# ITERATIVE APPROXIMATION OF COMMON FIXED POINTS FOR FAMILIES OF SOME CLASSES OF NONLINEAR OPERATORS

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MARCH, 2016

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**Ph.D/SCI/21808/2012-2013**

**A THESIS SUBMITTED TO THE SCHOOL OF POST GRADUATE STUDIES AHMADU BELLO UNIVERSITY, ZARIA**

**IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF DOCTOR OF PHILOSOPHY (Ph.D) DEGREE IN MATHEMATICS**

**DEPARTMENT OF MATHEMATICS**
**AHMADU BELLO UNIVERSITY, ZARIA,**
**NIGERIA**

**MARCH, 2016**

# DECLARATION

I declare that the work in this thesis titled **Iterative approximation of common fixed points for families of some classes of nonlinear operators** has been performed by me in the Department of Mathematics under the supervision of Dr. B. Ali, Prof. B. Sani and Prof. J. Singh. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at any University.

Godwin Chidi UGWUNNADI ——————- ————

(Student) Signature Date

# CERTIFICATION

This thesis titled ITERATIVE APPROXIMATION OF COMMON FIXED POINTS FOR FAMILIES OF SOME CLASSES OF NONLINEAR OPERATORS by Godwin Chidi UGWUNNADI meets the regulations governing the award of the degree of Doctor of Philosophy (Ph.D) in Mathematics of Ahmadu Bello University, Zaria, and is approved for its contribution to knowledge and literary presentation.

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# DEDICATION

To my beloved Mother 'Mrs. Ugwunnadi, Nwamaka' and my Father the slim lion in Ugwunnadi's kindom 'Chief Michael Ojobor Ugwunnadi' (of blessed memory). Each of whom I owe more than I can express.

# ACKNOWLEDGEMENT

I would like to start by thanking God, The Almighty for making all these possible. Scientific research and developing a PhD thesis can be accomplished only with remarkable guidance, which I have received from Dr. Bashir Ali, He has shared his great knowledge and expertise with me throughout this endeavour, and also offered me an extraordinary example. For all his effort and patience, I must kindly thank him, assuring him of my deep gratitude and consideration.

I would also like to express my gratitude to other members of my supervisory committee, Prof. B. Sani and Prof. J. Singh, for their careful reading of this manuscript and for their important observations and suggestions. I also thank the members of my PhD academic committee for their scientific support and direction.

At the same time, I thank all the members of Department of Mathematics, Ahmadu Bello University, Zaria and Michael Okpara University of Agriculture, Umudike for their immense contributions toward the successful completion of this work.

I must thank Prof. H. M. Ogbu (of blessed memory) and Prof. V. E. Asor for their constant support and strong believe, in varying degrees, that the bright light is and should fluoresce.

I reserve for the end, the most beautiful gratitude to all my family members and my beloved Eunice Nkiru Omeje for always being there for me and all those who helped me in one way or the other towards the completion of this work whose names are not mentioned here.

May God The Almighty reward you all, Amen.

# ABSTRACT

In this research, some iterative methods used to approximate common fixed points of some nonlinear operators in Banach space are introduced and studied. In particular, modified iterative schemes for fixed point of family of asymptotically nonexpansive mappings are introduced and studied. Therefrom, strong convergence theorem for members of this family of mapping which is also in some cases solution of some variational inequality problems is established. Furthermore, a hybrid projection method is studied and strong convergence theorem for a family of generalized asymptotically quasi- $\phi$-strict pseudocontractive mappings in a real reflexive, strictly convex and smooth Banach space with Kadec-Klee property and whose dual also possesses Kadec Klee property is proved. Also established this thesis is a strong convergence theorem for infinite family of uniformly $L-$Lipschitzian total quasi-$\phi$-asymptotically nonexpansive multi-valued mappings. This is done using a generalized $f$-projection operator in a real uniformly convex and uniformly smooth Banach space using a new iterative scheme. Thus a new iterative scheme by hybrid method is introduced and its convergence to a common element in the set of fixed points of a finite family of closed quasi-Bregman strictly pseudocontractive mappings is proved.

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# CHAPTER ONE

# INTRODUCTION

## 1.1 Background of the Study

The study in this thesis falls within the general area of Nonlinear functional analysis and applications, an area which has been of active research interest, especially within the past 40 years. We devote our attention to a special class of problems in this area; 'iterative approximation of common fixed points of families of some class of nonlinear operators'. If we consider a differential equation $\frac{du}{dt} + Au(t) = 0$ that describes an evolution system where $A$ is an accretive map from a normed space $E$ to itself. At equilibrium state, $\frac{du}{dt} = 0$ and so a solution of $Au = 0$ describes the equilibrium or stable state of the system. This is very desirable in many applications, for example ecology, economics, physics, to name but a few.

Consequently, considerable research efforts have been devoted to methods of solving the equation $Au = 0$ where $A$ is accretive. Generally $A$ is nonlinear, so there is it is difficult to get general closed form solution of this equation. The standard technique for finding solution of this equation is to introduce an operator $T : E \to E$ defined by $T := I - A$, where $I$ is the identity map on $E$. Such a map $T$ is called pseudo-contraction (or pseudo-contractive). It is then clear that any zero of $A$ is a fixed point of $T$ (where a point $x \in E$ is called a fixed point of $T$ if and only if $Tx = x$). As a result of this, the study of fixed point theory for pseudo-contractive maps has attracted the interest of numerous scientists and has become a nourishing area of research for numerous mathematicians Chidume (2009).

A very important subclass of the class of pseudocontractive mappings is the class of nonexpansive mappings (where a mapping $T : E \to E$ is called nonexpansive if and only if for any $x, y \in E$, $||Tx - Ty|| \leq ||x - y||$). The class

of nonexpansive mappings includes (as special case) the class of contractions (where a mapping $T : E \to E$ is called a contraction if and only if there exists a constant $\gamma \in (0, 1)$ such that for any $x, y \in E,\ ||Tx - Ty|| \leq \gamma||x - y||$). In general setting of a metric space $(E, \rho)$, a mapping $T : E \to E$ is called a contraction if and only if there exists a constant $\gamma \in (0, 1)$ such that for any $x, y \in E$,

$$\rho(Tx, Ty) \leq \gamma\rho(x, y) \tag{1.1.1}$$

Interest in, and the importance of the class of nonexpansive mappings will become evident in the sequel. One of the most important fixed point theorems in applications is the classical contraction mapping principle, or, in other words, the Banach-Cacciopoli fixed point theorem is stated in the following:

**Theorem 1.1.1** (Banach, 1922) and (Caccioppoli, 1930) Let $(E, \rho)$ be a complete metric space and let $T : (E, \rho) \to (E, \rho)$ be a contraction, then $T$ has a unique fixed point in $E$. Moreover, starting with arbitrary $x_0 \in E$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n = T^{n+1}x_0, \quad n \geq 0 \tag{1.1.2}$$

converges to the unique fixed point.

The iterative technique (1.1.2) is due to Picard (1890), see also Chidume (2009) for more detailed information. A mapping $T$ satisfying (3.3.3) is called a strict contraction. If $\gamma = 1$ in the relation (3.3.3), then $T$ is called nonexpansive. If however, $\gamma$ is an arbitrary fixed positive constant, then $T$ is called a Lipschitz map or a $\gamma-$Lipschitzian map. For the iterative formula, it is observed that if the condition $\gamma < 1$ on the operator $T$ is weakened to $\gamma = 1$, the sequence $\{x_n\}$ defined by (1.1.2) may fail to converge to a fixed point of $T$ if it exists. To see this, we consider the following example:

**Example 1.1.1** Let $E = [0, 1]$, and $T : E \to E$ be defined by $Tx = 1 - x$, for all $x \in E$. Then $T$ is nonexpansive, and unique fixed point of $T$ is $\frac{1}{2}$. But, for any $x_0 = a \neq \frac{1}{2}$, the picard iteration (1.1.2) yields an oscillatory sequence $a, 1 - a, a, 1 - a, \cdots$ which does not converge.

It is, however, observed that if in this example, the scheme of Krasnoselskii (1955), given by

$$x_0 \in E = [0, 1], x_{n+1} = \frac{1}{2}(x_n + Tx_n),\ n \geq 0, \tag{1.1.3}$$

is employed instead of Picard scheme, then the sequence $\{x_n\}_{n\geq 0}$ generated by (1.1.3) converges to the fixed point.

In general, if $E$ is a normed linear space and $T : E \to E$ is a nonexpansive mapping, a generalization of (1.1.3) which has proved successful in the approximation of fixed points (when they exist) of $T$ was given by Schaefer in 1957 as follows

$$x_0 \in K,\ x_{n+1} = (1 - \alpha)x_n + \alpha Tx_n,\ n \geq 0,\ \alpha \in (0, 1), \tag{1.1.4}$$

where $K \subseteq E$ is convex and nonempty. However, the most general iterative formula for approximation of fixed points of nonexpansive mappings which is called the *Mann iterative formula* due to Mann (1953), is the following:

$$x_0 \in K,\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n,\ n \geq 0, \tag{1.1.5}$$

where $\{\alpha_n\}$ is a real sequence in the interval (0,1) satisfying the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$ and (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

The recursion formula (1.1.4) is consequently called the Krasnoselskii-Mann (KM) formula for finding fixed points of nonexpansive mappings. This iterative process has become very important and applicable as noted below.

(i) Many well-known algorithms in signal processing and image reconstruction are iterative in nature. A wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the Krasnoselskii-Mann iteration procedure, for particular choices of the (nonexpansive) operator, Byne (2004). Apart from being an obvious generalization of the contraction mappings, nonexpansive maps are important, as has been observed by Bruck (1980), mainly for the following two reasons: Nonexpansive maps are intimately connected with the monotonicity methods developed since the early 1960s and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.

(ii) Nonexpansive mappings appear in applications as the transition operators for initial value problems of differential inclusions of the form $0 \in \frac{du}{dt} + T(t)u$, where the operators $\{T(t)\}$ are, in general, set-valued, accretive and minimally continuous.

Nonexpansive and their generalized mapping have been studied, and are still being studied, extensively by numerous authors (see for instance, Goebel and Kirk (1972), Bauschke (1996), Browder (1965), Bruck (1973), Chidume and Ali (2007), Chidume and Chidume (2006), Chidume et al. (2005), Ali and Minjibir (2012), Kimura and Takahashi (2009), Homaeipour and Razani (2011), Reich and Sabach (2011), and a host of others).

## 1.2 Statement of the Problem

Many problems in nonlinear analysis can be formulated as a problem of finding a fixed point of a certain mapping or a common fixed point of a family of mappings. Many problems for finding a common element of the set of fixed points of nonexpansive , asymptotically nonexpansive, relatively nonexpansive mapping. Among them, the work of Goebel and Kirk (1972), Bauschke (1996), Browder (1965), Bruck (1973), Chidume and Ali (2007), Chidume and Chidume (2006), Chidume et al. (2005), Ali and Minjibir (2012), Kimura and Takahashi (2009), Homaeipour and Razani (2011), Reich and Sabach (2011) is worth mentioning.

Also, it is well known that equilibrium problems and their generalizations have been important tools for solving problems arising in the fields of linear or nonlinear programming, variational inequalities, complementary problems, optimization problems, fixed point problems and have been widely applied to physics, structural analysis, management science and economics and so on (see, for instance Blum and Oettli (1994), Moudafi (2010). One of the most important and interesting topics in the theory of equilibria is to develop efficient and implementable algorithms for solving equilibrium problems and their generalizations (Alber and Reich (1994), Blum and Oettli (1994) and Hao (2011), Moudafi (2010), Qin and Kang (2010) and the references therein). Since the equilibrium problems have very close connections with both the fixed point problems and the variational inequalities problems, finding the common elements of these problems has drawn many people's attention and has become one of the hot topics in the related fields in the past few years ( Cruz and Iusem (2012), Plubtieng and Punpaeng (2008), Qin et al. (2009), Qin and Su (2007), Takahashi and Takahashi (2007), Takahashi and Zembayashi (2009) and the references therein).

Bregman (1967), discovered an elegant and effective technique for using the

so-called Bregman distance function $D_f$ see, (3.7.1) in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique has been applied in various ways in order to design and analyze not only iterative algorithms for solving feasibility and optimization problems, but also algorithms for solving variational inequalities, for approximating equilibria, for computing fixed points of nonlinear mappings and so on (see, for instance Bauschke et al. (2001), Chen et al. (2013), Takahashi and Takahashi (2007), Takahashi and Zembayashi (2008) and the references therein).

In 2006, Butnariu and Resmerita presented Bregman-type iterative algorithms and studied the convergence of the Bregman-type iterative method of solving some nonlinear operator equations.

Motivated by the above results, using the notion of asymptotically nonexpansive mapping, generalize and $f-$ generalize projection and Bregman projection, it is our purpose in this work to study a new modified iterative schemes and prove their strong convergence theorem

## 1.3 Significance of the Study

The basic significance of this study is to provide solutions to numerous problems in engineering, ecology, physics, optimization and economics which can be reduced to finding a common fixed point of some nonlinear operator including solution of variational inequality and equilibrium problems. Our results can be used to solve all such problems if all conditions are satisfied.

## 1.4 Aim and Objectives

### 1.4.1 Aim

The aim of this work is to formulate some new iterative approximation schemes for solving some nonlinear problems and establish their strong convergence to common fixed points of family of some nonlinear mappings which may also be solutions of some variational inequality problems and/or systems of equilibrium problems.

### 1.4.2 Objectives

The specific objectives of this work are to:

(i) introduce modified iterative schemes and prove their convergence to common fixed point of family of asymptotically nonexpansive mappings,

(ii) introduce a modified iterative scheme and prove its convergence to common fixed point of family of generalized asmptotically quasi-$\phi$-strict pseudo-contraction mapping,

(iii) introduce a modified iterative scheme and prove its convergence to common fixed point of infinite family of total quasi-$\phi$-asymptotically nonexpansive multi-valued mappings using $f$-projection,

(iv) introduce a modified iterative scheme and prove its convergence to common fixed point of family of closed quasi-Bregman strictly pseudo-contraction mapping and system of equilibrium problems.

## 1.5 Scope and Limitation

This research work is abstract and theoretically based. Even though our results hold in more general Banach spaces than Hilbert, but with restrictions to some convexity and smoothness requirements on them. All our results hold in Hilbert spaces. There are some limitations also imposed on some of the operators studied.

## 1.6 Research Methodology

The method of research adopted in this thesis is by consulting necessary and relevant literature on approximation of fixed points for some nonlinear operators. We then use techniques and results in Mathematical analysis, theories, lemmas and propositions in chapter three to establish new results.

## 1.7 Outline of Study

After the introductory chapter, the work contains four others including summary, conclusion and recommendation. A literature review is first presented followed by the discussion of the comparison between the existing results and the results obtained. The outline of the work chapter by chapter are as follows:

**Chapter 3:** in these chapter, theory and methods used in this work are presented.

**Chapter 4:** in these chapter, we studied new recursion formulas and strong convergence theorems for:

(i) family of asymptotically nonexpansive mapping and solutions to some

variational inequality problems,

(ii) family of generalized asymptotically quasi-$\phi$-strict pseudo-contractive mappings,

(iii) infinite family of uniformly $L-$Lipschitzian total quasi-$\phi$-asymptotically nonexpansive multi-valued mappings using a generalized $f-$projection operator,

(iv) finite family of closed quasi-Bregman strictly pseudocontractive mapping and system of equilibrium problem.

# CHAPTER TWO

# LITERATURE REVIEW

## 2.1 Iterative Algorithm for Nonexpansive Mappings

Let $K$ be a closed convex subset of a normed linear space $E$ and let $T : K \to K$ a nonexpansive mapping. For a sequence $\{\lambda_n\}$ of real numbers in [0,1] and an arbitrary $u \in K$, let the sequence $\{x_n\}$ in $K$ be iteratively defined by

$$x_0 \in K,\ x_{n+1} = \lambda_n u + (1 - \lambda_n)Tx_n,\ n \geq 0. \tag{2.1.1}$$

The recursion (2.1.1) was first introduced by Halpern (1967) with $u = 0$, in the framework of a real Hilbert space $H$. Under appropriate conditions on the domain of $T$, that is if $T$ is positive homogeneous (i.e., $T(tx) = tTx$), for any $t \geq 0$ and $x \in H$ and some restrictions on the parameter $\{\lambda_n\}$ ($\lambda_n = n^{-a}$, for some $a \in (0, 1)$ and for all $n \in \mathbb{N}$), Halpern proved strong convergence of $\{x_n\}$ to a fixed point of $T$. Under a different restriction on the parameter $\{\lambda_n\}$, Lions (1977) improved the result of Halpern, still in real Hilbert spaces. Lion proved strong convergence of $\{x_n\}$ to a fixed point of $T$ where the real sequence $\{\lambda_n\}$ satisfies the following conditions:

$$C1 : \lim_{n\to\infty} \lambda_n = 0;\ C2 : \sum_{n=1}^{\infty} \lambda_n = \infty;\ C3 : \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n^2} = 0.$$

Reich (1980) proved that the result of Halpern remains true when $E$ is uniformly smooth. It was observed that both Halpern's and Lion's conditions on the real sequence $\{\lambda_n\}$ excluded the canonical choice $\lambda_n = \frac{1}{n+1}$. Xu (2002) improved the result of Lion in two fold. First, Xu weakened the condition C3 by removing the square in the denominator so that the canonical choice

of $\lambda_n = \frac{1}{n+1}$ is possible. Secondly, Xu proved the strong convergence of the scheme (2.1.1) in the framework of real uniformly smooth Banach space. Xu (2002) also remarked that Halpern (1967) observed that conditions (C1) and (C2) are necessary for the strong convergence of algorithm (2.1.1) for the non-expansive mappings $T : K \to K$. It is however not clear if they are sufficient. This brought about the following question:

**Question:** Are the conditions $C1 : \lim_{n\to\infty} \lambda_n = 0$ and $C2 : \sum_{n=1}^{\infty} \lambda_n = \infty$ sufficient for the strong convergence of algorithm (2.1.1) for all nonexpansive mappings $T : K \to K$?

Chidume and Chidume (2006) modified the recursion formula (2.1.1) by introducing an auxiliary operator that has the same set of fixed points as $T$. With the help of this operator, they proved that, under suitable conditions, conditions C1 and C2 are sufficient for the modified iterative algorithm to converge strongly to a fixed point of $T$ even in the more general setting where $E$ is a real Banach space with uniformly Gâteaux differentiable norm.

Also many mathematicians have studied different iterative processes for approximation of fixed points of different classes of mappings which in some sense generalized nonexpansive mappings.

Moudafi (2000) introduced the viscosity approximation method for nonexpansive mappings. Let $f$ be a contraction on $H$, starting with an arbitrary $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \ n \geq 0, \tag{2.1.2}$$

where $\{\alpha_n\}$ is a sequence in (0,1). Moudafi proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}_{n\geq 0}$ generated by (2.1.2) strongly converges to the unique solution $x^*$ in $F$ of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \text{ for all } x \in F.$$

Xu (2003), proved, under some conditions on the real sequence $\{\alpha_n\}_{n\geq 0}$, that the sequence $\{x_n\}$ defined by $x_0 \in H$ (chosen arbitrarily),

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n,\ n \geq 0, \tag{2.1.3}$$

converges strongly to $x^* \in F$ which is the unique solution of the minimization problem

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where $A$ is a strongly positive bounded linear operator (i.e. $\exists\ \bar{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \bar{\gamma}||x||^2,\ \forall\ x \in H$).

Combining the iterative methods (2.1.2) and (2.1.3), Marino and Xu (2006), studied the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n,\ n \geq 0. \tag{2.1.4}$$

The authors proved that if the sequence $\{\alpha_n\}_{n\geq 0}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}_{n\geq 0}$ generated by (2.1.4) converges strongly to $x^* \in F$ which solves the variational inequality problem

$$\langle (\gamma f - A)x^*, x - x^*\rangle \leq 0\ \forall x \in F,$$

and is also the optimality condition for the minimization problem

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where $h$ is a potential function of $\gamma f$ (i.e. $h'(x) = \gamma f(x)$ for $x \in H$).

On the other hand, Yamada (2001) introduced the following hybrid iterative method:

$$x_{n+1} = Tx_n - \lambda_n \mu GTx_n,\ n \geq 0, \tag{2.1.5}$$

where $G$ is a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator with $\kappa > 0, \eta > 0$ and $0 < \mu < 2\eta/\kappa^2$. Under some appropriate conditions, Yamada proved that the sequence $\{x_n\}_{\geq 0}$ generated by (2.1.5) converges strongly to the unique solution of the variational inequality problem

$$\langle Gx^*, x - x^* \rangle \geq 0, \ \forall x \in F.$$

Recently, combining (2.1.4) and (2.1.5), Tian (2010) considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)T(x_n), \tag{2.1.6}$$

and proved that the sequence $\{x_n\}_{\geq 0}$ generated by (2.1.6) converges strongly to the unique solution $x^* \in F$ of the variational inequality problem

$$\langle (\gamma f - \mu G)x^*, x - x^* \rangle \leq 0, \ \forall x \in F.$$

Ali et al. (2011) extended the result of Tian (2010) to $q$-uniformly smooth Banach space whose duality mapping is weakly sequentially continuous. Under some assumptions on $\{\alpha_n\}, \gamma, \mu$ and $G$, they proved that the sequence $\{x_n\}$ generated by (2.1.6) converges strongly to the unique solution $x^* \in F$ of the variational inequality problem

$$\langle (\gamma f - \mu G)x^*, j(x - x^*) \rangle \leq 0, \ \forall x \in F.$$

## 2.2 Approximation of Common Fixed Points

Ali (2009) studied a new scheme for approximation of a common fixed point of family of nonexpansive mappings in a real $q$-uniformly smooth Banach space

which is also uniformly convex. He proved the following theorem.

**Theorem 2.2.1** (Ali, 2009). *Let $E$ be a real $q$-uniformly smooth Banach space which is also uniformly convex. Let $K$ be a closed, convex and nonempty subset of $E$. For $\alpha > 0$, let $T_i : K \to K$ $i \in \mathbb{N}$ be a family of nonexpansive mappings and let $A : K \to E$ be an $\alpha$-inverse strongly accretive mapping. Let $P_K$ be a nonexpansive projection of $E$ onto $K$. For some real number $\delta \in (0,1)$ and $\lambda \in (0, (\frac{q\alpha}{d_q})^{\frac{1}{q-1}})$ define a sequence $\{x_n\}$ iteratively by $x_1, u \in K$,*

$$x_{n+1} = \alpha_n u + (1-\delta)(1-\alpha_n)x_n + \delta \sum_{i\geq 1} \sigma_{in} T_i P_K(x_n - \lambda A x_n),\ n \geq 1, \quad (2.2.1)$$

*where $\{\alpha_n\}_{\geq 1}$ and $\{\sigma_{in}\}_{\geq 1}$ are real sequences in $(0,1)$ satisfying the following conditions: (i) $\lim \alpha_n = 0$, (ii) $\sum \alpha_n = \infty$, (iii) $\sum_{i\geq 1} \sigma_{in} = 1 - \alpha_n$ and $\lim_{n\to\infty} \sum_{i\geq 1} |\sigma_{i,n+1} - \sigma_{in}| = 0$. Let $F := [\cap_{i=1}^{\infty} F(T_i)] \cap VI(K, A) \neq \emptyset$. If either the duality map $j$ of $E$ admits weak sequential continuity or for at least one $i \in \mathbb{N}$, $T_i P_K(I - \lambda A)$ is demicompact, then $\{x_n\}$ converges strongly to some element in $F$.*

Motivated by the above results, in chapter four theorem 4.1.1 and 4.1.5 of this thesis, we introduce a new modified iterative scheme for approximation of common fixed points of family of asymptotically nonexpansive mappings and solutions of some variational inequality problems, and also prove strong convergence theorems that extend, generalize and improve some recent results.

Yao et al. (2009) studied a modified iteration algorithm $\{x_n\}_{n\geq 1}$ generated from an arbitrary $x_1 \in H$ by

$$\begin{cases} v_n = (1-t_n)x_n \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n T v_n, \end{cases} \quad (2.2.2)$$

where $\{t_n\}_{n\geq 1}$ and $\{\alpha_n\}_{n\geq 1}$ are real sequences in $(0,1)$ satisfying some appropriate conditions. They proved strong convergence of the modified algorithm to a fixed point of a nonexpansive mapping $T : H \to H$ and $F(T) \neq \emptyset$.
Clearly, the modified Mann iteration algorithm reduces to the normal Mann iteration algorithm when $t_n \equiv 0$.
Recently Osilike et al. (2013) modified the algorithm (2.2.2) and proved the following theorem:

**Theorem 2.2.2** (Osilike et al., 2013). *Let $K$ be a nonempty closed convex subset of a real Hilbert space and let $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_{n\geq 1} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $F(T) \neq \emptyset$ and let $\{t_n\}_{n\geq 1}$ and $\{\alpha_n\}_{n\geq 1}$ be sequences in (0,1) satisfying the following conditions:*

*(c1)* $\lim_{n\to\infty} t_n = 0$;

*(c2)* $\sum_{n=1}^{\infty} t_n = \infty$;

*(c3)* $\lim_{n\to\infty} \frac{1}{t_n}(k_n - 1) = 0$;

*(c4)* $\alpha_n \in [a,b]\ \forall\ n \geq 1$ *and for some* $a, b \in (0,1)$.

*Then the modified average iteration sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by*

$$\begin{cases} v_n = P_K\Big((1-t_n)x_n\Big) \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n T^n v_n, \end{cases} \tag{2.2.3}$$

*converges strongly to a fixed point of $T$.*

In chapter four theorem 4.1.7 of this thesis we modify the algorithm (2.2.2) and prove strong convergence of both implicit and explicit of the modified algorithm to a common fixed point of a family of uniformly asymptotically regular asymptotically nonexpansive mappings in a reflexive Banach space

with a uniformly Gâteaux differentiable norm.

Let $T_i : E \to E$, $i = 1, 2, 3, ...N$ be a finite family of nonexpansive maps. For $n \in \mathbb{N}$, define a map $W_n : E \to E$ by

$$\begin{aligned} U_{1,n} &= \gamma_{n,1}T_1 + (1 - \gamma_{1,n})I, \\ U_{2,n} &= \gamma_{2,n}T_2U_{1,n} + (1 - \gamma_{2,n})I \\ &\vdots \\ W_n = U_{N,n} &= \gamma_{N,n}T_NU_{N-1,n} + (1 - \gamma_{N,n})I, \end{aligned}$$

where $I = U_{0,n}$ and $\{\gamma_{i,n}\}_i^N \subseteq [0, 1]$. The mapping $W_n$ here is called the $W$ mapping generated by $T_i, T_2, ..., T_N$ and $\{\gamma_{i,n}\}_{n\geq 1}$, $i \in \{1, 2, ..., N\}$.

.Shang et al. (2007) introduced a composite iterative scheme as follows: given $x_0 = x \in C$ arbitrarily chosen,

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n)W_n x_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \end{aligned}$$

where $f$ is a contraction, and $A$ is a strongly positive linear and bounded operator on Hilbert space.

Kangtunyakarn and Suantai (2009), introduced and studied the following scheme for approximation of common fixed point of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$, for $n \in \mathbb{N}$;

$$\begin{aligned} U_{1,n} &= \gamma_{1,n}T_1 + (1 - \gamma_{1,n})I, \\ U_{2,n} &= \gamma_{2,n}T_2U_{1,n} + (1 - \gamma_{2,n})U_{1,n}, \\ &\vdots \\ K_n = U_{N,n} &= \gamma_{N,n}T_NU_{N-1,n} + (1 - \gamma_{N,n})U_{N-1,n}. \end{aligned} \tag{2.2.4}$$

The mapping $K_n$ here is called the $K$ mapping generated by $T_i, T_2, ..., T_N$ and $\{\gamma_{i,n}\}_{n\geq 1}$, $i \in \{1, 2, ..., N\}$.

Singthong and Suantai (2010) studied the convergence of the following composite scheme $x_0 \in C$,

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n)K_n x_n, \\ x_{n+1} &= P_C(\alpha_n \gamma f(x_n)) + (I - \alpha_n A)y_n, \end{aligned} \tag{2.2.5}$$

where $C$ is a nonempty, closed convex subset of Hilbert space $H$, $f : C \to C$ is a contraction, and $A$ is a strongly positive linear and bounded operator on $H$.

Recently, Ali et al. (2013) introduced a modified iterative scheme for approximation of common fixed point of a finite family $\{T_i\}_{i=1}^N$ of nonexpansive mappings for $n \in \mathbb{N}$ and a sequence $\{\gamma_{i,n}\}$, $i \in \{1, 2, \cdots, N\}$, as follows:

$$\begin{aligned} U_{1,n} &= \gamma_{1,n}T_1 + (1-\gamma_{1,n})I, \\ U_{2,n} &= \gamma_{2,n}T_2U_{1,n} + (1-\gamma_{2,n})U_{1,n}, \\ &\vdots \\ U_{N-1,n} &= \gamma_{N-1,n}T_{N-1}U_{N-2,n} + (1-\gamma_{N-1,n})U_{N-2,n} \\ K_n = U_{N,n} &= \gamma_{N,n}T_NU_{N-1,n} + (1-\gamma_{N,n})I. \end{aligned} \tag{2.2.6}$$

and proved strong convergence of both implicit and explicit schemes to a common fixed point of a finite family of nonexpansive mappings which is also a unique solution of some variational inequality problem in a framework of a Banach space much more general than Hilbert space. They proved the following theorems:

**Theorem 2.2.3** (Ali et al., 2013). *Let $E$ be a reflexive and strictly convex real Banach space with a uniformly Gâteaux differentiable norm. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $E$ into itself and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $f : E \to E$ be a contraction with constant $\alpha \in (0, 1)$. Let $G : E \to E$*

*be an $\eta-$ strongly accretive and $\mu-$ strictly pseudocontractive mapping with $\eta+\mu>1$ and let $\tau=1-\sqrt{\frac{1-\eta}{\mu}}$. Let $\gamma$ be a real number satisfying $0<\gamma<\frac{\tau}{\alpha}$ and let $K:E\to E$ be as in (2.2.6). Given $\beta\in(0,1)$, then for any $t\in(0,1)$, let $\{z_t\}_{t\in(0,1)}$ be a path defined by*

$$z_t = t\gamma f(z_t) + (I - tG)[\beta z_t + (1-\beta)Kz_t]. \tag{2.2.7}$$

*Then $\{z_t\}$ converges strongly to a common fixed point of the family say $p$ which is a unique solution of the variational inequality*

$$\langle (G-\gamma f)p, j(q-p)\rangle \geq 0,\ \forall\ q\in F. \tag{2.2.8}$$

**Theorem 2.2.4** (Ali et al., 2013). *Let $E$ be a reflexive and strictly convex real Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$. Let $G:E\to E$ be an $\eta$-strongly accretive and $\mu$-strictly pseudocontractive mapping with $\eta+\mu>1$ and let $f:E\to E$ be a contraction with coefficient $\alpha\in(0,1)$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $E$ into itself and $F=\bigcap_{i=1}^N F(T_i)\neq\emptyset$. Let $K_n$ be as in (2.2.6). Assume that $0<\gamma<\frac{\tau}{2\alpha}$, where $\tau:=(1-\sqrt{\frac{1-\eta}{\mu}})$ and let $x_0\in C$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

$(C1)$ $\alpha_n\to 0$ *as* $n\to\infty$;

$(C2)$ $\Sigma_{n=0}^\infty\alpha_n=\infty$

$(C3)$ $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;

$(C4)$ $\Sigma_{n=1}^\infty|\gamma_{i,n}-\gamma_{i,n-1}|<\infty$, *for all* $i=1,2,3,...,N$ *and* $\{\gamma_{i,n}\}_{i=1}^N\subset[a,b]$, *where* $0<a\leq b<1$;

$(C5)$ $\Sigma_{n=1}^\infty|\alpha_{n+1}-\alpha_n|<\infty$;

$(C6)$ $\Sigma_{n=1}^\infty|\beta_{n+1}-\beta_n|<\infty$.

*If* $\{x_n\}_{n=1}^{\infty}$ *is a sequence defined by,*

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n)K_n x_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n G)y_n, \quad n \geq 0, \end{aligned} \tag{2.2.9}$$

*then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in F$*, which also solves the variational inequality* (2.2.8)*.*

In chapter four theorem 4.1.10 of this thesis, we continue the study of the above problems Ali et al. (2013), and prove a new convergence theorem for approximation of common fixed point of finite family $\{T_i\}_{i=1}^{N}$ of asymptotically nonexpansive mappings in Banach spaces which is also a unique solution of some variational inequality problem.

## 2.3 Iterative Algorithm for Generalized Projection Mappings

Matsushita and Takahashi (2005) proved weak and strong convergence theorems for fixed point of a single relative nonexpansive mapping in a uniformly convex and uniformly smooth real Banach space by using the following scheme

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \ n \geq 1, \end{cases}$$

where $\Pi_{C_n \cap Q_n}$ is the generalized metric projection from $E$ onto $C_n \cap Q_n$, $J$ is the normalized duality mapping on $E$ and $\{\alpha_n\}_{n\geq 1}$ is a real sequence in (0,1).

Various generalizations of relatively nonexpansive mappings were introduced and studied by numerous authors see for example Ali and Minjibir (2012), Kimura and Takahashi (2009), Li et al. (2010), Nilsrakoo (2011), Noor (2000), Plubtieng and Punpaeng (2008), (just to mention but a few) and the references contained in them. Chang et al. (2010) obtained a strong convergence theorem for an infinite family of quasi-$\phi$-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space with Kadec-Klee property. This was achieved by using an iterative algorithm for an infinite family of closed and quasi-$\phi$-asymptotically nonexpansive mappings $T_i : C \to C$ with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ and $C$ closed and convex nonempty subset of a uniformly smooth and strictly convex real Banach space $E$

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J T_i^n x_n), \\ u_n \in C \text{ such that}, \\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \ \forall y \in C, , \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \ n \geq 0 \end{cases}$$

where $f : C \times C \to \mathbb{R}$ is a bifunction. The bifunction is satisfying conditions (A1)-(A4) with $EP(f)$ the set of solutions of the equilibrium problem. The mapping $J : E \to E^*$ is the normalized duality mapping, $\xi_n = \sup_{u \in \mathfrak{F}}(k_n - 1)\phi(u, x_n)$ where $\mathfrak{F} := \cap_{i=1}^{\infty} F(T_i) \cap EP(f)$, $\Pi_{\cap_{i=1}^{\infty} F(T_i) \cap EP(f)}$ is the generalized projection of $E$ onto $\cap_{i=1}^{\infty} F(T_i) \cap EP(f)$. The sequence $\{r_n\}$ is a real sequence in $[a, \infty)$ where $a$ is some positive real number, and for each $n \geq 0, \{\alpha_{n,i}\}$ is a sequence in $[0, 1]$.

In another development, Zhou and Gao (2010) introduced and studied the class

of quasi-strictly psudo-contractive mappings related to the function $\phi$. They proved a strong convergence theorem by hybrid projection algorithm for fixed point of a closed and quasi-strict pseudo-contraction in a reflexive and strictly convex real Banach space $E$. They actually proved the following theorem.

**Theorem 2.3.1** (Zhou and Gao, 2010). *Let $E$ be a real reflexive, strictly convex and smooth real Banach space such that $E$ and $E^*$ have Kadec-Klee property. Assume that $C$ is a nonempty closed convex subset of $E$. Let $T : C \to C$ be a closed and quasi-strict pseudo-contraction. Define a sequence $\{x_n\}$ in $C$ by the following algorithm;*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ C_{n+1} = \{z \in C_n : \phi(x_n, Tx_n) \leq \frac{1}{1-k}\langle x_n - z, Jx_n - JTx_n\rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \ n \geq 0, \end{cases} \tag{2.3.1}$$

*where $k \in [0, 1)$. Then $\{x_n\}$ converges strongly to $p_0 = \Pi_{F(T)}x_0$.*

Qin et al. (2012) introduced and studied the class of asymptotically strict quasi-$\phi$-pseudocontractions in the intermediate sense. They proved the following Theorem.

**Theorem 2.3.2** (Qin et al., 2012). *Let $E$ be a uniformly convex and smooth Banach space. Let $C$ be a nonempty closed and convex subset of $E$. Let $\Delta$ be an index set and $T_i : C \to C$, $i \in \Delta$ be a family of asymptotically strict-pseudocontraction in the intermediate sense with sequences $\{\mu_{i,n}\}$ such that $\mu_{i,n} \to 0$ as $n \to \infty$ for each $i \in \Delta$. Assume that $\bigcap_{i\in\Delta} F(T_i) \neq \emptyset$. For each $i \in \Delta$ assume that $T_i$ is closed and asymptotically regular on $C$. Let $F(T_i)$ be bounded*

*and* $\{x_n\}$ *be a sequence generated as follows;*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ C_{1,i} = C, \\ C_1 = \underset{i\in\Delta}{\cap} C_{1,i}, \\ C_{n+1,i} = \{z \in C_{n,i} : \phi(x_n, T_i^n) \leq \frac{2}{1-k_i}\langle x_n - z, Jx_n - JT_i^n x_n\rangle + \mu_{n,i}\frac{M_{n,i}}{1-k_i} + \frac{\zeta_{n,i}}{1-k_i}\}, \\ C_{n+1} = \cap_{i\in\Delta} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall n \geq 0, \end{cases}$$

*where* $M_{n,i} = sup\{\phi(p, x_n) : p \in F(T_i)\}$ *and*

$$\zeta_{n,i} = Max\Big\{0, \sup_{p\in F(T), x\in C}\Big(\phi(p, T^n x) - (1+u_n)\phi(p, x) - k\phi(x, T^n x)\Big)\Big\}.$$

*Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\underset{i\in\Delta}{\cap} F(T_i)} x_0$.

