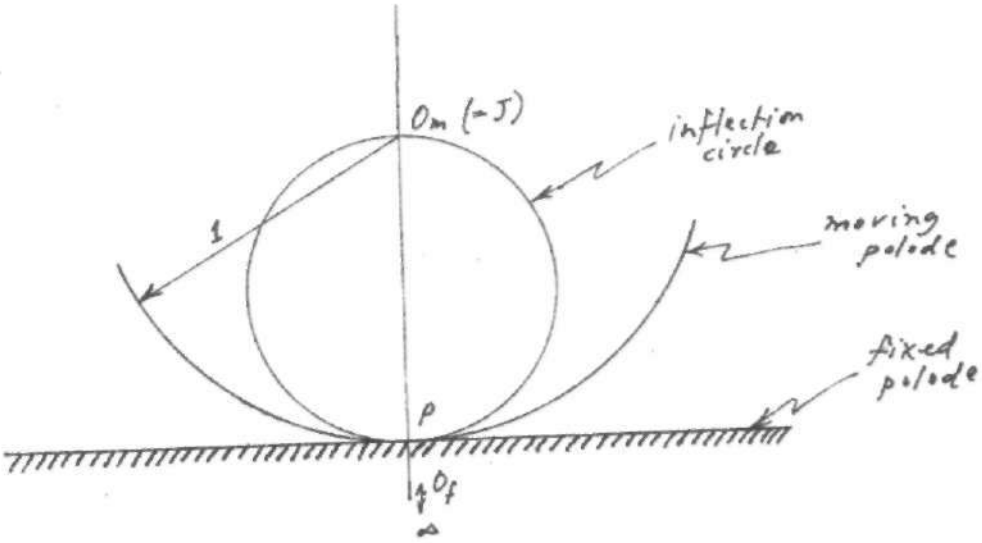


fig. 10·21: Polodes for Ex (10·02).

Solution: The inflection circle is first determined.

Writing the Euler-Savory equation;

$$\frac{1}{PJ} = \frac{1}{PO_m} - \frac{1}{PO_f}$$

$$\frac{1}{PJ} = \frac{1}{1} - \frac{1}{\infty}$$

$$\therefore \quad PJ = 1.$$

Polode curvature equation must also be satisfied;

$$\frac{2(PJ)}{R_f} = \left[3\frac{PA}{\bar{A}A} - 1\right](1 - K) + \left[3\frac{PB}{\bar{B}B} - 1\right](1 + K) \text{---(1054)}$$

where $$K = \frac{\sin(\gamma_b + \gamma_a)}{\sin(\gamma_b - \gamma_a)}$$

Also, the Euler-Savory equation is to be satisfied;

$$\bar{A}A = \frac{PA^2}{PA - PJ\cos\gamma_a} \text{----------(1055)}$$

$$\bar{B}B = \frac{PB^2}{PB - PJ\cos\gamma_b} \text{----------(1056)}$$

There are now three equations with six unknowns; PA, PB, $\bar{A}A$, $\bar{B}B$, γ_a, γ_b.

Assuming, $\gamma_a = 30^\circ$, $\gamma_b = -30^\circ$, PA = PB; from equations (1054), (1055), (1056),

$$K = 0.$$

$$PA = \frac{3\sqrt{3}}{4} = PB$$

$$\bar{A}A = \bar{B}B = \frac{9\sqrt{3}}{4}$$

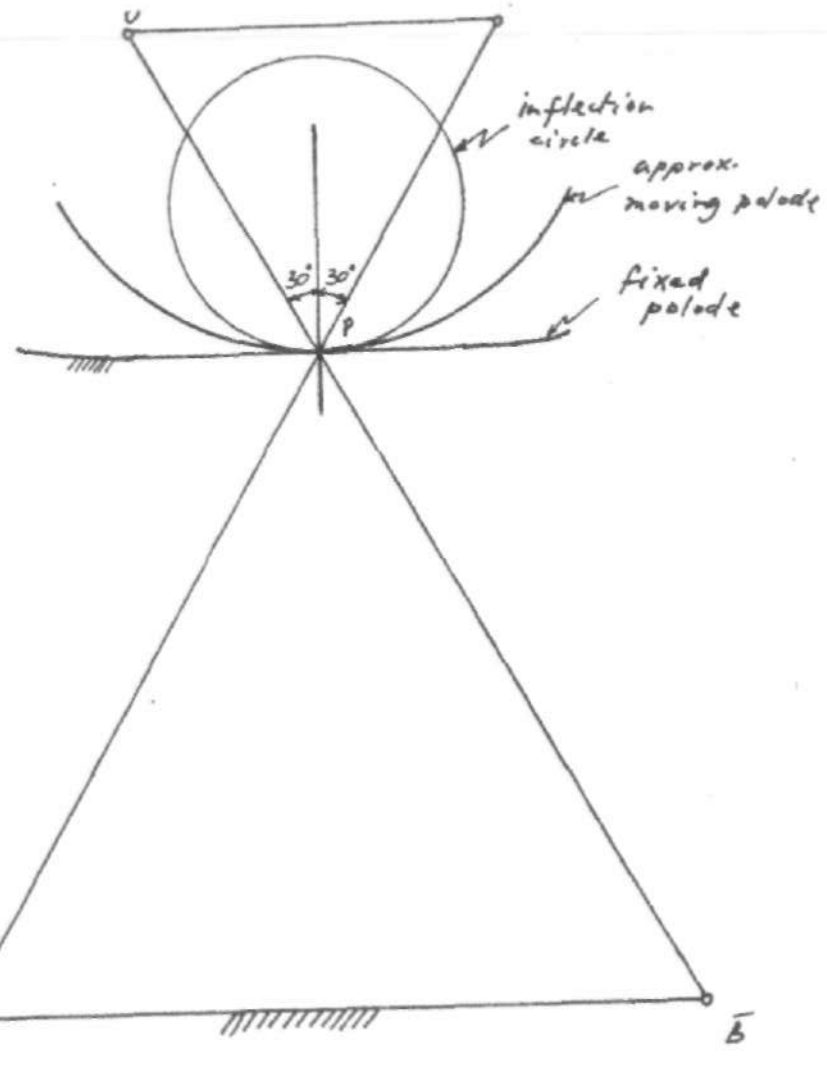


fig. 10.22: 4 - bar mechanism derived for Ex. (10.02).