In chapter four theorem 4.2.1 of this thesis, we introduce a new modified iterative scheme and prove strong convergence theorem for family of generalized asymptotically quasi-$\phi$-strict pseudo-contraction in a real reflexive, strictly convex and smooth Banach space $E$ in which both $E$ and $E^*$ have Kadec-Klee property.

Using properties of generalized $f$-projection operator in a uniformly smooth real Banach space which is also uniformly convexLi et al. (2010) introduced the following hybrid iterative scheme for approximation of fixed point of a relatively nonexpansive mapping $T$:

$$\begin{cases} x_0 \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \ n \geq 0. \end{cases}$$

They proved a strong convergence of the scheme to an element in the fixed point set of $T$.

Homaeipour and Razani (2011) proved weak and strong convergence theorems for a single relatively nonexpansive multi-valued mapping in a uniformly convex and uniformly smooth real Banach space.

Tang and Chang (2012), introduced a new hybrid iterative scheme for approximation of fixed point of a total quasi-$\phi$-asymptotically nonexpansive multi-valued mapping in a uniformly smooth and strictly convex real Banach space with Kadec-Klee property. They actually proved the following theorem.

**Theorem 2.3.3** (Tang and Chang, 2012) *Let $E$ be a uniformly smooth and strictly convex real Banach space with Kadec-Klee property, and $C$ be a nonempty closed and convex subset of $E$. Let $T : C \to 2^C$ be a closed and total quasi-$\phi$-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{v_n\}$ and $\{\mu_n\}$ and strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu_1 = 0, v_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$. Let and $\{x_n\}_{n\geq 0}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, C_0 = C \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n) \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jw_n) \\ C_{n+1} = \{v \in C_n : \phi(v, Jy_n) \leq \phi(v, Jx_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ n \geq 0, \end{cases} \tag{2.3.2}$$

*where $w_n \in T^n x_n, \ \forall n \geq 0, \ \xi_n = v_n \sup_{p\in F(T)} \zeta(\phi(p, x_n)) + \mu_n, \ \Pi_{C_n+1}$ is the generalized projection of $E$ onto $C_{n+1} \ \forall \ n \geq 1, \ \{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$ are sequences in $[0, 1]$ satisfying the following conditions:*

*(a)* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;

*(b)* $0 \leq \alpha_n \leq \alpha < 1$ *for some* $\alpha \in (0, 1) \ \forall \ n \geq 1$. *If $F(T)$ is a nonempty and bounded subset of $C$, then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.*

More recently, Zhang et al. (2013) introduced the notion of weak relatively nonexpansive mappings. They studied a new hybrid algorithm for fixed point of multivalued weak relatively nonexpansive mappings. They proved the following Theorem.

**Theorem 2.3.4** (Zhang et al., 2013). *Let $E$ be a uniformly convex and uniformly smooth real Banach space, let $C$ be a nonempty closed convex subset of $E$, let $T : C \to C$ be a weak relatively multivalued nonexpansive mapping. Assume that $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ such that $\alpha_n+\beta_n+\gamma_n = 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $0 < \gamma \leq \gamma_n \leq 1 \ \forall \ n \geq 1$ for some constant $\gamma \in (0, 1)$. Define a sequence $\{x_n\}$ in $C$ by the following algorithm:*

$$\begin{cases} x_0 \in C_0 = C \\ y_n = J^{-1}(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n Jz_n), z_n \in Tx_n \\ C_n = \{z \in C_{n-1} : \phi(z, y_n) \leq (1 - \alpha_n)\phi(z, x_n) + \alpha_n\phi(z, x_0), n \geq 1\}, \\ x_{n+1} = \Pi_{C_n} x_0. \end{cases}$$

*Then $\{x_n\}_{n\geq 0}$ converges strongly to $q = \Pi_{F(T)} x_0$.*

In chapter four theorem 4.3.2 of this thesis, we introduce a new modified iterative scheme and prove strong convergence theorem for infinite family of total quasi-$\phi$-asymptotically nonexpansive multi-valued mappings in a real uniformly convex and uniformly smooth Banach space using a generalized $f-$projection operator. Our result improve and unify several recent important results announced by numerous authors.

## 2.4 Iterative Algorithm for Bregman Projection Mappings

In 2006, Butnariu and Resmerita presented Bregman-type iterative algorithms and studied the convergence of the Bregman-type iterative method for solving some nonlinear operator equations.

By using the Bregman projection, Reich and Sabach (2011) proposed algorithms for finding common fixed points of finitely many Bregman firmly nonexpansive operators in a reflexive Banach space.

$$\begin{cases} x_0 \in E \\ y_n^i = T_i(x_n + e_n^i) \\ C_n^i = \{z \in E : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\} \\ C_n := \bigcap_{i=1}^{N} C_n^i \\ Q_n = \{z \in E : \langle z - x_n, \nabla f(x_0) - \nabla f(x_n)\rangle \leq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}^f x_0, n \geq 0. \end{cases} \tag{2.4.1}$$

Under some suitable conditions, they proved that the sequence generated by (2.4.1) converges strongly to $\bigcap_{i=1}^{N} F(T_i)$ and applied the result for the solution of convex feasibility and equilibrium problems.

Chen et al. (2011), introduced the concept of weak Bregman relatively nonexpansive mappings in a reflexive Banach space and gave an example to illustrate the existence of a weak Bregman relatively nonexpansive mapping and the difference between a weak Bregman relatively nonexpansive mapping and a Bregman relatively nonexpansive mapping. They also proved strong convergence of the sequences generated by the constructed algorithms with errors for finding a fixed point of weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings under some suitable conditions.

Zhu et al. (2013) proved a strong convergence theorem for finding a common

element of the set of solutions of an equilibrium problem and the set of fixed points of a Bregman strongly nonexpansive mapping in a real reflexive Banach space. This was achieved by using the shrinking projection method. In particular, they proved the following:

**Theorem 2.4.1** (Zhu et al., 2013) *Let $C$ be a nonempty, closed, and convex subset of a real reflexive Banach space $E$ and $f : E \to \mathbb{R}$ strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of $E$. Let $g$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $T$ be a Bregman strongly nonexpansive mapping from $C$ into itself such that $\hat{F}(T) = F(T)$ and $G = F(T) \bigcap EP(g) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C,\ C_0 = C$ and*

$$\begin{cases} y_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n), \\ u_n \in C \text{ such that} \\ g(u_n, y) + \langle \nabla f(u_n) - \nabla f(y_n), y - u_n \rangle \geq 0,\ \forall\ y \in C, \\ C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq D_f(z, x_n)\}, \\ x_{n+1} = P^f_{C_{n+1}} x. \end{cases}$$

*for every $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $P^f_{F(T)\cap EP(g)}x$, where $P^f_{F(T)\cap EP(g)}$ is the Bregman projection of $E$ onto $F(T) \cap EP(g)$.*

Motivated and inspired by the above works, in chapter four theorem 4.4.3 of this thesis, we introduce a new iterative algorithm and prove strong convergence theorem for finite family of closed quasi-Bregman strictly pseudocontractive mapping and system of equilibrium problem in a real reflexive Banach space. Our result improve and unify several recent important results announced by numerous authors.

# CHAPTER THREE

# THEORY OF METHODS

The method of research adopted in this thesis is by consulting necessary and relevant literature on approximation of fixed points for some nonlinear operators. We then use techniques and results in Mathematical analysis to establish our new results.

## 3.1 Geometry of Banach Spaces

This section gathers some basic definitions and geometrical properties of Banach spaces which can be found in Chidume (2009). We also surveyed basic results of convex analysis and connect them to characterize those spaces in which some iteration methods are defined and proved to converge to solutions of fixed point problems for certain nonlinear mappings. Let start by defining the duality pairing.

**Definition 3.1.1** Let $E$ be a real normed space. The space $E^*$ of all linear continuous functionals on $E$ is called the *dual* space of $E$. For $f \in E^*$ and $x \in E$ the value of $f$ at $x$ is denoted by $\langle x, f\rangle$ and is called the duality pairing.

**Element of Convex Analysis**

We briefly introduce some elements of convex analysis which can be found in many texts on the general theory of convex analysis.

**Definition 3.1.2** A function $f : E \to \mathbb{R}$ is said to be

(i) *proper* if and only if its effective, $dom(f) = \{x \in E : f(x) < \infty\}$, is nonempty;

(ii) *convex* if and only if for all $\lambda \in [0,1]$ and $x, y \in dom(f)$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$;

(iii) *lower semicontinuous* at $x_0$ if and only if for all $x \in E$, $f(x_0) \leq \liminf_{x \to x_0} f(x)$. We say that $f$ is lower semicontinuous on $dom(f)$ if it is so at every $x_0 \in dom(f)$;

(iv) *Gâteaux differentiable* at $x_0 \in dom(f)$ if there exists an $\nabla f \in E^*$ such that for all $y \in dom(f)$

$$\lim_{t \to 0^+} \frac{f(x_0 + ty) - f(x_0)}{t} = \langle y, \nabla f \rangle;$$

(v) *Fréchet differentiable* at $x_0 \in dom(f)$ if it is Gâteaux differentiable and

$$\lim_{t \to 0^+} \sup_{x||=1} \Big| \frac{f(x_0 + ty) - f(x_0)}{t} - \langle y, \nabla f \rangle \Big|;$$

(vi) Let $x \in \text{int dom} f$, the subdifferential of $f$ at $x$ is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y),\ \forall y \in E\},$$

where the Fenchel conjugate of $f$ is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in E\}.$$

We know that the Young-Fenchel inequality holds:

$$\langle x^*, x \rangle \leq f(x) + f^*(x^*),\ \forall x \in E,\ x^* \in E^*.$$

(vii) A function $f$ on $E$ is coercive ( Hiriart-Urruty and Lemarchal (1993)) if

the sublevel set of $f$ is bounded; equivalently,

$$\lim_{||x||\to+\infty} f(x) = +\infty,$$

and $f$ is said be strongly coercive if

$$\lim_{||x||\to+\infty} \frac{f(x)}{||x||} = +\infty.$$

**Classes of Banach spaces**

A Banach space $E$ is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. The space $E$ is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{||x+ty|| - ||x||}{t}$$

exists for all $x, y \in U := \{z \in E : ||z|| = 1\}$. In this case, the norm of $E$ is said to be *Gâteaux* differentiable. A Banach space $E$

(i) is uniformly convex if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in X$ with $||x|| = 1, ||y|| = 1$ and $||x - y|| \geq 1$, then $\left\|\frac{1}{2}(x+y)\right\| \leq 1 - \delta$;

(ii) is said to have *Kadec-Klee property* if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$ implies $||x_n - x|| \to 0$ as $n \to \infty$.

Let $J : E \to E^{**}$ be a mapping from $E$ into $E^{**}$, defined for each $x \in E$ by $J(x) = \phi_x$ where $\phi_x : E^* \to \mathbb{R}$ is given by $\phi_x(f) = \langle f, x\rangle$ for each $f \in E^*$. Thus $\langle J(x), f\rangle \equiv \langle f, x\rangle$. The mapping $J$ defined above is called canonical map (or canonical embedding) of $E$ into $E^{**}$, and the space $E$ is said to be embedded in $E^{**}$.

**Definition 3.1.3** Let $E$ be a normed linear space and let $J$ be the canonical embedding of $E$ into $E^{**}$. If $J$ is onto, then $E$ is called reflexive. Thus, a reflexive Banach space is one in which the canonical embedding is onto.

## 3.2 Duality Mapping

Let $E$ be a real Banach space and $E^*$ be the dual space of $E$. A mapping $\varphi : [0,\infty) \to [0,\infty)$ is called a guage function if it is strictly increasing, continuous and $\varphi(0) = 0$. Let $\varphi$ be a gauge function, a generalized duality mapping with respect to $\varphi, J_\varphi : E \to 2^{E^*}$ is defined for all $x \in E$, by

$$J_\varphi x = \{x^* \in E^* : \langle x, x^* \rangle = ||x||\varphi(||x||), ||x^*|| = \varphi(||x||)\},$$

where $\langle ., . \rangle$ denotes the duality pairing between element of $E$ and that of $E^*$. If $\varphi(t) = t$, then $J_\varphi$ is simply called the normalized duality mapping and is denoted by $J$ and

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

For any $x \in E$, an elements of $J_\varphi x$ is denoted by $j_\varphi x$.

The space $E$ is said to have weakly (sequentially) continuous duality map if there exists a gauge function $\varphi$ such that $J_\varphi$ is single valued and (sequentially) continuous from $E$ with weak topology to $E^*$ with weak$^*$ topology; that is if $\{x_n\} \subset E, x_n \rightharpoonup x$, then $J_\varphi(x_n) \rightharpoonup^* J_\varphi(x)$.

In a normed space $E$, apart from convergence of a sequence $\{x_n\}_{n=1}^\infty$ in $E$, i.e. convergence in the topology induced by the norm on $E$ where a sequence $\{x_n\}_{n=1}^\infty$ in $E$ is said to converge strongly to a point $a \in E$ if and only if $||x_n - a|| \to 0$ as $n \to \infty$ (i.e., for all $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0,\ ||x_n - a|| < \epsilon$). We also have the concept of weak convergence of a $\{x_n\}_{n=1}^\infty$ in $E$, i.e. convergence in the weak topology of $E$, where a sequence $\{x_n\}_{n=1}^\infty$ in $E$ is said to converge weakly (or rather in the weak topology) to $a \in E$ if and only if for any $f \in E^*,\ \langle x_n, f\langle \to \langle x, f\rangle$ as $n \to \infty$. Whenever $\{x_n\}_{n=1}^\infty$ converges strongly to $a \in E$, we write $x_n \to a$ as $n \to \infty$ and whenever $\{x_n\}_{n=1}^\infty$ converges weakly to $a \in E$ we write $x_n \rightharpoonup a$ as $n \to \infty$

**Remark:** Any weakly convergence sequence $\{x_n\}$ in a Banach space is bounded. Further, if $x_n \rightharpoonup a$, then $||a|| \leq \liminf ||x_n||$. The following properties are well known (Cioranescu, 1990; Reich and Sabach, 2009; Takahashi, 2000) for details:

(i) if $E$ is smooth, then $J$ is single-valued;

(ii) if $E$ is strictly convex, then $J$ is one-to-one

(iii) if $E$ is reflexive, then $J$ is surjective;

(iv) if $E$ is smooth and reflexive, then $J^{-1} : E^* \to 2^E$ is single-valued and demi-continuous (i.e. if $\{x_n^*\} \subset E^*$ such that $x_n^* \to x^*$, then $J^{-1}(x_n^*) \rightharpoonup J^{-1}(x^*)$);

(v) $E$ is uniformly smooth if and only if $E^*$ is uniformly convex;

(vi) if $E$ is uniformly convex then it is strictly convex, reflexive and has Kadec Klee property.

A normed space $E$ is said to satisfy *Opial's condition* (Lin et al., 1995) if for any sequence $\{x_n\}$ in $E$, such that $x_n \rightharpoonup x$ as $n \to \infty$, we obtain that for any $y \in E$ with $y \neq x$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

All Hilbert spaces and $l_p$ spaces, $1 < p < \infty$ satisfy Opial's condition.

The space $E$ is said to have *uniform Opial's condition* (Lin et al., 1995), if and only if for each $c > 0$, there exists an $r = r(c) > 0$ such that

$$1 + r \leq \liminf_{n\to\infty} \|x + x_n\|$$

for each $x \in E$ with $\|x\| \geq c$ and each sequence $\{x_n\}$ such that $x_n \rightharpoonup 0$ as $n \to \infty$, and $\underset{n\to\infty}{liminf}\|x_n\| \geq 1$.

The space $E$ is said to satisfy local uniform Opial's condition (Lin et al., 1995) if and only for any sequence $\{x_n\}$ in $E$ ssuch that $x_n \rightharpoonup 0$ with $\underset{n\to\infty}{liminf}\|x_n\| \geq 1$ and any $c > 0$, there exists $r > 0$ such that

$$1 + r \leq \underset{n\to\infty}{liminf}\|x + x_n\|$$

for all $x \in E$ with $\|x\| \geq c$.

**Remark 3.2.1** (Lin et al., 1995) Observed that uniform Opial's condition implies local uniform Opial's condition which in turn implies Opial's condition. It is also known that every Banach space with weakly sequentially continuous duality mapping satisfies uniform Opial's condition. Every $l_p$ space, $(1 < p < \infty)$ has a weakly sequentially continuous duality map.

## 3.3 Equilibrium Problem

Let $K$ be a nonempty subset of a natural linear space and $f : K \times K \to \mathbb{R}$ be some nonlinear map. An equilibrium problem is search for $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0 \text{ for all } y \in K. \tag{3.3.1}$$

To make the data of an equilibrium problem more precise, we assume $X$ is a real topological vector space; $K \subset X$ is a closed convex set and $f : K \times K \to \mathbb{R}$ satisfies $f(x, x) = 0$ for all $x \in K$. Then we consider the equilibrium problem find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0 \ \forall \ y \in K. \tag{3.3.2}$$

In the examples below $E^*$, the topological dual of $E$, should be topologized in such a way that the canonical bilinear form $\langle ., . \rangle$ is contained on $X^* \times X$. The function $f : K \times K \to \mathbb{R}$ will be called monotone if and only if

$$f(x, y) + f(y, x) \leq 0 \ \forall \ x, y \in K.$$

Examples for problem (3.6.3)

(i) Optimization: Let $\varphi : K \to \mathbb{R}$ be a map, we will like to find

$$\bar{x} \in K \ \textit{such that} \ \varphi(\bar{x}) \leq \varphi(y) \ \forall y \in K \tag{3.3.3}$$

we write $\min\{\varphi(x) : x \in K\}$ for this problem we set

$$f(x, y) := \varphi(y) - \varphi(x)$$

problem (3.3.3) coincides with (3.6.3). The function $f$ is monotone in this case.

(ii) Saddle Point: Let $\varphi : K_1 \times K_2 \to \mathbb{R}$. Then $(\bar{x_1}, \bar{x_2})$ is called a saddle point of $\varphi$ if and only if

$$(\bar{x_1}, \bar{x_2}) \in K_1 \times K_2, \varphi(\bar{x_1}, y_2) \leq \varphi(y_1, \bar{x_2}) \ \forall (y_1, y_2) \in K_1 \times K_2. \tag{3.3.4}$$

Set $K = K_1 \times K_2$ and define $f : K \times K \to \mathbb{R}$ by

$$f((x_1, x_2), (y_1, y_2)) := \varphi(y_1, x_2) - \varphi(x_1, y_2).$$

Then $\bar{x} = (\bar{x_1}, \bar{x_2})$ is a solution of (3.6.3) if and only if, it satisfies (4.4.3). $f$ is monotone in this case.

(iii) Fixed Point: Let $H = H^*$ be a Hilbert space. Let $T : K \to K$ be a given

mapping. We want find $\bar{x} \in K$ such that

$$\bar{x} = T\bar{x} \tag{3.3.5}$$

Set $f(x, y) := \langle x - Tx, y - x \rangle$.

Then $\bar{x}$ solve (3.6.3) if and only if, $\bar{x}$ is a solution of (3.3.5). Indeed (3.3.5) implies (3.6.3) is obvious. find if (3.6.3) is satisfied, then choose $\bar{y} := T\bar{x}$ to obtain

$$0 \leq f(\bar{x}, \bar{y}) = -||\bar{x} - T\bar{x}||^2,$$

hence $\bar{x} = T\bar{x}$. So (3.6.3) implies (3.3.5). In this case $T$ is monotone if and only if,

$$\langle Tx - Ty, x - y \rangle \leq ||x - y||^2 \ \forall x, y \in K,$$

hence in particular if $T$ is nonexpansive.

## 3.4 Variational Inequalities Problems

The rapid development of the theory of variational inequalities and the prolific growth of its application made evident to mathematicians the new way of comprehension and which brought them to the level of research in many subject. Without attempting to formulate any precise definitions, let us discuss several examples.

**Example 3.4.1** *Let $f$ be a smooth real valued function on the closed interval $I = [a, b]$. We seek the points $x_0 \in I$ for which*

$$f(x_0) = \min_{x \in I} f(x)$$

*Three case can occur*

*(i) if* $a < x_0 < b$, *then* $f'(x_0) = 0$;

*(ii) if* $x_0 = a$, *then* $f'(x_0) \geq 0$; *and*

*(iii) if* $x_0 = b$, $f'(x_0) \leq 0$

*These statements may be summarized by writing*

$$f'(x_0)(x - x_0) \geq 0 \ \forall x \in I$$

*such an inequality will be refereed to as a variational inequality.*

**Example 3.4.2** *Let* $f$ *be a smooth real valued function defined on the closed convex subset* $K$ *of Euclidean* $N$ *dimensional space* $\mathbb{R}^N$. *Again we shall characterise the point* $x_0 \in K$ *such that*

$$f(x_0) = \min_{x \in K} f(x)$$

*Assume* $x_0$ *is a point where the minimum is achieved and let* $x \in K$. *Since* $K$ *is convex, the segment* $(1-t)x_0 + tx = x_0 + t(x - x_0), 0 \leq t \leq 1$, *lies in* $K$. *Let* $\phi$ *be a real valued function such that*

$$\phi(t) = f(x_0 + t(x - x_0)), 0 \leq t \leq 1 \tag{3.4.1}$$

*then at* $t = 0$ *we have that* $\phi(0) = f(x_0) = \min_{x \in K} f(x)$ *as in Example 3.4.1, from* (3.4.1)*we have*

$$\phi'(0) = gradf(x_0)(x - x_0) \geq 0 \ for \ any \ x \in K,$$

*where gradf means gradient of* $f$. *Consequently, the point* $x_0$ *satisfies the*

*variational inequality*

$$x_0 \in K : gradf(x_0)(x - x_0) \geq 0 \ \forall x \in K.$$

*If $K$ is bounded, the existence of at least one $x_0$ is immediate.*

**Example 3.4.3** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$ and let $\Psi$ be a given function on $\bar{\Omega} = \Omega \cup \partial\Omega$ satisfying*

$$\max_{\Omega} \Psi \geq 0 \text{ and } \Psi \leq 0 \text{ on } \partial\Omega$$

*Define*

$$K = \{v \in C^1(\bar{\Omega}) : v \geq \Psi \text{ on } \Omega \text{ and } v = 0 \text{ on } \partial\Omega\},$$

*a convex set of functions which we assume is not empty, we seek a function $u \in K$ for which*

$$\int_\Omega |gradu|^2 dx = \min_{v \in K} \int_\Omega |gradv|^2 dx$$

*Assume such a $u$ exist, we argue analogously to our previous example relying again on the convexity of $K$. For any $v \in K$, the sequence $u + t(v - u) \in K,\ 0 \leq t \leq 1$, whence the function*

$$\Phi(t) = \int_\Omega |grad(u + t(v - u))|^2 dx, 0 \leq t \leq 1,$$

*attains its minimum at $t = 0$. This implies that $\Phi'(0) \geq 0$, which lead to the variational inequality*

$$u \in K : \int_\Omega gradugrad(v - u)dx \geq 0 \ \forall v \in K.$$

*The point set*

$$I = \{x \in \Omega : u(x) = \Psi(x)\}$$

*which is called the set of coincidence. Its presence distinguishes $u$. One inter-*

*prets u as the height function of the equilibrium position of a thin membrane constrained to lie above the body*

$$\{(x_1, x_{N+1}) : x_{N+1} \leq \Phi(x), x \in \Omega\}$$

*and with fixed height zero on* $\partial\Omega$.

**Example 3.4.4** *With* $\Omega$ *and* $K$ *as in Example 3.3.5, we ask for the element* $u \in K$ *of least area, namely*

$$\int_\Omega \sqrt{1 + |gradu|^2}dx = \min_{v \in K} \int_\Omega \sqrt{1 + |gradv|^2}dx$$

*The associated inequality is*

$$u \in K : \int_\Omega \frac{gradugrad(v-u)}{\sqrt{1 + |gradv|^2}}dx \geq 0 \ \forall v \in K.$$

*Among the examples considered, the first two may be solved by means of calculus for they depend on a finite number of variation. The other example are similar to problem of the calculus of variation since the competing functions lie in an infinite dimensional space.*

## 3.5 Nonlinear Mappings

This subsection deals with definitions of nonlinear mappings which will play important roles in this work. We proceed as follows: Let $E$ be a normed space. A mapping $T : D(T) \subseteq E \to R(T) \subseteq E$ (where $D(T)$ and $R(T)$ denote the domain and range of $T$ respectively) is said to be L- Lipschitzian if and only if there exists a constant $L > 0$ such that for all $x, y \in E$, $||Tx - Ty|| \leq L||x - y||$. The mapping $T$ is said to be uniformly $L - Lipschitzian$ if there exists $L \geq 0$

such that for all $x, y \in E$

$$||T^n x - T^n y|| \leq L||x - y||. \tag{3.5.1}$$

A mapping $T : D(T) \to R(T)$ is called *asymptotically nonexpansive* if and only if there exists a sequence $\{k_n\}$in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for all $x, y \in D(T), n \in \mathbb{N}$

$$||T^n x - T^n y|| \leq k_n||x - y|| \quad \text{for all} \quad n \in N. \tag{3.5.2}$$

It is clear that every asymptotically nonexpansive mapping is uniformly L- Lipschitzian with $L = \sup_{n\geq 1}\{k_n\}$. It is well known (Goebel and Kirk, 1972) that the class of nonexpansive mappings is a proper subclass of class of asymptotically nonexpansive mappings. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk (1972), as an important generalization of the class of nonexpansive mappings.

A mapping $T : D(T) \to R(T)$ is said to be asymptotically regular if and only if for all $x \in D(T)$

$$\lim_{n\to\infty} ||T^{n+1}x - T^n x|| = 0.$$

It is said to be uniformly asymptotically regular if for any bounded subset $C$ of $D(T)$,

$$\lim_{n\to\infty} \sup_{x\in C} ||T^{n+1}x - T^n x|| = 0.$$

A mapping $G : D(T) \subseteq E \to R(T) \subseteq E$ is said to be $\delta$-strongly accretive if and only if there exists a $\delta > 0$ such that for all $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Gx - Gy, j(x - y)\rangle \geq \delta||x - y||^2 \tag{3.5.3}$$

and the mapping $G$ is called $\lambda$-strictly pseudocontractive if and only if there

exists $\lambda > 0$ such that for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Gx - Gy, j(x-y)\rangle \leq ||x-y||^2 - \lambda||(I-G)x - (I-G)y||^2. \qquad (3.5.4)$$

Let $E$ be a real Banach space and let $\delta, \lambda$ and $\tau$ be positive real numbers satisfying $\delta + \lambda > 1$ and $\tau \in (0,1)$. Let $G : E \to E$ be a $\delta$-strongly accretive and $\lambda$-strictly pseudocontractive mapping then, (see (Piri and Vaezi, 2011)) for $x, y \in E$,

$$||(I-G)x - (I-G)y|| \leq (\sqrt{(1-\delta)/\lambda})||x-y||$$

and

$$||(I-\beta G)x - (I-\beta G)y|| \leq [1-\beta(1-\sqrt{(1-\delta)/\lambda})]||x-y||,$$

that is, $(I-G)$ and $(I-\beta G)$ are contractive mappings.

Let $E$ be a real Banach space and let $K$ be a closed subset of $E$. A mapping $T : K \to K$ is said to be demi-closed at the origin if for any sequence $\{x_n\}$ in $K$, the conditions $x_n \rightharpoonup q \in K$ and $Tx_n \to 0 \in K$ as $n \to \infty$ we have that $Tq = 0$.

**Theorem 3.5.1** (Lin et al., 1995) Suppose $E$ is a Banach space satisfying the local uniform Opial's condition, $C$ is a nonempty weakly compact convex subset of $E$, and $T : C \to C$ is an asymptotically nonexpansive mapping. Then (I-T) is demiclosed at zero.

**Lemma 3.5.2** (Chidume, 2009) Let $E$ be a real normed space. Then

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle,$$

for all $x, y \in E$ and $j(x+y) \in J(x+y)$.

**Lemma 3.5.3** (Suzuki, 2005) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $E$ and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose that $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integer $n \geq 1$ and $\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$. Then, $\lim_{n\to\infty} ||y_n - x_n|| = 0$.

**Lemma 3.5.4** (Xu, 2002) Let $\{a_n\}$ be a sequence of nonegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n,\ n \geq 0$$

where, (i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$;; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

**Lemma 3.5.5** (Xu, 2003) Let $E$ be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$ and $\lambda \in [0,1]$, then there exists a continuous strictly increasing convex function

$$g : [0, 2r] \to \mathbb{R},\ g(0) = 0$$

such that for every $x, y \in B_r(0)$, the following inequality holds:

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)g(||x-y||).$$

**Lemma 3.5.6** (Chang et al., 2001) Let $E$ be a uniformily convex Banach space, $K$ a nonempty closed convex subset of $E$ and $T : K \to K$ be an asymptotically nonexpansive mapping, then $I - T$ is demiclosed at zero.

**Lemma 3.5.7** ((Piri and Vaezi, 2011) and (Ali, 2011)) Let $E$ be a real Banach space and $G : E \to E$ be a mapping.

**(i)** If $G$ is $\eta - strongly$ accretive and $\mu - strictly$ pseudo-contractive with $\eta + \mu > 1$, then $I - G$ is contractive with constant $\sqrt{\frac{1-\eta}{\mu}}$.

**(ii)** If $G$ is $\eta - strongly$ accretive and $\mu - strictly$ pseudo-contractive with $\eta + \mu > 1$, then for any fixed number $\kappa \in (0,1)$, $I - \kappa G$ is contractive with constant $1 - \kappa\Big(1 - \sqrt{\frac{1-\eta}{\mu}}\Big)$.

## 3.6 Generalized Projection

Let $E$ be a normed space and $\phi : E \times E \to [0, \infty)$ denote the Lyapunov functional defined for all $x, y \in E$ by

$$\phi(x,y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2,$$

it is obvious from the definition of the function $\phi$ that

$$(||x|| - ||y||)^2 \leq \phi(x,y) \leq (||x|| + ||y||)^2. \tag{3.6.1}$$

and also

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle,\ \forall x, y, z \in E. \tag{3.6.2}$$

Let $E$ be a smooth, strictly convex and reflexive Banach space, let $C$ be a nonempty, closed and convex subset of $E$ and $T : C \to C$ be a mapping. A point $p \in C$ is said to be a fixed point of $T$ if and only $Tp = p$ and it is called an asymptotic fixed point of $T$ if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset C$ such that $x_n \rightharpoonup p$ in $C$ and $||x_n - Tx_n|| \to 0$ as $n \to \infty$. The set of all fixed points of $T$ is denoted by $F(T)$ and the set of all asymptotic fixed points of $T$ is denoted by $\widehat{F(T)}$.

**Definition 3.6.1** (i) A mapping $T : C \to C$ is said to be *relatively* non-expansive, (Matsushita and Takahashi, 2005) if and only if $F(T) \neq$

$\emptyset,\ F(T) = \widehat{F(T)}$ and

$$\phi(p, Tx) \leq \phi(p, x),\ \forall x \in\ C,\ p \in F(T).$$

(ii) A mapping $T : C \to C$ is said to be quasi-$\phi$-asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$, such that $k_n \to 1$ as $n \to \infty$ and

$$\phi(p, T^n x) \leq k_n \phi(p, x),\ \forall n \geq 1,\ x \in\ C,\ p \in F(T).$$

(iii) A mapping $T : C \to C$ is said to be generalized quasi-$\phi$-asymptotically nonexpansive if and only if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{u_n\}$ and $\{v_n\}$ with $u_n \to 0,\ v_n \to 0$ (as $n \to \infty$) such that for all $x \in C,\ p \in F(T)$

$$\phi(p, T^n x) \leq (1 + u_n)\phi(p, x) + v_n,\ \forall\ n \geq 1.$$

(iv) $T$ is said to be quasi-$\phi$-strictly pseudocontractive, Zhou and Gao (2010) if and only if $F(T) \neq \emptyset$ and there exists a constant $k \in (0, 1]$ such that

$$\phi(p, Tx) \leq \phi(p, x) + k\phi(x, Tx)\ \forall\ x \in C \text{ and } p \in F(T).$$

In particular $T$ is said to be quasi-$\phi$-nonexpansive if $k = 0$.

(v) A mapping $T : C \to C$ is said to be asymptotically quasi-$\phi$-strict pseudocontractive if and only if $F(T) \neq \emptyset$ and there exist $k \in [0, 1)$ and a nonnegative real sequence $\{u_n\}$ with $u_n \to 0$ (as $n \to \infty$) such that for all $x \in C,\ p \in F(T)$,

$$\phi(p, T^n x) \leq (1 + u_n)\phi(p, x) + k\phi(x, T^n x),\ \forall\ n \geq 1.$$

(vi) A mapping $T : C \to C$ is said to be generalized asymptotically quasi-$\phi$-strictly pseudo-contractive if and only if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ and nonnegative real sequences $\{u_n\}$ and $\{v_n\}$ with $u_n \to 0,\ v_n \to 0$ (as $n \to \infty$) such that for all $x \in C,\ p \in F(T)$

$$\phi(p, T^n x) \leq (1 + u_n)\phi(p, x) + k\phi(x, T^n x) + v_n,\ \forall\ n \geq 1. \quad (3.6.3)$$

Using (3.6.2) we get that (3.6.3) is equivalent to

$$\phi(x, T^n x) \leq \frac{1}{1-k}[u_n\phi(p, x) + v_n + 2\langle x - p, Jx - JT^n x\rangle], \quad (3.6.4)$$

(vii) A mapping $T : C \to C$ is said to be asymptotically quasi-$\phi$-strict pseudo-contractive in the intermediate sense, see Qin et al. (2012), if and only if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ and nonnegative real sequence $\{u_n\}$ with $u_n \to 0$ (as $n \to \infty$) such that

$$\limsup_{n\to\infty} \sup_{p\in F(T), x\in C} \Big(\phi(p, T^n x) - (1 + u_n)\phi(p, x) - k\phi(x, T^n x)\Big) \leq 0. \quad (3.6.5)$$

Setting $\zeta_n := Max\Big\{0, \sup_{p\in F(T), x\in C} \Big(\phi(p, T^n x) - (1+u_n)\phi(p, x) - k\phi(x, T^n x)\Big)\Big\}$, we trivially have $\zeta_n \to 0$ as $n \to \infty$ and so (3.6.5) become

$$\phi(p, T^n x) \leq (1 + u_n)\phi(p, x) + k\phi(x, T^n x) + \zeta_n,\ \forall\ n \geq 1.$$

Thus, every asymptotically quasi-$\phi$-strictly pseudo-contractive in the intermediate sense is a generalized asymptotically quasi-$\phi$-strict pseudo-contractive mapping.

It is known that if $C$ is a nonempty closed convex subset of a real Hilbert space $H$ and $P_C : H \to C$ is the metric projection of $H$ onto $C$, then $P_C$ is nonexpansive, that is for any $x, y \in H, ||P_C x - P_C y|| \leq ||x - y||$. This

fact actually characterizes Hilbert space and consequently, it is not available in more general Banach space. In this connection, Alber (1996) introduced a generalized projection operator $\Pi_C$ in a space more general than Hilbert space which is an analogue of the metric projection in Hilbert space.

Let $C$ be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, for all $x \in E$, the *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$ as $y$ varies in $C$, that is, $\Pi_C x = \bar{x}$ implies

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x).$$

Existence and uniqueness of the operator $\Pi_C$ follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping $J$ Alber (1996); Alber and Reich (1994); Cioranescu (1990); Kamimura and Takahashi (2002); Takahashi (2000).

In Hilbert space $\phi(y, x) = ||y - x||^2$ and $\Pi_C$ coincides with the metric projection $P_C$.

It is also known that if $E$ is a reflexive, strictly convex and smooth Banach space, then $x, y \in E, \phi(x, y) = 0$ if and only if $x = y$.

**Lemma 3.6.1** (Alber, 1996), Let $C$ be a nonempty, closed and convex subset of a smooth Banach space $E$, and $x \in E$. Then $x_0 = \Pi_C x$ if and only if

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

**Lemma 3.6.2** (Alber, 1996), Let $E$ be a reflexive, strictly convex and smooth Banach space, $C$ a nonempty, closed and convex subset of $E$, and $x \in E$. Then

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

**Generalized $f$-projection**

Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varphi) = ||\xi||^2 - 2\langle \xi, \varphi \rangle + ||\varphi||^2 + 2\rho f(\xi), \tag{3.6.6}$$

where $\xi \in C, \varphi \in E^*, \rho$ is a positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. From the definition of $G$ and $f$, it is easy to observe the following properties:

(i) $G(\xi, \varphi)$ is convex and continuous with respect to $\varphi$ when $\xi$ is fixed;

(ii) $G(\xi, \varphi)$ is convex and lower semi-continuous with respect to $\xi$ when $\varphi$ is fixed.

**Definition 3.6.2** (Wu and Huang, 2006) Let $E$ be a real Banach space with its dual $E^*$. Let $C$ be a nonempty closed and convex subset of $E$. A mapping $\Pi_C^f : E^* \to 2^C$ defined by

$$\Pi_C^f \varphi = \{u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi)\}, \ \forall \ \varphi \in E^*,$$

is called a generalized $f$-projection operator.

Recall that $J$ is single valued in a smooth real Banach space $E$. This implies that for each $x \in E$ there exists a unique element $\varphi \in E^*$ such that $\varphi = Jx$. Using this in (3.6.6), we obtain, in a real smooth Banach space,

$$G(\xi, Jx) = ||\xi||^2 - 2\langle \xi, Jx \rangle + ||x||^2 + 2\rho f(\xi). \tag{3.6.7}$$

With this, the generalized $f$-projection operator in a smooth real Banach space can be defined as follows.