10.12. Polodes For The Relative Motion Of The Cranks:

The relative motion between the cranks in a 4-bar mechanism is the same as the motion of their polodes.

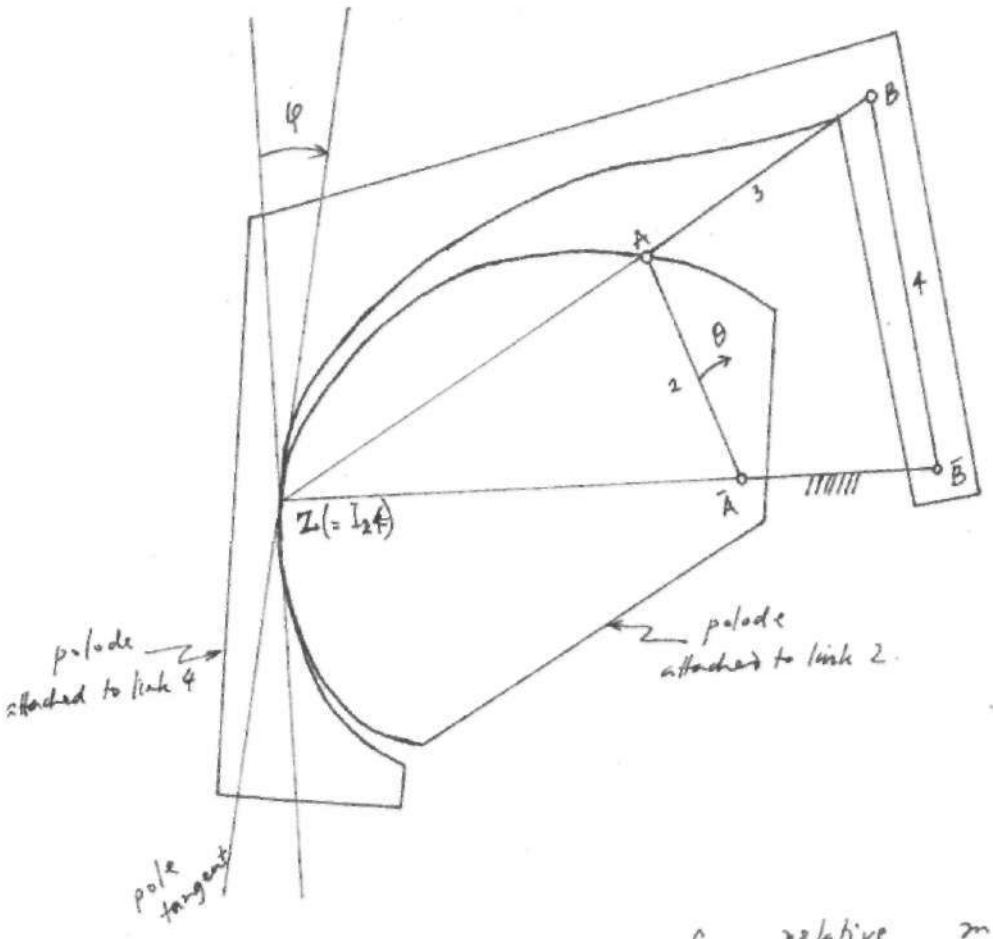


fig. 10·23: Polodes for relative motion of cranks in a 4-bar mechanism.

Fig. (10.24) shows a portion of the polode of link 4 relative to link 2 for a small angular displacement of link 2, $\delta\theta$.

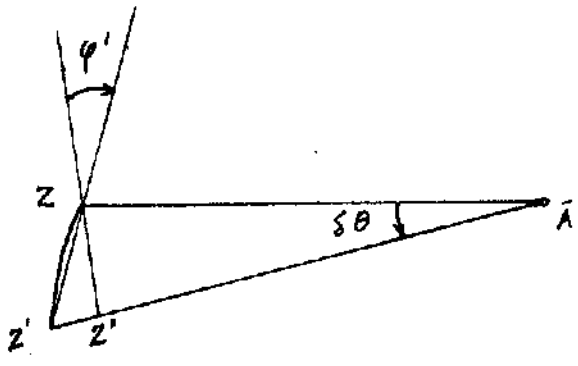


fig. 10.24.

Let $Z\bar{A} = R$, $\frac{\omega_4}{\omega_2} = n$, $\bar{AB} = a$.

From Fig. (10.24);

$$Z'\bar{A} = R + \delta R$$

$$\delta R = R\,\delta\theta \tan\varphi'$$

As $\delta\theta$ tends towards zero and angle φ' tends towards angle φ, angle ZZ"Z' tends to 90°; hence $\frac{dR}{d\theta} = R \tan\varphi$.

$$n = \frac{R}{R + a} \qquad \text{----------(1057)}$$

$$\therefore R = \frac{na}{1 - n} \qquad \text{----------(1058)}$$

Consequently, $\frac{dR}{d\theta} = \frac{a\frac{dn}{d\theta}}{(1 - n)^2}$ ----------(1059)

Therefore, $\tan\varphi = \frac{(dn/d\theta)}{n(1 - n)}$ ----------(1060)

Ex.(10.03): Design a 4-bar mechanism with constant crank angular velocity of 10 rad/s and follower angular velocity of 5 rad/s with an angular acceleration of 20 rad/s^2. The mechanism should have a fixed link length of 1 m.

Solution: $n = \frac{\omega_4}{\omega_2} = \frac{1}{2}$

$$R = \frac{na}{1 - n} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1 \text{ m.}$$

$$\tan \varphi = \frac{(dn/d\theta)}{n(1 - n)}$$

$$\frac{dn}{d\theta} = \frac{dn}{dt} \cdot \frac{dt}{d\theta}$$

$$= \frac{d(\frac{\omega_4}{\omega_2})}{dt} \cdot \frac{1}{\omega_2}$$

$$= \frac{\alpha_4 - n\alpha_2}{\omega_2^2}$$

$$\therefore \frac{dn}{d\theta} = \frac{\alpha_4}{\omega_2^2} \quad \text{since } \alpha_2 = 0.$$

$$= \frac{20}{100} = \frac{1}{5}$$

$$\therefore \tan \varphi = \frac{1/5}{\frac{1}{2} \cdot 2} = \frac{4}{5}; \varphi = 38.66 \text{ degrees.}$$