**Definition 3.6.3** Let $E$ be a real Banach space and $C$ a nonempty closed and convex subset of $E$. We say that $\Pi_C^f : E \to 2^C$ is a generalized $f$-projection

operator if and only if

$$\Pi_C^f x = \{u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx)\},\ \forall x \in E.$$

For the generalized $f$-projection operator, Wu and Huang (2006) proved the following basic properties:

**Lemma 3.6.3** (Wu and Huang, 2006) Let $E$ be a reflexive real Banach space with its dual space $E^*$. Let $C$ be a nonempty closed and convex subset of $E$, then the following statements hold:
(i) $\Pi_C^f \varphi$ is a nonempty closed convex subset of $C$ for all $\varphi \in E^*$,
(ii) If $E$ is smooth, then for all $\varphi \in E^*, x \in \Pi_C^f \varphi$ if and only if

$$\langle x - y, \varphi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0 \forall\ y \in C;$$

(iii) If $E$ is strictly convex and $f : C \to \mathbb{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ with $tx \in C$ where $x \in C$), then $\Pi_C^f$ is a single valued mapping.

Fan et al. (2009) showed that the condition $f$ is positive homogeneous which appeared in Lemma 3.6.3 can be dropped. In fact they proved the following:

**Lemma 3.6.4** (Fan et al., 2009) Let $E$ be a reflexive real Banach space with its dual space $E^*$ and $C$ a nonempty closed and convex subset of $E$. If $E$ is strictly convex, then $\Pi_C^f$ is a single valued mapping.

The following definition follows from the relations between $\phi$ and $G$.

A multi-valued mapping $T : C \to 2^C$ is

(i) relatively nonexpansive if $F(T) \neq \emptyset,\ F(T) = \widehat{F(T)}$ and

$$G(p, Jw) \leq G(p, Jx),\ \forall x \in C,\ p \in F(T),\ x \in C,\ \text{and } w \in Tx.$$

(ii) called *total quasi* $-\ \phi -$ *asymptotically nonexpansive* with respect to $f$ if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{u_n\}$ and $\{v_n\}$ with $u_n \to 0, v_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that for all $x \in C,\ p \in F(T)$

$$G(p, Jw_n) \leq G(p, Jx) + u_n\zeta(G(p, Jx)) + v_n, ,\ \forall n \geq 1,\ w_n \in T^n x. \quad (3.6.8)$$

**Lemma 3.6.5** (Li et al., 2010) Let $E$ be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous convex functional. Then there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that for all $x \in E$

$$f(x) \geq \langle x, x^* \rangle + \alpha.$$

**Lemma 3.6.6** (Li et al., 2010) Let $C$ be a nonempty, closed and convex subset of a smooth and reflexive Banach space $E$. Then the following statements hold:
(i) for all $x \in E\ \Pi_C^f x$ is a nonempty, closed and convex subset of $C$;
(ii) for all $x \in E,\ \hat{x} \in \Pi_C^f x$ if and only if

$$\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(\hat{x}) \geq 0,\ \forall\ y \in C;$$

(iii) if $E$ is strictly convex, then $\Pi_C^f x$ is a single valued mapping for each $x \in E$.

**Lemma 3.6.7** (Li et al., 2010) Let $C$ be a nonempty, closed and convex subset of a smooth and reflexive Banach space $E$. Let $x \in E$ and $\hat{x} \in \Pi_C^f x$, then

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx),\ \forall\ y \in C.$$

**Lemma 3.6.8** (Kamimura and Takahashi, 2002) Let $C$ be a nonempty closed and convex subset of a smooth uniformly convex Banach space $E$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in $E$ such that either $\{x_n\}_{n=1}^{\infty}$ or $\{y_n\}_{n=1}^{\infty}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

**Lemma 3.6.9** (Chang et al., 2010) Let $E$ be a uniformly convex real Banach space. For arbitrary $r > 0$, let

$$B_r(0) := \{x \in E : ||x|| \leq r\}.$$

Then, for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function

$$g : [0, 2r] \to \mathbb{R},\ g(0) = 0$$

such that for any positive integers $i, j$ with $i < j$, the following inequality holds:

$$\Big\| \sum_{n=1}^{\infty} \lambda_n x_n \Big\|^2 \leq \sum_{n=1}^{\infty} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

## 3.7 Bregman Projection

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \text{dom} f \times \text{int dom} f \to [0, +\infty)$ is defined as follows:

$$D_f(y, x) := f(y) - f(x) - \langle y - x, \nabla f(x) \rangle \tag{3.7.1}$$

called the Bregman distance with respect to $f$ ( Censor and Lent (1981)). The geometric significance of the Bregman distance in 2-dimension is illustrated by

Figure 1.1.1 below. The Bregman distance is the vertical distance at $y$ between the graph of $f$ and the tangent to the graph of $f$ in $x$.

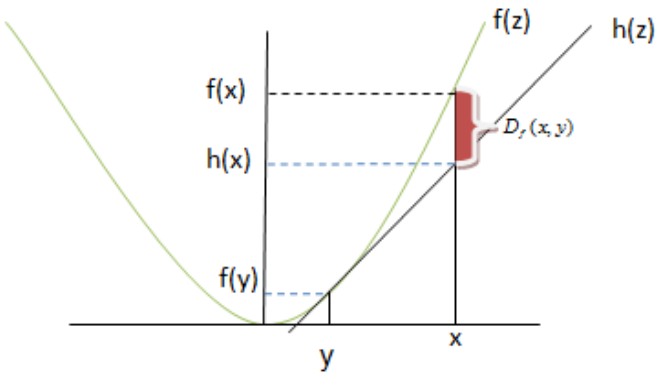



Figure 3.7.1: The significance of the Bregman distance

from the Figure, we get

$$D_f(x,y) = f(x) - h(x) \tag{3.7.2}$$

but, at $y$, $f(y) = h(y)$ and since the gradient of a straight line is unique at any point on the line, so

$$\begin{aligned} \nabla f(y) = \nabla h(y) &= \frac{h(x) - h(y)}{x - y} \\ &= \frac{h(x) - f(y)}{x - y} \end{aligned}$$

which implies

$$h(x) = f(y) + (x - y)\nabla f(y)$$

using this in (3.7.2), we obtain

$$D_f(x,y) = f(x) - f(y) - (x - y)\nabla f(y).$$

It is obvious from the definition of $D_f$ that

$$D_f(z,x) = D_f(z,y) + D_f(y,x) + \langle x - y, \nabla f(y) - \nabla f(x)\rangle. \tag{3.7.3}$$

Recall that the Bregman projection (Bregman, 1967) of $x \in \text{int dom} f$ onto the nonempty closed and convex set $C \subset \text{dom} f$ is the necessarily unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

A mapping $T$ is said to be Bregman firmly nonexpansive ( see (Reich and Sabach, 2011)) if for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle Tx - Ty, \nabla f(x) - \nabla f(y) \rangle$$

or equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

Let $f : E \to \mathbb{R}$, a mapping $T : C \to C$ is said to be Bregman relatively nonexpansive Matsushita and Takahashi (2005) if $F(T) \neq \emptyset, \hat{F}(T) = F(T)$ and $D_f(p, T(x)) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$. The map $T : C \to C$ is said to be Bregman weak relatively nonexpansive if $F(T) \neq \emptyset, \tilde{F}(T) = F(T)$ and $D_f(p, T(x)) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$. $T$ is said to be quasi-Bregman relatively nonexpansive if $F(T) \neq \emptyset$, and $D_f(p, T(x)) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$.

In Reich and Sabach (2010), quasi-Bregman relatively nonexpansive is called left quasi-Bregman relatively nonexpansive. A map $T : C \to C$ is called right quasi-Bregman relatively nonexpansive by Martin-Marquez et al. (2012) if $F(T) \neq \emptyset$, and $D_f(T(x), p) \leq D_f(x, p)$ for all $x \in C$ and $p \in F(T)$. $T$ is said to be quasi-Bregman strict pseudo-contractive if there exists a constant $k \in [0, 1)$ and $F(T) \neq \emptyset$ such that $D_f(p, Tx) \leq D_f(p, x) + kD_f(x, Tx)$ for all $x \in C$ and $p \in F(T)$. In particular, $T$ is said to be quasi-Bregman relatively nonexpansive if $k = 0$ and $T$ is said to be quasi-Bregman pseudo contractive

if $k = 1$.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \text{dom} f \times \text{int dom} f \to [0, +\infty)$. Concerning the Bregman projection, the following are well known.

**Lemma 3.7.1** (Reich and Sabach, 2009) If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of $E$, then $\nabla f$ is uniformly continuous on bounded subsets of $E$ from the strong topology of $E$ to the strong topology of $E^*$.

**Definition 3.7.1** (Bauschke et al., 2001) The function $f$ is said to be:

(i) essentially smooth, if $\partial f$ is both locally bounded and single-valued on its domain.

(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and $f$ is strictly convex on every convex subset $\text{dom}\partial f$.

(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

**Proposition 3.7.2** Let $E$ be a reflexive Banach space. Then we have

(i) $f$ is essentially smooth if and only if $f^*$ is essentially strictly convex (see (Bauschke et al., 2001), Theorem 5.4).

(ii) $(\partial f)^{-1} = \partial f^*$ ( Bonnans and Shapiro (2000))

(iii) $f$ is Legendre if and only if $f^*$ is Legendre, (Bauschke et al. (2001), Corollary 5.5).

(iv) If $f$ is Legendre, then $\nabla f$ is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, ran $\nabla f$ = dom $\nabla f^*$ = int dom $f^*$ and ran $\nabla f^*$ = dom $f$ = int dom $f$, ( Bauschke et al. (2001), Theorem 5.10).

Examples of Legendre functions were given in Bauschke et al. (2001), Bauschke and Borwein (1997). One important and interesting Legendre function is $\frac{1}{p}||\cdot||^p$ $(1 < p < \infty)$ when $E$ is a smooth and strictly convex Banach space. In this case the gradient $\nabla f$ of $f$ coincides with the generalized duality mapping of $E$, i.e., $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces. In the rest of this chapter, we always assume that $f : E \to (-\infty, +\infty]$ is Legendre.

**Lemma 3.7.3** (Butnariu and Resmerita (2006)) Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then

(a) $z = P_C^f(x)$ if and only if $\langle y - z, \nabla f(x) - \nabla f(z)\rangle \leq 0,\ \forall y \in C.$

(b) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x),\ \forall x \in E,\ y \in C.$

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The modulus of total convexity of $f$ at $x \in \text{int dom} f$ is the function $v_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ define by

$$v_f(x, t) := \inf\{D_f(y, x) : y \in \text{dom} f, ||y - x|| = t\}.$$

The function $f$ is called totally convex at $x$ if and only if and only if $v_f(x, t) > 0$ whenever $t > 0$. The function $f$ is called totally convex if it is totally convex at any point $x \in \text{int dom} f$ and is said to be totally convex on bounded sets if $v_f(B, t) > 0$ for any nonempty bounded subset $B$ of $E$ and $t > 0$, where the modulus of total convexity of the function $f$ on the set $B$ is the function $v_f : \text{int dom} f \times [0, +\infty) \to [0, +\infty]$ defined by

$$v_f(B, t) := \inf\{v_f(x, t) : x \in B \cap \text{dom} f\}.$$

**Lemma 3.7.4** (Resmerita, 2004) If $x \in \text{dom}f$, then the following statements are equivalent:

(i) The function $f$ is totally convex at $x$;

(ii) For any sequence $\{y_n\} \subset \text{dom}f$,

$$\lim_{n\to+\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to+\infty} ||y_n - x|| = 0.$$

Recall that the function $f$ is called sequentially consistent by Butnariu and Resmerita (2006) if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $E$ such that the first one is bounded

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to+\infty} ||y_n - x_n|| = 0.$$

**Lemma 3.7.5** (Butnariu and Iusem, 2000) *The function $f$ is totally convex on bounded sets if and only if the function $f$ is sequentially consistent.*

**Lemma 3.7.6** (Reich and Sabach, 2010) Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.

**Lemma 3.7.7** (Reich and Sabach, 2010) Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$ and let $C$ be a nonempty, closed and convex subset of $E$. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to $C$. If $D_f(x_n, x_0) \leq D_f(P_C^f(x_0), x_0)$ for any $n \in \mathbb{R}$, then $\{x_n\}$ converges strongly to $P_C^f(x_0)$.

**Lemma 3.7.8** (Reich and Sabach, 2010) Let $E$ be a reflexive real Banach space, $f : E \to (-\infty, +\infty]$ be a proper lower semi-continuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak$^*$ lower semi-continuous and convex

function. Thus, for all $z \in E$, we have

$$D_f(z, \nabla f^*(\sum_{i=1}^{N} t_i \nabla f(x_i))) \leq \sum_{i=1}^{N} t_i D_f(z, x_i). \tag{3.7.4}$$

In order to solve the equilibrium problem, let us assume that a bifunction $g : C \times C \to \mathbb{R}$ satisfies the following condition ( Blum and Oettli (1994)).

(A1) $g(x, x) = 0, \ \forall x \in C.$

(A2) $g$ is monotone, i.e., $g(x, y) + g(y, x) \leq 0, \ \forall x, y \in C.$

(A3) $\limsup_{t \downarrow 0} g(x + t(z - x), y) \leq g(x, y) \ \forall x, z, y \in C.$

(A4) The function $y \mapsto g(x, y)$ is convex and lower semi-continuous.

The resolvent of a bifunction $g$ ( Combettes and Hirstoaga (2005)) is the operator $Res_g^f : E \to 2^C$ defined by

$$Res_g^f(x) = \{z \in C : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \ \forall y \in C\}. \tag{3.7.5}$$

From (Lemma 1, in Reich and Sabach (2010)) if $f : (-\infty, +\infty]$ is a strongly coercive and Gâteaux differentiable function, and $g$ satisfies conditions (A1)-(A4), then $\text{dom}(Res_g^f) = E$. The following lemma gives some characterization of the resolvent $Res_g^f$.

**Lemma 3.7.9** (Reich and Sabach, 2010) Let $E$ be a real reflexive Banach space and $C$ be a nonempty closed convex subset of $E$. Let $f : E \to (-\infty, +\infty]$ be a Legendre function. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions (A1)-(A4). Then, the followings hold:

(i) $Res_g^f$ is single-valued;

(ii) $Res_g^f$ is a Bregman firmly nonexpansive operator;

(iii) $F(Res_g^f) = EP(g)$;

(iv) $EP(g)$ is closed and convex subset of $C$;

(v) for all $x \in E$ and for all $q \in F(Res_g^f)$, we have

$$D_f(q, Res_g^f(x)) + D_f(Res_g^f(x), x) \leq D_f(q, x). \tag{3.7.6}$$

# CHAPTER FOUR

# MAIN RESULTS

## 4.1 Families of Asymptotically Nonexpansive Mappings

In this section, we introduce a new modified iterative scheme for approximation of common fixed points of family of asymptotically nonexpansive mappings and solutions of some variational inequality problems and prove strong convergence theorem that extend and generalize some recent results. The result of this section particularly, is applicable in $l_p$ and/or $L_p$ spaces $(1 < p < \infty)$. The following theorems

**Theorem 4.1.1** *Let $E$ be a real Banach space whose duality map is weakly sequentially continuous. Let $G : E \to E$ be an $\eta$-strongly accretive and $\mu$-strictly pseudocontractive with $\eta + \mu > 1$ and let $f : E \to E$ be a contraction with coefficient $\alpha \in (0,1)$. Let $\{T_i\}_{i=1}^{\infty}$ be a family of uniformly asymptotically regular asymptotically nonexpansive self mappings of $C$ with sequences $\{v_{in}\}$ such that $v_{in} \to 0$ as $n \to \infty$ for each $i \geq 1$ and $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ . Assume that $\gamma \in \left(0, \min\{\frac{\tau}{2\alpha}, \eta\}\right)$, where $\tau := (1 - \sqrt{\frac{1-\eta}{\mu}})$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\sigma_{i,n}\}_{n=1}^{\infty}, i = 1, 2, ...$ be sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

*(C1)* $\lim_{n\to\infty} \beta_n = 0$ *and* $\lim_{n\to\infty} \frac{v_n}{\beta_n} = 0$, *where* $v_n := \sup_{i\geq 1}\{v_{in}\}$

*(C2)* $\Sigma_{n=0}^{\infty} \beta_n = \infty$

*(C3)* $\lim_{n\to\infty} \alpha_n = 0$ *and (C4)* $\sum_{i=1}^{\infty} \sigma_{in} := 1 - \alpha_n$

*For some fixed $\delta \in (0,1)$ and arbitrary $x_0 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence*

*defined iteratively by $x_0 \in C$ chosen arbitrarily,*

$$\begin{cases} y_n = \beta_n \gamma f(x_n) + (I - \beta_n G)x_n, \\ x_{n+1} = [1 - \delta(1-\alpha_n)]x_n + \delta \sum_{i\geq 1} \sigma_{in} T_i^n y_n \quad n \geq 0. \end{cases} \tag{4.1.1}$$

*Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F$, where $p$ is the unique solution of the variational inequality problem*

$$\langle \gamma f(p) - Gp, j(q-p) \rangle \leq 0, \ \forall \ q \in F. \tag{4.1.2}$$

**Proof.** From the choice of $\gamma$, $(G - \gamma f)$ is strongly accretive, then the variational inequality (4.1.2) has a unique solution in $F$. We show that $\{x_n\}$ is bounded. Let $p \in F$, since $\frac{v_n}{\beta_n} \to 0$ as $n \to \infty$, it implies that $\frac{v_n}{(1+v_n)\beta_n} \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\frac{v_n}{(1+v_n)\beta_n} < \frac{\tau - \gamma\alpha}{2}$, for all $n \geq n_0$. We now estimate as follows

$$\begin{aligned} ||y_n - p|| &= ||\beta_n(\gamma f(x_n) - Gp) + (I - \beta_n G)(x_n - p)|| \\ &\leq \beta_n ||\gamma f(x_n) - Gp|| + (1 - \beta_n \tau)||x_n - p|| \\ &\leq \Big(1 - \beta_n(\tau - \gamma\alpha)\Big)||x_n - p|| + \beta_n ||\gamma f(p) - Gp||. \end{aligned} \tag{4.1.3}$$

Using (4.1.3), we obtain that

$$\begin{aligned} ||x_{n+1} - p|| &= ||[1 - \delta(1-\alpha_n)](x_n - p) + \delta \sum_{i\geq 1} \sigma_{in}(T_i^n y_n - p)|| \\ &\leq [1 - \delta(1-\alpha_n)]||x_n - p|| + \delta(1-\alpha_n)(1+v_n)||y_n - p|| \\ &\leq \Big[1 - \delta(1-\alpha_n) + \delta(1-\alpha_n)(1+v_n)[1 - \beta_n(\tau - \gamma\alpha)]\Big]||x_n - p|| \\ &\quad + \delta(1-\alpha_n)(1+v_n)\beta_n||\gamma f(p) - Gp|| \\ &= \Big[1 + \delta(1-\alpha_n)v_n - \beta_n\delta(1-\alpha_n)(1+v_n)(\tau - \gamma\alpha)]\Big]||x_n - p|| \\ &\quad + \delta(1-\alpha_n)(1+v_n)\beta_n||\gamma f(p) - Gp|| \end{aligned}$$

$$
\begin{aligned}
&\leq \Big[1-\beta_n\delta(1-\alpha_n)(1+v_n)\Big((\tau-\gamma\alpha)-\frac{v_n}{(1+v_n)\beta_n}\Big)\Big]||x_n-p|| \\
&\quad +\beta_n\delta(1-\alpha_n)(1+v_n)\Big((\tau-\gamma\alpha)-\frac{v_n}{(1+v_n)\beta_n}\Big)\frac{2||\gamma f(p)-Gp||}{\tau-\gamma\alpha} \\
&\leq \max\Big\{||x_n-p||, \frac{2||\gamma f(p)-Gp||}{\tau-\gamma\alpha}\Big\}
\end{aligned}
$$

By induction, we obtain

$$
||x_n-p|| \leq \max\Big\{||x_{n_0}-p||, \frac{2||\gamma f(p)-Gp||}{\tau-\gamma\alpha}\Big\} \ \forall\ n \geq n_0.
$$

Hence $\{x_n\}$ is bounded. Also $\{f(x_n)\}, \{G(x_n)\}, \{y_n\}, \{T_i^n x_n\}$ and $\{T_i^n y_n\}$ are all bounded.

Define two sequences by $\gamma_n := (1-\delta)\alpha_n + \delta$ and $z_n := \frac{1}{\gamma_n}(x_{n+1} - x_n + \gamma_n x_n)$. From the recursion formula (4.1.1), observe that

$$
z_n = \frac{1}{\gamma_n}(\delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n + \alpha_n x_n)
$$

which implies that

$$
\begin{aligned}
z_{n+1}-z_n &= \frac{1}{\gamma_{n+1}}\Big[\delta\sum_{i\geq 1}\sigma_{in+1}T_i^{n+1}y_{n+1}+\alpha_{n+1}x_{n+1}\Big] \\
&\quad -\frac{1}{\gamma_n}\Big[\delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n+\alpha_n x_n\Big] \\
&= \frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}\Big(T_i^{n+1}y_{n+1}-T_i^{n+1}y_n\Big) \\
&\quad +\frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}\Big(T_i^{n+1}y_n-T_i^n y_n\Big) \\
&\quad +\Big(\frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}-\frac{\delta\sum_{i\geq 1}\sigma_{in}}{\gamma_n}\Big)T_i^n y_n \\
&\quad +\frac{\alpha_{n+1}}{\gamma_{n+1}}x_{n+1}-\frac{\alpha_n}{\gamma_n}x_n
\end{aligned}
$$

therefore

$$\begin{aligned}
||z_{n+1}-z_n|| &\leq \frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}\Big\|T_i^{n+1}y_{n+1}-T_i^{n+1}y_n\Big\| \\
&+\frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}\Big\|T_i^{n+1}y_n-T_i^{n}y_n\Big\| \\
&+\Big|\frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}-\frac{\delta\sum_{i\geq 1}\sigma_{in}}{\gamma_{n}}\Big||T_i^{n}y_n|| \\
&+\frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}||+\frac{\alpha_n}{\gamma_n}||x_n|| \\
&\leq \frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})\|y_{n+1}-y_n\| \\
&+\frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}\Big\|T_i^{n+1}y_n-T_i^{n}y_n\Big\| \\
&+\Big|\frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}-\frac{\delta\sum_{i\geq 1}\sigma_{in}}{\gamma_{n}}\Big||T_i^{n}y_n|| \\
&+\frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}||+\frac{\alpha_n}{\gamma_n}||x_n||. \qquad (4.1.4)
\end{aligned}$$

but

$$\begin{aligned}
y_{n+1}-y_n &= \beta_{n+1}\gamma\Big(f(x_{n+1})-f(x_n)\Big)+\Big(\beta_{n+1}-\beta_n\Big)\gamma f(x_n) \\
&+\Big((I-\beta_{n+1}G)x_{n+1}-(I-\beta_{n+1}G)x_n\Big) \\
&+\Big((I-\beta_{n+1}G)x_n-(I-\beta_nG)x_n\Big),
\end{aligned}$$

so that

$$\begin{aligned}
||y_{n+1}-y_n|| &\leq \beta_{n+1}\gamma\alpha||x_{n+1}-x_n||+|\beta_{n+1}-\beta_n|||\gamma f(x_n)|| \\
&+(1-\beta_{n+1}\tau)||x_{n+1}-x_n||+|\beta_{n+1}-\beta_n|||G(x_n)|| \\
&= [1-\beta_{n+1}(\tau-\gamma\alpha)]||x_{n+1}-x_n|| \\
&+|\beta_{n+1}-\beta_n|\big[||\gamma f(x_n)||+||G(x_n)||\big] \qquad (4.1.5)
\end{aligned}$$

Using (4.1.5) in (4.1.4), we obtain that

$$
\begin{aligned}
&||z_{n+1} - z_n|| - ||x_{n+1} - x_n|| \\
&\leq \Big(\frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})[1-\beta_{n+1}(\tau-\gamma\alpha)] - 1\Big)||x_{n+1} - x_n|| \\
&\quad + \frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})|\beta_{n+1} - \beta_n|\big[||\gamma f(x_n)|| + ||G(x_n)||\big] \\
&\quad + \frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n - T_i^n y_n\Big\|}{\gamma_{n+1}} \\
&\quad + \Big|\frac{\delta\sum_{i\geq 1}\sigma_{in+1}||T_i^n y_n||}{\gamma_{n+1}} - \frac{\delta\sum_{i\geq 1}\sigma_{in}||T_i^n y_n||}{\gamma_n}\Big| \\
&\quad + \frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}|| + \frac{\alpha_n}{\gamma_n}||x_n|| \\
&\leq \Big(\frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})[1-\beta_{n+1}(\tau-\gamma\alpha)] - 1\Big)||x_{n+1} - x_n|| \\
&\quad + \frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})|\beta_{n+1} - \beta_n|\big[||\gamma f(x_n)|| + ||G(x_n)||\big] \\
&\quad + \frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n - T_i^n y_n\Big\|}{\gamma_{n+1}} \\
&\quad + \Big|\frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}} - \frac{\delta(1-\alpha_n)}{\gamma_n}\Big|M^* \\
&\quad + \frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}|| + \frac{\alpha_n}{\gamma_n}||x_n||.
\end{aligned}
$$

for some $M^* > 0$ and this implies

$$\limsup_{n\to\infty}(||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \leq 0,$$

and by Lemma 3.5.3, we have

$$\lim_{n\to\infty}||z_n - x_n|| = 0.$$

Hence

$$||x_{n+1} - x_n|| = (1-\gamma_n)||z_n - x_n|| \to 0 \text{ as } n \to \infty. \tag{4.1.6}$$

From the recursion formula (4.1.1), we obtain

$$\delta \sum_{i\geq 1} \sigma_{in} ||T_i^n y_n - x_n|| = ||x_{n+1} - x_n|| \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that for each $i \geq 1$,

$$\lim_{n\to\infty} ||T_i^n y_n - x_n|| = 0. \tag{4.1.7}$$

Also, from the recursion formula (4.1.1), we obtain

$$||y_n - x_n|| = \beta_n ||\gamma f(x_n) - G(x_n)|| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.1.8}$$

which implies that for each $i \geq 1$,

$$||T_i^n y_n - y_n|| \leq ||T_i^n y_n - x_n|| + ||x_n - y_n|| \to 0 \text{ as } n \to \infty. \tag{4.1.9}$$

From (4.1.6) and (4.1.8), for each $i \geq 1$, we obtain

$$\begin{aligned} ||T_i^n x_n - x_n|| &\leq ||T_i^n x_n - T_i^n y_n|| + ||T_i^n y_n - x_n|| \\ &\leq (1 + v_n)||x_n - y_n|| + ||T_i^n y_n - x_n|| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.1.10}$$

Therefore

$$\begin{aligned} ||T_i x_n - x_n|| &\leq ||T_i x_n - T_i^{n+1} x_n|| + ||T_i^{n+1} x_n - T_i^{n+1} x_{n+1}|| \\ &\quad + ||T_i^{n+1} x_{n+1} - x_{n+1}|| + ||x_{n+1} - x_n|| \\ &\leq L_i ||x_n - T_i^n x_n|| + (2 + v_{n+1})||x_{n+1} - x_n|| \\ &\quad + ||T_i^{n+1} x_{n+1} - x_{n+1}||, \end{aligned}$$

for each $i \geq 1$, also by using (4.1.7) and (4.1.10), we obtain

$$\lim_{n\to\infty} ||T_i x_n - x_n|| = 0 \text{ for each } i \geq 1. \tag{4.1.11}$$

we also have

$$\begin{aligned} ||T_i y_n - y_n|| &\leq ||T_i y_n - T_i x_n|| + ||T_i x_n - x_n|| + ||x_n - y_n|| \\ &\leq (1 + L_i)||y_n - x_n|| + ||T_i x_n - x_n||. \end{aligned}$$

This implies that,

$$\lim_{n\to\infty} ||T_i y_n - y_n|| = 0 \text{ for each } i \geq 1. \tag{4.1.12}$$

Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma f(p) - G(p), j(y_n - p)\rangle = \lim_{k\to\infty} \langle \gamma f(p) - G(p), j(y_{n_k} - p)\rangle \tag{4.1.13}$$

and assume without loss of generality that $y_{n_k} \rightharpoonup z \in E$. By Remark 3.2.1 and Theorem 3.5.1, $(I - T_i)$ is demiclosed at zero for each $i \geq 1$, so $z \in F$. Since the duality map of $E$ is weakly sequentially continuous, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \langle \gamma f(p) - G(p), j(y_n - p)\rangle &= \lim_{k\to\infty} \langle \gamma f(p) - G(p), j(y_{n_k} - p)\rangle \\ &\leq \langle \gamma f(p) - G(p), j(z - p)\rangle \leq 0 \end{aligned} \tag{4.1.14}$$

We now conclude by showing that $x_n \to p$ as $n \to \infty$. Since $\frac{v_n}{\beta_n} \to 0$ as $n \to \infty$, if we denote by $w_n$ the value $2v_n + v_n^2$, it implies that $\frac{w_n}{(1+w_n)\beta_n} \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\frac{w_n}{(1+w_n)\beta_n} \leq \frac{\tau - 2\gamma\alpha}{2}$, for all $n \geq n_0$. From recursion formula (4.1.1), we obtain

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||[1 - \delta(1 - \alpha_n)](x_n - p) + \delta \sum_{i\geq 1} \sigma_{in}(T_i^n y_n - p)||^2 \\ &\leq [1 - \delta(1 - \alpha_n)]||x_n - p||^2 + \delta \sum_{i\geq 1} \sigma_{in}||T_i^n y_n - p||^2 \\ &\leq [1 - \delta(1 - \alpha_n)]||x_n - p||^2 + \delta(1 - \alpha_n)(1 + v_n)^2||y_n - p||^2 \\ &\leq [1 - \delta(1 - \alpha_n)]||x_n - p||^2 \\ &\quad + \delta(1 - \alpha_n)(1 + w_n)\Big[||\beta_n(\gamma f(x_n) - Gp) + (I - \beta_n G)(x_n - p)||^2\Big] \end{aligned}$$

$$
\begin{aligned}
&\leq [1-\delta(1-\alpha_n)]||x_n-p||^2+\delta(1-\alpha_n)(1+w_n)\Big[(1-\beta_n\tau)||x_n-p||^2\\
&\quad +2\beta_n\langle\gamma f(x_n)-Gp, j(y_n-p)\rangle\Big]\\
&\leq \Big[1-\delta(1-\alpha_n)+\delta(1-\alpha_n)(1+w_n)(1-\beta_n\tau)\Big]||x_n-p||^2\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\alpha||x_n-p||||y_n-p||\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\langle\gamma f(p)-Gp, j(y_n-p)\rangle\\
&\leq \Big[1-\delta(1-\alpha_n)+\delta(1-\alpha_n)(1+w_n)(1-\beta_n\tau)\Big]||x_n-p||^2\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\alpha||x_n-p||^2\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\alpha||x_n-p||||y_n-x_n||\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\langle\gamma f(p)-Gp, j(y_n-p)\rangle\\
&\leq \Big[1-\delta(1-\alpha_n)+\delta(1-\alpha_n)(1+w_n)(1-\beta_n[\tau-2\gamma\alpha])\Big]||x_n-p||^2\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\alpha||x_n-p||||y_n-x_n||\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\langle\gamma f(p)-Gp, j(y_n-p)\rangle\\
&= \Big[1-\beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\alpha)-\frac{w_n}{(1+w_n)\beta_n}\Big)\Big]||x_n-p||^2\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\langle\gamma f(p)-Gp, j(y_n-p)\rangle\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\alpha||x_n-p||||y_n-x_n||
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[1-\beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\alpha)-\frac{w_n}{(1+w_n)\beta_n}\Big)\Big]||x_n-p||^2\\
&\quad +\beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\alpha)-\frac{w_n}{(1+w_n)\beta_n}\Big)\times\\
&\quad \frac{2[\langle\gamma f(p)-Gp, j(y_n-p)\rangle+\gamma\alpha||x_n-p||||y_n-x_n||]}{\Big((\tau-2\gamma\alpha)-(w_n/(1+w_n)\beta_n)\Big)}.
\end{aligned}
$$

Observe that $\sum\beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\alpha)-\frac{w_n}{(1+w_n)\beta_n}\Big)=\infty$ and

$$
\limsup\Big(\frac{2[\langle\gamma f(p)-Gp, j(y_n-p)\rangle+\gamma\alpha||x_n-p||||y_n-x_n||]}{\Big((\tau-2\gamma\alpha)-(w_n/(1+w_n)\beta_n)\Big)}\Big)\leq 0
$$

Applying Lemma 3.5.4, we obtain $||x_n-p||\to 0$ as $n\to\infty$. This completes the proof. □

The following corollaries follow from Theorem 4.1.1.

**Corollary 4.1.2** *Let $E, G, f, F, C, \{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be as in Theorem 4.1.1. Let $\{T_i\}_{i=1}^{\infty}$ be a family of nonexpansive self mappings of $C$. Assume that $\gamma \in \left(0, \min\{\frac{\tau}{2\alpha}, \eta\}\right)$, where $\tau := (1 - \sqrt{\frac{1-\eta}{\mu}})$. For some fixed $\delta \in (0,1)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by $x_0 \in C$ chosen arbitrarily,*

$$\begin{cases} x_{n+1} = [1 - \delta(1-\alpha_n)]x_n + \delta \sum_{i\geq 1} \sigma_{in} T_i y_n, \\ y_n = \beta_n \gamma f(x_n) + (I - \beta_n G)x_n, \quad n \geq 0. \end{cases} \tag{4.1.15}$$

*Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F$, where $p$ is a solution of the variational inequality problem* (4.1.2).

**Corollary 4.1.3** *Let $E = H$ a real Hilbert space and $G, f, F, C, \{\alpha_n\}_{n=1}^{\infty}, \{T_i\}_{i=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be as in Theorem 4.1.1. Let $\{T_i\}_{i=1}^{\infty}$ be a family of nonexpansive self mappings of $C$. Assume that $\gamma \in \left(0, \min\{\frac{\tau}{2\alpha}, \eta\}\right)$, where $\tau := (1 - \sqrt{\frac{1-\eta}{\mu}})$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined as in* (4.1.15). *Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F$, where $p$ is a solution of the variational inequality problem* (4.1.2).

**Theorem 4.1.4** *Let $E$ be a reflexive real Banach space with a uniformly Gâteaux differentiable norm. Let $G : E \to E$ be an $\eta$-strongly accretive and $\mu$-strictly pseudocontractive with $\eta + \mu > 1$ and let $f : E \to E$ be a contraction with coefficient $\beta \in (0,1)$. Let $\{T_i\}_{i=1}^{\infty}$ be family of uniformly asymptotically regular asymptotically nonexpansive mappings of $E$ into itself with sequences $\{v_{in}\}$ such that $v_{in} \to 0$ as $n \to \infty$ for each $i \geq 1$ and $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $0 < \gamma < \frac{\tau}{2\beta}$, where $\tau := (1 - \sqrt{\frac{1-\eta}{\mu}})$. Let $\{\alpha_n\}_{n=1}^{\infty}$ $\{\beta_n\}_{n=1}^{\infty}$ and $\{\sigma_{i,n}\}_{n=1}^{\infty}, i = 1, 2, 3, ...$ be sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

*(C1)* $\lim_{n\to\infty} \beta_n = 0$ *and* $\lim_{n\to\infty} \frac{v_n}{\beta_n} = 0$, *where* $v_n := \sup_{i\geq 1}\{v_{in}\}$

*(C2)* $\Sigma_{n=0}^{\infty} \beta_n = \infty$

*(C3)* $\lim_{n\to\infty} \alpha_n = 0$ *and (C4)* $\sum_{i=1}^{\infty} \sigma_{in} = 1 - \alpha_n$

*For some fixed $\delta \in (0,1)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined by $x_0 \in E$ chosen arbitrarily,*

$$\begin{cases} y_n = \beta_n \gamma f(x_n) + (I - \beta_n G)x_n, \\ x_n = [1 - \delta(1 - \alpha_n)]x_n + \delta \sum_{i\geq 1} \sigma_{in} T_i^n y_n \quad n \geq 0. \end{cases} \tag{4.1.16}$$

*then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F$, which also solves the following variational inequality:*

$$\langle \gamma f(p) - Gp, j(q - p)\rangle \leq 0, \qquad \forall\ q \in F. \tag{4.1.17}$$

**Proof.** First, we show that $\{x_n\}$ defined by (4.1.16) is well defined. For all $n \in \mathbb{N}$, let us define the mapping

$$T_n^f x := [1 - \delta(1 - \alpha_n)]x + \delta \sum_{n\geq 1} \sigma_{in} T_i^n [\beta_n \gamma f(x) + (I - \beta_n G)x].$$

Indeed, for all $x, y \in E$, we have

$$\begin{aligned}
||T_n^f x - T_n^f y|| &\leq [1-\delta(1-\alpha_n)]||x-y|| + \delta \sum_{i\geq 1} \sigma_{in} ||T_i^n[\beta_n \gamma f(x) + (I-\beta_n)x] \\
&\quad -T_i^n[\beta_n \gamma f(y) + (I-\beta_n)y]|| \\
&\leq [1-\delta(1-\alpha_n)]||x-y|| + \delta(1-\alpha_n)(1+v_n) \\
&\quad \times[\beta_n\gamma||f(x)-f(y)|| + ||(I-\beta_n G)x - (I-\beta_n G)y||] \\
&\leq [1-\delta(1-\alpha_n)]||x-y|| + \delta(1-\alpha_n)(1+v_n) \\
&\quad \times[\beta_n\gamma\beta||x-y|| + (1-\tau\beta_n)||x-y||] \\
&= \Big(1-\delta(1-\alpha_n) + \delta(1-\alpha_n)(1+v_n)[1-\beta_n(\tau-\gamma\beta)]\Big)||x-y|| \\
&= \Big(1-\delta(1-\alpha_n)[\beta_n(1+v_n)(\tau-\gamma\beta) - v_n]\Big)||x-y||.
\end{aligned}$$

Since, $\lim_{n\to\infty}[\beta_n(1+v_n)(\tau-\gamma\beta)-v_n] = 0$, then there exists $n_0 \in \mathbb{N}$ such that $[\beta_n(1+v_n)(\tau-\gamma\beta)-v_n] < \epsilon \in (0,1)$ for all $n \geq n_0$. Thus, for $n \geq n_0$, we have

$$\delta(1-\alpha_n)[\beta_n(1+v_n)(\tau-\gamma\beta)-v_n] < \delta(1-\alpha_n)\epsilon < 1,$$

therefore, for $n \geq n_0$, we obtain

$$1-\delta(1-\alpha_n)[\beta_n(1+v_n)(\tau-\gamma\beta)-v_n] < 1.$$

Hence,

$$||T_n^f x - T_n^f y|| < ||x-y||.$$

Thus, $\{x_n\}$ defined by (4.1.16) is well defined.

Therefore, by the contraction mapping principle, there exists a unique fixed point $x_n \in E$ of $T_n^f$ which satisfies (4.1.16).

Now, from the choice of the parameter $\gamma$, it is easy to see that the mapping $(G-\gamma f) : E \to E$ is strongly accretive and so the variational inequality (4.1.17) has unique solution in $F$. Let $p \in F$, since $\frac{v_n}{\beta_n} \to 0$ as $n \to \infty$, it implies that $\frac{v_n}{(1+v_n)\beta_n} \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\frac{v_n}{(1+v_n)\beta_n} < \frac{\tau-\gamma\beta}{2}$,

for all $n \geq n_0$.

$$\begin{aligned} ||y_n - p|| &= ||\beta_n(\gamma f(x_n) - Gp) + (I - \beta_n G)(x_n - p)|| \\ &\leq \beta_n||\gamma f(x_n) - Gp|| + (1 - \beta_n\tau)||x_n - p|| \\ &\leq \Big(1 - \beta_n(\tau - \gamma\beta)\Big)||x_n - p|| + \beta_n||\gamma f(p) - Gp||. \quad (4.1.18) \end{aligned}$$

From (4.1.16), we obtain

$$\begin{aligned} ||x_n - p|| &= ||[1 - \delta(1 - \alpha_n)](x_n - p) + \delta\sum_{i\geq 1}\sigma_{in}(T_i^n y_n - p)|| \\ &\leq [1 - \delta(1 - \alpha_n)]||x_n - p|| + \delta(1 - \alpha_n)(1 + v_n)||y_n - p|| \\ &\leq \Big[1 - \delta(1 - \alpha_n) + \delta(1 - \alpha_n)(1 + v_n)[1 - \beta_n(\tau - \gamma\beta)]\Big]||x_n - p|| \\ &\quad +\delta(1 - \alpha_n)(1 + v_n)\beta_n||\gamma f(p) - Gp|| \\ &= \Big[1 + \delta(1 - \alpha_n)v_n - \beta_n\delta(1 - \alpha_n)(1 + v_n)(\tau - \gamma\beta)]\Big]||x_n - p|| \\ &\quad +\delta(1 - \alpha_n)(1 + v_n)\beta_n||\gamma f(p) - Gp|| \\ &\leq \Big[1 - \beta_n\delta(1 - \alpha_n)(1 + v_n)\Big((\tau - \gamma\beta) - \frac{v_n}{(1 + v_n)\beta_n}\Big)\Big]||x_n - p|| \\ &\quad +\beta_n\delta(1 - \alpha_n)(1 + v_n)\Big((\tau - \gamma\beta) - \frac{v_n}{(1 + v_n)\beta_n}\Big)\frac{2||\gamma f(p) - Gp||}{\tau - \gamma\beta}. \end{aligned}$$

Therefore,

$$||x_n - p|| \leq \frac{2||\gamma f(p) - Gp||}{\tau - \gamma\beta}, \quad \forall n \geq n_0.$$

Hence $\{x_n\}$ is bounded. Also $\{f(x_n)\}, \{G(x_n)\}, \{y_n\}, \{T_i^n x_n\}$ and $\{T_i^n y_n\}$ are all bounded for each $i \geq 1$.

Also from (4.1.16),

$$||y_n - x_n|| = \beta_n||\gamma f(x_n) - G(x_n)|| \to 0 \quad \text{as} \quad n \to \infty. \qquad (4.1.19)$$

which implies that

$$\delta\sum_{i\geq 1}\sigma_{in}||T_i^n y_n - y_n|| = \delta(1 - \alpha_n)||x_n - y_n|| \to 0 \quad \text{as} \quad n \to \infty,$$

hence, for each $i \geq 1$, we have

$$\lim_{n\to\infty} ||T_i^n y_n - y_n|| = 0. \tag{4.1.20}$$

Since

$$\begin{aligned} ||T_i y_n - y_n|| &\leq ||T_i y_n - T_i(T_i^n y_n)|| + ||T_i(T_i^n y_n) - T_i^n y_n|| + ||T_i^n y_n - y_n|| \\ &\leq (L_i + 1)||y_n - T_i^n y_n|| + ||T_i^{n+1} y_n - T_i^n y_n|| \end{aligned} \tag{4.1.21}$$

From (4.1.20) and asymptotically regularity of $T_i$, for each $i \geq 1$ we obtain

$$\lim_{n\to\infty} ||T_i y_n - y_n|| = 0. \tag{4.1.22}$$

Also

$$\begin{aligned} ||T_i^n x_n - x_n|| &\leq ||T_i^n x_n - T_i^n y_n|| + ||T_i^n y_n - y_n|| + ||y_n - x_n|| \\ &\leq (v_n + 1)||y_n - x_n|| + ||T_i^n y_n - y_n||, \end{aligned}$$

from (4.1.19) and (4.1.20), for each $i \geq 1$, we obtain

$$\lim_{n\to\infty} ||T_i^n x_n - x_n|| = 0. \tag{4.1.23}$$

Following the same argument of (4.1.21) and (4.1.22) and using (4.1.23), we obtain

$$\lim_{n\to\infty} ||T_i x_n - x_n|| = 0, \text{ for each } \quad i \geq 1. \tag{4.1.24}$$

Define a map $\phi : E \to \mathbb{R}$ by

$$\phi(y) := \mu_n ||y_n - y||^2, \ \forall y \in E.$$

Then, $\phi(y) \to \infty$ as $||y|| \to \infty$, $\phi$ is continuous and convex, so since $E$ is reflexive, there exists $q \in E$ such that $\phi(q) = \min_{u \in E} \phi(u)$. Hence, the set

$$K^* := \{y \in E : \phi(y) = \min_{u \in E} \phi(u)\} \neq \emptyset.$$

Since $\lim_{n\to\infty} ||y_n - T_i y_n|| = 0$, $\lim_{n\to\infty} ||y_n - T_i^m y_n|| = 0$, for any $m \geq 1$ and each $i \geq 1$, by induction. Now let $v \in K^*$, we have

$$\begin{aligned} \lim_{n\to\infty} \phi(T_i v) &= \lim_{n\to\infty} \mu_n ||y_n - T_i v||^2 \\ &= \lim_{n\to\infty} \mu_n ||y_n - T_i y_n + T_i y_n - T_i v||^2 \\ &\leq \lim_{n\to\infty} \mu_n [(1 + v_n)||y_n - v||]^2 = \lim_{n\to\infty} \phi(v), \end{aligned}$$

and hence $T_i v \in K^*$.

Now let $z \in F$, then $z = T_i z$. Since $K^*$ is a closed convex set, there exists a unique $v^* \in K^*$ such that

$$||z - v^*|| = \min_{u \in K^*} ||z - u||.$$

But

$$\lim_{n\to\infty} ||z - T_i v^*|| = \lim_{n\to\infty} ||T_i z - T_i v^*|| \leq \lim_{n\to\infty} (1 + v_n)||z - v^*||,$$

which implies $v^* = T_i v^*$ and so $K^* \cap F \neq \emptyset$.

Without loss of generality, assume that $q := (G - \gamma f)p \in K^* \cap F(T)$. Let $t \in (0, 1)$. Then, it follows that $\phi(p) \leq \phi(p - t(G - \gamma f)p)$ and using Lemma 3.5.2, we obtain

$$||y_n - p + t(G - \gamma f)p||^2 \leq ||y_n - p||^2 + 2t\langle (G - \gamma f)p, j(y_n - p + t(G - \gamma f)p)\rangle$$

which implies that

$$\mu_n\langle(\gamma f - G)p, j(y_n - p + t(G - \gamma f)p)\rangle \leq 0.$$

Moreover,

$$\begin{aligned}\mu_n\langle(\gamma f - G)p, j(y_n - p)\rangle &= \mu_n\langle(\gamma f - G)p, j(y_n - p) - j(y_n - p + t(G - \gamma f)p)\rangle \\ &\quad +\mu_n\langle(\gamma f - G)p, j(y_n - p + t(G - \gamma f)p)\rangle \\ &\leq \mu_n\langle(\gamma f - G)p, j(y_n - p) - j(y_n - p + t(G - \gamma f)p)\rangle.\end{aligned}$$

Since $j$ is norm-to-weak$^*$ uniformly continuous on bounded subsets of $E$, we have that

$$\mu_n\langle(\gamma f - G)p, j(y_n - p)\rangle \leq 0. \tag{4.1.25}$$

Now, from (4.1.16), we obtain

$$\begin{aligned}||y_n - p||^2 &= ||\beta_n\gamma f(x_n) + (I - \beta_n G)x_n - p||^2 \\ &= ||(I - \beta_n G)x_n - (I - \beta_n G)p + \beta_n(\gamma f(x_n) - Gp)||^2 \\ &\leq ||(I - \beta_n G)x_n - (I - \beta_n G)p||^2 + 2\beta_n\langle\gamma f(x_n) - Gp, j(y_n - p)\rangle \\ &\leq (1 - \tau\beta_n)^2||x_n - p||^2 + 2\beta_n\langle\gamma f(x_n) - \gamma f(p), j(y_n - p)\rangle \\ &\quad +2\beta_n\langle\gamma f(p) - Gp, j(y_n - p)\rangle \\ &\leq (1 - \tau\beta_n)||x_n - p||^2 + 2\beta_n\gamma\beta||x_n - p||||y_n - p|| \\ &\quad +2\beta_n\langle\gamma f(p) - Gp, j(y_n - p)\rangle \\ &\leq [1 - \beta_n(\tau - \gamma\beta)]||x_n - p||^2 + \beta_n\gamma\beta||y_n - p||^2 \\ &\quad +2\beta_n\langle\gamma f(p) - Gp, j(y_n - p)\rangle\end{aligned}$$

Therefore

$$||y_n - p||^2 \leq \frac{1-\beta_n(\tau-\gamma\beta)}{1-\beta_n\gamma\beta}||x_n-p||^2 + \frac{2\beta_n}{1-\beta_n\gamma\beta}\langle \gamma f(p) - G(p), j(y_n-p)\rangle$$
$$= \Big(1 - \frac{\beta_n[\tau-2\gamma\beta]}{1-\beta_n\gamma\beta}\Big)||x_n-p||^2 + \frac{2\beta_n}{1-\beta_n\gamma\beta}\langle \gamma f(p) - G(p), j(y_n-p)\rangle \quad (4.1.26)$$

From (4.1.16) and (4.1.26) by denoting $w_n$ for $2v_n + v_n^2$, we obtain

$$\begin{aligned}
||x_n-p||^2 &= ||[1-\delta(1-\alpha_n)](x_n-p) + \delta\sum_{i\geq 1}\sigma_{in}(T_i^n y_n - p)||^2 \\
&\leq [1-\delta(1-\alpha_n)]||x_n-p||^2 + \delta\sum_{i\geq 1}\sigma_{in}||T_i^n y_n - p||^2 \\
&\leq [1-\delta(1-\alpha_n)]||x_n-p||^2 + \delta(1-\alpha_n)(1+v_n)^2||y_n-p||^2 \\
&= [1-\delta(1-\alpha_n)]||x_n-p||^2 + \delta(1-\alpha_n)(1+w_n)||y_n-p||^2 \\
&\leq [1-\delta(1-\alpha_n)]||x_n-p||^2 \\
&\quad + \delta(1-\alpha_n)(1+w_n)\Big(1-\frac{\beta_n[\tau-2\gamma\beta]}{1-\beta_n\gamma\beta}\Big)||x_n-p||^2 \\
&\quad + \delta(1-\alpha_n)(1+w_n)\frac{2\beta_n}{1-\beta_n\gamma\beta}\langle\gamma f(p)-G(p), j(y_n-p)\rangle \\
&= \Big[1-\delta(1-\alpha_n)(1+w_n) \\
&\quad \times \frac{\beta_n}{1-\beta_n\gamma\beta}\Big([\tau-2\gamma\beta] - \frac{w_n(1-\beta_n\gamma\beta)}{(1+w_n)\beta_n}\Big)\Big]||x_n-p||^2 \\
&\quad + \delta(1-\alpha_n)(1+w_n)\frac{2\beta_n}{1-\beta_n\gamma\beta}\langle\gamma f(p)-G(p), j(y_n-p)\rangle.
\end{aligned}$$

Since $\frac{v_n}{\beta_n} \to 0$ as $n \to \infty$, it implies that $\frac{w_n(1-\beta_n\gamma\beta)}{(1+w_n)\beta_n} \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\frac{w_n(1-\beta_n\gamma\beta)}{(1+w_n)\beta_n} \leq \frac{\tau-2\gamma\beta}{2}$, for all $n \geq n_0$. This implies

$$||x_n-p||^2 \leq \frac{2\langle\gamma f(p)-G(p), j(y_n-p)\rangle}{\Big([\tau-2\gamma\beta] - \frac{w_n(1-\beta_n\gamma\beta)}{(1+w_n)\beta_n}\Big)}$$

and so from (4.1.25), we obtain

$$\mu_n||x_n-p||^2 \leq 0.$$

Thus there exists a subsequence say $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = p$, from (4.1.19) we obtain $\lim_{k\to\infty} y_{n_k} = p$. Next, we show that $p$ solves the variational inequality (4.1.17). Indeed, from the relation (4.1.16)

$$y_n = \beta_n \gamma f(x_n) + (I - \beta_n G)x_n$$

we get

$$(G(y_n) - \gamma f(x_n)) = -\frac{1}{\beta_n}(I - \beta_n G)(y_n - x_n).$$

So, for any $z \in F$ we obtain

$$\begin{aligned}
\langle (G(y_n) - \gamma f(x_n)), j(y_n - z)\rangle &= -\frac{1}{\beta_n}\langle (I - \beta_n G)(y_n - x_n), j(y_n - z)\rangle \\
&= -\frac{1}{\beta_n}\langle y_n - x_n, j(y_n - z)\rangle + \langle G(y_n - x_n), j(y_n - z)\rangle \\
&= -\frac{1}{\beta_n \delta(1-\alpha_n)}\langle \delta \sum_{i\geq 1} \sigma_{in}(y_n - T_i^n y_n), j(y_n - z)\rangle \\
&\quad + \langle G(y_n - x_n), j(y_n - z)\rangle \\
&= -\frac{1}{\beta_n \delta(1-\alpha_n)}\langle \delta \sum_{i\geq 1} \sigma_{in}(y_n - T_i^n y_n) - \delta \sum_{i\geq 1} \sigma_{in}(I - T_i^n)z, j(y_n - z)\rangle \\
&\quad + \langle G(y_n - x_n), j(y_n - z)\rangle \\
&= -\frac{1}{\beta_n \delta(1-\alpha_n)}\langle \delta(1-\alpha_n)(y_n - z) - \delta \sum_{i\geq 1} \sigma_{in}(T_i^n y_n - T_i^n z), j(y_n - z)\rangle \\
&\quad + \langle G(y_n - x_n), j(y_n - z)\rangle \\
&= -\frac{1}{\beta_n \delta(1-\alpha_n)}\Big(\delta(1-\alpha_n)\langle y_n - z, j(y_n - z)\rangle \\
&\quad - \delta \sum_{i\geq 1} \sigma_{in}\langle T_i^n y_n - T_i^n z, j(y_n - z)\rangle\Big) \\
&\quad + \langle G(y_n - x_n), j(y_n - z)\rangle \\
&\leq -\frac{1}{\beta_n}||y_n - z||^2 + \frac{1}{\beta_n}(1 + v_n)||y_n - z||^2 \\
&\quad + ||G||||y_n - x_n||||y_n - z||.
\end{aligned}$$

Therefore

$$\langle (G(y_n) - \gamma f(x_n)), j(y_n - z)\rangle \leq \frac{v_n}{\beta_n}||y_n - z||^2 + ||G||||y_n - x_n||||y_n - z||.$$

and so

$$\langle (G(y_{n_k}) - \gamma f(x_{n_k})), j(y_{n_k} - z)\rangle \leq \frac{v_{n_k}}{\beta_{n_k}} ||y_{n_k} - z||^2 + ||G||||y_{n_k} - x_{n_k}||||y_{n_k} - z||.$$

Taking the limit as $k \to \infty$ on both sides of the above inequality, and using the fact that $x_{n_k} \to p$ and $y_{n_k} \to p$ as $k \to \infty$ we obtain $\langle (G(p) - \gamma f(p)), j(p - z)\rangle \leq 0 \ \forall z \in F$ . This implies that $p \in F$ is a solution of the variational inequality (4.1.17). Now assume there exists another subsequence of $\{x_n\}$ and $\{y_n\}$ say $\{x_{n_j}\}$ and $\{y_{n_j}\}$ respectively such that $\lim\limits_{j\to\infty} x_{n_j} = p^* = \lim\limits_{j\to\infty} y_{n_j}$. Then, using (4.1.20) and (4.1.24) we have $p^* \in F$. Repeating the above argument with $p$ replaced by $p^*$ we can easily obtain that $p^*$ also solved the variational inequality (4.1.17). By uniqueness of the solution of the variational inequality, we obtained that $p = p^*$. This completes the proof. □

**Theorem 4.1.5** *Let $E$ be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let $G : E \to E$ be an $\eta$-strongly accretive and $\mu$-strictly pseudocontractive with $\eta + \mu > 1$ and let $f : E \to E$ be a contraction with coefficient $\beta \in (0, 1)$. Let $\{T_i\}_{i=1}^{\infty}$ be a family of uniformly asymptotically regular asymptotically nonexpansive self mappings of $E$ with sequences $\{v_{in}\}$ such that $v_{in} \to 0$ as $n \to \infty$ for each $i \geq 1$ and $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ . Assume that $\gamma \in \left(0, \min\{\frac{\tau}{2\beta}, \eta\}\right)$, where $\tau := (1 - \sqrt{\frac{1-\eta}{\mu}})$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\sigma_{in}\}_{n=1}^{\infty}, i = 1, 2, 3, ...$ be sequences in $(0, 1)$, and suppose that the following conditions are satisfied:*

*(C1)* $\lim\limits_{n\to\infty} \beta_n = 0$ *and* $\lim\limits_{n\to\infty} \frac{v_n}{\beta_n} = 0$, *where* $v_n := \sup\limits_{i\geq 1}\{v_{in}\}$

*(C2)* $\Sigma_{n=0}^{\infty} \beta_n = \infty$

*(C3)* $\lim\limits_{n\to\infty} \alpha_n = 0$ *and (C4)* $\sum_{i=1}^{\infty} \sigma_{in} = 1 - \alpha_n \ \forall \ n \in \mathbb{N}$.

*For some fixed $\delta \in (0, 1)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by $x_0 \in E$*

*chosen arbitrarily,*

$$\begin{cases} y_n = \beta_n \gamma f(x_n) + (I - \beta_n G)x_n, \\ x_{n+1} = [1 - \delta(1 - \alpha_n)]x_n + \delta \sum_{i \geq 1} \sigma_{in} T_i^n y_n, \quad n \geq 0. \end{cases} \tag{4.1.27}$$

*Then,* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in F$*, where* $p$ *is the unique solution of the variational inequality problem*

$$\langle \gamma f(p) - Gp, j(q - p) \rangle \leq 0, \ \forall q \in F. \tag{4.1.28}$$

**Proof.** From the choice of $\gamma$, $(G - \gamma f)$ is strongly accretive, then the variational inequality (4.1.28) has a unique solution in $F$. Now we show that $\{x_n\}$ is bounded. Let $p \in F$, since $\frac{v_n}{\beta_n} \to 0$ as $n \to \infty$, it implies that $\frac{v_n}{(1+v_n)\beta_n} \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\frac{v_n}{(1+v_n)\beta_n} < \frac{\tau - \gamma\beta}{2}$, for all $n \geq n_0$.

$$\begin{aligned} ||y_n - p|| &= ||\beta_n(\gamma f(x_n) - Gp) + (I - \beta_n G)(x_n - p)|| \\ &\leq \beta_n ||\gamma f(x_n) - Gp|| + (1 - \beta_n \tau)||x_n - p|| \\ &\leq \Big(1 - \beta_n(\tau - \gamma\beta)\Big)||x_n - p|| + \beta_n ||\gamma f(p) - Gp||. \end{aligned} \tag{4.1.29}$$

Using (4.1.29), we obtain

$$\begin{aligned} ||x_{n+1} - p|| &= ||[1 - \delta(1 - \alpha_n)](x_n - p) + \delta \sum_{i \geq 1} \sigma_{in}(T_i^n y_n - p)|| \\ &\leq [1 - \delta(1 - \alpha_n)]||x_n - p|| + \delta(1 - \alpha_n)(1 + v_n)||y_n - p|| \\ &\leq \Big[1 - \delta(1 - \alpha_n) + \delta(1 - \alpha_n)(1 + v_n)[1 - \beta_n(\tau - \gamma\beta)]\Big]||x_n - p|| \\ &\quad + \delta(1 - \alpha_n)(1 + v_n)\beta_n ||\gamma f(p) - Gp|| \end{aligned}$$

$$
\begin{aligned}
&= \Big[1+\delta(1-\alpha_n)v_n-\beta_n\delta(1-\alpha_n)(1+v_n)(\tau-\gamma\beta)]\Big]||x_n-p|| \\
&\quad +\delta(1-\alpha_n)(1+v_n)\beta_n||\gamma f(p)-Gp|| \\
&\leq \Big[1-\beta_n\delta(1-\alpha_n)(1+v_n)\Big((\tau-\gamma\beta)-\frac{v_n}{(1+v_n)\beta_n}\Big)\Big]||x_n-p|| \\
&\quad +\beta_n\delta(1-\alpha_n)(1+v_n)\Big((\tau-\gamma\beta)-\frac{v_n}{(1+v_n)\beta_n}\Big)\frac{2||\gamma f(p)-Gp||}{\tau-\gamma\beta} \\
&\leq \max\Big\{||x_n-p||,\frac{2||\gamma f(p)-Gp||}{\tau-\gamma\beta}\Big\}
\end{aligned}
$$

By induction, we obtain

$$
||x_n-p||\leq \max\Big\{||x_{n_0}-p||,\frac{2||\gamma f(p)-Gp||}{\tau-\gamma\beta}\Big\}\ \forall n\geq n_0.
$$

Hence $\{x_n\}$ is bounded. Also $\{f(x_n)\},\{G(x_n)\},\{y_n\},\{T_i^n x_n\}$ and $\{T_i^n y_n\}$ are all bounded.

Define two sequences by $\gamma_n := (1-\delta)\alpha_n+\delta$ and $z_n := \frac{1}{\gamma_n}(x_{n+1}-x_n+\gamma_n x_n)$. From the recursion formula (4.1.27), observe that

$$
z_n=\frac{1}{\gamma_n}(\delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n+\alpha_n x_n)
$$

which implies

$$
\begin{aligned}
z_{n+1}-z_n &= \frac{1}{\gamma_{n+1}}\Big[\delta\sum_{i\geq 1}\sigma_{in+1}T_i^{n+1}y_{n+1}+\alpha_{n+1}x_{n+1}\Big] \\
&\quad -\frac{1}{\gamma_n}\Big[\delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n+\alpha_n x_n\Big] \\
&= \frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}\Big(T_i^{n+1}y_{n+1}-T_i^{n+1}y_n\Big) \\
&\quad +\frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}\Big(T_i^{n+1}y_n-T_i^n y_n\Big) \\
&\quad +\Big(\frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}-\frac{\delta\sum_{i\geq 1}\sigma_{in}}{\gamma_n}\Big)T_i^n y_n \\
&\quad +\frac{\alpha_{n+1}}{\gamma_{n+1}}x_{n+1}-\frac{\alpha_n}{\gamma_n}x_n
\end{aligned}
$$

therefore

$$\begin{aligned}
||z_{n+1}-z_n|| &\leq \frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}\Big\|T_i^{n+1}y_{n+1}-T_i^{n+1}y_n\Big\| \\
&+\frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}\Big\|T_i^{n+1}y_n-T_i^{n}y_n\Big\| \\
&+\Big|\frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}-\frac{\delta\sum_{i\geq 1}\sigma_{in}}{\gamma_{n}}\Big||T_i^{n}y_n|| \\
&+\frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}||+\frac{\alpha_n}{\gamma_n}||x_n|| \\
&\leq \frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})\|y_{n+1}-y_n\| \\
&+\frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}\Big\|T_i^{n+1}y_n-T_i^{n}y_n\Big\| \\
&+\Big|\frac{\delta\sum_{i\geq 1}\sigma_{in+1}}{\gamma_{n+1}}-\frac{\delta\sum_{i\geq 1}\sigma_{in}}{\gamma_{n}}\Big||T_i^{n}y_n|| \\
&+\frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}||+\frac{\alpha_n}{\gamma_n}||x_n||. \qquad (4.1.30)
\end{aligned}$$

but

$$\begin{aligned}
y_{n+1}-y_n &= \beta_{n+1}\gamma\Big(f(x_{n+1})-f(x_n)\Big)+\Big(\beta_{n+1}-\beta_n\Big)\gamma f(x_n) \\
&+\Big((I-\beta_{n+1}G)x_{n+1}-(I-\beta_{n+1}G)x_n\Big) \\
&+\Big((I-\beta_{n+1}G)x_n-(I-\beta_nG)x_n\Big),
\end{aligned}$$

so that

$$\begin{aligned}
||y_{n+1}-y_n|| &\leq \beta_{n+1}\gamma\beta||x_{n+1}-x_n||+|\beta_{n+1}-\beta_n|||\gamma f(x_n)|| \\
&+(1-\beta_{n+1}\tau)||x_{n+1}-x_n||+|\beta_{n+1}-\beta_n|||G(x_n)|| \\
&= [1-\beta_{n+1}(\tau-\gamma\beta)]||x_{n+1}-x_n|| \\
&+|\beta_{n+1}-\beta_n|\big[||\gamma f(x_n)||+||G(x_n)||\big] \qquad (4.1.31)
\end{aligned}$$

Using (4.1.31) in (4.1.30), we obtain

$$
\begin{aligned}
&||z_{n+1} - z_n|| - ||x_{n+1} - x_n|| \\
&\leq \Big(\frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})[1-\beta_{n+1}(\tau-\gamma\beta)] - 1\Big)||x_{n+1} - x_n|| \\
&\quad + \frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})|\beta_{n+1} - \beta_n|\big[||\gamma f(x_n)|| + ||G(x_n)||\big] \\
&\quad + \frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n - T_i^n y_n\Big\|}{\gamma_{n+1}} \\
&\quad + \Big|\frac{\delta\sum_{i\geq 1}\sigma_{in+1}||T_i^n y_n||}{\gamma_{n+1}} - \frac{\delta\sum_{i\geq 1}\sigma_{in}||T_i^n y_n||}{\gamma_n}\Big| \\
&\quad + \frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}|| + \frac{\alpha_n}{\gamma_n}||x_n||
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big(\frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})[1-\beta_{n+1}(\tau-\gamma\beta)] - 1\Big)||x_{n+1} - x_n|| \\
&\quad + \frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})|\beta_{n+1} - \beta_n|\big[||\gamma f(x_n)|| + ||G(x_n)||\big] \\
&\quad + \frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n - T_i^n y_n\Big\|}{\gamma_{n+1}} \\
&\quad + \Big|\frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}} - \frac{\delta(1-\alpha_n)}{\gamma_n}\Big|M^* \\
&\quad + \frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}|| + \frac{\alpha_n}{\gamma_n}||x_n||.
\end{aligned}
$$

for some $M^* > 0$ and this implies

$$\limsup_{n\to\infty}(||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \leq 0,$$

and from Lemma 3.5.3, we have

$$\lim_{n\to\infty}||z_n - x_n|| = 0.$$

Hence

$$||x_{n+1} - x_n|| = (1-\gamma_n)||z_n - x_n|| \to 0 \text{ as } n \to \infty. \tag{4.1.32}$$

From the recursion formula (4.1.27), we obtain

$$\delta \sum_{i\geq 1} \sigma_{in} ||T_i^n y_n - x_n|| = ||x_{n+1} - x_n|| \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that for each $i \geq 1$,

$$\lim_{n\to\infty} ||T_i^n y_n - x_n|| = 0. \tag{4.1.33}$$

Also, from the recursion formula (4.1.27), we obtain

$$||y_n - x_n|| = \beta_n ||\gamma f(x_n) - G(x_n)|| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.1.34}$$

which implies that for each $i \geq 1$,

$$||T_i^n y_n - y_n|| \leq ||T_i^n y_n - x_n|| + ||x_n - y_n|| \to 0 \text{ as } n \to \infty. \tag{4.1.35}$$

From (4.1.32) and (4.1.34), for each $i \geq 1$, we obtain

$$\begin{aligned} ||T_i^n x_n - x_n|| &\leq ||T_i^n x_n - T_i^n y_n|| + ||T_i^n y_n - x_n|| \\ &\leq (1 + v_n)||x_n - y_n|| + ||T_i^n y_n - x_n|| \to 0 \end{aligned} \tag{4.1.36}$$

as $n \to \infty$. Therefore

$$\begin{aligned} ||T_i x_n - x_n|| &\leq ||T_i x_n - T_i^{n+1} x_n|| + ||T_i^{n+1} x_n - T_i^{n+1} x_{n+1}|| \\ &\quad + ||T_i^{n+1} x_{n+1} - x_{n+1}|| + ||x_{n+1} - x_n|| \\ &\leq L_i ||x_n - T_i^n x_n|| + (2 + v_{n+1})||x_{n+1} - x_n|| \\ &\quad + ||T_i^{n+1} x_{n+1} - x_{n+1}||, \end{aligned}$$

for each $i \geq 1$, also by using (4.1.33) and (4.1.36), we obtain

$$\lim_{n\to\infty} ||T_i x_n - x_n|| = 0 \text{ for each } i \geq 1. \tag{4.1.37}$$

we also have

$$\begin{aligned} ||T_i y_n - y_n|| &\leq ||T_i y_n - T_i x_n|| + ||T_i x_n - x_n|| + ||x_n - y_n|| \\ &\leq (1 + L_i)||y_n - x_n|| + ||T_i x_n - x_n||. \end{aligned}$$

This implies that,

$$\lim_{n\to\infty} ||T_i y_n - y_n|| = 0 \text{ for each } i \geq 1. \tag{4.1.38}$$

Next we show that

$$\limsup_{n\to\infty} \langle (\gamma f - G)p, j(y_n - p)\rangle \leq 0.$$

For each $m \geq 0$, let $z_m \in E$ be the unique fixed point of the contraction mapping, such that $z_m = [1 - \delta(1 - \alpha_n)]z_m + \delta \sum_{i\geq 1} \sigma_{im} T_i^m y_m$, where $y_m = \beta_m \gamma f(z_m) + (I - \beta_m G)z_m$ (see Theorem 4.1.4) and that $p = \lim_{m\to\infty} z_m$, so that

$$\begin{aligned} ||z_m - y_n||^2 &\leq [1 - \delta(1 - \alpha_m)]||z_m - y_n||^2 + \delta \sum_{i\geq 1} \sigma_{im} ||T_i^m y_m - y_n||^2 \\ &\leq [1 - \delta(1 - \alpha_m)]||z_m - y_n||^2 \\ &\quad + \delta \sum_{i\geq 1} \sigma_{im} \Big[ ||T_i^m y_m - T_i^m y_n|| + ||T_i^m y_n - y_n|| \Big]^2 \\ &\leq [1 - \delta(1 - \alpha_m)]||z_m - y_n||^2 + \delta \sum_{i\geq 1} \sigma_{im} \Big[ ||T_i^m y_m - T_i^m y_n||^2 \\ &\quad + 2||T_i^m y_m - T_i^m y_n||||T_i^m y_n - y_n|| + ||T_i^m y_n - y_n||^2 \Big]^2 \\ &\leq [1 - \delta(1 - \alpha_m)]||z_m - y_n||^2 + \delta \sum_{i\geq 1} \sigma_{im} \Big[ (1 + w_m)||y_m - y_n||^2 \\ &\quad + 2(1 + v_m)||y_m - y_n||||T_i^m y_n - y_n|| + ||T_i^m y_n - y_n||^2 \Big]^2 \\ &= [1 - \delta(1 - \alpha_m)]||z_m - y_n||^2 \\ &\quad + \delta(1 - \alpha_m)(1 + w_m)||y_m - y_n||^2 \\ &\quad + \delta \sum_{i\geq 1} \sigma_{im} [(1 + v_m)||y_m - y_n|| + ||T_i^m y_n - y_n||]||T_i^m y_n - y_n|| \end{aligned}$$

$$
\begin{aligned}
&= [1-\delta(1-\alpha_m)]||z_m - y_n||^2 \\
&\quad +\delta(1-\alpha_m)(1+w_m)||\beta_m(\gamma f(z_m) - G(z_m)) + z_m - y_n||^2 \\
&\quad +\delta\sum_{i\geq 1}\sigma_{im}[(1+v_m)||y_m - y_n|| + ||T_i^m y_n - y_n||]||T_i^m y_n - y_n|| \\
&\leq [1-\delta(1-\alpha_m)]||z_m - y_n||^2 \\
&\quad +\delta(1-\alpha_m)(1+w_m)\Big[||z_m - y_n||^2 \\
&\quad +2\beta_m\langle\gamma f(z_m) - G(z_m), j(y_m - y_n)\rangle\Big] \\
&\quad +\delta\sum_{i\geq 1}\sigma_{im}[(1+v_m)||y_m - y_n|| + ||T_i^m y_n - y_n||]||T_i^m y_n - y_n|| \\
&= ||z_m - y_n||^2 + \delta(1-\alpha_m)w_m||z_m - y_n||^2 \\
&\quad +2\beta_m\delta(1-\alpha_m)(1+w_m)\langle\gamma f(z_m) - G(z_m), j(y_m - y_n)\rangle \\
&\quad +\delta\sum_{i\geq 1}\sigma_{im}[(1+v_m)||y_m - y_n|| + ||T_i^m y_n - y_n||]||T_i^m y_n - y_n||.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\langle\gamma f(z_m) - G(z_m), j(y_n - y_m)\rangle &\leq \frac{w_m}{2\beta_m(1+w_m)}||z_m - y_n||^2 \\
&+\frac{\delta\sum_{i\geq 1}\sigma_{im}[(1+v_m)||y_m - y_n|| + ||T_i^m y_n - y_n||]||T_i^m y_n - y_n||}{2\beta_m(1+v_m)}.
\end{aligned}
$$

Now, taking limit superior as $n\to\infty$ firstly, and then as $m\to\infty$, we have

$$
\limsup_{m\to\infty}\limsup_{n\to\infty}\langle\gamma f(z_m) - Gz_m, j(y_n - z_m)\rangle \leq 0 \tag{4.1.39}
$$

Moreover, we note that

$$
\begin{aligned}
\langle\gamma f(p) - Gp, j(y_n - p)\rangle &= \langle\gamma f(p) - Gp, j(y_n - p)\rangle \\
&\quad -\langle\gamma f(p) - Gp, j(y_n - z_m)\rangle \\
&\quad +\langle\gamma f(p) - Gp, j(y_n - z_m)\rangle - \langle\gamma f(p) - Gz_m, j(y_n - z_m)\rangle \\
&\quad +\langle\gamma f(p) - Gz_m, j(y_n - z_m)\rangle - \langle\gamma f(z_m) - Gz_m, j(y_n - z_m)\rangle \\
&\quad +\langle\gamma f(z_m) - Gz_m, j(y_n - z_m)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle \gamma f(p) - Gp, j(y_n - p) - j(y_n - z_m)\rangle \\
&\quad + \langle Gz_m - Gp, j(y_n - z_m)\rangle \\
&\quad + \langle \gamma f(z_m) - \gamma f(p), j(y_n - z_m)\rangle \\
&\quad + \langle \gamma f(z_m) - Gz_m, j(y_n - z_m)\rangle \qquad (4.1.40)
\end{aligned}
$$

Taking limit superior as $n \to \infty$ in (4.1.40), we have

$$
\begin{aligned}
&\limsup_{n\to\infty} \langle \gamma f(p) - Gp, j(y_n - p)\rangle \\
&\quad \leq \limsup_{n\to\infty} \langle \gamma f(p) - Gp, j(y_n - p) - j(y_n - z_m)\rangle \\
&\quad + ||Gz_m - Gp|| \limsup_{n\to\infty} ||y_n - z_m|| \\
&\quad + ||\gamma f(z_m) - \gamma f(p)|| \limsup_{n\to\infty} ||y_n - z_m|| \\
&\quad + \limsup_{n\to\infty} \langle \gamma f(z_m) - Gz_m, j(y_n - z_m)\rangle \\
&\leq \limsup_{n\to\infty} \langle \gamma f(p) - Gp, j(y_n - p) - j(y_n - z_m)\rangle \\
&\quad + \Big((1 + \frac{1}{\mu}) + \beta\gamma\Big) \|z_m - p\| \limsup_{n\to\infty} ||y_n - z_m|| \\
&\quad + \limsup_{n\to\infty} \langle \gamma f(z_m) - Gz_m, j(y_n - z_m)\rangle
\end{aligned}
$$

By Theorem 4.1.4, $z_m \to p \in F$ as $m \to \infty$ and the fact that $E$ has a uniformly Gâteaux differentiable norm implies that $j$ is $norm - to - weak^*$ uniformly continuous on bounded subset of $E$, so we conclude using (4.1.85) that

$$
\begin{aligned}
\limsup_{n\to\infty} \langle \gamma f(p) - Gp, j(y_n - p)\rangle &\leq \limsup_{m\to\infty} \limsup_{n\to\infty} \langle \gamma f(z_m) - Gz_m, j(y_n - p)\rangle \\
&\leq 0
\end{aligned}
$$

We now conclude by showing that $x_n \to p$ as $n \to \infty$. Since $\frac{v_n}{\beta_n} \to 0$ as $n \to \infty$. If we denote by $w_n$ the value $2v_n + v_n^2$, it implies that $\frac{w_n}{(1+w_n)\beta_n} \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\frac{w_n}{(1+w_n)\beta_n} \leq \frac{\tau - 2\gamma\beta}{2}$, for all $n \geq n_0$. From

recursion formula (4.1.27), we obtain that

$$
\begin{aligned}
||x_{n+1}-p||^2 &= |||[1-\delta(1-\alpha_n)](x_n-p)+\delta\sum_{i\geq 1}\sigma_{in}(T_i^n y_n-p)||^2 \\
&\leq [1-\delta(1-\alpha_n)]||x_n-p||^2+\delta\sum_{i\geq 1}\sigma_{in}||T_i^n y_n-p||^2 \\
&\leq [1-\delta(1-\alpha_n)]||x_n-p||^2+\delta(1-\alpha_n)(1+v_n)^2||y_n-p||^2 \\
&\leq [1-\delta(1-\alpha_n)]||x_n-p||^2 \\
&\quad +\delta(1-\alpha_n)(1+w_n)\Big[||\beta_n(\gamma f(x_n)-Gp)+(I-\beta_n G)(x_n-p)||^2\Big] \\
&\leq [1-\delta(1-\alpha_n)]||x_n-p||^2+\delta(1-\alpha_n)(1+w_n)\Big[(1-\beta_n\tau)||x_n-p||^2 \\
&\quad +2\beta_n\langle\gamma f(x_n)-Gp, j(y_n-p)\rangle\Big] \\
&\leq \Big[1-\delta(1-\alpha_n)+\delta(1-\alpha_n)(1+w_n)(1-\beta_n\tau)\Big]||x_n-p||^2 \\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\beta||x_n-p||||y_n-p|| \\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\langle\gamma f(p)-Gp, j(y_n-p)\rangle \\
&\leq \Big[1-\delta(1-\alpha_n)+\delta(1-\alpha_n)(1+w_n)(1-\beta_n\tau)\Big]||x_n-p||^2 \\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\beta||x_n-p||^2 \\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\beta||x_n-p||||y_n-x_n|| \\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\langle\gamma f(p)-Gp, j(y_n-p)\rangle \\
&\leq \Big[1-\delta(1-\alpha_n)+\delta(1-\alpha_n)(1+w_n)(1-\beta_n[\tau-2\gamma\beta])\Big]||x_n-p||^2 \\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\beta||x_n-p||||y_n-x_n|| \\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\langle\gamma f(p)-Gp, j(y_n-p)\rangle \\
&= \Big[1-\beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\beta)-\frac{w_n}{(1+w_n)\beta_n}\Big)\Big]||x_n-p||^2 \\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\langle\gamma f(p)-Gp, j(y_n-p)\rangle \\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\beta||x_n-p||||y_n-x_n||
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \Big[1-\beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\beta)-\frac{w_n}{(1+w_n)\beta_n}\Big)\Big]||x_n-p||^2 \\
& +\beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\beta)-\frac{w_n}{(1+w_n)\beta_n}\Big)\times \\
& \frac{2[\langle\gamma f(p)-Gp, j(y_n-p)\rangle+\gamma\alpha||x_n-p||||y_n-x_n||]}{\Big((\tau-2\gamma\beta)-(w_n/(1+w_n)\beta_n)\Big)}.
\end{aligned}
$$

Observe that $\sum\beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\beta)-\frac{w_n}{(1+w_n)\beta_n}\Big)=\infty$ and

$$
\limsup\Big(\frac{2[\langle\gamma f(p)-Gp, j(y_n-p)\rangle+\gamma\beta||x_n-p||||y_n-x_n||]}{\Big((\tau-2\gamma\beta)-(w_n/(1+w_n)\beta_n)\Big)}\Big)\leq 0
$$

Applying Lemma 3.5.4, we obtain $||x_n-p||\to 0$ as $n\to\infty$. This completes the proof.