Writing the Euler-Savary equation for the fixed link $\bar{AB}$,

$$\frac{1}{ZJ^1} = \frac{1}{Z\bar{B}} - \frac{1}{Z\bar{A}} = \frac{1}{2} - \frac{1}{1} = -\frac{1}{2}$$

$$\therefore ZJ^1 = -2.$$

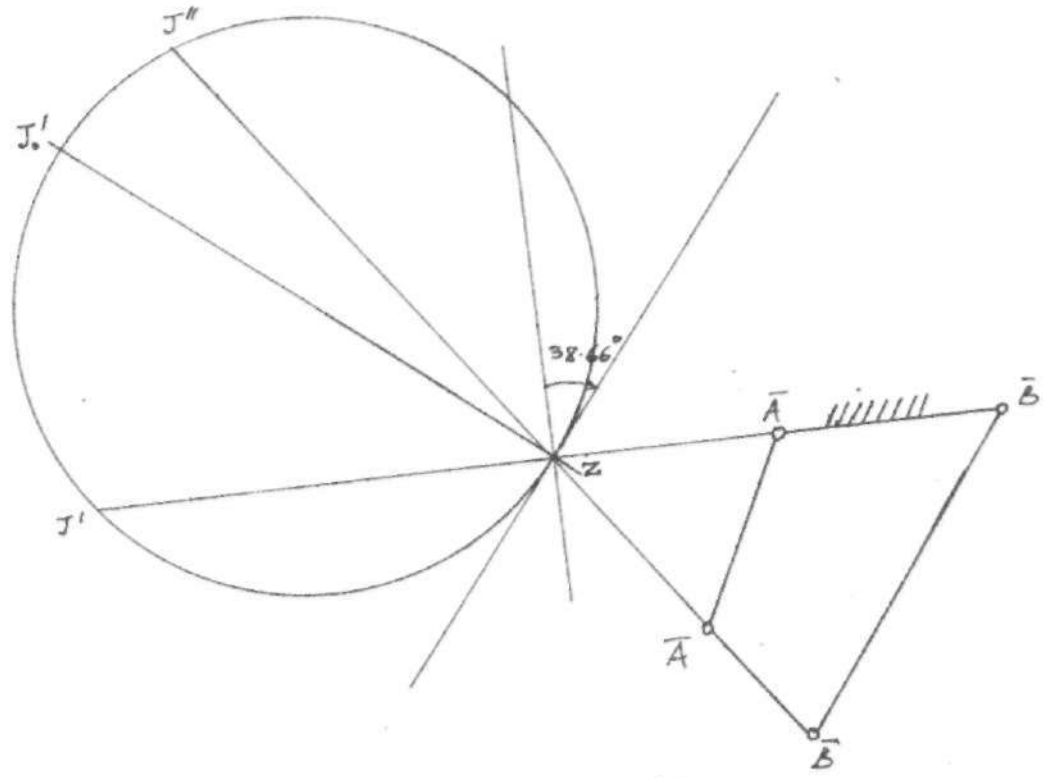


fig. 10·25 : 4 – bar mechanism with $n = \frac{1}{2}$ and $\alpha_4 = 20$ rad/s.

The slope of the coupler AB can be chosen and the position of one of the moving points, A or B.

Similarly, $\frac{1}{ZJ''} = \frac{1}{ZB} - \frac{1}{ZA} - \frac{1}{2.5} = \frac{1}{ZB} - \frac{1}{1}$

$\therefore \quad \frac{1}{ZB} = \frac{1}{1} - \frac{2}{5} = \frac{3}{5}$

$\therefore \quad ZB = \frac{5}{3}$

10.13. Radius Of Curvature Of Polode Attached To Link 2:

Generally, $$R_f = \frac{\left[R^2 + (dR/d\theta)^2\right]^{\frac{3}{2}}}{R^2 + 2\left(\frac{dR}{d\theta}\right)^2 - R\left(\frac{d^2R}{d\theta^2}\right)} \qquad \text{---------- (1061)}$$

Differentiating equation (1059) with respect to θ,

$$\frac{d^2R}{d\theta^2} = \frac{a}{(1-n)^2}\left[(1-n)\frac{d^2n}{d\theta^2} + 2\left(\frac{dn}{d\theta}\right)^2\right] \qquad \text{---------- (1062)}$$

Substituting for R, $\frac{dR}{d\theta}$ and $\frac{d^2R}{d\theta^2}$ in equation (1061),

$$R_f = \frac{\left[\frac{n^2a^2}{(1-n)^2} + \frac{a^2\left(\frac{dn}{d\theta}\right)^2}{(1-n)^4}\right]^{\frac{3}{2}}}{\frac{n^2a^2}{(1-n)^2} + \frac{2a^2\left(\frac{dn}{d\theta}\right)^2}{(1-n)^4} - \frac{na^2}{(1-n)^4}\left[(1-n)\frac{d^2n}{d\theta^2} + 2\left(\frac{dn}{d\theta}\right)^2\right]}$$

$$= \frac{a\left[n^2(1-n)^2 + a^2\left(\frac{dn}{d\theta}\right)^2\right]^{\frac{3}{2}}}{(1-n)^2\left[n^2(1-n)^2 + 2\left(\frac{dn}{d\theta}\right)^2 - n\left\{(1-n)\frac{d^2n}{d\theta^2} + 2\left(\frac{dn}{d\theta}\right)^2\right\}\right]}$$

$$= \frac{a\left[n^2(1-n)^2 + a^2\left(\frac{dn}{d\theta}\right)^2\right]^{\frac{3}{2}}}{(1-n)^3\left[n^2(1-n) + 2\left(\frac{dn}{d\theta}\right) - n\frac{d^2n}{d\theta^2}\right]} \qquad \text{-------- (1063)}$$

Equation (1063) gives the expression for the radius of curvature of the polode attached to link 2.

Ex.(10.04): Design a 4-bar mechanism to replace circular gears for a limited range. (n = - 1).