□

**Concluding remarks**

In conclusion, the results in this section extend the result of Ali (2009) from family of nonexpansive mappings to family of asymptotically nonexpansive mapping.

**Theorem 4.1.6** *Let $E$ be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let $\{T_i\}_{i=1}^{\infty}$ be a family of uniformly asymptotically regular asymptotically nonexpansive self mappings of $E$ with sequences $\{v_{in}\}$ such that $v_{in}\to 0$ as $n\to\infty$ for each $i\geq 1$ and $F=\bigcap_{i=1}^{\infty}F(T_i)\neq\emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ , $\{\beta_n\}_{n=1}^{\infty}$ and $\{\sigma_{in}\}_{n=1}^{\infty}, i=1,2,3,..$ be sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

*(C1)* $\lim_{n\to\infty}\alpha_n=0$ *and* $\lim_{n\to\infty}\frac{v_n}{\alpha_n}=0$, *where* $v_n:=\sup_{i\geq 1}\{v_{in}\}$

*(C2)* $\Sigma_{n=0}^{\infty}\alpha_n=\infty$, *and (c3)* $\sum_{i=1}^{\infty}\sigma_{in}=1-\alpha_n,\ \forall\ n\in\mathbb{N}$

*(c4)* $\beta_n\in[a,b]\ \forall n\geq 1$ *and for some* $a,b\in(0,1)$.

*For some fixed $\delta\in(0,1)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by $x_0\in C$*

*chosen arbitrarily,*

$$\begin{cases} y_n = (1-\alpha_n)x_n \\ x_n = [1-\delta(1-\beta_n)]y_n + \delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n, \quad n \geq 0. \end{cases} \tag{4.1.41}$$

*Then,* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in F$.

**Proof.** First, we show that $\{x_n\}$ defined by (4.1.41) is well defined. For all $n \in \mathbb{N}$, let define the mapping

$$T_n^{\delta}x := [1-\delta(1-\beta_n)](1-\alpha_n)x + \delta\sum_{i\geq 1}\sigma_{in}T_i^n(1-\alpha_n)x.$$

Indeed, for all $x, y \in E$, we have

$$\begin{aligned} ||T_n^{\delta}x - T_n^{\delta}y|| &\leq [1-\delta(1-\beta_n)](1-\alpha_n)||x-y|| \\ &\quad +\delta\sum_{i\geq 1}\sigma_{in}||T_i^n(1-\alpha_n)x - T_i^n(1-\alpha_n)y|| \\ &\leq [1-\delta(1-\beta_n)](1-\alpha_n)||x-y|| \\ &\quad +\delta(1-\beta_n)(1+v_n)(1-\alpha_n)||x-y|| \\ &\leq [1-\alpha_n+\delta(1-\beta_n)v_n]||x-y|| \\ &\leq \Big(1-\alpha_n[1-\delta(1-\beta_n)v_n/\alpha_n]\Big)||x-y||. \end{aligned}$$

Since, $\lim_{n\to\infty}\delta(1-\beta_n)v_n/\alpha_n = 0$, then there exists $n_0 \in \mathbb{N}$ such that $\delta(1-\beta_n)v_n/\alpha_n < 1/2$ for all $n \geq n_0$. Therefore, for $n \geq n_0$. we have

$$1-\alpha_n[1-\delta(1-\beta_n)v_n/\alpha_n] < 1.$$

Hence,

$$||T_n^{\delta}x - T_n^{\delta}y|| \leq ||x-y||, \quad n \geq n_0.$$

Thus, $\{x_n\}$ defined by (4.1.41) is well defined. Therefore, by contraction mapping principle, there exists a unique fixed point $x_n \in E$ of $T_n^f$ for each $n \geq 0$

such that (4.1.41) holds.

Let $p \in F$, then from (4.1.41), we obtain

$$\begin{aligned}
||x_n - p|| &\leq [1-\delta(1-\beta_n)]||y_n - p|| + \delta \sum_{i\geq 1} \sigma_{in}||T_i^n y_n - p|| \\
&\leq [1-\delta(1-\beta_n)]||y_n - p|| + \delta(1-\beta_n)(1+v_n)||y_n - p|| \\
&= [1+\delta(1-\beta_n)v_n]||y_n - p|| \\
&\leq [1+\delta(1-\beta_n)v_n]\Big((1-\alpha_n)||x_n - p|| + \alpha_n||p||\Big) \\
&= [1+\delta(1-\beta_n)v_n](1-\alpha_n)||x_n - p|| \\
&\quad +\alpha_n[1+\delta(1-\beta_n)v_n]||p|| \\
&= [1-\alpha_n+\delta(1-\alpha_n)(1-\beta_n)v_n]||x_n - p|| \\
&\quad +\alpha_n[1+\delta(1-\beta_n)v_n]||p|| \\
&\leq [1-\alpha_n+\delta(1-\beta_n)v_n]||x_n - p|| \\
&\quad +\alpha_n[1+\delta(1-\beta_n)v_n]||p|| \\
&= \Big(1-\alpha_n[1-\delta(1-\beta_n)v_n/\alpha_n]\Big)||x_n - p|| \\
&\quad +\alpha_n[1+\delta(1-\beta_n)v_n]||p||.
\end{aligned}$$

Therefore

$$||x_n - p|| \leq \frac{[1+\delta(1-\beta_n)v_n]||p||}{1-\delta(1-\beta_n)v_n/\alpha_n}.$$

Since $\delta(1-\beta_n)v_n \to 0$ and $\delta(1-\beta_n)v_n/\alpha_n \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\delta(1-\beta_n)v_n < 1/2$ and $\delta(1-\beta_n)(v_n/\alpha_n) < 1/2$ respectively for all $n \geq n_0$.

Hence $||x_n - p|| \leq 3||p||$, for all $n \geq n_0$. Thus $\{x_n\}$ is bounded, which implies that $\{y_n\}$ is also bounded. From (4.1.41), we also obtain

$$||y_n - x_n|| = \alpha_n||x_n|| \to 0 \quad \text{as} \quad n \to \infty \tag{4.1.42}$$

which implies

$$\sum_{i\geq 1}\sigma_{in}||T_i^n y_n - y_n|| = ||x_n - y_n|| \to 0 \quad \text{as} \quad n \to \infty$$

hence

$$||T_i^n y_n - y_n|| = ||x_n - y_n|| \to 0 \quad \text{as} \quad n \to \infty \tag{4.1.43}$$

for each $i \geq 1$. Therefore

$$\begin{aligned} ||T_i^n x_n - x_n|| &= ||T_i^n x_n - T_i^n y_n|| + ||T_i^n y_n - y_n|| + ||y_n - x_n|| \\ &\leq (2 + v_n)||x_n - y_n|| + ||T_i^n y_n - y_n|| \end{aligned}$$

From (4.1.42) and (4.1.43), we obtain

$$||T_i^n x_n - x_n|| \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad i \geq 1. \tag{4.1.44}$$

For each $i \geq 1$, using the asymptotic regularity of $T_i$, we obtain

$$\begin{aligned} \lim_{n\to\infty}||T_i x_n - x_n|| &= \lim_{n\to\infty}||x_n - T_i^n x_n|| + \lim_{n\to\infty}||T_i^n x_n - T_i^{n+1} x_n|| \\ &\quad + \lim_{n\to\infty}||T_i^{n+1} x_n - T_i x_n|| \\ &\leq (1 + L)\lim_{n\to\infty}||x_n - T_i^n x_n|| \\ &\quad + \lim_{n\to\infty}||T_i^{n+1} x_n - T_i^n x_n|| = 0 \end{aligned} \tag{4.1.45}$$

where $L = \sup_{i\geq 1} L_i$, hence

$$\begin{aligned} \lim_{n\to\infty}||y_n - T_i y_n|| &\leq \lim_{n\to\infty}||y_n - x_n|| + \lim_{n\to\infty}||x_n - T_i x_n|| \\ &\quad + \lim_{n\to\infty}||T_i x_n - T_i y_n|| \\ &\leq (1 + L)\lim_{n\to\infty}||x_n - y_n|| \\ &\quad + \lim_{n\to\infty}||x_n - T_i x_n|| = 0 \end{aligned} \tag{4.1.46}$$

We next show that $x_n \to p$ (as $n \to \infty$). Indeed, define a map $\phi : E \to \mathbb{R}$ by

$$\phi(y) := \mu_n ||y_n - y||^2, \ \forall y \in E.$$

Then, $\phi(y) \to \infty$ as $||y|| \to \infty$, $\phi$ is continuous and convex, so that $E$ is reflexive and there exists $q \in E$ such that $\phi(q) = \min_{u \in E} \phi(u)$. Hence, the set

$$K^* := \{y \in E : \phi(y) = \min_{u \in E} \phi(u)\} \neq \emptyset.$$

Since $\lim_{n\to\infty} ||y_n - T_i y_n|| = 0$, $\lim_{n\to\infty} ||y_n - T_i^m y_n|| = 0$, for any $m \geq 1$ and each $i \geq 1$, by induction. Now let $v \in K^*$, we have

$$\begin{aligned} \lim_{n\to\infty} \phi(T_i v) &= \lim_{n\to\infty} \mu_n ||y_n - T_i v||^2 \\ &= \lim_{n\to\infty} \mu_n ||y_n - T_i y_n + T_i y_n - T_i v||^2 \\ &\leq \lim_{n\to\infty} \mu_n [(1 + v_n)||y_n - v||]^2 = \lim_{n\to\infty} \phi(v), \end{aligned}$$

and hence $T_i v \in K^*$.

Now let $z \in F$, then $z = T_i z$. Since $K^*$ is a closed convex set, there exists a unique $v^* \in K^*$ such that

$$||z - v^*|| = \min_{u \in K^*} ||z - u||.$$

But

$$\lim_{n\to\infty} ||z - T_i v^*|| = \lim_{n\to\infty} ||T_i z - T_i v^*|| \leq \lim_{n\to\infty} (1 + v_n)||z - v^*||,$$

which implies $v^* = T_i v^*$ and so $K^* \cap F \neq \emptyset$.

Let $p \in K^* \cap F$ and $t \in (0, 1)$, then it follows that $\phi(p) \leq \phi(p - tp)$ and using Lemma 3.5.2, we obtain

$$||y_n - p + tp||^2 \leq ||y_n - p||^2 + 2t\langle p, j(y_n - p + tp)\rangle$$

which implies that

$$\mu_n\langle -p, j(y_n - p + tp)\rangle \leq 0.$$

Moreover

$$\begin{aligned}\mu_n\langle -p, j(y_n - p)\rangle &= \mu_n\langle -p, j(y_n - p) - j(y_n - p + tp)\rangle \\ &\quad +\mu_n\langle -p, j(y_n - p + tp)\rangle \\ &\leq \mu_n\langle -p, j(y_n - p) - j(y_n - p + tp)\rangle.\end{aligned}$$

Since $j$ is norm-to-weak$^*$ uniformly continuous on bounded subsets of $E$, we have

$$\mu_n\langle -p, j(y_n - p)\rangle \leq 0. \tag{4.1.47}$$

Since $\delta(1-\beta_n)v_n \to 0$ and $\delta(1-\beta_n)v_n/\alpha_n \to 0$ as $n \to \infty$, if we denote by $w_n$ the value of $2v_n + v_n^2$, it implies that $\delta(1-\beta_n)w_n \to 0$ and $\delta(1-\beta_n)w_n/\alpha_n \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\delta(1-\beta_n)w_n < 1/2$ and $\delta(1-\beta_n)(w_n/\alpha_n) < 1/2$, for all $n \geq n_0$. From recursion formula (4.1.41), we obtain

$$\begin{aligned}||x_n - p||^2 &= ||[1-\delta(1-\beta_n)](y_n - p) + \delta\sum_{i\geq 1}\sigma_{in}(T_i^n y_n - p)||^2 \\ &\leq [1-\delta(1-\beta_n)]||y_n - p||^2 + \delta\sum_{i\geq 1}\sigma_{in}||T_i^n y_n - p||^2 \\ &\leq [1-\delta(1-\beta_n)]||y_n - p||^2 + \delta(1-\beta_n)(1+v_n)^2||y_n - p||^2 \\ &= [1-\delta(1-\beta_n) + \delta(1-\beta_n)(1+w_n)]||y_n - p||^2 \\ &= [1+\delta(1-\beta_n)w_n]||(1-\alpha_n)(x_n - p) - \alpha_n p||^2 \\ &\leq [1+\delta(1-\beta_n)w_n]\Big((1-\alpha_n)||x_n - p||^2 + 2\alpha_n\langle -p, j(y_n - p)\rangle\Big) \\ &= [1+\delta(1-\beta_n)w_n](1-\alpha_n)||x_n - p||^2 \\ &\quad +2\alpha_n[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n - p)\rangle\end{aligned}$$

$$
\begin{aligned}
&= [1-\alpha_n+\delta(1-\alpha_n)(1-\beta_n)w_n] \\
&\quad +2\alpha_n[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n-p)\rangle \\
&\leq \Big[1-\alpha_n\Big(1-\delta(1-\beta_n)w_n/\alpha_n\Big)\Big] \\
&\quad +2\alpha_n[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n-p)\rangle.
\end{aligned}
$$

Therefore

$$||x_n-p||^2 \leq \frac{2[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n-p)\rangle}{(1-\delta(1-\beta_n)w_n/\alpha_n)}$$

hence

$$\mu_n||x_n-p||^2 \leq 3\mu_n\langle -p, j(y_n-p)\rangle. \tag{4.1.48}$$

Therefore, from (4.1.47) we obtain $\mu_n||x_n-p|| \leq 0$. Hence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ as $k \to \infty$. To complete the proof, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \to z$ as $j \to \infty$. From (4.1.48) we obtain

$$\mu_n||z-p||^2 \leq 0.$$

which implies that $z = p$ and hence $\{x_n\}$ converges strongly to $p \in F$ as $n \to \infty$. This completes the proof.

□

**Theorem 4.1.7** *Let $E$ be a real reflexive Banach space with a uniformly Gâteaux differentiable norm, $K$ a nonempty closed convex subset of $E$. Let $\{T_i\}_{i=1}^{\infty}$ be a family of uniformly asymptotically regular asymptotically nonexpansive self mappings of $E$ with sequences $\{v_{in}\}$ such that $v_{in} \to 0$ as $n \to \infty$ for each $i \geq 1$ and $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\sigma_{in}\}_{n=1}^{\infty}, i = 1, 2, 3, ...$ be sequences in $(0, 1)$, and suppose that the following conditions are satisfied:*

*(C1)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \frac{v_n}{\alpha_n} = 0$, *where* $v_n := \sup_{i\geq 1}\{v_{in}\}$ *and* $\sum_{n=1}^{\infty} v_n < \infty$

*(C2)* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\sum_{n\geq 1}|\sigma_{i,n+1} - \sigma_{in}| = 0$

*(C3)* $\sum_{n=1}^{\infty}\beta_n < \infty$ *and (C4)* $\sum_{i=1}^{\infty}\sigma_{in} = 1 - \alpha_n,\ \forall\ n \in \mathbb{N}$.

*For some fixed* $\delta \in (0,1)$*, let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence defined iteratively by* $x_0 \in C$ *chosen arbitrarily,*

$$\begin{cases} y_n = (1-\alpha_n)x_n \\ x_{n+1} = [1-\delta(1-\beta_n)]y_n + \delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n, \quad n \geq 0. \end{cases} \tag{4.1.49}$$

*Then,* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in F$.

**Proof.** Let $p \in F$ be arbitrary, we obtain from (4.1.49)

$$\begin{aligned} ||x_{n+1} - p|| &= ||[1-\delta(1-\beta_n)](y_n - p) + \delta\sum_{i\geq 1}\sigma_{in}(T_i^n y_n - p)|| \\ &\leq [1-\delta(1-\beta_n)]||y_n - p|| + \delta(1-\beta_n)(1+v_n)||y_n - p|| \\ &= [1-\delta(1-\beta_n) + \delta(1-\beta_n)(1+v_n)]||y_n - p|| \\ &= [1+\delta(1-\beta_n)v_n]||(1-\alpha_n)(x_n - p) - \alpha_n p|| \\ &\leq [1+\delta(1-\beta_n)v_n]\Big((1-\alpha_n)||x_n - p|| + \alpha_n||p||\Big) \\ &\leq [1+\delta(1-\beta_n)v_n]\max\{||x_n - p||, ||p||\} \\ &\ \ \vdots \\ &\leq \prod_{j=1}^{n}[1+\delta(1-\beta_j)v_j]\max\{||x_1 - p||, ||p||\}. \end{aligned} \tag{4.1.50}$$

Since $\sum_{n=1}^{\infty}v_n < \infty$. it follows from (4.1.50) that $\{x_n\}$ is bounded. Hence $\{y_n\}$ is also bounded. Furthermore, it follows from (4.1.49) that

$$||y_n - x_n|| = \alpha_n||x_n|| \to 0 \quad \text{as} \quad n \to \infty \tag{4.1.51}$$

Define two sequences by $\gamma_n := (1-\delta)\beta_n + \delta$ and $z_n := \frac{1}{\gamma_n}(x_{n+1} - x_n + \gamma_n x_n)$. From the recursion formula (4.1.49), observe that

$$z_n = \frac{1}{\gamma_n}([1-\delta(1-\beta_n)](y_n - x_n) + \delta \sum_{i\geq 1} \sigma_{in} T_i^n y_n + \beta_n x_n)$$

which implies

$$\begin{aligned} z_{n+1} - z_n &= \frac{1}{\gamma_{n+1}}\Big[[1-\delta(1-\beta_{n+1})](y_{n+1} - x_{n+1}) \\ &\quad + \delta \sum_{i\geq 1} \sigma_{i,n+1} T_i^{n+1} y_{n+1} + \beta_{n+1} x_{n+1}\Big] \\ &\quad - \frac{1}{\gamma_n}\Big[[1-\delta(1-\beta_n)](y_n - x_n) + \delta \sum_{i\geq 1} \sigma_{in} T_i^n y_n + \beta_n x_n\Big] \\ &= \frac{1}{\gamma_{n+1}}\Big[[1-\delta(1-\beta_{n+1})](y_{n+1} - x_{n+1})\Big] \\ &\quad - \frac{1}{\gamma_n}\Big[[1-\delta(1-\beta_n)](y_n - x_n)\Big] \end{aligned}$$

$$\begin{aligned} &+ \frac{\delta \sum_{i\geq 1} \sigma_{i,n+1}\Big(T_i^{n+1} y_{n+1} - T_i^{n+1} y_n\Big)}{\gamma_{n+1}} \\ &+ \frac{\delta \sum_{i\geq 1} \sigma_{in+1}\Big(T_i^{n+1} y_n - T_i^n y_n\Big)}{\gamma_{n+1}} \\ &+ \Big(\frac{\delta \sum_{i\geq 1} \sigma_{i,n+1} T_i^n y_n}{\gamma_{n+1}} - \frac{\delta \sum_{i\geq 1} \sigma_{in} T_i^n y_n}{\gamma_n}\Big) \\ &+ \frac{\beta_{n+1}}{\gamma_{n+1}} x_{n+1} - \frac{\beta_n}{\gamma_n} x_n \end{aligned}$$

therefore

$$\begin{aligned} ||z_{n+1} - z_n|| &= \frac{[1-\delta(1-\beta_{n+1})]\Big\|y_{n+1} - x_{n+1}\Big\|}{\gamma_{n+1}} \\ &\quad + \frac{[1-\delta(1-\beta_n)]\Big\|y_n - x_n\Big\|}{\gamma_n} \\ &\quad + \frac{\delta \sum_{i\geq 1} \sigma_{i,n+1}\Big\|T_i^{n+1} y_{n+1} - T_i^{n+1} y_n\Big\|}{\gamma_{n+1}} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n-T_i^n y_n\Big\|}{\gamma_{n+1}}\\
&+\Big\|\frac{\delta\sum_{i\geq 1}\sigma_{i,n+1}T_i^n y_n}{\gamma_{n+1}}-\frac{\delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n}{\gamma_n}\Big\|\\
&+\frac{\beta_{n+1}}{\gamma_{n+1}}||x_{n+1}||+\frac{\beta_n}{\gamma_n}||x_n||\\
\leq\ &\frac{[1-\delta(1-\beta_{n+1})]\Big\|y_{n+1}-x_{n+1}\Big\|}{\gamma_{n+1}}\\
&+\frac{[1-\delta(1-\beta_n)]\Big\|y_n-x_n\Big\|}{\gamma_n}\\
&+\frac{\delta(1-\beta_n)(1+v_{n+1})\Big\|y_{n+1}-y_n\Big\|}{\gamma_{n+1}}\\
&+\frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n-T_i^n y_n\Big\|}{\gamma_{n+1}}\\
&+\frac{\delta}{\gamma_{n+1}\gamma_n}\Big\|\sum_{i\geq 1}\Big(\gamma_n\sigma_{i,n+1}-\gamma_{n+1}\sigma_{in}\Big)T_i^n y_n\Big\|\\
&+\frac{\beta_{n+1}}{\gamma_{n+1}}||x_{n+1}||+\frac{\beta_n}{\gamma_n}||x_n||. \qquad (4.1.52)
\end{aligned}
$$

But

$$
\begin{aligned}
y_{n+1}-y_n &= (1-\alpha_{n+1})x_{n+1}-(1-\alpha_n)x_n\\
&= (1-\alpha_{n+1})(x_{n+1}-x_n)+(\alpha_{n+1}-\alpha_n)x_n
\end{aligned}
$$

so that

$$
||y_{n+1}-y_n||=(1-\alpha_{n+1})||x_{n+1}-x_n||+|\alpha_{n+1}-\alpha_n|||x_n||. \qquad (4.1.53)
$$

Using (4.1.53) in (4.1.52), we obtain

$$
\begin{aligned}
||z_{n+1}-z_n||-||x_{n+1}-x_n|| \leq\ &\frac{[1-\delta(1-\beta_{n+1})]\Big\|y_{n+1}-x_{n+1}\Big\|}{\gamma_{n+1}}\\
&+\frac{[1-\delta(1-\beta_n)]\Big\|y_n-x_n\Big\|}{\gamma_n}\\
&+\Big(\frac{\delta(1-\beta_n)}{\gamma_{n+1}}(1+v_{n+1})(1-\alpha_{n+1})-1\Big)||x_{n+1}-x_n||
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\delta(1-\beta)}{\gamma_{n+1}}|\alpha_{n+1}-\alpha_n|||x_n|| \\
&+\frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n-T_i^n y_n\Big\|}{\gamma_{n+1}} \\
&+\frac{\delta}{\gamma_{n+1}\gamma_n}\Big\|\sum_{i\geq 1}\Big(\gamma_n\sigma_{i,n+1}-\gamma_{n+1}\sigma_{in}\Big)T_i^n y_n\Big\| \\
&+\frac{\beta_{n+1}}{\gamma_{n+1}}||x_{n+1}||+\frac{\beta_n}{\gamma_n}||x_n|| \\
\leq\ &\frac{[1-\delta(1-\beta_{n+1})]\Big\|y_{n+1}-x_{n+1}\Big\|}{\gamma_{n+1}} \\
&+\frac{[1-\delta(1-\beta_n)]\Big\|y_n-x_n\Big\|}{\gamma_n} \\
&+\Big(\frac{\delta(1-\beta_n)}{\gamma_{n+1}}(1+v_{n+1})(1-\alpha_{n+1})-1\Big)||x_{n+1}-x_n|| \\
&+\frac{\delta(1-\beta)}{\gamma_{n+1}}|\alpha_{n+1}-\alpha_n|||x_n|| \\
&+\frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n-T_i^n y_n\Big\|}{\gamma_{n+1}} \\
&+\frac{\delta M^*}{\gamma_{n+1}\gamma_n}\Big[\gamma_n\sum_{i\geq 1}|\sigma_{i,n+1}-\sigma_{in}|+|\gamma_n-\gamma_{n+1}|(1-\alpha_n)\Big] \\
&+\frac{\beta_{n+1}}{\gamma_{n+1}}||x_{n+1}||+\frac{\beta_n}{\gamma_n}||x_n||.
\end{aligned}
$$

for some $M^*>0$ and this implies

$$\limsup_{n\to\infty}(||z_{n+1}-z_n||-||x_{n+1}-x_n||)\leq 0,$$

and by Lemma 3.5.3, we have

$$\lim_{n\to\infty}||z_n-x_n||=0.$$

Hence

$$||x_{n+1}-x_n||=(1-\gamma_n)||z_n-x_n||\to 0 \text{ as } n\to\infty. \tag{4.1.54}$$

From the recursion formula (4.1.49), we obtain

$$\delta \sum_{i\geq 1} \sigma_{in} ||T_i^n y_n - y_n|| = ||x_{n+1} - x_n|| \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that for each $i \geq 1$,

$$\lim_{n\to\infty} ||T_i^n y_n - y_n|| = 0. \tag{4.1.55}$$

for each $i \geq 1$. Therefore

$$\begin{aligned} ||T_i^n x_n - x_n|| &\leq ||T_i^n x_n - T_i^n y_n|| + ||T_i^n y_n - y_n|| + ||y_n - x_n|| \\ &\leq (2 + v_n)||x_n - y_n|| + ||T_i^n y_n - y_n|| \end{aligned}$$

From (4.1.51) and (4.1.55), we obtain

$$||T_i^n x_n - x_n|| \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad i \geq 1. \tag{4.1.56}$$

For each $i \geq 1$, using the asymptotic regularity $T_i$, we obtain

$$\begin{aligned} \lim_{n\to\infty} ||T_i x_n - x_n|| &\leq \lim_{n\to\infty} ||x_n - T_i^n x_n|| + \lim_{n\to\infty} ||T_i^n x_n - T_i^{n+1} x_n|| \\ &\quad + \lim_{n\to\infty} ||T_i^{n+1} x_n - T_i x_n|| \\ &\leq (1 + L) \lim_{n\to\infty} ||x_n - T_i^n x_n|| + \lim_{n\to\infty} ||T_i^{n+1} x_n - T_i^n x_n|| = 0 \end{aligned} \tag{4.1.57}$$

hence

$$\begin{aligned} \lim_{n\to\infty} ||y_n - T_i y_n|| &\leq \lim_{n\to\infty} ||y_n - x_n|| + \lim_{n\to\infty} ||x_n - T_i x_n|| \\ &\quad + \lim_{n\to\infty} ||T_i x_n - T_i y_n|| \\ &\leq (1 + L) \lim_{n\to\infty} ||x_n - y_n|| + \lim_{n\to\infty} ||x_n - T_i x_n|| = 0 \end{aligned} \tag{4.1.58}$$

For each $m \geq 0$, let $z_m \in E$ be the unique fixed point of the contraction mapping

$$z_m := [1 - \delta(1 - \alpha_m)](1 - \beta_m)z_m + \delta \sum_{i \geq 1} \sigma_{im} T_i^m (1 - \beta_n) z_m$$

on $E$, for $i \geq 1$ (see Theorem 4.1.6). Then we obtain by letting $y_m = (1-\alpha_m)z_m$ and $w_m$ denoted by $2v_m + v_m^2$

$$\begin{aligned}
||z_m - y_n||^2 &= ||[1 - \delta(1 - \beta_m)](y_m - y_n) + \delta \sum_{i \geq 1} \sigma_{im}(T_i^m y_m - y_n)||^2 \\
&\leq [1 - \delta(1 - \beta_m)]||y_m - y_n||^2 + \delta \sum_{i \geq 1} \sigma_{im}||T_i^m y_m - y_n||^2 \\
&\leq [1 - \delta(1 - \beta_m)]||y_m - y_n||^2 \\
&\quad + \delta \sum_{i \geq 1} \sigma_{im} \Big[ ||T_i^m y_m - T_i^m y_n|| + ||T_i^m y_n - y_n|| \Big]^2 \\
&\leq [1 - \delta(1 - \beta_m)]||y_m - y_n||^2 \\
&\quad + \delta \sum_{i \geq 1} \sigma_{im} \Big[ (1 + v_m)||y_m - y_n|| + ||T_i^m y_n - y_n|| \Big]^2 \\
&\leq [1 - \delta(1 - \beta_m)]||y_m - y_n||^2 \\
&\quad + \delta(1 - \beta_m)(1 + v_m)^2 ||y_m - y_n||^2 \\
&\quad + 2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n||||T_i^m y_n - y_n|| \\
&\quad + \delta \sum_{i \geq 1} \sigma_{im}||T_i^m y_n - y_n||^2 \\
&= [1 + \delta(1 - \beta_m)w_m]||y_m - y_n||^2 \\
&\quad + 2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n||||T_i^m y_n - y_n|| \\
&\quad + \delta \sum_{i \geq 1} \sigma_{im}||T_i^m y_n - y_n||^2
\end{aligned}$$

$$\begin{aligned}
&= [1 + \delta(1 - \beta_m)w_m]||(1 - \alpha_m)(z_m - y_n) - \alpha_m y_n||^2 \\
&\quad + 2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n||||T_i^m y_n - y_n|| \\
&\quad + \delta \sum_{i \geq 1} \sigma_{im}||T_i^m y_n - y_n||^2
\end{aligned}$$

$$
\begin{aligned}
&\leq [1+\delta(1-\beta_m)w_m]\Big[(1-\alpha_m)^2||z_m-y_n||^2+2\alpha_m\langle -y_n, j(z_m-y_n)\rangle\Big]\\
&\quad +2\delta\sum_{i\geq 1}\sigma_{im}(1+v_m)||y_m-y_n||||T_i^m y_n-y_n||\\
&\quad +\delta\sum_{i\geq 1}\sigma_{im}||T_i^m y_n-y_n||^2\\
&\leq [1+\delta(1-\beta_m)w_m](1-\alpha_m)^2||z_m-y_n||^2\\
&\quad +2[1+\delta(1-\beta_m)w_m]\alpha_m\langle -z_m, j(z_m-y_n)\rangle\\
&\quad +2\alpha_m[1+\delta(1-\beta_m)w_m]||z_m-y_n||^2\\
&\quad +2\delta\sum_{i\geq 1}\sigma_{im}(1+v_m)||y_m-y_n||||T_i^m y_n-y_n||\\
&\quad +\delta\sum_{i\geq 1}\sigma_{im}||T_i^m y_n-y_n||^2\\
&\leq [1+\delta(1-\beta_m)w_m](1+\alpha_m^2)||z_m-y_n||^2\\
&\quad +2\alpha_m[1+\delta(1-\beta_m)w_m]\langle -z_m, j(z_m-y_n)\rangle\\
&\quad +2\delta\sum_{i\geq 1}\sigma_{im}(1+v_m)||y_m-y_n||||T_i^m y_n-y_n||\\
&\quad +\delta\sum_{i\geq 1}\sigma_{im}||T_i^m y_n-y_n||^2\\
&\leq \Big[1+\alpha_m\Big\{\delta(1-\beta_m)(w_m/\alpha_m)+\alpha_m[\delta(1-\beta_m)w_m]\Big\}\Big]||z_m-y_n||^2\\
&\quad +2\alpha_m[1+\delta(1-\beta_m)w_m]\langle -z_m, j(z_m-y_n)\rangle\\
&\quad +2\delta\sum_{i\geq 1}\sigma_{im}(1+v_m)||y_m-x_n||||T_i^m y_n-y_n||\\
&\quad +\delta\sum_{i\geq 1}\sigma_{im}||T_i^m y_n-y_n||^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\langle -z_m, j(y_n-z_m)\rangle \;\leq\; & \frac{\Big[\delta(1-\beta_m)(w_m/\alpha_m)+\alpha_m[\delta(1-\beta_m)w_m]\Big]||z_m-y_n||^2}{2[1+\delta(1-\beta_m)w_m]}\\
& +\frac{2\delta\sum_{i\geq}\sigma_{im}(1+v_m)||y_m-y_n||||T_i^m y_n-y_n||}{2\alpha_m[1+\delta(1-\beta_m)w_m]}\\
& +\frac{\delta\sum_{i\geq 1}\sigma_{im}||T_i^m y_n-y_n||^2}{2\alpha_m[1+\delta(1-\beta_m)w_m]}
\end{aligned}
$$

Now, taking limit superior as $n \to \infty$ firstly, and then as $m \to \infty$, we have

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \langle -z_m, j(y_n - z_m)\rangle \leq 0 \tag{4.1.59}$$

But by Theorem 4.1.7, $z_m \to p$ as $m \to \infty$ and the fact that $E$ has a uniformly Gâteaux differentiable norm implies that $j$ is norm-to-weak$^*$ uniformly continuous on bounded sets. Thus, since

$$\begin{aligned}
\langle -p, j(y_n - p)\rangle &= \langle -p, j(y_n - p) - j(y_n - z_m)\rangle + \langle z_m - p, j(y_n - z_m)\rangle \\
&\quad + \langle -z_m, j(y_n - z_m)\rangle \\
&\leq \langle -p, j(y_n - p) - j(y_n - z_m)\rangle + ||z_m - p||||y_n - z_m|| \\
&\quad + \langle -z_m, j(y_n - z_m)\rangle
\end{aligned}$$

we get

$$\begin{aligned}
\limsup_{n\to\infty} \langle -p, j(y_n - p)\rangle &\leq \limsup_{m\to\infty} \limsup_{n\to\infty} \langle -z_m, j(y_n - z_m)\rangle \\
&\leq 0
\end{aligned}$$

Finally, we prove that $x_n \to p$ as $n \to \infty$. Since $\delta(1 - \beta_n)v_n \to 0$ and $\delta(1 - \beta_n)v_n/\alpha_n \to 0$ as $n \to \infty$, if we denote by $w_n$ the value of $2v_n + v_n^2$, it implies that $\delta(1 - \beta_n)w_n \to 0$ and $\delta(1 - \beta_n)w_n/\alpha_n \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\delta(1 - \beta_n)w_n < 1/2$ and $\delta(1 - \beta_n)(w_n/\alpha_n) < 1/2$, for all $n \geq n_0$. From recursion formula (4.1.49), we obtain

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||[1 - \delta(1 - \beta_n)](y_n - p) + \delta \sum_{i\geq 1} \sigma_{in}(T_i^n y_n - p)||^2 \\
&\leq [1 - \delta(1 - \beta_n)]||y_n - p||^2 + \delta \sum_{i\geq 1} \sigma_{in}||T_i^n y_n - p||^2 \\
&\leq [1 - \delta(1 - \beta_n)]||y_n - p||^2 + \delta(1 - \beta_n)(1 + v_n)^2||y_n - p||^2 \\
&= [1 - \delta(1 - \beta_n) + \delta(1 - \beta_n)(1 + w_n)]||y_n - p||^2
\end{aligned}$$

$$
\begin{aligned}
&= [1+\delta(1-\beta_n)w_n]||(1-\alpha_n)(x_n-p)-\alpha_n p||^2 \\
&\leq [1+\delta(1-\beta_n)w_n]\Big((1-\alpha_n)||x_n-p||^2 \\
&\quad +2\alpha_n\langle -p, j(y_n-p)\rangle\Big) \\
&= [1+\delta(1-\beta_n)w_n](1-\alpha_n)||x_n-p||^2 \\
&\quad +2\alpha_n[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n-p)\rangle \\
&= [1-\alpha_n+\delta(1-\alpha_n)(1-\beta_n)w_n] \\
&\quad +2\alpha_n[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n-p)\rangle \\
&\leq \Big[1-\alpha_n\Big(1-\delta(1-\beta_n)w_n/\alpha_n\Big)\Big] \\
&\quad +\alpha_n\Big(1-\delta(1-\beta_n)w_n/\alpha_n\Big) \\
&\quad \times\frac{2[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n-p)\rangle}{1-\delta(1-\beta_n)w_n/\alpha_n}.
\end{aligned}
$$

Observe that $\sum \alpha_n(1-\delta(1-\beta_n)w_n/\alpha_n)=\infty$ and

$$
\limsup_{n\to\infty}\Big(\frac{2[1+\delta(1-\beta_n)w_n]\langle -p, j(y_n-p)\rangle}{1-\delta(1-\beta_n)w_n/\alpha_n}.\Big)\leq 0.
$$

Applying Lemma 3.5.4, we obtain $||x_n-p||\to 0$ as $n\to\infty$. This completes the proof. □

**Concluding remarks**

In conclusion, the result in this section extends the result of Osilike et al. (2013) from asymptotically nonexpansive mappings in Hilbert space to family of asymptotically nonexpansive mapping in Banach space.

We introduce a new iterative scheme is introduced and a new strong convergence theorem for approximation of common fixed points of family of uniformly asymptotically regular asymptotically nonexpansive mappings, which is also a unique solution of some variational inequality problem is proved in the framework of a real Banach space. We proceed as follows:

**Lemma 4.1.8** *Let $C$ be a nonempty closed convex subset of a uniformly con-*

*vex real Banach space $E$. Let $\{T_k\}_{k=1}^N$ be finite family of uniformly asymtotically regular asymptotically nonexpansive mappings of $C$ into itself with sequences $\{\rho_{n,k}\} \subset [1,\infty)$, let $\{\gamma_{n,k}\}_{k=1}^N$ be a sequence in $(0,1)$ such that $0 < \liminf_{n\to\infty} \gamma_{k,n} \leq \limsup_{n\to\infty} \gamma_{k,n} < 1$ and $\lim_{n\to\infty} |\gamma_{k,n} - \gamma_{k.n-1}| = 0$ $\forall k \in \{1,2,3,...,N\}$. Let $K_n$ be a mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\gamma_{1,n}, \gamma_{2,n}, \gamma_{3,n}, ..., \gamma_{N,n}$ as follows;*

$$\begin{aligned} U_{1,n} &= \gamma_{1,n}T_1^n + (1-\gamma_{1,n})I, \\ U_{2,n} &= \gamma_{2,n}T_2^n U_{1,n} + (1-\gamma_{2,n})U_{1,n}, \\ &\vdots \\ U_{N-1,n} &= \gamma_{N-1,n}T_{N-1}^n U_{N-2,n} + (1-\gamma_{N-1,n})U_{N-2,n}, \\ K_n = U_{N,n} &= \gamma_{N,n}T_N^n U_{N-1,n} + (1-\gamma_{N,n})I. \end{aligned} \tag{4.1.60}$$

*Then, the following hold:*

*(i)* $\|K_n x - K_n y\| \leq (1+v_n)\|x-y\|$, *where* $v_n = \rho_{N,n}(1+\lambda_{N-1,n}) - 1$, *and* $\{\lambda_{N,n}\}$ *is some sequence in* $[0,\infty)$, *with* $\lambda_{N,n} \to 0$ *as* $n \to \infty$.

*(ii) If* $\lim_{n\to\infty} \|T_k^{n+1}U_{k-1,n}z_n - T_k^n U_{k-1,n}z_n\| = 0$,
*then* $\lim_{n\to\infty} \|K_{n+1}z_n - K_n z_n\| = 0$, *for every bounded sequence* $\{z_n\}$ *in* $E, k = 1,2,...,N$;

*(iii) For every bounded sequence* $\{z_n\}$ *in* $C$ *such that* $\lim_{n\to\infty} ||K_n z_n - z_n|| = 0$, *we have* $\lim_{n\to\infty} ||T_k z_n - z_n|| = 0$ *for any* $k \in \{1,2,3,\cdots,N\}$. *Furthermore , we have* $w_w(z_n) \subset \cap_{k=1}^N F(T_k)$ *and* $F(K_n) = \bigcap_{k=1}^N F(T_k)$.

**Proof.** (i) Let $x, y \in C$ then from (4.1.60), if $N = 1$ the result follows. Assume $N \neq 1$ and $U_{0,n} = I$ (identity map), then for $k \in \{1, 2, ..., N-1\}$, we have

$$\begin{aligned}
\|U_{k,n}x - U_{k,n}y\| &\leq \gamma_{k,n}\|T_k^n U_{k-1,n}x - T_k^n U_{k-1,n}y\| \\
&+ (1-\gamma_{k,n})\|U_{k-1,n}x - U_{k-1,n}y\| \\
&\leq [\gamma_{k,n}\rho_{k,n} + (1-\gamma_{k,n})]\|U_{k-1,n}x - U_{k-1,n}y\| \\
&= [1 + \gamma_{k,n}(\rho_{k,n} - 1)]\|U_{k-1,n}x - U_{k-1,n}y\| \\
&\leq [1 + \gamma_{k,n}(\rho_{k,n} - 1)][\gamma_{k-1,n}\|T_{k-1}^n U_{k-2,n}x - T_{k-1}^n U_{k-2,n}y\| \\
&\quad +(1-\gamma_{k-1,n})\|U_{k-2,n}x - U_{k-2,n}y\|] \\
&\leq [1 + \gamma_{k,n}(\rho_{k,n} - 1)][1 + \gamma_{k-1,n}(\rho_{k-1,n} - 1)]\|U_{k-2,n}x - U_{k-2,n}y\| \\
&\vdots \quad \vdots \\
&\leq [1 + \gamma_{k,n}(\rho_{k,n} - 1)][1 + \gamma_{k-1,n}(\rho_{k-1,n} - 1)]\cdots[1 + \gamma_{2,n}(\rho_{2,n} - 1)] \\
&\quad \|U_{1,n}x - U_{1,n}y\| \\
&\leq [1 + \gamma_{k,n}(\rho_{k,n} - 1)][1 + \gamma_{k-1,n}(\rho_{k-1,n} - 1)]\cdots[1 + \gamma_{2,n}(\rho_{2,n} - 1)] \\
&\quad [1 + \gamma_{1,n}(\rho_{1,n} - 1)]\|x - y\| \\
&= \prod_{j=1}^{k}[1 + \gamma_{j,n}(\rho_{j,n} - 1)]\|x - y\| \\
&= (1 + \lambda_{n,k})\|x - y\|,
\end{aligned}$$

where $\prod_{j=1}^{k}\Big(1 + \gamma_{j,n}(\rho_{j,n} - 1)\Big) = (1 + \lambda_{k,n})$, observe that $\lim_{n\to\infty}\lambda_{k,n} = 0$. Then,

$$\begin{aligned}
\|K_n x - K_n y\| &= \|U_{N,n}x - U_{N,n}y\| \\
&\leq \gamma_{N,n}\|T_N^n U_{N-1,n}x - T_N^n U_{N-1,n}y\| + (1-\gamma_{N,n})\|x - y\| \\
&\leq \gamma_{N,n}\rho_{N,n}\|U_{N-1,n}x - U_{N-1,n}y\| + (1-\gamma_{N,n})\|x - y\| \\
&\leq \gamma_{N,n}\rho_{N,n}(1 + \lambda_{N-1,n})\|x - y\| + (1-\gamma_{N,n})\|x - y\| \\
&= [1 + \gamma_{N,n}(\rho_{N,n}(1 + \lambda_{N-1,n}) - 1)]\|x - y\| \\
&\leq [1 + (\rho_{N,n}(1 + \lambda_{N-1,n}) - 1)]\|x - y\| \\
&= (1 + v_n)\|x - y\|,
\end{aligned}$$

where $v_n = \rho_{N,n}(1+\lambda_{N-1,N})-1$, observe that $\lim_{n\to\infty} v_n = 0$.

Next we show (ii). For $k \in \{2,3,...,N-1\}$ and any bounded sequence $\{z_n\} \subset E$, letting $\delta_{k,n+1} := [1+\gamma_{k,n+1}(\rho_{k,n+1}-1)]$,
$M_{k,n} := [\|T_k^{n+1}U_{k,n}z_n\| + \|U_{k,n}z_n\|]$ and $P_{k,n} := \|T_{k-1}^{n+1}U_{k,n}z_n - T_{k-1}^{n}U_{k,n}z_n\|$, we have

$$
\begin{aligned}
\|U_{k,n+1}z_n - U_{k,n}z_n\| = \|&\gamma_{k,n+1}T_k^{n+1}U_{k-1,n+1}z_n \\
&-\gamma_{k,n+1}T_k^{n+1}U_{k-1,n}z_n \qquad (4.1.61)\\
&+[\gamma_{k,n+1}-\gamma_{k,n}]T_k^{n+1}U_{k-1,n}z_n \\
&+\gamma_{k,n}[T_k^{n+1}U_{k-1,n}z_n - T_k^{n}U_{k-1,n}z_n] \\
&+(1-\gamma_{k,n+1})(U_{k-1,n+1}z_n - U_{k-1,n}z_n) \\
&+[(1-\gamma_{k,n+1})-(1-\gamma_{k,n})]U_{k-1,n}z_n\| \\
\leq\ & [1+\gamma_{k,n+1}(\rho_{k,n+1}-1)]\|U_{k-1,n+1}z_n - U_{k-1,n}z_n\| \\
&+|\gamma_{k,n+1}-\gamma_{k,n}|\Big[\|T_k^{n+1}U_{k-1,n}z_n\| + \|U_{k-1,n}z_n\|\Big] \\
&+\gamma_{k,n}\|T_k^{n+1}U_{k-1,n}z_n - T_k^{n}U_{k-1,n}z_n\| \\
\leq\ & \big[1+\gamma_{k,n+1}(\rho_{k,n+1}-1)\big]\Big[\big[1+\gamma_{k-1,n+1}(\rho_{k-1,n+1}-1)\big]\|U_{k-2,n+1}z_n - U_{k-2,n}z_n\| \\
&+|\gamma_{k-1,n+1}-\gamma_{k-1,n}|\big[\|T_{k-1}^{n+1}U_{k-2,n}z_n\| + \|U_{k-2,n}z_n\|\big] \\
&+\gamma_{k-1,n}\|T_{k-1}^{n+1}U_{k-2,n}z_n - T_{k-1}^{n}U_{k-2,n}z_n\|\Big] \\
&+|\gamma_{k,n+1}-\gamma_{k,n}|\big[\|T_k^{n+1}U_{k-1,n}z_n\| + \|U_{k-1,n}z_n\|\big] \\
&+\gamma_{k,n}\|T_k^{n+1}U_{k-1,n}z_n - T_k^{n}U_{k-1,n}z_n\| \\
=\ & \delta_{k,n+1}\delta_{k-1,n+1}\|U_{k-2,n+1}z_n - U_{k-2,n}z_n\| \\
&+\delta_{k,n+1}|\gamma_{k-1,n+1}-\gamma_{k-1,n}|M_{k-2,n} \\
&+\delta_{k,n+1}\gamma_{k-1,n}P_{k-2,n} \\
&+|\gamma_{k,n+1}-\gamma_{k,n}|M_{k-1,n} \\
&+\gamma_{k,n}P_{k-1,n}
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \delta_{k,n+1}\delta_{k-1,n+1}\Big[\delta_{k-2,n+1}\|U_{k-3,n+1}z_n - U_{k-3,n}z_n\| \\
& +|\gamma_{k-2,n+1}-\gamma_{k-2,n}|[\|T_{k-2}^{n+1}U_{k-3,n}z_n\| + \|U_{k-3,n}z_n\|] \\
& +\gamma_{k-1,n}\|T_{k-2}^{n+1}U_{k-3,n}z_n - T_{k-2}^{n}U_{k-2,n}z_n\|\Big] \\
& +\delta_{k,n+1}|\gamma_{k-1,n+1}-\gamma_{k-1,n}|M_{k-2,n} \\
& +\delta_{k,n+1}\gamma_{k-1,n}P_{k-2,n} \\
& +|\gamma_{k,n+1}-\gamma_{k,n}|M_{k-1,n} \\
& +\gamma_{k,n}P_{k-1,n} \\
=\ & \delta_{k,n+1}\delta_{k-1,n+1}\delta_{k-2,n+1}\|U_{k-3,n+1}z_n - U_{k-3,n}z_n\| \\
& +\delta_{k,n+1}\delta_{k-1,n+1}|\gamma_{k-2,n+1}-\gamma_{k-2,n}|M_{k-3,n} \\
& +\delta_{k,n+1}\delta_{k-1,n+1}\gamma_{k-1,n}P_{k-3,n} \\
& +\delta_{k,n+1}|\gamma_{k-1,n+1}-\gamma_{k-1,n}|M_{k-2,n} \\
& +\delta_{k,n+1}\gamma_{k-1,n}P_{k-2,n} \\
& +|\gamma_{k,n+1}-\gamma_{k,n}|M_{k-1,n} \\
& +\gamma_{k,n}P_{k-1,n} \\
\leq & \\
\vdots\ & \vdots \\
\leq\ & \delta_{k,n+1}\delta_{k-1,n+1}\delta_{k-2,n+1}\cdots\delta_{3,n+1}\delta_{2,n+1}\|U_{1,n+1}z_n - U_{1,n}z_n\| \\
& +\Big(\delta_{k,n+1}\delta_{k-1,n+1}\delta_{k-2,n+1}\cdots\delta_{3,n+1}\gamma_{2,n+1}P_{1,n}+\cdots \\
& +\delta_{k,n+1}\delta_{k-1,n+1}\gamma_{k-2,n+1}P_{k-3,n} \\
& +\delta_{k,n+1}\gamma_{k-1,n+1}P_{k-2,n}+\gamma_{k,n+1}P_{k-1,n}\Big) \\
& +\Big(\delta_{k,n+1}\delta_{k-1,n+1}\delta_{k-2,n+1}\cdots\delta_{3,n+1}|\gamma_{2,n+1}-\gamma_{2,n}|M_{1,n}+\cdots \\
& +\delta_{k,n+1}\delta_{k-1,n+1}|\gamma_{k-2,n+1}-\gamma_{k-2,n}|M_{k-3,n} \\
& +\delta_{k,n+1}|\gamma_{k-1,n+1}-\gamma_{k-1,n}|M_{k-2,n} \\
& +|\gamma_{k,n+1}-\gamma_{k,n}|M_{k-1,n}\Big) \\
=\ & \|U_{1,n+1}z_n - U_{1,n}z_n\|\prod_{j=2}^{k}\delta_{j,n+1} \\
& +\sum_{i=2}^{k}\gamma_{i,n+1}P_{i-1,n}\prod_{j=i+1}^{k}\delta_{j,n+1}+\sum_{i=2}^{k}|\gamma_{i,n+1}-\gamma_{i,n}|M_{i-1,n}\prod_{j=i+1}^{k}\delta_{j,n+1}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big[\gamma_{1,n+1}||T_1^{n+1}z_n - T_1^n z_n|| + |\gamma_{1,n+1}-\gamma_{1,n}|(||z_n|| + ||T_1^n z_n||)\Big]\prod_{j=2}^{k}\delta_{j,n+1} \\
&\quad + \sum_{i=2}^{k-1}\gamma_{i,n+1}P_{i-1,n}\prod_{j=i+1}^{k}\delta_{j,n+1} + \sum_{i=2}^{k}|\gamma_{i,n+1}-\gamma_{i,n}|M_{i-1,n}\prod_{j=i+1}^{k}\delta_{j,n+1} \\
&= \sum_{i=1}^{k}\gamma_{i,n+1}P_{i,n}\prod_{j=i+1}^{k}\delta_{j,n+1} + \sum_{i=1}^{k}|\gamma_{i,n+1}-\gamma_{i,n}|M_{i,n}\prod_{j=i+1}^{k}\delta_{j,n+1}
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\|K_{n+1}z_n - K_n z_n\| &= \|U_{N,n+1}z_n - U_{N,n}z_n\| \\
&\leq \gamma_{N,n+1}||T_N^{n+1}U_{N-1,n+1}z_n - T_N^{n+1}U_{N-1,n}z_n|| \\
&\quad +\gamma_{N,n+1}||T_N^{n+1}U_{N-1,n+1}z_n - T_N^{n}U_{N-1,n}z_n|| \\
&\quad +|\gamma_{N,n+1}-\gamma_{N,n}|[||T_N^n U_{N-1,n}z_n|| + ||z_n||] \\
&\leq \gamma_{N,n+1}\rho_{N,n+1}||U_{N-1,n+1}z_n - U_{N-1,n}z_n|| \\
&\quad +\gamma_{N,n+1}||T_N^{n+1}U_{N-1,n+1}z_n - T_N^{n}U_{N-1,n}z_n|| \\
&\quad +|\gamma_{N,n+1}-\gamma_{N,n}|[||T_N^n U_{N-1,n}z_n|| + ||z_n||] \\
&\leq \rho_{N,n+1}\Big[\sum_{i=1}^{N-1}\gamma_{i,n+1}P_{i,n}\prod_{j=i+1}^{N-1}\delta_{j,n+1} \\
&\quad + \sum_{i=1}^{N-1}|\gamma_{i,n+1}-\gamma_{i,n}|M_{i,n}\prod_{j=i+1}^{N-1}\delta_{j,n+1}\Big] \\
&\quad +\gamma_{N,n+1}||T_N^{n+1}U_{N-1,n+1}z_n - T_N^{n}U_{N-1,n}z_n|| \\
&\quad +|\gamma_{N,n+1}-\gamma_{N,n}|\big[||T_N^n U_{N-1,n}z_n|| + ||z_n||\big]. \qquad (4.1.62)
\end{aligned}
$$

Therefore

$$
\lim_{n\to\infty}||K_{n+1}z_n - K_n z_n|| = 0. \qquad (4.1.63)
$$

Hence (ii) is satisfied.

Next, we show (iii), let $\{z_n\}$ be a bounded sequence in $E$ such that

$\lim_{n\to\infty}||K_n z_n - z_n|| = 0$, then for $x^* \in \cap_{k=1}^{N} F(T_k)$, we obtain

$$\begin{aligned}
||K_n z_n - x^*||^2 &\leq \gamma_{N,n}||T_N^n U_{N-1,n} z_n - x^*||^2 + (1-\gamma_{N,n})||z_n - x^*||^2 \\
&\leq \gamma_{N,n}\rho_{N,n}^2||U_{N-1,n} z_n - x^*||^2 + (1-\gamma_{N,n})||z_n - x^*||^2 \qquad (4.1.64) \\
&\leq \gamma_{N,n}\rho_{N,n}^2\big[\gamma_{N-1,n}||T_{N-1}^n U_{N-2,n} z_n - x^*||^2 \\
&\quad +(1-\gamma_{N-1,n})||U_{N-2,n} z_n - x^*||^2\big] \\
&\quad +(1-\gamma_{N,n})||z_n - x^*||^2
\end{aligned}$$

$$\begin{aligned}
&\leq \gamma_{N,n}\rho_{N,n}^2\big[\gamma_{N-1,n}\rho_{N-1,n}^2||U_{N-2,n} z_n - x^*||^2 \\
&\quad +(1-\gamma_{N-1,n})||U_{N-2,n} z_n - x^*||^2\big] \\
&\quad +(1-\gamma_{N,n})||z_n - x^*||^2 \\
&= \gamma_{N,n}\rho_{N,n}^2\Big([1+\gamma_{N-1,n}(\rho_{N-1,n}^2-1)]||U_{N-2,n} z_n - x^*||^2\Big) \\
&\quad +(1-\gamma_{N,n})||z_n - x^*||^2 \\
&\leq \gamma_{N,n}\rho_{N,n}^2\big[1+\gamma_{N-1,n}(\rho_{N-1,n}^2-1)\big]\big[1+\gamma_{N-2,n}(\rho_{N-2,n}^2-1)\big]\cdots \\
&\quad \times[1+\gamma_{1,n}(\rho_{1,n}^2-1)]||z_n - x^*||^2 + (1-\gamma_{N,n})||z_n - x^*||^2 \\
&= \Big(1+\gamma_{N,n}\Big\{\rho_{N,n}^2[1+\gamma_{N-1,n}(\rho_{N-1,n}^2-1)][1+\gamma_{N-2,n}(\rho_{N-2,n}^2-1)]\cdots \\
&\quad \times[1+\gamma_{1,n}(\rho_{1,n}^2-1)]-1\Big\}\Big)||z_n - x^*||^2 \\
&= (1+\vartheta_n)||z_n - x^*||^2, \qquad (4.1.65)
\end{aligned}$$

where $\vartheta_n := \gamma_{N,n}\Big\{\rho_{N,n}^2[1+\gamma_{N-1,n}(\rho_{N-1,n}^2-1)][1+\gamma_{N-2,n}(\rho_{N-2,n}^2-1)]\cdots[1+\gamma_{2,n}(\rho_{2,n}^2-1)][1+\gamma_{1,n}(\rho_{1,n}^2-1)]-1\Big\}$ and observe that $\lim_{n\to\infty}\vartheta_n = 0$.

Then by using Lemma 3.5.5, (4.1.64) and (4.1.65), we have

$$\begin{aligned}
||K_n z_n - x^*||^2 &= ||\gamma_{N,n}(T_N^n U_{N-1,n} z_n - x^*) + (1-\gamma_{N,n})(z_n - x^*)||^2 \\
&\leq \gamma_{N,n}||T_N^n U_{N-1,n} z_n - x^*||^2 + (1-\gamma_{N,n})||z_n - x^*||^2 \\
&\quad -\gamma_{N,n}(1-\gamma_{N,n})g(||T_N^n U_{N-1,n} z_n - z_n||) \\
&\leq (1+\vartheta_n)||z_n - x^*||^2 - \gamma_{N,n}(1-\gamma_{N,n})g(||T_N^n U_{N-1,n} z_n - z_n||),
\end{aligned}$$

from this we obtain

$$\begin{aligned}
\gamma_{N,n}(1-\gamma_{N,n})g(||T_N^n U_{N-1,n}z_n - z_n||) &\leq ||z_n - x^*||^2 - ||K_n z_n - x^*||^2 + \vartheta_n||z_n - x^*||^2 \\
&= (||z_n - x^*|| - ||K_n z_n - x^*||)(||z_n - x^*|| + ||K_n z_n - x^*||) + \vartheta_n||z_n - x^*||^2 \\
&\leq ||z_n - K_n z_n||(||z_n - x^*|| + ||K_n z_n - x^*||) + \vartheta_n||z_n - x^*||^2 \\
&\leq (||z_n - K_n z_n|| + \vartheta_n)M_0 \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

for some $M_0 > 0$. Thus, by the property of $g$, we obtain

$$\lim_{n\to\infty} ||T_N^n U_{N-1,n}z_n - z_n|| = 0. \tag{4.1.66}$$

Moreover,

$$\begin{aligned}
||z_n - x^*||^2 &\leq (||z_n - T_N^n U_{N-1,n}z_n|| + ||T_N^n U_{N-1,n}z_n - x^*||)^2 \\
&= ||z_n - T_N^n U_{N-1,n}z_n||(||z_n - T_N^n U_{N-1,n}z_n|| + 2||T_N^n U_{N-1,n}z_n - x^*||) \\
&\quad + ||T_N^n U_{N-1,n}z_n - x^*||^2 \\
&\leq ||z_n - T_N^n U_{N-1,n}z_n||M_1 + \rho_{N,n}^2||U_{N-1,n}z_n - x^*||^2 \quad (\text{for some } M_1 > 0) \\
&\leq ||z_n - T_N^n U_{N-1,n}z_n||M_1 + \rho_{N,n}^2[\gamma_{N-1,n}||T^{N-1,n}U_{N-2,n}z_n - x^*||^2 \\
&\quad + (1-\gamma_{N-1,n})||U_{N-2,n}z_n - x^*||^2 \\
&\quad - \gamma_{N-1,n}(1-\gamma_{N-1,n})g(||T_{N-1}^n U_{N-2,n}z_n - U_{N-2,n}z_n||)]
\end{aligned}$$

$$\begin{aligned}
&\leq ||z_n - T_N^n U_{N-1,n}z_n||M_1 + \rho_{N,n}^2[\gamma_{N-1,n}\rho_{N-1,n}^2||U_{N-2,n}z_n - x^*||^2 \\
&\quad + (1-\gamma_{N-1,n})||U_{N-2,n}z_n - x^*||^2 \\
&\quad - \gamma_{N-1,n}(1-\gamma_{N-1,n})g(||T_{N-1}^n U_{N-2,n}z_n - U_{N-2,n}z_n||)] \\
&\leq ||z_n - T_N^n U_{N-1,n}z_n||M_1 + \rho_{N,n}^2[(1+\gamma_{N-1,n}(\rho_{N-1,n}^2 - 1))||U_{N-2,n}z_n - x^*||^2 \\
&\quad - \gamma_{N-1,n}(1-\gamma_{N-1,n})g(||T_{N-1}^n U_{N-2,n}z_n - U_{N-2,n}z_n||)] \\
&\leq ||z_n - T_N^n U_{N-1,n}z_n||M_1 + (1+\vartheta_n)||z_n - x^*||^2 \\
&\quad - \rho_{N,n}^2\gamma_{N-1,n}(1-\gamma_{N-1,n})g(||T_{N-1}^n U_{N-2,n}z_n - U_{N-2,n}z_n||)],
\end{aligned}$$

$$
\begin{aligned}
g(||T_{N-1}^n U_{N-2,n} z_n - U_{N-2,n} z_n||) &\leq \frac{\Big(||z_n - T_N^n U_{N-1,n} z_n|| M_1 + \vartheta_n ||z_n - x^*||^2\Big)}{\rho_{N,n}^2 \gamma_{N-1,n}(1-\gamma_{N-1,n})} \\
&\leq \frac{\Big(||z_n - T_N^n U_{N-1,n} z_n|| + \vartheta_n\Big) M}{\rho_{N,n}^2 \gamma_{N-1,n}(1-\gamma_{N-1,n})},
\end{aligned}
$$

for some $M > 0$. Thus, using property of $g$,

$$
\lim_{n\to\infty} ||T_{N-1}^n U_{N-2,n} z_n - U_{N-2,n} z_n|| = 0. \tag{4.1.67}
$$

Continuing in this fashion we observe that for $k \in \{2, 3, 4, \cdots, N-1\}$

$$
\lim_{n\to\infty} ||T_k^n U_{k-1,n} z_n - U_{k-1,n} z_n|| = 0, \tag{4.1.68}
$$

and

$$
\lim_{n\to\infty} ||T_1^n z_n - z_n|| = 0. \tag{4.1.69}
$$

Also

$$
\begin{aligned}
||U_{k,n} z_n - z_n|| &\leq ||U_{k,n} z_n - T_k^n U_{k-1,n} z_n|| + ||T_k^n U_{k-1,n} z_n - U_{k-1,n} z_n|| \\
&\quad + ||U_{k-1,n} z_n - T_{k-1}^n U_{k-2,n} z_n|| \\
&\quad + ||T_{k-1}^n U_{k-2,n} z_n - U_{k-2,n} z_n|| \\
&\quad + \cdots + ||T_2^n U_{1,n} z_n - U_{1,n} z_n|| + ||U_{1,n} z_n - z_n|| \\
&\leq (1-\gamma_{k,n}) ||U_{k-1,n} z_n - T_k^n U_{k-1,n} z_n|| \\
&\quad + ||T_k^n U_{k-1,n} z_n - U_{k-1,n} z_n|| \\
&\quad + (1-\gamma_{k-1,n}) ||U_{k-2,n} z_n - T_{k-1}^n U_{k-2,n} z_n|| \\
&\quad + \cdots + (1-\gamma_{2,n}) ||U_{1,n} z_n - T_2^n U_{1,n} z_n|| \\
&\quad + \gamma_{n,1} ||T_1^n z_n - z_n|| \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Thus

$$\begin{aligned} ||T_k^n U_{k-1,n} z_n - z_n|| &\leq ||T_k^n U_{k-1,n} z_n - U_{k-1,n} z_n|| \\ &\quad + ||U_{k-1,n} z_n - z_n|| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

So for any $k \in \{1, 2, 3, \cdots, N\}$, we obtain

$$\begin{aligned} ||z_n - T_k^n z_n|| &\leq ||z_n - T_k^n U_{k-1,n} z_n|| + ||T_k^n U_{k-1,n} z_n - T_k^n z_n|| \\ &\leq ||z_n - T_k^n U_{k-1,n} z_n|| \\ &\quad + \rho_{n,k} ||U_{k-1,n} z_n - z_n|| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{4.1.70}$$

Hence

$$\begin{aligned} ||T_k z_n - z_n|| &\leq ||T_k z_n - T_k(T_k^n) z_n|| + ||T_k(T_k^n) z_n - T_k^n z_n|| + ||T_k^n z_n - z_n|| \\ &\leq (L_k + 1)||z_n - T_k^n z_n|| + ||T_k^{n+1} z_n - T_k^n z_n||. \end{aligned}$$

Therefore, from (4.1.70), for each $k \in \{1, 2, 3, \cdots, N\}$, we obtain

$$\lim_{n \to \infty} ||T_k z_n - z_n|| = 0. \tag{4.1.71}$$

Moreover, by Lemma 3.5.6, we have $w_w(x_n) \subset \cap_{k=1}^N F(T_k)$, also since $\cap_{k=1}^N F(T_k) \subset F(K_n)$ is obvious, we only need to show that $F(K_n) \subset \cap_{k=1}^N F(T_k)$. Let $z^* \in F(K_n)$, and $z_n = z^*$, then, we have $||z^* - T_k z^*|| = 0$ for each $k \in \{1, 2, 3, \cdots, N\}$ that is $z^* = T_k z^*$, for each $k \in \{1, 2, 3, \cdots, N\}$, so that $z^* \in \cap_{k=1}^N F(T_k)$. Hence (iii) is satisfied. □

**Theorem 4.1.9** *Let $E$ be a real uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$. Let $G : E \to E$ be an $\eta$-strongly accretive and $\mu$-strictly pseudocontractive with $\eta + \mu > 1$ and let $f : E \to E$ be a contraction with coefficient $\alpha \in (0, 1)$. Let $\{T_i\}_{i=1}^N$ be a family of uniformly asymptotically regular asymptotically nonex-*

*pansive self mappings of $C$ into itself and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $K_n$ be as in Lemma 4.1.8. Assume that $0 < \gamma < \frac{\tau}{\alpha}$, where $\tau := (1 - \sqrt{\frac{1-\eta}{\mu}})$ and let $x_0 \in C$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

*(C1) $\alpha_n \to 0$ and $\frac{v_n}{\alpha_n} \to 0$ as $n \to \infty$, where $v_n$ is as in (i) of Lemma 4.1.8;*

*(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$*

*If $\{x_n\}_{n=1}^{\infty}$ is a sequence defined by,*

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n G)[\beta_n x_n + (1 - \beta_n) K_n x_n], \quad n \geq 0, \quad (4.1.72)$$

*then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F$, which also solves the following variational inequality:*

$$\langle \gamma f(p) - Gp, j(q - p) \rangle \leq 0, \qquad \forall q \in F. \qquad (4.1.73)$$

**Proof.** First, we show that $\{x_n\}$ defined by (4.1.72) is well defined. For all $n \in \mathbb{N}$, let us define the mapping

$$T_n^f x := \alpha_n \gamma f(x) + (I - \alpha_n G)[\beta_n x + (1 - \beta_n) K_n x].$$

Indeed, for all $x, y \in E$, we have

$$\begin{aligned} ||T_n^f x - T_n^f y|| &= ||\alpha_n \gamma (f(x) - f(y)) + (1 - \alpha_n G)[\beta_n (x - y) + (1 - \beta_n)(K_n x - K_n y)]|| \\ &\leq \alpha_n \gamma \alpha ||x - y|| + (1 - \alpha_n \tau)[\beta_n ||x - y|| + (1 - \beta_n)(1 + v_n)||x - y||] \\ &\leq [\alpha_n \gamma \alpha + (1 - \alpha_n \tau)(1 + v_n)]||x - y|| \\ &= \Big(1 - \alpha_n[(\tau - \gamma\alpha) - (1 - \alpha_n \tau)(v_n/\alpha_n)]\Big)||x - y||. \end{aligned}$$

Since, $\lim_{n\to\infty}(1-\alpha_n\tau)v_n/\alpha_n \to 0$, then there exists $n_0 \in \mathbb{N}$ such that $(1-\alpha_n\tau)v_n/\alpha_n < (\tau-\gamma\alpha)/2$ for all $n \geq n_0$. Therefore, for $n \geq n_0$, we have

$$1-\alpha_n[(\tau-\gamma\alpha)-(1-\alpha_n\tau)(v_n/\alpha_n)] < 1-\alpha_n[(\tau-\gamma\alpha)-(\tau-\gamma\alpha)/2] < 1.$$

Hence,

$$||T_n^f x - T_n^f y|| < ||x-y||.$$

Thus, $\{x_n\}$ defined by (4.1.72) is well defined. Therefore, by the contraction mapping principle, there exists a unique fixed point $x_n \in C$ of $T_n^f$ which satisfies (4.1.72).