Solution:

$$n = -1$$

$$\frac{dn}{d\theta} = 0$$

$$\frac{d^2n}{d\theta^2} = 0$$

$$Z\bar{A} = R = \frac{na}{1-n}$$

$$Z\bar{B} = R + a = \frac{a}{1-n}$$

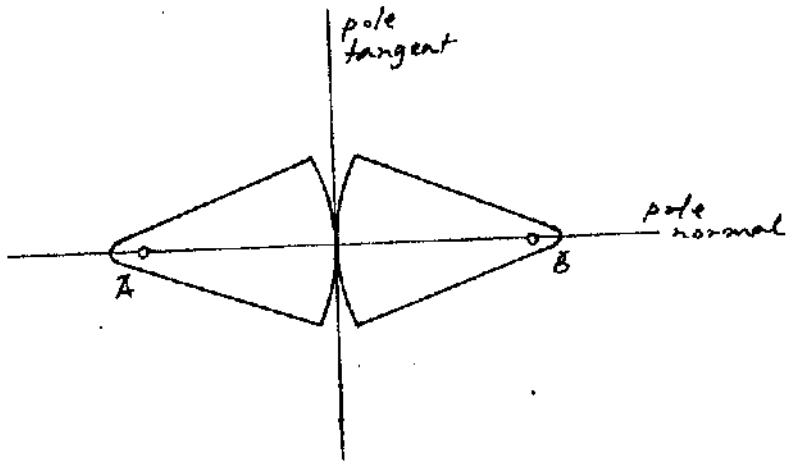


fig. 10.26: Gears to be replaced by a 4-bar mechanism.

Writing the Euler-Savary equation,

$$\frac{1}{\bar{ZB}} - \frac{1}{\bar{ZA}} = \frac{1}{ZJ_o^1} \qquad \text{----------(1064)}$$

From equation (1026), $\bar{AB}$ is the pole normal.

Substituting in equation (1064),

$$\frac{1-n}{a} - \frac{1-n}{na} = \frac{1}{ZJ_o^1}$$

$$\therefore \quad ZJ_o^1 = -\frac{na}{(1-n)^2}$$

Substituting in equation (1063),

$$R_f = \frac{a(4)^{\frac{3}{2}}}{8(2)} = \frac{a}{2}$$

$$\therefore \quad a = 2R_f.$$

Equation for radius of curvature then becomes

$$\frac{2\,ZJ_o^1}{R_f} = \left[3\frac{ZB}{AB} - 1\right](1-K) + \left[3\frac{Z\bar{B}}{\bar{AB}} - 1\right](1+K) \text{---(1065)}$$

where $K = \dfrac{\sin(\gamma_{\bar{a}\bar{b}} + \gamma_{ab})}{\sin(\gamma_{\bar{a}\bar{b}} - \gamma_{ab})}$

$\gamma_{\bar{a}\bar{b}}$ is the angle between the pole normal and the link

$\bar{AB}$ = 0 degrees.

γ_{ab} is the angle between the pole normal and the link AB

and $R = \dfrac{na}{1-n}$ $\qquad \therefore \quad K = -1$

Substituting in equation (1065),

$$-\frac{2na}{(1-n)^2}\cdot\frac{(1-n)}{na}=\left[3\frac{ZB}{AB}-1\right]\cdot 2$$

$$\therefore\quad 3\frac{ZB}{AB}=\frac{2-n}{1-n}$$

Euler-Savary equation must also be satisfied for link 3,

$$\frac{1}{ZB}-\frac{1}{ZA}=\frac{1}{ZJ'_{ab}}\qquad\text{----------(1066)}$$

and $ZA = ZB - AB$

and $ZJ_{ab}^{1} = ZJ_{o}^{1}\cos\gamma_{ab} = \frac{na}{(1-n)^2}\cos\gamma_{ab}$

Equation (1066) then becomes,

$$ZJ_{ab}^{1} = -\frac{(ZA)(ZB)}{ZB-ZA}$$

$$= -\frac{(ZA)(ZB)}{AB}$$

$$= -\frac{ZB}{AB}\left[ZB-AB\right]$$

$$\therefore\quad ZB=\frac{3na\cos\gamma_{ab}}{(1-2n)(1-n)}$$

$$\text{and}\quad AB=\frac{9na\cos\gamma_{ab}}{(1-2n)(2-n)}$$

Let $\gamma_{ab} = -45^{\circ}$,

$$\therefore\quad ZB=\frac{3x-1\ x\ 1x-\sqrt{2}/2}{3\ x\ 2}=\frac{\sqrt{2}}{4}$$

$$\text{and}\quad AB=\frac{9x-1\ x\ 1x-\sqrt{2}/2}{3\ x\ 3}=\frac{\sqrt{2}}{2}$$

The derived 4-bar mechanism is shown in Fig. (10.27).

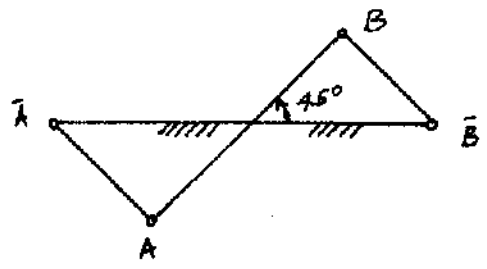


fig. 10.27: 4-bar mechanism designed to replace circular gears with speed ratio -1, for a limited range.

10.14. Stationary Curvature Of Point Paths:

The path of a point in the coupler of a 4-bar mechanism has stationary curvature when the rate of change of parth curvature with respect to the distance along the path is zero.

$$\frac{d\left(\frac{1}{\rho}\right)}{ds} = 0 \text{ , where } \rho \text{ is the radius of curvature.}$$

Considering the rolling of the polodes as illustrated in Fig. (10.28), the equation for the particle path with stationary curvature can be derived.

$$\frac{d(1/\rho)}{d\sigma} = 0 \quad \text{when} \quad \frac{d(1/\rho)}{ds} = 0$$

except when $\frac{d\sigma}{ds} = 0$.

s - distance along the path

σ - displacement on the polode.