From the choice of the parameter $\gamma$, it is easy to see that the mapping $(G-\gamma f) : E \to E$ is strongly accretive and so the variational inequality (4.1.73) has unique solution in $F$. Let $p \in F$ then,

$$\begin{aligned}\|x_n-p\|^2 &= \alpha_n\langle\gamma f(p)-Gp, j(x_n-p)\rangle + \langle(I-\alpha_n G)[\beta_n x_n+(1-\beta_n)K_n x_n]\\ &\quad -(I-\alpha_n G)p, j(x_n-p)\rangle + \alpha_n\langle\gamma f(x_n)-\gamma f(p), j(x_n-p)\rangle\\ &\leq [1-\alpha_n(\tau-\gamma\alpha)+(1-\alpha_n\tau)v_n]\|x_n-p\|^2 + \alpha_n\langle(\gamma f-G)p, j(x_n-p)\rangle.\end{aligned}$$

Let $d_n = (1-\alpha_n\tau)(v_n/\alpha_n)$. Since, $\lim_{n\to\infty}(1-\alpha_n\tau)v_n/\alpha_n = 0$, then there exists $n_0 \in \mathbb{N}$ such that $(1-\alpha_n\tau)v_n/\alpha_n < (\tau-\gamma\alpha)/2$ for all $n \geq n_0$.

$$\|x_n-p\|^2 \leq \frac{\langle(\gamma f-G)p, j(x_n-p)\rangle}{(\tau-\gamma\alpha)-d_n}, \tag{4.1.74}$$

that is $||x_n-p|| \leq \frac{2||\gamma f(p)-G(p)||}{\tau-\gamma\alpha}$, for all $n \geq n_0$. Thus $\{x_n\}$ is bounded implies that $\{f(x_n)\}$, $\{G(x_n)\}$ and $\{K_n(x_n)\}$ are also bounded. From (4.1.72) we also

obtain

$$\|x_n - K_n x_n\| \leq \beta_n \|x_n - K_n x_n\| + \alpha_n \|\gamma f(x_n) - G(\beta_n x_n + (1-\beta_n) K_n(x_n))\|$$

and hence

$$\|x_n - K_n x_n\| \leq \frac{\alpha_n}{1-\beta_n} \|\gamma f(x_n) - G(\beta_n x_n + (1-\beta_n) K_n(x_n))\| \ \to 0 \quad (4.1.75)$$

as $n \to \infty$. Since $\{x_n\}$ is bounded, using (4.1.75), it follows from (iii) of Lemma 4.1.8 that $F = F(K_n)$. We claim that the set $\{x_n\}$ is sequentially compact. Indeed, define a map $\phi : E \to \mathbb{R}$ by

$$\phi(y) := \mu_n ||x_n - y||^2, \ \forall y \in E.$$

Then, $\phi(y) \to \infty$ as $||y|| \to \infty$, $\phi$ is continuous and convex, so as $E$ is reflexive, there exists $q \in E$ such that $\phi(q) = \min_{u \in E} \phi(u)$. Hence, the set

$$K^* := \{y \in E : \phi(y) = \min_{u \in E} \phi(u)\} \neq \emptyset$$

since $\lim_{n\to\infty} ||x_n - K_n x_n|| = 0$, $\lim_{n\to\infty} ||x_n - K_n^m x_n|| = 0$, for any $m \geq 1$ by induction. Now let $v \in K^*$, we have

$$\begin{aligned} \lim_{n\to\infty} \phi(K_n v) &= \lim_{n\to\infty} \mu_n ||x_n - K_n v||^2 \\ &= \lim_{n\to\infty} \mu_n ||x_n - K_n x_n + K_n x_n - K_n v||^2 \\ &\leq \lim_{n\to\infty} \mu_n [(1+v_n)||x_n - v||]^2 = \lim_{n\to\infty} \phi(v), \end{aligned}$$

and hence $K_n v \in K^*$.

Now let $z \in F$, then $z = K_n z$. Since $K^*$ is a closed convex set, there exists a unique $v^* \in K^*$ such that

$$||z - v^*|| = \min_{u \in K^*} ||z - u||.$$

But

$$\lim_{n\to\infty}||z - K_n v^*|| \;=\; \lim_{n\to\infty}||K_n z - K_n v^*|| \leq \lim_{n\to\infty}(1+v_n)||z - v^*||,$$

which implies $v^* = K_n v^*$ and so $K^* \cap F \neq \emptyset$.

Let $p \in K^* \cap F$ and $\epsilon \in (0,1)$. Then, it follows that $\phi(p) \leq \phi(p - \epsilon(G - \gamma f)p)$ and using Lemma 3.5.2, we obtain

$$||x_n - p + \epsilon(G-\gamma f)p||^2 \leq ||x_n - p||^2 + 2\epsilon\langle(G-\gamma f)p, j(x_n - p + \epsilon(G-\gamma f)p)\rangle$$

which implies

$$\mu_n\langle(\gamma f - G)p, j(x_n - p + \iota(G-\gamma f)p)\rangle \leq 0.$$

Moreover,

$$\begin{aligned}\mu_n\langle(\gamma f - G)p, j(x_n - p)\rangle &= \mu_n\langle(\gamma f - G)p, j(x_n - p) - j(x_n - p + \epsilon(G-\gamma f)p)\rangle \\ &\quad + \mu_n\langle(\gamma f - G)p, j(x_n - p + \epsilon(G-\gamma f)p)\rangle \\ &\leq \mu_n\langle(\gamma f - G)p, j(x_n - p) - j(x_n - p + \epsilon(G-\gamma f)p)\rangle.\end{aligned}$$

Since $j$ is $norm - to - weak^*$ uniformly continuous on bounded subsets of $E$, we have

$$\mu_n\langle(\gamma f - G)p, j(x_n - p)\rangle \leq 0.$$

It follows from (4.1.74) that

$$\|x_n - p\|^2 \leq \frac{\langle(\gamma f - G)p, j(x_n - p)\rangle}{(\tau - \gamma\alpha) - d_n},$$

and so

$$\mu_n \|x_n - p\|^2 \leq 0.$$

Thus there exist a subsequence say $\{x_{n_l}\}$ of $\{x_n\}$ such that $\lim_{l\to\infty} x_{n_l} = p$.

Define $S_n$ as $S_n x := \beta_n x + (1-\beta_n)K_n x$, then $\lim_{l\to\infty} S_n x_{n_l} = p$ and $S_n p = p$. Thus for any $z \in F$, using (4.1.72) we have

$$\begin{aligned}
\langle G(x_{n_l}) - \gamma f(x_{n_l}), j(x_{n_l} - z)\rangle &= \frac{-1}{\alpha_{n_l}}\langle (I - S_n)x_{n_l} - (I - S_n)p, j(x_{n_l} - z)\rangle \\
&\quad + \langle Gx_{n_l} - GS_n x_{n_l}, j(x_{n_l} - z)\rangle \\
&\leq \langle Gx_{n_l} - GS_n x_{n_l}, j(x_{n_l} - z)\rangle \\
&\leq (1 + \frac{1}{\mu})||x_{n_l} - S_n x_{n_l}||||x_{n_l} - z||, \quad (4.1.76)
\end{aligned}$$

since $\langle (I - S_n)x_{n_l} - (I - S_n)p, j(x_{n_l} - z)\rangle \geq 0$ and $G$ is Lipschitzian. Using the fact that $\|x_{n_l} - S_n x_{n_l}\| = (1 - \beta_{n_l})\|x_{n_l} - K_{n_l} x_{n_l}\| \to 0$ as $l \to \infty$, we have $\|x_{n_l} - S_n x_{n_l}\| \to 0$ as $l \to \infty$. From (4.1.76), taking limit as $l \to \infty$ we obtain

$$\langle (G - \gamma f)p, j(p - z)\rangle \leq 0.$$

Hence $p$ is the unique solution of the variational inequality (4.1.73). Now assume there exists another subsequence of $\{x_n\}$ say $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = p^*$. Then, using (4.1.75) we have $p^* \in F$. Repeating the above argument with $p$ replaced by $p^*$ we can easily see that $p^*$ also solves the variational inequality (4.1.73). By uniqueness of the solution of the variational inequality, we obtain $p = p^*$ and this completes the proof. □

**Theorem 4.1.10** *Let $E$ be a real, uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$. Let $G : E \to E$ be an $\eta$-strongly accretive and $\mu$-strictly pseudocontractive with*

*$\eta + \mu > 1$ and let $f : E \to E$ be a contraction with coefficient $\alpha \in (0,1)$. Let $\{T_i\}_{i=1}^N$ be family of uniformly asymptotically regular asymptotically nonexpansive self mappings of $C$ into itself and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $K_n$ be as in Lemma 4.1.8. Assume that $0 < \gamma < \frac{\tau}{2\alpha}$, where $\tau := (1 - \sqrt{\frac{1-\eta}{\mu}})$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

*(C1)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \frac{v_n}{\alpha_n} = 0$, *where $v_n$ is as in (i) of Lemma 4.1.8;*

*(C2)* $\Sigma_{n=0}^\infty \alpha_n = \infty$

*(C3)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

*Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by letting $x_0 \in C$ arbitrary and,*

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n) K_n x_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n G) y_n, \quad n \geq 0, \end{aligned} \tag{4.1.77}$$

*then, the following hold*

*(a) $\{x_n\}_{n=1}^\infty$ is bounded;*

*(b) $\lim_{n\to\infty} ||K_n x_n - x_n|| = 0$;*

*(c) $F(K_n) = F$;*

*(d) $\{x_n\}_{n=1}^\infty$ converges strongly to $p \in F$, where $p$ is a solution of the variational inequality:*

$$\langle \gamma f(p) - Gp, j(q-p) \rangle \leq 0, \qquad \forall q \in F. \tag{4.1.78}$$

**Proof.** First, we show that the sequence $\{x_n\}_{n=1}^\infty$ is bounded. Let $u \in F$ then, since $(1 - \alpha_n \tau)(v_n/\alpha_n) \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that

$(1-\alpha_n\tau)(v_n/\alpha_n) < (\tau-\gamma\alpha)/2$ for all $n \geq n_0$. Hence, for $n \geq n_0$, we have the following.

$$\begin{aligned}\|y_n - u\| &\leq \beta_n\|x_n - u\| + (1-\beta_n)\|K_n x_n - u\| \\ &\leq \beta_n\|x_n - u\| + (1-\beta_n)(1+v_n)\|x_n - u\| \\ &\leq (1+v_n)\|x_n - u\|, \end{aligned} \tag{4.1.79}$$

so that,

$$\begin{aligned}\|x_{n+1} - u\| &= \|\alpha_n\gamma f(x_n) + (I-\alpha_n G)y_n - u\| \\ &= \|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(u) + \alpha_n\gamma f(u) - \alpha_n G(u) + \alpha_n G(u) + (I-\alpha_n G)y_n - u\| \\ &\leq \alpha_n\gamma\|f(x_n) - f(u)\| + \alpha_n\|\gamma f(u) - G(u)\| + \|(I-\alpha_n G)y_n - (I-\alpha_n G)u\| \\ &\leq \alpha_n\gamma\|f(x_n) - f(u)\| + \alpha_n\|\gamma f(u) - G(u)\| + (1-\alpha_n\tau)\|y_n - u\| \\ &\leq \alpha_n\gamma\alpha\|x_n - u\| + \alpha_n\|\gamma f(u) - G(u)\| + (1-\alpha_n\tau)(1+v_n)\|x_n - u\| \\ &= \Big[1 - \alpha_n\Big((\tau-\alpha\gamma) - (1-\alpha_n\tau)\frac{v_n}{\alpha_n}\Big)\Big]\|x_n - u\| \\ &\quad +\alpha_n\Big((\tau-\alpha\gamma) - (1-\alpha_n\tau)\frac{v_n}{\alpha_n}\Big)\frac{2\|\gamma f(u) - G(u)\|}{\tau-\alpha\gamma} \\ &\leq \max\Big\{\|x_n - u\|, \frac{2\|\gamma f(u) - G(u)\|}{\tau-\alpha\gamma}\Big\}. \end{aligned}$$

Thus by induction, we have

$$\|x_n - u\| \leq \max\Big\{\|x_0 - u\|, \frac{2\|\gamma f(u) - G(u)\|}{\tau-\alpha\gamma}\Big\},\ \forall n \geq n_0. \tag{4.1.80}$$

Hence, $\{x_n\}$ is bounded. As such $\{y_n\}$, $\{Gy_n\}$ and $\{f(x_n)\}$ are also bounded. Next, we show that $\lim_{n\to\infty}||x_{n+1} - x_n|| = 0$.

Let $z_n := \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$, which implies

$$\begin{aligned} z_n &= \frac{\alpha_n\gamma f(x_n) + (I-\alpha_n G)y_n - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_n(\gamma f(x_n) - Gy_n) + y_n - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_n(\gamma f(x_n) - Gy_n) + (1-\beta_n)K_n x_n}{1-\beta_n} \\ &= \frac{\alpha_n(\gamma f(x_n) - Gy_n)}{1-\beta_n} + K_n x_n \end{aligned}$$

then

$$\begin{aligned} z_{n+1} - z_n &= \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - Gy_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n(\gamma f(x_n) - Gy_n)}{1-\beta_n} \\ &\quad + K_{n+1}x_{n+1} - K_n x_n. \end{aligned}$$

Hence, by letting $M = \sup_n(||\gamma f(x_n)|| + ||Gy_n||)$, we obtain

$$\begin{aligned} ||z_{n+1} - z_n|| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(||\gamma f(x_{n+1})|| + ||Gy_{n+1}||) + \frac{\alpha_n}{1-\beta_n}(||\gamma f(x_n)|| + ||Gy_n||) \\ &\quad + ||K_{n+1}x_{n+1} - K_n x_n|| \\ &\leq \Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\Big)M + ||K_{n+1}x_{n+1} - K_{n+1}x_n|| \\ &\quad + ||K_{n+1}x_n - K_n x_n|| \\ &\leq \Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\Big)M + (1+v_{n+1})||x_{n+1} - x_n|| \\ &\quad + ||K_{n+1}x_n - K_n x_n|| \end{aligned}$$

Therefore

$$\begin{aligned} ||z_{n+1} - z_n|| - ||x_{n+1} - x_n|| &\leq \Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\Big)M + v_{n+1}||x_{n+1} - x_n|| \\ &\quad + ||K_{n+1}x_n - K_n x_n|| \end{aligned}$$

which implies

$$\limsup_{n\to\infty}(||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \leq 0.$$

Hence, by Lemma 3.5.3, we obtain

$$\lim_{n\to\infty} ||z_n - x_n|| = 0$$

thus

$$||x_{n+1} - x_n|| = (1-\beta_n)||z_n - x_n|| \to 0 \quad \text{as} \quad n \to \infty.$$

From (4.1.77) it follows that,

$$\begin{aligned} \|x_{n+1} - y_n\| &= \|\alpha_n \gamma f(x_n) + (I - \alpha_n G)y_n - y_n\| \\ &\leq \|\alpha_n \gamma f(x_n)\| + \|(I - \alpha_n G)y_n - y_n\| \\ &= \alpha_n \Big\{ \|\gamma f(x_n)\| + \|G(y_n)\| \Big\}, \end{aligned}$$

we have $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. As

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|,$$

we also get

$$\|x_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{4.1.81}$$

On the other hand, we obtain

$$\begin{aligned} \|K_n x_n - x_n\| &\leq \|x_n - y_n\| + \|y_n - K_n x_n\| \\ &= \|x_n - y_n\| + \|(\beta_n x_n + (1-\beta_n)K_n x_n) - K_n x_n\| \\ &= \|x_n - y_n\| + \beta_n \|x_n - K_n x_n\|, \end{aligned} \tag{4.1.82}$$

which implies that $(1-\beta_n)\|K_n x_n - x_n\| \leq \|x_n - y_n\|$. From condition $(C3)$ and (4.1.81) we obtain

$$\|K_n x_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{4.1.83}$$

Hence (b) is satisfied.

Next, we show that (c) is satisfied, that is $F(K_n) = \cap_{i=1}^{N} F(T_i)$, but from (a), (b) above and (iii) of Lemma 4.1.8, (c) is satisfied.

Next, we show that

$$\limsup_{n\to\infty} \langle (\gamma f - G)p, j(x_n - p)\rangle \leq 0, \tag{4.1.84}$$

where $p$ is the unique solution of the variational inequality (4.1.73). Let $z_m = \alpha_m \gamma f(z_m) + (1 - \alpha_m G)y_m$, where $y_m = \beta_m z_m + (1-\beta_m)K_m z_m$ and $\{\alpha_m\}$, $\{\beta_m\}$ satisfy the condition of Theorem 4.1.9. Then it follows from Theorem 4.1.9 that $p = \lim_{m\to\infty} z_m$, so that

$$z_m - x_n = \alpha_m(\gamma f(z_m) - Gz_m) + \alpha_m(Gz_m - Gy_m) + y_m - x_n$$

Hence

$$\begin{aligned}
||z_m - x_n||^2 &= \alpha_m \langle \gamma f(z_m) - Gz_m, j(z_m - x_n)\rangle \\
&\quad + \alpha_m \langle Gz_m - Gy_m, j(z_m - x_n)\rangle \\
&\quad + \langle y_m - x_n, j(z_m - x_n)\rangle \\
&\leq \alpha_m \langle \gamma f(z_m) - Gz_m, j(z_m - x_n)\rangle \\
&\quad + \alpha_m ||Gz_m - Gy_m|| ||z_m - x_n|| \\
&\quad + ||y_m - x_n|| ||z_m - x_n|| \\
&\leq \alpha_m \langle \gamma f(z_m) - Gz_m, j(z_m - x_n)\rangle \\
&\quad + \alpha_m (1 + \frac{1}{\mu}) ||z_m - y_m|| ||z_m - x_n|| \\
&\quad + ||y_m - x_n|| ||z_m - x_n||
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \alpha_m\langle\gamma f(z_m) - Gz_m, j(z_m - x_n)\rangle \\
& +\alpha_m(1+\frac{1}{\mu})(1-\beta_m)||z_m - K_m z_m||||z_m - x_n|| \\
& +||x_n - z_m||^2 + (1-\beta_m)[v_m||z_m - x_n|| \\
& +||K_m x_n - x_n||]||z_m - x_n||.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\langle\gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle \ \leq\ & (1+\frac{1}{\mu})(1-\beta_m)||z_m - K_m z_m||||z_m - x_n|| \\
+\ & (1-\beta_m)[v_m/\alpha_m]||z_m - x_n||^2 \\
+\ & \frac{||K_m x_n - x_n||||z_m - x_n||}{\alpha_m}.
\end{aligned}
$$

Now, taking limit superior as $n \to \infty$ firstly, and then as $m \to \infty$, we have

$$
\limsup_{m\to\infty} \limsup_{n\to\infty} \langle\gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle \leq 0 \tag{4.1.85}
$$

Moreover, we note that

$$
\begin{aligned}
& \langle\gamma f(p) - Gp, j(x_n - p)\rangle \qquad (4.1.86) \\
=\ & \langle\gamma f(p) - Gp, j(x_n - p)\rangle - \langle\gamma f(p) - Gp, j(x_n - z_m)\rangle \\
& +\langle\gamma f(p) - Gp, j(x_n - z_m)\rangle - \langle\gamma f(p) - Gz_m, j(x_n - z_m)\rangle \\
& +\langle\gamma f(p) - Gz_m, j(x_n - z_m)\rangle - \langle\gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle \\
& +\langle\gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle
\end{aligned}
$$

$$
\begin{aligned}
=\ & \langle\gamma f(p) - Gp, j(x_n - p) - j(x_n - z_m)\rangle \\
& +\langle Gz_m - Gp, j(x_n - z_m)\rangle \\
& +\langle\gamma f(z_m) - \gamma f(p), j(x_n - z_m)\rangle \\
& +\langle\gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle \qquad (4.1.87)
\end{aligned}
$$

Taking limit superior as $n \to \infty$ in (4.1.86), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(x_n - p)\rangle \\
&\leq \limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(x_n - p) - j(x_n - z_m)\rangle \\
&\quad + ||Gz_m - Gp||\limsup_{n\to\infty}||x_n - z_m|| \\
&\quad + ||\gamma f(z_m) - \gamma f(p)||\limsup_{n\to\infty}||x_n - z_m|| \\
&\quad + \limsup_{n\to\infty}\langle \gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle \\
&\leq \limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(x_n - p) - j(x_n - z_m)\rangle \\
&\quad + \Big((1 + \frac{1}{\mu}) + \alpha\gamma\Big)\|z_m - p\|\limsup_{n\to\infty}||x_n - z_m|| \\
&\quad + \limsup_{n\to\infty}\langle \gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle
\end{aligned}$$

By Theorem 4.1.9, $z_m \to p \in F$ as $m \to \infty$.

Since $j$ is $norm - to - weak^*$ uniformly continuous on bounded subset of $E$, we obtain

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(x_n - p) - j(x_n - z_m)\rangle = 0,$$

therefore, from (4.1.85) we obtain

$$\limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(x_n - p)\rangle \leq 0$$

Finally, we show that (d) is satisfied, since $\lim_{n\to\infty}(v_n/\alpha_n) = 0$. If we denote by $\sigma_n$ the value of $2v_n + v_n^2$ then, clearly $\lim_{n\to\infty}(\sigma_n/\alpha_n) = 0$. Let $N_0 \in \mathbb{N}$ be large enough such that $(1 - \alpha_n\tau)(\sigma_n/\alpha_n) < (\tau - 2\gamma\alpha)/2$, for all $n \geq N_0$. Then, using

the recursion formula (4.1.77) and for all $n \geq N_0$, we obtain.

$$\begin{aligned}
||x_{n+1}-p||^2 &= ||\alpha_n\gamma f(x_n) - \alpha_n G(p) + (1-\alpha_n G)y_n - (1-\alpha_n G)p||^2 \\
&\leq ||(1-\alpha_n G)y_n - (1-\alpha_n G)p||^2 + 2\alpha_n\langle \gamma f(x_n) - G(p), j(x_{n+1}-p)\rangle \\
&\leq (1-\alpha_n\tau)^2||y_n-p||^2 + 2\alpha_n\langle \gamma f(x_n) - \gamma f(p), j(x_{n+1}-p)\rangle \\
&+ 2\alpha_n\langle \gamma f(p) - G(p), j(x_{n+1}-p)\rangle \\
&\leq (1-\alpha_n\tau)||y_n-p||^2 + 2\alpha_n\gamma\alpha||x_n-p||||x_{n+1}-p|| \\
&+ 2\alpha_n\langle \gamma f(p) - G(p), j(x_{n+1}-p)\rangle \\
&\leq (1-\alpha_n\tau)[\beta_n + (1-\beta_n)(1+v_n)^2]||x_n-p||^2 \\
&+ \alpha_n\gamma\alpha||x_n-p||^2 + \alpha_n\gamma\alpha||x_{n+1}-p||^2 \\
&+ 2\alpha_n\langle \gamma f(p) - G(p), j(x_{n+1}-p)\rangle \\
&\leq (1-\alpha_n\tau)[1+\sigma_n]||x_n-p||^2 \\
&+ \alpha_n\gamma\alpha||x_n-p||^2 + \alpha_n\gamma\alpha||x_{n+1}-p||^2 \\
&+ 2\alpha_n\langle \gamma f(p) - G(p), j(x_{n+1}-p)\rangle \\
&= \Big(1-\alpha_n[(\tau-\alpha\gamma) - (1-\alpha_n\tau)(\sigma_n/\alpha_n)]\Big)||x_n-p||^2 \\
&+ \alpha_n\gamma\alpha||x_{n+1}-p||^2 + 2\alpha_n\langle \gamma f(p) - G(p), j(x_{n+1}-p)\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
||x_{n+1}-p||^2 &\leq \Big(1-\alpha_n[\frac{(\tau-2\alpha\gamma) - (1-\alpha_n\tau)(\sigma_n/\alpha_n)}{1-\alpha_n\alpha\gamma}]\Big)||x_n-p||^2 \\
&+ \frac{2\alpha_n[(\tau-2\alpha\gamma) - (1-\alpha_n\tau)(\sigma_n/\alpha_n)]\langle \gamma f(p) - G(p), j(x_{n+1}-p)\rangle}{(1-\alpha_n\gamma\alpha)[(\tau-2\alpha\gamma) - (1-\alpha_n\tau)(\sigma_n/\alpha_n)]}.
\end{aligned}$$

Observe that $\sum \alpha_n[(\tau-2\alpha\gamma) - (1-\alpha_n\tau)(\sigma_n/\alpha_n)] = \infty$ and

$$\limsup\Big(\frac{2\alpha_n\langle \gamma f(p) - G(p), j(x_{n+1}-p)\rangle}{(1-\alpha_n\gamma\alpha)[(\tau-2\alpha\gamma) - (1-\alpha_n\tau)(\sigma_n/\alpha_n)]}\Big) \leq 0$$

Consequently, applying Lemma 3.5.4, we conclude that $x_n \to p$ as $n \to \infty$. □

The following corollaries follow from Theorem 4.1.10.

**Corollary 4.1.11** *Let $E$ be a real uniformly convex Banach space whose duality mapping $J$ is weakly sequentially continuous.Let $G : H \to H$, $f : E \to E$, $\{T_i\}_{i=1}^N$ $F$, $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ be as in Theorem 4.1.10, then $\{x_n\}_{n=1}^\infty$ converges strongly to $p \in F$, which is also the unique solution of the variational inequality*

$$\langle \gamma f(p) - Gp, j(q-p) \rangle \leq 0, \ \forall q \in F$$

**Corollary 4.1.12** *Let $H$ be a real Hilbert space, $\{z_t\}_{t\in(0,1)}$, be as in Theorem 4.1.10. Then $\{z_t\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^N$ say $p$ which is a unique solution of the variational inequality*

$$\langle (G - \gamma f)p, q - p \rangle \geq 0, \ \forall q \in F.$$

**Corollary 4.1.13** *Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $G : H \to H$, $f : E \to E$, $\{T_i\}_{i=1}^N$ $F$, $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ be as in Theorem 4.1.10, then $\{x_n\}_{n=1}^\infty$ converges strongly to $p \in F$, which is also the unique solution of the variational inequality*

$$\langle \gamma f(p) - Gp, q - p \rangle \leq 0, \ \forall q \in F$$

### 4.1.1 Concluding remarks

In conclusion, the results in this section extend the result of Ali et al. (2013) from finite of nonexpansive mappings to finite family of asymptotically nonexpansive mappings.

## 4.2 Generalized Asymptotically Quasi-Phi Strict Pseudocontractive Mappings

In this section, we introduce new recursion formulas and prove strong convergence theorem for family of generalized asymptotically quasi-$\phi$-strict pseudocontractive mappings in a real reflexive, strictly convex and smooth Banach space with Kadec-Klee property, whose dual $E^*$ also possesses Kdec Klee property. The following result is obtained:

**Theorem 4.2.1** *Let $E$ be a reflexive, strictly convex and smooth Banach space such that both $E$ and $E^*$ have Kadec-Klee property. Let $C$ be a nonempty, closed, and convex subset of $E$. Let $\Delta$ be an index set, and $T_i : C \to C$ be a closed, uniformly asymptotically regular and generalized asymptotically quasi-$\phi$-strictly pseudocontractive mapping with sequences $\{u_{i,n}\}$ and $\{v_{i,n}\}$, for every $i \in \Delta$. Assume that $F := \cap_{i\in\Delta} F(T_i) \neq \emptyset$ . Let $\{x_n\}$ be a sequence generated in the following manner;*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ C = C_1 = C_{i,0}\ i \in \Delta, \\ y_{i,n} = J^{-1}(\alpha_{n,i}Jx_n + (1-\alpha_{n,i})J(T_i^n x_n)), \\ C_{i,n+1} = \{z \in C_n : \phi(x_n, y_{i,n}) \leq \frac{1}{1-k_i}[u_{i,n}M + v_{i,n}] \\ +\frac{2k_i}{1-k_i}\langle x_n - z, Jx_n - JT_i^n x_n\rangle + 2\langle x_n - z, Jx_n - Jy_{n,i}\rangle\}, \\ C_{n+1} = \cap_{i\in\Delta} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0,\ \forall n \geq 0, \end{cases} \tag{4.2.1}$$

*where $M := \sup_{z\in F, n\geq 1} \phi(z, x_n)$; $\{k_i\}_{i\in\Delta} \subset [0,1)$ and $\{\alpha_{i,n}\}$ are sequences in $(0,1]$ such that $\liminf_{n\to\infty}(1-\alpha_{i,n}) > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Delta}F(T_i)}x_0$.*

**Proof.** We start by showing that for every $i \in \Delta$, $F(T_i)$ is closed and convex. This proves that $\Pi_{\cap_{i\in\Delta}F(T_i)}x_0$ is well defined, for $x_0 \in E$. The closedness of $\bigcap_{i\in\Delta} F(T_i)$ easily followed from the closedness of $T_i$ for each $i \in \Delta$. We only

need to show that $\cap_{i\in\Delta}F(T_i)$ is convex. Let $p_{i,1}, p_{i,2} \in F(T_i)$, and $p_i = t_i p_{i,1} + (1-t_i)p_{i,2}$, where $t_i \in (0,1)$, for every $i \in \Delta$. Then, we show that $p_i = T_i p_i$. Indeed, from the definition of $T_i$ and (3.6.4) we have

$$\phi(p_i, T_i^n p_i) \leq \frac{1}{1-k_i}[u_{i,n}\phi(p_{i,1}, p_i) + v_{i,n} + 2\langle p_i - p_{i,1}, Jp_i - JT_i^n p_i\rangle] \quad (4.2.2)$$

and

$$\phi(p_i, T_i^n p_i) \leq \frac{1}{1-k_i}[u_{i,n}\phi(p_{i,2}, p_i) + v_{n,i} + 2\langle p_i - p_{i,2}, Jp_i - JT_i^n p_i\rangle]. \quad (4.2.3)$$

Multiplying (4.2.2) and (4.2.3) by $t_i$ and $(1-t_i)$ respectively, and adding the results we get

$$\begin{aligned}\phi(p_i, T_i^n p_i) &\leq \frac{1}{1-k_i}[t_i u_{i,n}\phi(p_{i,1}, p_i) + (1-t_i)u_{i,n}\phi(p_{i,2}, p_i) + v_{i,n} \\ &\quad +2\langle p_i - [t_i p_{i,1} + (1-t_i)p_{i,2}], Jp_i - JT_i^n p_i\rangle] \\ &= \frac{1}{1-k_i}[t_i u_{i,n}\phi(p_{i,1}, p_i) + (1-t_i)u_{i,n}\phi(p_{i,2}, p_i) + v_{i,n}]\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\phi(p_i, T_i^n p_i) = 0$$

from (3.6.1), we have

$$\lim_{n\to\infty}||T_i^n p_i|| = ||p_i||. \quad (4.2.4)$$

It follows that

$$\lim_{n\to\infty}||JT_i^n p_i|| = ||Jp_i||. \quad (4.2.5)$$

Since $E^*$ is reflexive, we may, without loss of generality, assume that $J(T_i^n p_i) \rightharpoonup e^{i,*} \in E^*$. But $E$ is also reflexive, then we have $J(E) = E^*$. This shows that

there exists an element say $e^i \in E$ such that $Je^i = e^{i,*}$. Therefore

$$\begin{aligned}\phi(p_i, T_i^n p_i) &= ||p_i||^2 - 2\langle p_i, J(T_i^n p_i)\rangle + ||T_i^n p_i||^2 \\ &= ||p_i||^2 - 2\langle p_i, J(T_i^n p_i)\rangle + ||JT_i^n p_i||^2.\end{aligned}$$

Taking $\underset{n\to\infty}{liminf}$ of both sides of the equality above, we obtain

$$\begin{aligned}0 &\geq ||p_i||^2 - 2\langle p_i, e^{i,*})\rangle + ||e^{i,*}||^2 \\ &= ||p_i||^2 - 2\langle p_i, Je^i\rangle + ||Je^i||^2 \\ &= ||p_i||^2 - 2\langle p_i, Je^i\rangle + ||e^i||^2 \\ &= \phi(p_i, e^i).\end{aligned}$$

This implies that $p_i = e^i$, that is $Jp_i = e^{i,*}$. It follows that $J(T_i^n p_i) \rightharpoonup Jp_i \in E^*$. Using (4.2.5) and the Kadec-Klee property of $E^*$, we obtain

$$\lim_{n\to\infty} ||J(T_i^n p_i) - Jp_i|| = 0.$$

Since $J^{-1} : E^* \to E$ is demicontinuous, we see that $T_i^n p_i \rightharpoonup p_i$. By the Kadec-Klee property of $E$ together with (4.2.4) we get $T_i^n p_i \to p_i$ as $n \to \infty$. Hence

$$T_i T_i^n p_i = T_i^{n+1} p_i \to p_i \text{ as } n \to \infty.$$

As $T_i$ is closed, we have $p_i \in F(T_i)$, for every $i \in \Delta$. Hence for every $i \in \Delta$, $F(T_i)$ is convex, and cosequently $\cap_{i\in\Delta} F(T_i)$ is convex.

Next we show that $C_n$ is closed and convex for all $n \geq 1$. It suffices to show for any fixed but arbitrary $i \in \Delta$, that $C_{i,n}$ is closed and convex for each $n \geq 1$. This can be proved by induction on $n$. It is obvious that $C_{i,1} = C$ is closed and convex. Assume that $C_{i,h}$ is closed and convex for some $h \geq 1$. We now

prove that $C_{i,h+1}$ is closed and convex . For $z \in C_{i,h+1}$, one can obtain that

$$\begin{aligned}\phi(x_h, y_{i,h}) &\leq \frac{1}{1-k_i}[u_{i,h}M + v_{i,h}] + \frac{2k_i}{1-k_i}\langle x_h - z, Jx_h - JT_i^h x_h\rangle \\ &+ 2\langle x_h - z, Jx_h - Jy_{h,i}\rangle,\end{aligned}$$

using the fact that $\langle ., Jx_h - JT_i^h x_h\rangle$ and $\langle ., Jx_h - Jy_{i,h}\rangle$ are continuous and linear in $E$, for every $i \in \Delta$, then $C_{i,h+1}$ is closed and convex and consequently $C_n$ is closed and convex $\forall n \geq 1$.

We now show that $\cap_{i\in\Delta}F(T_i) \subset C_n$ for every $n \geq 1$. It suffices to show that $\cap_{i\in\Delta}F(T_i) \subset C_{n,i}$ for every $n \geq 1$ and for every $i \in \Delta$. Note that $\cap_{i\in\Delta}F(T_i) \subset C_{i,1} = C$. Suppose that $\cap_{i\in\Delta}F(T_i) \subset C_{i_0,h}$, for some $h \geq 1$ and for arbitrary fixed $i_0 \in \Delta$. Then, for all $w \in \cap_{i\in\Delta}F(T_i) \subset C_{i_0,h}$ we have

$$\begin{aligned}\phi(w, y_{i_0,h}) &= \phi(w, J^{-1}(\alpha_{i_0,h}Jx_h + (1-\alpha_{i_0,h})J(T_{i_0}^h x_h))) \\ &= ||w||^2 - 2\langle w, \alpha_{i_0,h}Jx_h + (1-\alpha_{h,i_0})J(T_{i_0}^h x_h)\rangle \\ &+ ||\alpha_{i_0,h}Jx_h + (1-\alpha_{i_0,h})J(T_{i_0}^h x_h)||^2 \\ &\leq ||w||^2 - 2\alpha_{i_0,h}\langle w, Jx_h\rangle - 2(1-\alpha_{i_0,h})\langle w, J(T_{i_0}^h x_h)\rangle \\ &+ \alpha_{i_0,h}||x_h||^2 + (1-\alpha_{i_0,h})||T_{i_0}^h x_h||^2 \\ &= \alpha_{i_0,h}\phi(w, x_h) + (1-\alpha_{i_0,h})\phi(w, T_{i_0}^h x_h) \\ &\leq \alpha_{i_0,h}\phi(w, x_h) + (1-\alpha_{i_0,h})[(1+u_{i_0,h})\phi(w, x_h) \\ &+ k_{i_0}\phi(x_h, T_{i_0}^h x_h) + v_{i_0,h}] \\ &\leq (1+u_{i_0,h})\phi(w, x_h) + k_{i_0}\phi(x_h, T_{i_0}^h x_h) + v_{i_0,h}. \qquad (4.2.6)\end{aligned}$$

Substituting (3.6.2) and (3.6.4) in (4.2.6) we obtain

$$\begin{aligned}\phi(x_h, y_{i_0,h}) &\leq \frac{1}{1-k_{i_0}}[u_{i_0,h}\phi(w, x_h) + v_{i_0,h}] + \frac{2k_{i_0}}{1-k_{i_0}}\langle x_h - w, Jx_h - JT_{i_0}^h x_h\rangle \\ &+ 2\langle x_h - w, Jx_h - Jy_{i_0,h}\rangle\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{1-k_{i_0}}[u_{i_0,h}M + v_{h,i_0}] + \frac{2k_{i_0}}{1-k_{i_0}}\langle x_h - w, Jx_h - JT_{i_0}^h x_h\rangle \\ &+ 2\langle x_h - w, Jx_h - Jy_{i_0,h}\rangle \end{aligned}$$

Now since $i_0 \in \Delta$ is arbitrary we get $w \in C_h$. This implies that $w \in C_{i,h+1}$ for each $i \in \Delta$ which in turn implies $w \in C_{h+1}$. By induction we obtain $\cap_{i\in\Delta} F(T_i) \subset C_n$ for every $n \geq 1$.

Next we show that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. Since $x_n = \Pi_{C_n} x_0$, we have

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0 \ \forall \ z \in C_n.$$

Now as $\cap_{i\in\Delta} F(T_i) \subset C_n$, we obtain

$$\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0 \ \forall \ w \in \cap_{i\in\Delta} F(T_i). \tag{4.2.7}$$

It follows from Lemma 3.6.2 that

$$\begin{aligned} \phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \\ &\leq \phi(\Pi_{\cap_{i\in\Delta} F(T_i)} x_0, x_0) - \phi(\Pi_{\cap_{i\in\Delta} F(T_i)} x_0, x_n) \\ &\leq \phi(\Pi_{\cap_{i\in\Delta} F(T_i)} x_0, x_0). \end{aligned}$$

This implies that the sequence $\{\phi(x_n, x_0)\}$ is bounded. It follows from (3.6.1) that the sequence $\{x_n\}$ is also bounded. Using the fact that $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0). \end{aligned} \tag{4.2.8}$$

Since $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we obtain $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \ \forall \ n \geq 1$. This shows that $\{\phi(x_n, x_0)\}$ is nondecreasing and hence

the limit $\lim_{n\to\infty}\phi(x_n, x_0)$ exists. Thus from (4.2.8) we have

$$\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0. \tag{4.2.9}$$

Now we show that $x_n \to p$ as $n \to \infty$, where $p$ is some point in $\cap_{i\in\Delta}F(T_i) \subset C_n$. Since $\{x_n\}$ is bounded and the space is reflexive, we may assume that $x_n \rightharpoonup p$ as $n \to \infty$. Noticing that $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \leq \phi(p, x_0)$ and since $C_n$ is closed and convex we have $p \in C_n$. On the other hand, we have from the weak lower semicontinuity of the norm function that

$$\begin{aligned}
\phi(p, x_0) &= ||p||^2 - 2\langle p, Jx_0\rangle + ||x_0||^2 \\
&\leq \liminf_{n\to\infty}(||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2) \\
&= \liminf_{n\to\infty}\phi(x_n, x_0) \\
&= \limsup_{n\to\infty}\phi(x_n, x_0) \\
&\leq \phi(p, x_0).
\end{aligned}$$

This implies that $\phi(x_n, x_0) \to \phi(p, x_0)$ as $n \to \infty$. Hence, $||x_n|| \to ||p||$ as $n \to \infty$. From the Kadec-Klee property of $E$, we have $x_n \to p$ as $n \to \infty$.