Writing the Euler-Savary equation,

$$O_c C = \frac{PC^2}{PC - PJ \cos \gamma_c}$$

or
$$P = \frac{r^2}{r - \delta \cos \gamma} \qquad \text{---------- (1067)}$$

$$\frac{1}{\rho} = \frac{r - \delta \cos \gamma}{r^2}$$

$$\frac{d(1/\rho)}{d} = \frac{r^2\left(\frac{dr}{d\sigma} + \delta \sin\gamma \frac{d\gamma}{d\sigma} - \frac{d\delta}{d\sigma} \cos\gamma\right)}{r^4}$$

$$- \frac{(r - \delta\cos\gamma)\ 2r \frac{dr}{d\sigma}}{r^4} \qquad \text{---------- (1968)}$$

but $\Delta r \simeq - \Delta\delta \sin \gamma$

and $\Delta\gamma \simeq - \frac{\Delta\sigma}{R_m} - \frac{\cos\gamma}{r}$

therefore $\frac{dr}{d\sigma} = - \sin \gamma \qquad \text{---------- (1069)}$

and $\frac{d\gamma}{d\sigma} = - \frac{1}{R_m} - \frac{\cos\gamma}{r} \qquad \text{---------- (1070)}$

Substituting in equation (1068) gives

$$\frac{d(1/\varrho)}{d\sigma} = \frac{3\epsilon \sin\gamma \cos\gamma}{r^2}\left[\frac{1}{M\cos\gamma} - \frac{1}{N\sin\gamma} - \frac{1}{r}\right] \quad \text{---(1071)}$$

For stationary curvature, $\frac{d(1/\varrho)}{d} = 0$.

Therefore, $\frac{1}{M\cos\gamma} - \frac{1}{N\sin\gamma} = \frac{1}{r}$ ----------(1072)

This is the equation of the curve of stationary curvature or the 'cubic of stationary curvature'.

In rectangular coordinates,

$$r = (x^2 + y^2)^{\frac{1}{2}}$$

$$x = r\cos\gamma, \quad \therefore \cos\gamma = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}$$

$$y = r\sin\gamma, \quad \therefore \sin\gamma = \frac{y}{(x^2 + y^2)^{\frac{1}{2}}}$$

The equation then becomes,

$$\frac{1}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{(x^2 + y^2)^{\frac{1}{2}}}{Mx} - \frac{(x^2 + y^2)^{\frac{1}{2}}}{Ny} \quad \text{----------(1073)}$$

$$1 = (x^2 + y^2)\left[\frac{Ny - Mx}{MNxy}\right] \quad \text{----------(1073a)}$$

or $MNxy = (x^2 + y^2)(Ny - Mx)$ ----------(1073b)

10.15. Graphical Construction:

The graphical construction for plotting points on the cubic of stationary curvature is shown in Fig. (10.29).

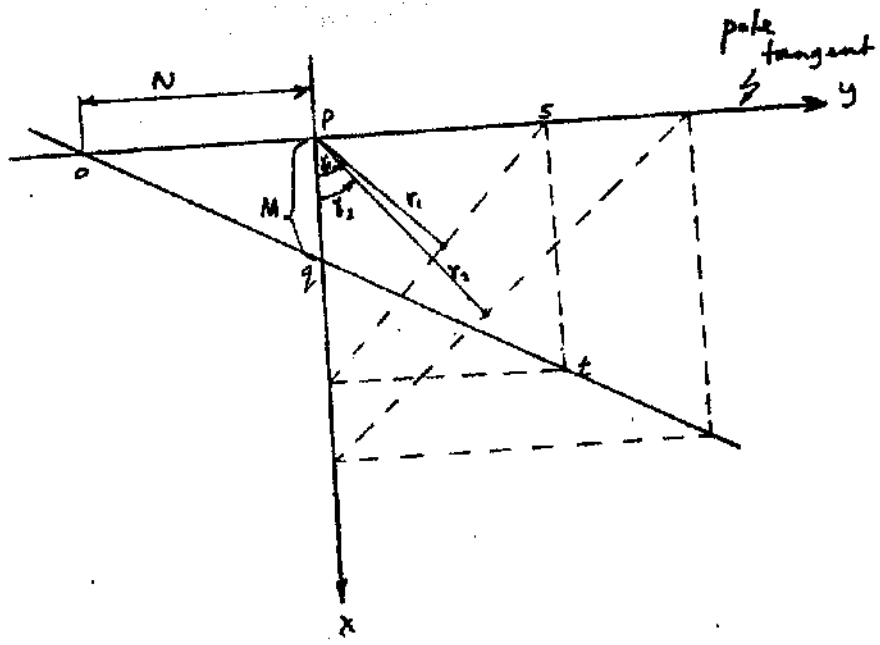


fig. 10.29: A graphical construction for plotting points of the cubic of stationary curvature.

From Fig. (10.29), triangles oPq and ost are similar, therefore,

$$\frac{os}{st} = \frac{N}{M} \qquad \text{---------- (1074)}$$

or $$\frac{N + r_1/\sin\gamma_1}{r_1/\cos\gamma_1} = \frac{N}{M} \qquad \text{---------- (1075)}$$

Rearranging equation (1075),

$$\frac{1}{r_1} = \frac{1}{M\cos\gamma_1} - \frac{1}{N\sin\gamma_1} \qquad \text{---------- (1076)}$$

Plotting The Cubic Of Stationary Curve For A 4-Bar Mechanism:

The cubic of stationary curvature crosses the axes only at the instantaneous centre, P; at which point the curve is tangent to both axes - the pole tangent and the pole normal. The slope of the asymptotes is M/N. The radius of curvature at γ = 0 or 180 degrees is $\frac{M}{2}$ and the radius of curvature at γ = 90 or 270 degrees is $-\frac{N}{2}$ (Fig. 10.30).

In a 4-bar mechanism, the cubic of stationary curvature passes through the fixed and moving pivots. This enables the value of N to be found.

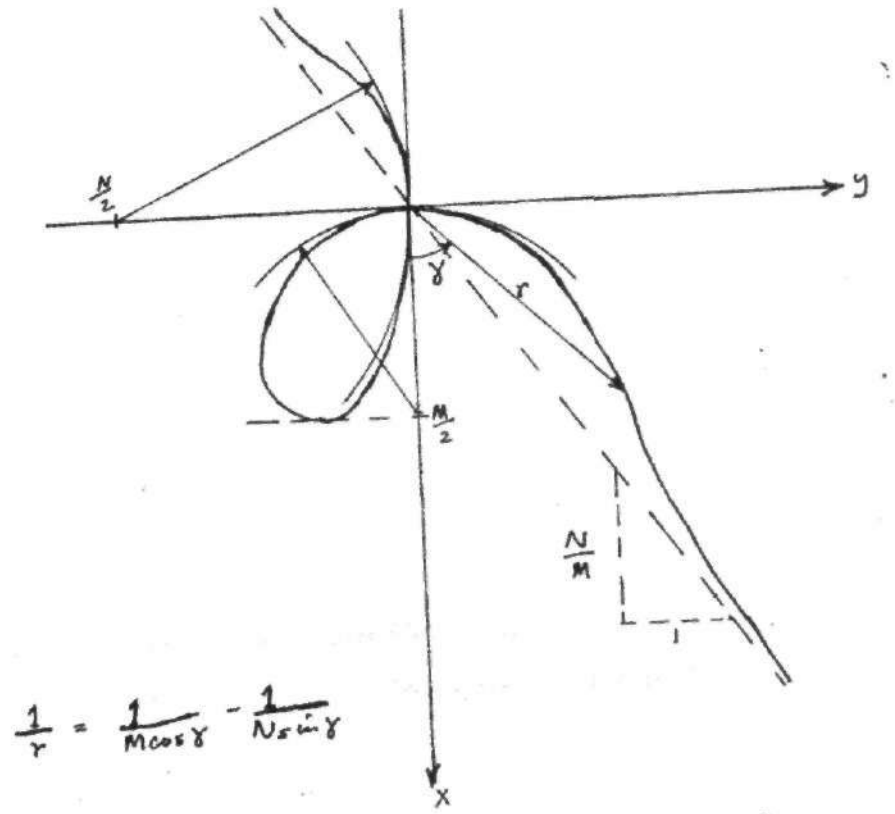


Fig. 10.30: Typical curve of stationary curvature.

Given the 4-bar mechanism configuration as shown in Fig. (10.31), the curve of stationary curvature can be plotted by determining the values of M and N for the given configuration and then using the equation of cubic of stationary curvature.

Points A and B, on the coupler, have stationary curvature and as such they lie on the curve of stationary curvature.

Writing the equation for cubic of stationary curvature for points A and B,

$$\frac{1}{r_a} = \frac{1}{M \cos \gamma_a} - \frac{1}{N \sin \gamma_a} \qquad \text{---------- (1077)}$$

$$\frac{1}{r_b} = \frac{1}{M \cos \gamma_b} - \frac{1}{N \sin \gamma_b} \qquad \text{---------- (1078)}$$

With $\gamma_a = 22^\circ$ and $\gamma_b = 75.5^\circ$, M and N are found to be 1.64 and - 3.95 respectively.

The curve of stationary curvature is plotted on the space diagram of the given mechanism in Fig. (10.31).

From the relation;

$\frac{1}{M} = \frac{1}{3}(\frac{1}{PJ} + \frac{1}{PO_m})$, the polode radii of curvature can be calculated once the value of the inflection diameter is determined.

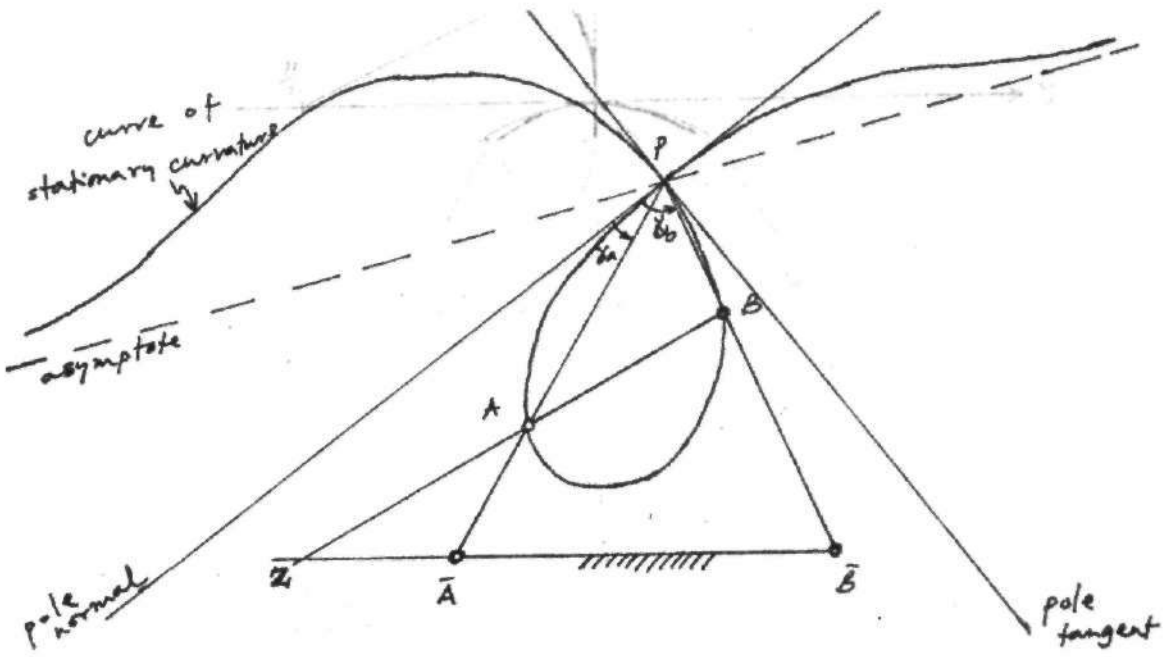


fig. 10·31. Curve of stationary curvature for the 4-bar mechanism, $\bar{A}AB\bar{B}$.

10.16. Pivot Point Curve:

The equation for the pivot point curve, the locus of the centre of curvature of points with stationary curvature can be found from the cubic of stationary curvature equation (equation 1076).

If a point D on the coupler of a 4-bar mechanism lies on the curve of stationary curvature, then

$$\frac{1}{r_d} = \frac{1}{M \cos \gamma_d} - \frac{1}{N \sin \gamma_d} \qquad \text{---------- (1079)}$$

Writing the Euler-Savary equation for D,

$$\frac{1}{PJ \cos \gamma_d} = \frac{1}{r_d} - \frac{1}{R_d} \qquad \text{---------- (1080)}$$

where $r_d = PD$

and $R_d = PO_d$

Substituting for $\frac{1}{r_d}$ in equation (1080),

$$\frac{1}{R_d} = \left(\frac{1}{M} - \frac{1}{PJ}\right) \frac{1}{\cos \gamma_d} - \frac{1}{N \sin \gamma_d}$$

but $\frac{1}{M} = \frac{1}{3}\left(\frac{1}{PJ} + \frac{1}{PO_m}\right)$

where $PO_m = -O_mP = -R_m$, the radius of curvature of moving polode.

$$\therefore \frac{1}{R_d} = \frac{1}{3}\left(\frac{1}{PO_m} - \frac{2}{PJ}\right) \frac{1}{\cos \gamma_d} - \frac{1}{N \sin \gamma_d} \qquad \text{--------- (1081)}$$

$$\text{or} \quad \frac{1}{R_d} = \frac{1}{M^* \cos \gamma_d} - \frac{1}{N^* \sin \gamma_d} \qquad \text{---------- (1082)}$$

where $\frac{1}{M^*} = \frac{1}{3}\left(\frac{1}{PO_m} - \frac{2}{PJ}\right)$

and $\frac{1}{N^*} = \frac{1}{N}$

Then, generally, the equation for the pivot point curve is given by

$$\frac{1}{R} = \frac{1}{M^* \cos \gamma} - \frac{1}{N^* \sin \gamma} \qquad \text{----------(1083)}$$

Since the equation for the pivot point curve is identical with that of the cubic of stationary curvature, their construction details are the same.

The plotting of both curves enables the design of adjustable coupler rest (dwell) mechanisms.

CHAPTER ELEVEN

COMPUTATIONAL METHODS

Due to the tediousness of the calculations, it is advisable to carry out the analytical procedures by the use of computer methods. Some of the analyses are explained for easy computer programming.

11.1. Input, Output Angles:

It is necessary to know the ranges that the cranks in in a 4-bar mechanism can attain so as to limit the analysis within those angular ranges, $\triangle\theta$ and $\triangle\phi$.

Before the input and output angles are found, it is necessary to ascertain the type of 4-bar mechanism. A 4-bar mechanism could be (i) a crank-rocker mechanism, (ii) a double-crank mechanism, or (iii) a double-rocker mechanism.

11.2. Transmission Angle, Time Ratio:

The relation between the transmission angle, μ, and the relative link lengths can be easily calculated by computer methods. The same goes for the time ratio, T.R. A sub programme to check these quantities can be inserted in the main programme and if the transmission angle or time ratio values are not good enough, another design will be found that is suitable.

11.3. Coordination Of Crank Motion:

The Freudenstein equation can be used to study the relationship between the input and output angles for a required range.

$$R_1 \cos \phi - R_2 \cos \theta + R_3 = \cos(\theta - \phi)$$

where $R_1 = \frac{a}{b}$ ----------(906)

$R_2 = \frac{a}{d}$

and $R_3 = \frac{a^2 + b^2 - c^2 + d^2}{2bd}$

From equation (906),

$$\text{Sin } \phi = \frac{ab^2 \sin 2\theta - bk^2 \sin \theta \pm (a - b \cos \theta)\sqrt{\Delta}}{2d(a^2 + b^2 - 2ab \cos \theta)} \quad ---(1101)$$

and

$$\cos \phi = \frac{-ak^2 + b(k^2 + 2a^2)\cos \theta - 2ab^2 \cos^2 \theta \pm b \sin \theta \sqrt{\Delta}}{2d(a^2 + b^2 - 2ab \cos \theta)} \quad ------(1102)$$

and also;

$$\frac{d\phi}{d\theta} = \frac{a \sin \theta \left[2a^2 - 2b^2 \cos 2\theta - k^2\right] + b \sin 2\theta(k^2 - a^2)}{2a \sin \theta \left[k^2 + 2b^2 \cos^2 \theta\right] + b \sin 2\theta(2a^2 + k^2)}$$

$$\frac{\pm (b - a \cos \theta)\sqrt{\Delta}}{\pm \frac{1}{b}(a^2 + b^2 - 2ab \cos \theta)\sqrt{\Delta}} \quad ----------(1103)$$

where $\Delta = 4d^2(a^2 + b^2) - k^4 + 4ab(k^2 - 2d^2)\cos \theta - 4a^2b^2 \cos^2 \theta$

and $k^2 = a^2 + b^2 - c^2 + d^2$

From equations (903) and (904),

$$\cos\gamma = \frac{a - b\cos\theta + d\cos\phi}{c}$$

$$\sin\gamma = \frac{d\sin\phi - b\sin\theta}{c}$$

11.4. Velocity And Acceleration:

From equations (323) to (329),

$$\omega_4 = \omega_2 \cdot \frac{b}{d}\frac{\sin(\theta - \gamma)}{\sin(\phi - \gamma)}$$

$$= \frac{\omega_2 b}{d}\frac{\sin\theta\cos\gamma - \sin\gamma\cos\theta}{\sin\phi\cos\gamma - \sin\gamma\cos\phi} \qquad \text{----------(1104)}$$

Using equation (1104) in conjunction with the relationships between γ, ϕ and θ in equations (1101) to (1103); the values of ω_4 can be deduced for varying values of θ.

Similarly, from equation (332),

$$\omega_3 = \omega_2 \frac{b}{c}\frac{\sin(\theta - \phi)}{\sin(\phi - \gamma)}$$

$$= \frac{\omega_2 b}{c}\frac{\sin\theta\cos\phi - \sin\phi\cos\theta}{\sin\phi\cos\gamma - \sin\gamma\cos\phi} \qquad \text{----------(1105)}$$

The values of ω_3 can then be found for varying values of ω_2 and θ, when equation (1105) is used in conjunction with equations (1101) to (1103).

The same holds for the angular acceleration values α_3, α_4. Using equations (340) and (344) in conjunction with the relationships between the angles θ, γ , ϕ; their values (α_3, α_4) can be got for varying values of ω_2, α_2 and θ.

11.5. Error Function:

Computational methods are very useful for the analysis of 4-bar function generators. Computational methods make the derivation of suitable 4-bar mechanisms for particular functions to be generated, a lot easier. It is also necessary to know the deviation (error) of the relationship of the input and output angular displacements of the designed 4-bar mechanism from the expected. In this aspect, computational methods are ideal in determining the error function over the required range.

For a given function,

$$y = f(x),\ x_i \leq x < x_f \qquad \text{----------(1106)}$$

where x_i and x_f are the initial and final value of x, the function is only generated at the precision points. At the other positions within the given range, there is an error.

$$y = Ky(\phi' - \phi_i') \qquad \text{----------(1107)}$$

where $K_y = \frac{\Delta y}{\Delta \phi'}$; Δy - range of y

$\Delta \phi'$ - range of ϕ'

and ϕ_i is the initial value of ϕ.

Also,

$$x = Kx(\theta' - \theta_i') \text{ ; where } Kx = \frac{\Delta x}{\Delta \theta'} \qquad \text{--------(1108)}$$

The relationship between the input and output angles, θ and ϕ respectively, is given by the Freudenstein equation. The relationship between y and x is expressed by the given function (equation 1106).

11.6. Path Curvature:

Using the Euler-Savary equation in conjunction with the geometric relations in a 4-bar mechanism, the curvature of any point on any of the links in the plane of the 4-bar mechanism can be found.

Writing the Euler-Savary equation for a point C on the coupler of a 4-bar mechanism,

$$\frac{1}{PC} - \frac{1}{PO_c} = \frac{1}{PJ_c} = \frac{1}{PJ \cos \gamma_c} \qquad \text{----------(1109)}$$

For the moving pivots A, B of a 4-bar mechanism,

$$\frac{1}{PA} - \frac{1}{P\bar{A}} = \frac{1}{PJ_a} = \frac{1}{PJ \cos \gamma_a} \qquad \text{----------(1110)}$$

$$\frac{1}{PB} - \frac{1}{P\bar{B}} = \frac{1}{PJ_b} = \frac{1}{PJ \cos \gamma_b} \qquad \text{----------(1111)}$$

The notations are as they appear in Fig. (1016).

For a particular configuration of 4-bar mechanism, the quantities PA, $P\bar{A}$, PB, $P\bar{B}$, PJ, γ_a, γ_b, all depend on the link length ratios. PJ is found and the centre of curvature is located for the given point, C, on the coupler.

The expression for the centrode curvature given in equation (1054) is also useful in finding the fixed and moving radii of curvature for any particular configuration of a 4-bar mechanism. Computational methods can then be used to design 4-bar mechanisms for given centrodes.

The expression for the cubic of stationary curvature given in equation (1072) is useful for deriving suitable coupler-dwell mechanisms. The values of M and N are found for the mechanism in the given configuration and the cubic of stationary curvature curve is plotted. A suitable point on the curve of stationary curvature is selected and the centre of curvature, found by using the Euler-Savary equation, is the point on the coupler with a dwell (rest).

CHAPTER TWELVE

SUMMARY

The study of 4-bar mechanisms undertaken had been considered both analytically and graphically, although more attention had been given to analytical methods as they yield more accurate results than the graphical analysis.

The classification of various 4-bar mechanisms is a necessary analysis as this enables us to know the various uses a particular 4-bar mechanism could be put to. Crank rocker mechanisms are useful for intermittent motion, quick-return motion, conversion of rotary motion to approximate reciprocating motion as in sewing machines, and generally for high speed functions. Double-rocker mechanisms are used mostly for systems with relatively slow crank speeds. Double-crank mechanisms are particularly useful in bringing about specific speed changes in the driven crank, if the input crank runs at a constant angular velocity. It is useful as a function-generator as the continuity in motion between the cranks is always assured.

The velocity relationships between links provides a useful angular displacement relationship. The acceleration study gives rise to the inertia forces generated in the links and it is

necessary that the forces generated in the links should be minimized in order not to have undue stresses that could damage the mechanism. To keep the stresses due to inertia forces down, the transmission angle, μ - the angle between the coupler and the output link - is kept as near 90 degrees as much as possible. A maximum of 50 degrees is considered good enough for $/90 - \mu/_{max}$.

As already stated, a coupler curve is the path generated by any point on the coupler. Coupler curves are polynominals of the sixth degree and this accounts for their often complex shapes. Coupler curves are very useful in the synthesis of 4-bar mechanisms; they are useful in path generation as in the jib crane (Fig.1.09); they are useful in coupler-dwell mechanisms, in which case a portion of the coupler curve approximates a circular path and a dwell occurs around that portion of the curve during the motion of the mechanism; they are sometimes useful for driving general stop mechanisms for intermittent motion, in the present instance, a symetrical coupler curve is utilized; they are also used for such other functions that the shape of the coupler curve can be suited to.

The study on position coordination gives the analytical relationships between the link lengths and the input angle (input range), the output angle (output range), the transmission angle, μ, the time ratio, T.R., in crank-rocker mechanisms. It also

gives the graphical analysis on crank coordination for angular displacements.

The section, finite link displacements, deals with methods of deriving suitable 4-bar mechanisms for given number of coupler-link displacements. It shows how a 4-bar mechanism is derived for path-generation.

Point position reduction makes the construction work done under position coordination and finite link displacement a lot easier. The centre-curve, which is the locus of the fixed pivots in a 4-bar mechanism, is known to pass through the poles and point position reduction method makes the pivot(s) coincide with the poles of some relative positions of the coupler in the 4-bar mechanisms. This method, therefore, eliminates the task of plotting the centre curve or the circle-point curve, as the case may be.

In a 4-bar mechanism, the motions of the input and output links are easily analyzed as they have their fixed rotation points attached to the links. The motion of the connecting link (coupler) is however more complex and no point on the coupler has a fixed point. The motion of the coupler however is the same as the rolling of two curves, (one on the other). These curves are termed polodes or centrodes. The polode of the coupler relative to the fixed link is the fixed polode and that of the fixed link relative to the coupler is the moving polode. Polodes however

are not restricted to the coupler link alone. Actually, a polode is the locus of the instantaneous centre of one link to the other. The motions of the cranks can also be studied through the rolling of their respective polodes. Equations of the polodes are also polynominals of the sixth order and at every part of their rolling motion, the curvature of the polodes differs. The section on path curvature deals on this aspect, giving relations for the curvature of any point on the coupler at a particular configuration. Useful relations such as the Euler-Savary equation, polode curvature equation, cubic of stationary curvature equation, are shown. Examples on the theory discussed are also shown and graphical constructions are also illustrated for the various aspects.

The study on coordination of crank motion gives the analytical methods for the theories expounded in graphical construction under position coordination. The Freudenstein equation, the starting base, is used in the fourth order approximation for the synthesis of 4-bar mechanism to generate given functions. Four-point and five-point approximation methods are shown for the synthesis of 4-bar mechanisms to mechanize given functions. The analytical methods provide a high degree of calculation accuracy than can be got by the graphical methods.

The section on computational methods gives a general guide-line on how the various theories and relations discussed so far, can be coordinated through computational analysis to give the derived design of a 4-bar mechanism.

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