Then we show that $p \in \cap_{i\in\Delta}F(T_i)$. Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$, we obtain from (4.2.1)

$$\begin{aligned}
\phi(x_n, y_{i,n}) &\leq \frac{1}{1-k_i}[u_{i,n}M + v_{i,n}] + \frac{2k_i}{1-k_i}\langle x_n - x_{n+1}, Jx_n - JT_i^n x_n\rangle \\
&+ 2\langle x_n - x_{n+1}, Jx_n - Jy_{i,n}\rangle
\end{aligned} \tag{4.2.10}$$

From (4.2.9) we have $x_{n+1} - x_n \to 0$ as $n \to \infty$. Taking limit of both sides of (4.2.10) as $n \to \infty$, we obtain

$$\lim_{n\to\infty}\phi(x_n, y_{i,n}) = 0. \tag{4.2.11}$$

In view of (3.6.1) we get

$$\lim_{n\to\infty}(||x_n|| - ||y_{i,n}||) = 0.$$

This in turn implies that

$$\lim_{n\to\infty}||y_{i,n}|| = ||p||$$

It follows that

$$\lim_{n\to\infty}||Jy_{i,n}|| = ||Jp||. \tag{4.2.12}$$

This implies that $\{Jy_{i,n}\}$ is bounded. Since both $E$ and $E^*$ are reflexive we may assume that $Jy_{i,n} \rightharpoonup y^{i,*} \in E^*$ for every $i \in \Delta$. By reflexivity of $E$, we have $J(E) = E^*$. This shows that there exists an element $y^i \in E$ such that $Jy^i = y^{i,*}$. It follows that

$$\begin{aligned}\phi(x_n, y_{i,n}) &= ||x_n||^2 - 2\langle x_n, Jy_{i,n}\rangle + ||y_{i,n}||^2 \\ &= ||x_n||^2 - 2\langle x_n, Jy_{i,n}\rangle + ||Jy_{i,n}||^2.\end{aligned}$$

Taking $\underset{n\to\infty}{liminf}$ of both sides of the equality above, we have

$$\begin{aligned}0 &\geq ||p||^2 - 2\langle p, y^{i,*}\rangle + ||y^{i,*}||^2 \\ &= ||p||^2 - 2\langle p, Jy^i\rangle + ||Jy^i||^2 \\ &= ||p||^2 - 2\langle p, Jy^i\rangle + ||y^i||^2 \\ &= \phi(p, y^i).\end{aligned}$$

Hence, $p = y^i$ which implies that $y^{i,*} = Jp$ for every $i \in \Delta$. It follows that $Jy_{n,i} \rightharpoonup Jp \in E^*$ for every $i \in \Delta$. From the Kadec-Klee property of $E^*$ together with (4.2.12) we obtain

$$\lim_{n\to\infty} Jy_{i,n} = Jp.$$

Since

$$||Jx_n - Jy_{i,n}|| \leq ||Jx_n - Jp|| + ||Jp - Jy_{i,n}||,$$

and using the fact that $Jx_n \to Jp$, it follows that

$$\lim_{n\to\infty} ||Jx_n - Jy_{i,n}|| = 0. \tag{4.2.13}$$

Note from the recursion formula (4.2.1), that

$$(1 - \alpha_{i,n})||J(T_i^n x_n) - Jx_n|| = ||Jx_n - Jy_{i,n}||$$

In view of the assumption that $\liminf_{n\to\infty}(1 - \alpha_{i,n}) > 0$, it follows that

$$\lim_{n\to\infty} ||J(T_i^n x_n) - Jx_n|| = 0. \tag{4.2.14}$$

From

$$||J(T_i^n x_n) - Jp|| \leq ||J(T_i^n x_n) - Jx_n|| + ||Jx_n - Jp||,$$

we have

$$\lim_{n\to\infty} ||J(T_i^n x_n) - Jp|| = 0. \tag{4.2.15}$$

From the demicontinuity of $J^{-1} : E^* \to E$ we have $T_i^n x_n \rightharpoonup p$, for every $i \in \Delta$. Since

$$\begin{aligned} \Big|||T_i^n x_n|| - ||p||\Big| &= \Big|||J(T_i^n x_n)|| - ||Jp||\Big| \\ &\leq ||J(T_i^n x_n) - Jp||, \end{aligned}$$

and from (4.2.15), we have

$$||T_i^n x_n|| \to ||p|| \text{ as } n \to \infty \text{ for every } i \in \Delta.$$

As $E$ possesses Kadec-Klee property, we obtain

$$\lim_{n\to\infty} ||T_i^n x_n - p|| = 0. \tag{4.2.16}$$

But

$$||T_i^{n+1} x_n - p|| \leq ||T_i^{n+1} x_n - T_i^n x_n|| + ||T_i^n x_n - p||.$$

It follows from the uniform asymptotic regularity of $T_i$, and (4.2.16), that

$$\lim_{n\to\infty} ||T_i^{n+1} x_n - p|| = 0, \tag{4.2.17}$$

that is, $T_i T_i^n x_n - p \to 0$ as $n \to \infty$. It follows from the closedness of $T_i$ that $T_i p = p$ for every $i \in \Delta$.

Finally, we show that $p = \Pi_{\cap_{i\in\Delta} F(T_i)} x_0$. Letting $n \to \infty$ in (4.2.7) we obtain

$$\langle p - w, Jx_0 - Jp\rangle \geq 0 \quad \forall\ w \in \cap_{i\in\Delta} F(T_i),$$

it follows from Lemma 3.6.1 that $p = \Pi_{\cap_{i\in\Delta} F(T_i)} x_0$. This completes the proof. □ The following corollaries follow from Theorem 4.2.1.

**Corollary 4.2.2** *Let $E$ be a real uniformly convex and uniformly smooth Banach space. Let $C$, $T_i : C \to C$, $\{u_{n,i}\}$, $\{v_{n,i}\}$, $\cap_{i\in\Delta} F(T_i)$, $\{\alpha_{n,i}\}$ and $\{x_n\}$ be as in Theorem 4.2.1, then $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Delta} F(T_i)} x_0$.*

**Corollary 4.2.3** *Let $E$ be a real reflexive, strictly convex and smooth Banach space such that both $E$ and $E^*$ have Kadec-Klee property. Let $C$ be a nonempty, closed, and convex subset of $E$. Let $T : C \to C$ be a closed, uniformly asymptotically regular and generalized asymptotically quasi-$\phi$-strictly pseudo-contraction mapping with sequences $\{u_n\}$ and $\{v_n\}$. Assume that $F :=$*

$F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated in the following manner;*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ C = C_1, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J(T^n x_n)), \\ C_{n+1} = \{z \in C_n : \phi(x_n, y_n) \leq \frac{1}{1-k}[u_n M + v_n] \\ +\frac{2k}{1-k}\langle x_n - z, Jx_n - JT^n x_n\rangle + 2\langle x_n - z, Jx_n - Jy_n\rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall n \geq 0 \end{cases}$$

*where* $M := \sup_{z \in F, n \geq 1} \phi(z, x_n)$; $k \in (0,1]$ *and* $\{\alpha_n\}$ *a sequence in* $(0,1]$ *such that* $\liminf_{n\to\infty}(1-\alpha_n) > 0$, *Then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)} x_0$.

**Corollary 4.2.4** *Let* $E$ *be a real reflexive, strictly convex and smooth Banach space such that both* $E$ *and* $E^*$ *have Kadec-Klee property. Let* $C$ *be a nonempty, closed, and convex subset of* $E$. *Let* $\Delta$ *be an index set, and* $T_i : C \to C$ *be a closed and quasi-*$\phi$*-strictly pseudo-contraction mapping. Assume that* $F := \cap_{i\in\Delta} F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated in the following manner;*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ C = C_1 = C_{i,0} \ i \in \Delta, \\ y_{i,n} = J^{-1}(\alpha_{n,i} Jx_n + (1-\alpha_{n,i})J(T_i x_n)), \\ C_{i,n+1} = \{z \in C_n : \phi(x_n, y_{i,n}) \leq \frac{2k_i}{1-k_i}\langle x_n - z, Jx_n - JT_i x_n\rangle \\ +2\langle x_n - z, Jx_n - Jy_{n,i}\rangle\}, \\ C_{n+1} = \cap_{i\in\Delta} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall n \geq 0 \end{cases}$$

*where* $\{k_i\}_{i\in\Delta} \subset [0,1)$ *and* $\{\alpha_{n,i}\}$ *are sequences in* $(0,1]$ *such that* $\liminf_{n\to\infty}(1-\alpha_{n,i}) > 0$ *for each* $i \in \Delta$. *Then,* $\{x_n\}$ *converges strongly to* $\Pi_{\cap_{i\in\Delta} F(T_i)} x_0$.

**Corollary 4.2.5** *Let* $E$ *be a real reflexive, strictly convex and smooth Banach space such that both* $E$ *and* $E^*$ *have Kadec-Klee property. Let* $C$ *be a nonempty, closed, and convex subset of* $E$. *Let* $\Delta$ *be an index set, and* $T_i : C \to C$ *be*

*a closed, uniformly asymptotically regular and asymptotically strict quasi-$\phi$-pseudocontraction in the intermediate sense with sequence $\{u_{n,i}\} \subset [0,\infty)$ such that $u_{n,i} \to 0$ as $n \to \infty$. Assume that $F := \cap_{i\in\Delta} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by* (4.2.1)*, where the sequence $\{v_{n,i}\}$ is given by*

$$v_{n,i} := Max\big\{0, \sup_{p\in F(T_i), x\in C} \big(\phi(p, T_i^n x) - (1+u_{n,i})\phi(p, x_n) - k_i\phi(x_n, T_i^n x_n)\big)\big\}$$

for each $i \in \Delta$. *Then, $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Delta} F(T_i)} x_0$.*

### 4.2.1 Concluding remarks

In conclusion, the result in this section extends the result of Qin et al. (2012), from the class of asymptotically strict quasi-$\phi$-pseudo-contraction in the intermediate sense to family of generalized asymptotically quasi-$\phi$-strict pseudo-contraction mapping.

## 4.3 Quasi Phi Asymptotically Nonexpansive Multi Valued Mappings

In this section, we introduce new recursion formulas and prove strong convergence theorem for infinite family of uniformly $L-$Lipschitzian total quasi-$\phi$-asymptotically nonexpansive multi-valued mappings using a generalized $f-$projection operator in a real uniformly convex and uniformly smooth Banach space. We begin with the following Lemma:

**Lemma 4.3.1** *Let $E$ be a real uniformly convex and uniformly smooth Banach space, and $C$ a nonempty closed and convex subset of $E$. Let $T : C \to N(C)$ be a closed and total quasi-$\phi$-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{u_n\}$ and $\{v_n\}$ and a strictly increasing*

*continuous function* $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $u_n \to 0, v_n \to 0$ *as* $n \to \infty$ *and* $\zeta(0) = 0$. *Let* $f : E \to \mathbb{R}$ *be a convex, lower semi continuous mapping with* $C \subset int(D(f))$. *Then* $F(T)$ *is closed and convex.*

**Proof.** First, we show that $F(T)$ is closed. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $F(T)$ such that $x_n \to p$ (as $n \to \infty$). Since $x_n \in Tx_n$, by the closure of $T$, we have $p \in Tp$, that is $p \in F(T)$. Thus, $F(T)$ is closed.

Next we show that $F(T)$ is convex. Let $x, y \in F(T)$, put $z = tx + (1-t)y$ for any $t \in (0,1)$, we prove that $z \in F(T)$. Indeed, let $\{w_n\}$ be a sequence generated by

$w_1 \in Tz, w_2 \in Tw_1 \subset T^2z, \ldots, w_n \in Tw_{n-1} \subset T^nz, \ldots$. By the convexity of $||.||^2$ and letting $\mu_n := t[u_n\zeta(G(x, Jz)) + v_n] + (1-t)[u_n\zeta(G(y, Jz)) + v_n]$, we have

$$\begin{aligned}
G(z, Jw_n) &= ||z||^2 - 2\langle z, Jw_n\rangle + ||w_n||^2 + 2\rho f(z) \\
&= ||z||^2 - 2t\langle x, Jw_n\rangle - 2(1-t)\langle y, Jw_n\rangle \\
&+ t||w_n||^2 + (1-t)||w_n||^2 + 2\rho f(z) \\
&= ||z||^2 + tG(x, Jw_n) + (1-t)G(y, Jw_n) + 2\rho f(z) \\
&- t||x||^2 - (1-t)||y||^2 - 2t\rho f(x) - 2(1-t)\rho f(y) \\
&\leq ||z||^2 + t[G(x, Jz) + u_n\zeta(G(x, Jz)) + v_n]
\end{aligned}$$

$$\begin{aligned}
&+ (1-t)[G(y, Jz) + u_n\zeta(G(y, Jz)) + v_n] + 2\rho f(z) \\
&- t||x||^2 - (1-t)||y||^2 - 2t\rho f(x) - 2(1-t)\rho f(y) \\
&= ||z||^2 - 2\langle z, Jz\rangle + ||z||^2 + 2\rho f(z) + \mu_n \\
&= G(z, Jz) + \mu_n.
\end{aligned}$$

Since $G(z, Jz) \leq G(z, Jw_n)$ and $\mu_n \to 0$ as $n \to \infty$, we have $G(z, Jw_n) \to G(z, Jz)$, which implies that $\phi(z, w_n) \to 0$ as $n \to \infty$. By Lemma 3.6.8 we

have $w_n \to z$ (as $n \to \infty$). Since $T$ is closed, we have $z \in Tz$ that is $z \in F(T)$. Therefore $F(T)$ is convex. □

**Theorem 4.3.2** *Let $E$ be a real uniformly convex and uniformly smooth Banach space, and $C$ a nonempty closed and convex subset of $E$. Let $T_i : C \to N(C)$, $i \in \mathbb{N}$ be an infinite family of uniformly $L_i-$Lipschitzian, closed and total quasi-$\phi$-asymptotically nonexpansive multi-valued mappings. Let $\{u_{ni}\}$ and $\{v_{ni}\}$ be nonnegative real sequences of strictly increasing continuous functions $\zeta_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that $u_{ni} \to 0, v_{ni} \to 0$ as $n \to \infty$, and $\zeta_i(0) = 0$, $i \in \mathbb{N}$. Let $F = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$, be bounded and $f : E \to \mathbb{R}$ be convex and lower semi continuous mapping with $C \subset int(D(f))$. Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence iteratively generated by $x_0 \in C, C_0 = C$,*

$$\begin{cases} z_n = J^{-1}(\alpha_{n0} J x_n + \sum_{i=1}^{\infty} \alpha_{ni} J w_n^{(i)}); \\ y_n = J^{-1}(\beta_n J z_n + (1-\beta_n) J x_n); \\ C_{n+1} = \{v \in C_n : G(v, Jy_n) \leq G(v, Jx_n) + \sigma_n\}; \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \ n \geq 0, \end{cases} \tag{4.3.1}$$

*where $\{\alpha_{ni}\}_{n=1}^{\infty}$ are sequences of positive real numbers, $w_n^{(i)} \in T_i^n x_n$, $i \in \mathbb{N}$, $\sigma_n := \sum_{i=1}^{\infty} \alpha_{ni} \Big( v_{ni} \sup_{p \in F} \zeta_i(G(p, Jx_n)) + u_{ni} \Big)$, and*

*(i) $\sum_{i=0}^{\infty} \alpha_{ni} = 1$; $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0 \ \forall i \in \mathbb{N}$;*

*(ii) $0 < \beta < \beta_n \leq 1$ for some $\beta \in (0,1)$.*

*Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\Pi_F^f x_0$.*

**Proof.** We first show that $C_n, \forall n \geq 0$ is closed and convex. It is obvious that $C_0 = C$ is closed and convex. Thus, we only need to show that $C_n$ is closed and convex for each $n \geq 1$. Since $G(v, Jy_n) \leq G(v, Jx_n) + \sigma_n$ is equivalent to $2\langle v, Jx_n - Jy_n \rangle \leq ||Jy_n||^2 - ||Jx_n||^2 + \sigma_n$, $n \in \mathbb{N}$, we obtain that $C_{n+1}$ is closed and convex for all $n \geq 0$.

We now show that $\{x_n\}$ is bounded and the limit $\lim_{n\to\infty} G(x_n, Jx_0)$ exists. Since $f : E \to \mathbb{R}$ is convex and lower semi-continuous, by Lemma 3.6.5, there exists $u^* \in E^*$ and $\alpha \in \mathbb{R}$ such that.

$$f(y) \geq \langle y, u^* \rangle + \alpha, \ \forall y \in E.$$

It follows that

$$\begin{aligned} G(x_n, Jx_0) &= ||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho f(x_n) \\ &\geq ||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho\langle x_n, u^*\rangle + 2\rho\alpha \\ &= ||x_n||^2 - 2\langle x_n, Jx_0 - \rho u^*\rangle + ||x_0||^2 + 2\rho\alpha \\ &\geq ||x_n||^2 - 2||x_n||||Jx_0 - \rho u^*|| + ||x_0||^2 + 2\rho\alpha \\ &= (||x_n|| - ||Jx_0 - \rho u^*||)^2 + ||x_0||^2 - ||Jx_0 - \rho u^*||^2 + 2\rho\alpha \end{aligned} \quad (4.3.2)$$

Since $x_n = \Pi^f_{C_n} x_0$, it follows from (4.3.2) that

$$G(u, Jx_0) \geq G(x_n, Jx_0) \geq (||x_n|| - ||Jx_0 - \rho u^*||)^2 + ||x_0||^2 - ||Jx_0 - \rho u^*||^2 + 2\rho\alpha$$

$\forall n \geq 0$ and for each $u \in F$. This implies that $\{x_n\}_{n=0}^{\infty}$ is bounded and so is $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$. By the construction of $C_n$, we have that $C_m \subset C_n$ and $x_n = \Pi^f_{C_n} x_0$ for any positive integer $m \geq n$. Then from Lemma 3.6.7, we obtain

$$\phi(x_m, x_n) + G(x_n, Jx_0) \leq G(x_m, Jx_0). \quad (4.3.3)$$

In particular,

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0)$$

In view of (3.6.1), we have

$$G(x_{n+1}, Jx_0) - G(x_n, Jx_0) \geq \phi(x_{n+1}, x_n) \geq (||x_{n+1}|| - ||x_n||)^2 \geq 0,$$

and so $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$ is nondecreasing. It follows that the limit $\lim_{n\to\infty} G(x_n, Jx_0)$ exists.

We now show that $F \subset C_n, \forall n \geq 0$. Since $\{x_n\}$ is bounded, so is $\{G(u, Jx_n)\}$ for any fixed $u \in C$. Let $B_r \subset \mathbb{R}$ be some ball of positive radius $r$ satisfying $\{G(u, Jx_n)\} \subset B_r$ for every $u \in F$. Also let $B$ be a set defined by $B := \{t \in B_r : t > 0\}$ and define a map $\zeta : \bar{B} \to \mathbb{R}^+$ by $\zeta(t) = \sup_{i\geq 1} \zeta_i(t)$.

For $n = 0$, we clearly have $F \subset C = C_0$. Now let $u \in F$, then we have

$$\begin{aligned}
G(u, Jy_n) &= G(u, \beta_n Jz_n + (1-\beta_n)Jx_n) \\
&= ||u||^2 - 2\beta_n\langle u, Jz_n\rangle - 2(1-\beta_n)\langle u, Jx_n\rangle \\
&+ ||\beta_n Jz_n + (1-\beta_n)Jx_n||^2 + 2\rho f(u) \\
&\leq ||u||^2 - 2\beta_n\langle u, Jz_n\rangle - 2(1-\beta_n)\langle u, Jx_n\rangle \\
&+ \beta_n||Jz_n||^2 + (1-\beta_n)||Jx_n||^2 + 2\rho f(u) \\
&= \beta_n G(u, Jz_n) + (1-\beta_n)G(u, Jx_n) \\
&= (1-\beta_n)G(u, Jx_n) + \beta_n G(u, \alpha_{n0}Jx_n + \sum_{i=1}^{\infty} \alpha_{ni} Jw_n^{(i)}) \\
&= (1-\beta_n)G(u, Jx_n) + \beta_n[||u||^2 - 2\langle u, \alpha_{n0}Jx_n + \sum_{i=1}^{\infty} \alpha_{ni} Jw_n^{(i)}\rangle \\
&+ ||\alpha_{n0}Jx_n + \sum_{i=1}^{\infty} \alpha_{ni} Jw_n^{(i)}||^2 + 2\rho f(u)] \\
&\leq (1-\beta_n)G(u, Jx_n) + \beta_n\alpha_{n0}G(u, Jx_n) + \beta_n \sum_{i=1}^{\infty} G(u, Jw_n^{(i)}) \\
&- \beta_n\alpha_{n0}\alpha_{ni} g(||Jx_n - Jw_n^{(i)}||) \\
&\leq (1-\beta_n)G(u, Jx_n) + \beta_n\alpha_{n0}G(u, Jx_n) \\
&+ \beta_n \sum_{i=1}^{\infty} \alpha_{ni}[G(u, Jx_n) + v_{ni}\zeta(G(u, Jx_n)) + u_{ni}]
\end{aligned}$$

$$
\begin{aligned}
&- \quad \beta_n\alpha_{n0}\alpha_{ni}g(||Jx_n - Jw_n^{(i)}||) \\
&\leq \quad G(u, Jx_n) + \beta_n \sum_{i=1}^{\infty} \alpha_{ni}[v_{ni}\sup_{u\in F}\zeta(G(u, Jx_n)) + u_{ni}] \\
&- \quad \beta_n\alpha_{n0}\alpha_{ni}g(||Jx_n - Jw_n^{(i)}||) \\
&= \quad G(u, Jx_n) + \sigma_n - \beta_n\alpha_{n0}\alpha_{ni}g(||Jx_n - Jw_n^{(i)}||) \qquad (4.3.4)
\end{aligned}
$$

Therefore

$$G(u, Jy_n) \leq G(u, Jx_n) + \sigma_n, \ \forall \ u \in F. \qquad (4.3.5)$$

i.e., $u \in C_{n+1}$ and so $F \subset C_{n+1}$ for all $n \geq 0$. Since $C_{n+1}$ is closed and convex and $F \subset C_n, \forall n \geq 0$, it follows that $\Pi^f_{C_{n+1}}x_0$ is well defined for all $n \geq 0$.
By the assumptions on $\{u_{ni}\}$ and $\{v_{ni}\}$ for each $i = 1, 2, ...$ we have

$$\sigma_n = \sum_{i=1}^{\infty} \alpha_{ni}\Big(v_{ni}\sup_{p\in F}\zeta(G(p, Jx_n)) + u_{ni}\Big) \to 0 \ (n \to \infty).$$

Now,(4.3.3) implies that

$$\phi(x_m, x_n) \leq G(x_m, Jx_0) - G(x_n, Jx_0), \qquad (4.3.6)$$

taking the limit as $m, n \to \infty$, we obtain

$$\lim_{n\to\infty} \phi(x_m, x_n) = 0.$$

It then follows from Lemma 3.6.8 that $||x_m - x_n|| \to 0$ as $m, n \to \infty$. Hence $\{x_n\}_{n=1}^{\infty}$ is Cauchy. As $E$ is a Banach space and $C$ is closed, then there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Now since $\phi(x_m, x_n) \to 0$ as $m, n \to \infty$ we have in particular

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$$

and this further implies that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{4.3.7}$$

By the fact that $x_{n+1} \in C_{n+1} \subset C_n$, it follows that

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \sigma_n,$$

hence

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

By Lemma 3.6.8 we obtain

$$\lim_{n\to\infty} ||x_{n+1} - y_n|| = 0.$$

So

$$||x_n - y_n|| \leq ||x_{n+1} - x_n|| + ||x_{n+1} - y_n||,$$

and this implies

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{4.3.8}$$

Since $J$ is uniformly norm-to-norm continuous on bounded sets we also have

$$\lim_{n\to\infty} ||Jx_n - Jy_n|| = 0. \tag{4.3.9}$$

As $0 < \beta \leq \beta_n < 1$, then

$$||Jz_n - Jx_n|| = \frac{1}{\beta_n}||Jx_n - Jy_n|| \to 0, \quad \text{as} \quad n \to \infty.$$

Since $J^{-1}$ is uniformly norm-to-norm continuous on bounded sets we obtain

$$\lim_{n\to\infty} ||z_n - x_n|| = 0,$$

so that $z_n \to p$ as $n \to \infty$. From (4.3.4), for any $u \in F$ and $w_n^{(i)} \in T_i^n x_n,\ i = 1, 2, ...,$ we have

$$G(u, Jy_n) \leq G(u, Jx_n) + \sigma_n - \beta_n \alpha_{n0} \alpha_{ni} g(||Jx_n - Jw_n^{(i)}||),$$

it follows that,

$$\beta_n \alpha_{n0} \alpha_{ni} g(||Jx_n - Jw_n^{(i)}||) \leq G(u, Jx_n) - G(u, Jy_n) + \sigma_n,$$

but

$$\begin{aligned} G(u, Jx_n) - G(u, Jy_n) &= ||x_n||^2 - ||y_n||^2 - 2\langle u, Jx_n - Jy_n\rangle \\ &\leq \Big|||x_n||^2 - ||y_n||^2\Big| + 2\Big|\langle u, Jx_n - Jy_n\rangle\Big| \\ &\leq \Big|||x_n|| - ||y_n||\Big|(||x_n|| + ||y_n||) + 2||u||||Jx_n - Jy_n|| \\ &\leq ||x_n - y_n||(||x_n|| + ||y_n||) + 2||u||||Jx_n - Jy_n|| \end{aligned}$$

from this, (4.3.8) and (4.3.9), we obtain $G(u, Jx_n) - G(u, Jy_n) \to 0$ as $n \to \infty$, using conditions (i) and (ii) and it follows that

$$\lim_{n\to\infty} g(||Jx_n - Jw_n^{(i)}||) = 0.$$

By property of $g$, we have

$$\lim_{n\to\infty} ||Jx_n - Jw_n^{(i)}|| = 0. \tag{4.3.10}$$

Since $J^{-1}$ is uniformly norm-to-norm continuous on bounded sets we obtain

$$\lim_{n\to\infty} ||x_n - w_n^{(i)}|| = 0. \tag{4.3.11}$$

As $w_n^{(i)} \in T_i^n x_n$ it implies that $T_i w_n^{(i)} \subset T_i^{n+1} x_n$, now let $s_{n+1}^{(i)} \in T_i w_n^{(i)}$ which implies that $s_{n+1}^{(i)} \in T_i^{n+1} x_n$. Then, since $T_i$ is uniformly $L_i$-Lipschitzian, $i \in \mathbb{N}$,

we have

$$\begin{aligned} ||s_{n+1}^{(i)} - w_n^{(i)}|| &\leq ||s_{n+1}^{(i)} - w_{n+1}^{(i)}|| + ||w_{n+1}^{(i)} - x_{n+1}|| + ||x_{n+1} - x_n|| + ||x_n - w_n^{(i)}|| \\ &\leq (L_i + 1)||x_{n+1} - x_n|| + ||w_{n+1}^{(i)} - x_{n+1}|| + ||x_n - w_n^{(i)}||. \end{aligned}$$

From this together with (4.3.7) and (4.3.11), we obtain $\lim_{n\to\infty} ||s_{n+1}^{(i)} - w_n^{(i)}|| = 0$ and so $s_{n+1}^{(i)} \to p$ as $n \to \infty$ for each $i \in \mathbb{N}$. In view of the closure of $T_i$, we have $p \in T_i p$, for each $i \in \mathbb{N}$, therefore $p \in F$.

Finally, we show that $p = \Pi_F^f x_0$. Since $F = \cap_{i=1}^{\infty} F(T_i)$ is closed and convex, by Lemma 3.6.6 $\Pi_F^f$ is single valued and so if we denote $v = \Pi_F^f x_0$, as $x_n = \Pi_{C_n}^f x_0$ and $v \in F \in C_n$, we have

$$G(x_n, Jx_0) \leq G(v, Jx_0), \ \forall \ n \geq 0.$$

We know that $G(\xi, J\varphi)$ is convex and lower semi-continuous with respect to $\xi$ when $\varphi$ is fixed and this implies that

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(v, Jx_0).$$

From the definition of $\Pi_F^f x_0$ and the fact that $p \in F$, we see that $p = v$. This completes the proof. □

The following important Corollaries follow from Theorem 4.3.2 and Lemma 4.3.1.

**Corollary 4.3.3** *Let $E$ be a real uniformly convex and uniformly smooth Banach space, and $C$ a nonempty closed and convex subset of $E$. Let $T_i : C \to N(C)$, $i = 1, 2, 3, ...$ be an infinite family of uniformly $L_i-$Lipschitzian, closed and relatively nonexpansive multi-valued mappings. Let $F = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $f : E \to \mathbb{R}$ be convex and lower semi continuous mapping with $C \subset$*

*int*$(D(f))$. *Suppose* $\{x_n\}_{n=0}^{\infty}$ *is iteratively generated by* $x_0 \in C, C_0 = C$,

$$\begin{cases} z_n = J^{-1}(\alpha_{n0}Jx_n + \sum_{i=1}^{\infty}\alpha_{in}Jw_n^{(i)}); \\ y_n = J^{-1}(\beta_n Jz_n + (1-\beta_n)Jx_n); \\ C_{n+1} = \{v \in C_n : G(v, Jy_n) \leq G(v, Jx_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \ n \geq 0, \end{cases} \tag{4.3.12}$$

*where* $\{\alpha_{ni}\}_{n=1}^{\infty}$ *are sequences of positive real numbers,* $w_n^{(i)} \in T_i x_n, \ i \in \mathbb{N}$ *and*

*(i)* $\sum_{i=0}^{\infty}\alpha_{ni} = 1$; $\liminf_{n\to\infty}\alpha_{n0}\alpha_{ni} > 0 \ \forall i \in \mathbb{N}$;

*(ii)* $0 < \beta < \beta_n \leq 1$ *for some* $\beta \in (0,1)$.

*Then,* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $\Pi_F^f x_0$.

**Corollary 4.3.4** *Let* $E$ *be a real uniformly convex and uniformly smooth Banach space, and* $C$ *be a nonempty closed and convex subset of* $E$. *Let* $T_i : C \to N(C)$, $i = 1, 2, 3, ...$ *be an infinite family of uniformly* $L_i-$*Lipschitzian, closed and weak relatively nonexpansive multi-valued mappings. Let* $F = \cap_{i=1}^{\infty}F(T_i) \neq \emptyset$ *and* $f : E \to \mathbb{R}$ *be convex and lower semi continuous mapping with* $C \subset$ *int*$(D(f))$. *Suppose* $\{x_n\}_{n=0}^{\infty}$ *is iteratively generated by* $x_0 \in C, C_0 = C$,

$$\begin{cases} z_n = J^{-1}(\alpha_{n0}Jx_n + \sum_{i=1}^{\infty}\alpha_{in}Jw_n^{(i)}); \\ y_n = J^{-1}(\beta_n Jz_n + (1-\beta_n)Jx_n); \\ C_{n+1} = \{v \in C_n : G(v, Jy_n) \leq G(v, Jx_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \ n \geq 0, \end{cases} \tag{4.3.13}$$

*where* $\{\alpha_{ni}\}_{n=1}^{\infty}$ *are sequences of positive real numbers,* $w_n^{(i)} \in T_i x_n, \ i \in \mathbb{N}$ *and*

*(i)* $\sum_{i=0}^{\infty}\alpha_{ni} = 1$; $\liminf_{n\to\infty}\alpha_{n0}\alpha_{ni} > 0 \ \forall i \in \mathbb{N}$;

*(ii)* $0 < \beta < \beta_n \leq 1$ *for some* $\beta \in (0,1)$.

*Then,* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $\Pi_F^f x_0$.

**Corollary 4.3.5** *Let $E$ be a real uniformly convex and uniformly smooth Banach space, and $C$ be a nonempty closed and convex subset of $E$. Let $T_i : C \to N(C)$, $i = 1, 2, 3, ...$ be an infinite family of uniformly $L_i-$Lipschitzian, closed and quasi-$\phi$-asymptotically nonexpansive multi-valued mappings with real sequences $\{k_{ni}\}$ $i = 1, 2, ...$ such that $k_{ni} \to 1$, as $n \to \infty$, $\forall i \in \mathbb{N}$. Let $F = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$, bounded and $f : E \to \mathbb{R}$ be convex and lower semi continuous mapping with $C \subset int(D(f))$. Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence iteratively generated by $x_0 \in C, C_0 = C$.*

$$\begin{cases} z_n = J^{-1}(\alpha_{n0} J x_n + \sum_{i=1}^{\infty} \alpha_{in} J w_n^{(i)}); \\ y_n = J^{-1}(\beta_n J z_n + (1 - \beta_n) J x_n); \\ C_{n+1} = \{v \in C_n : G(v, Jy_n) \leq G(v, Jx_n) + \sigma_n\}; \\ x_{n+1} = \Pi_{C_{n+1}}^{f} x_0, \ n \geq 0, \end{cases} \tag{4.3.14}$$

*where $\{\alpha_{ni}\}_{n=1}^{\infty}$ are sequences of positive real numbers, $w_n^{(i)} \in T_i^n x_n$, $i \in \mathbb{N}$, $\sigma_n := \sum_{i=1}^{\infty} \alpha_{ni} \Big( (k_{ni} - 1) \sup_{p \in F} G(p, Jx_n) \Big)$, and*

*(i) $\sum_{i=0}^{\infty} \alpha_{ni} = 1$; $\liminf_{n \to \infty} \alpha_{n0} \alpha_{ni} > 0$ $\forall i \in \mathbb{N}$;*

*(ii) $0 < \beta < \beta_n \leq 1$ for some $\beta \in (0, 1)$.*

*Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\Pi_F^f x_0$.*

### 4.3.1 Concluding remarks

In conclusion, the result in this section extends the result of Zhang et al. (2013), from weak relatively multi-valued nonexpansive mapping to infinite family of total quasi-$\phi$-asymptotically nonexpansive multi-valued mappings using $f$-projection.

## 4.4 Bregman Strictly Pseudocontractive Mapping and Equilibrium Problems

In this section, we introduce new recursion formula and prove strong convergence theorem for finite family of closed quasi-Bregman strictly pseudocontractive mapping and system of equilibrium problems in a real reflexive Banach space.

The next Lemmas will be useful in the proof of our main result.

**Lemma 4.4.1** Let $f : E \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and bounded on subsets of $E$, let $C$ be a nonempty, closed and convex subset of $E$ and let $T : C \to C$ be a quasi-Bregman strict pseudo contractive mapping with respect to $f$. Then, for any $x \in C, p \in F(T)$ and $k \in [0, 1)$ the following hold

$$D_f(x, Tx) \leq \frac{1}{1-k}\langle x - p, \nabla f(x) - \nabla f(Tx)\rangle. \tag{4.4.1}$$

**Proof.** Let $x \in C,\ p \in F(T)$ and $k \in [0, 1)$, by definition of $T$, we have

$$D_f(p, Tx) \leq D_f(p, x) + kD_f(x, Tx)$$

from (3.7.3), we obtain that

$$D_f(p, x) + D_f(x, Tx) + \langle x - p\nabla f(x) - \nabla f(Tx)\rangle \leq D_f(p, x) + kD_f(x, Tx)$$

which implies

$$D_f(x, Tx) \leq \frac{1}{1-k}\langle x - p, \nabla f(x) - \nabla f(Tx)\rangle.$$

This completes the proof. □

**Lemma 4.4.2** *Let $E$ be reflexive real Banach space and let $f : E \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of $E$, let $C$ be a nonempty, closed and convex subset of $E$ and let $T : C \to C$ be a quasi-Bregman strict pseudo contractive mapping with respect to $f$. Then, $F(T)$ is closed and convex.*

**Proof.** Let $F(T)$ be nonempty set. First we show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to z$ as $n \to \infty$, we need to show that $z \in F(T)$. From Lemma 4.4.1, we obtain

$$D_f(z, Tz) \leq \frac{1}{1-k}\langle z - x_n, \nabla f(z) - \nabla f(Tz)\rangle. \tag{4.4.2}$$

From (4.4.2), we have $D_f(z, Tz) \leq 0$ and from Bauschke et al. (2001), it follows that $Tz = z$. Therefore $F(T)$ is closed.

Next, we show that $F(T)$ is convex. Let $z_1, z_2 \in F(T)$, for any $t \in (0, 1)$, putting $z = tz_1 + (1-t)z_2$, we need to show that $z \in F(T)$. From Lemma 4.4.1, we obtain respectively

$$D_f(z, Tz) \leq \frac{1}{1-k}\langle z - z_1, \nabla f(z) - \nabla f(Tz)\rangle \tag{4.4.3}$$

and

$$D_f(z, Tz) \leq \frac{1}{1-k}\langle z - z_2, \nabla f(z) - \nabla f(Tz)\rangle \tag{4.4.4}$$

Multiplying (4.4.3) by $t$ and (4.4.4) by $(1-t)$ and adding the results, we obtain

$$D_f(z, Tz) \leq \frac{1}{1-k}\langle z - z, \nabla f(z) - \nabla f(Tz)\rangle \tag{4.4.5}$$

which implies, $D_f(z, Tz) \leq 0$, from Bauschke et al. (2001), it follows that $Tz = z$. Therefore $F(T)$ is also convex. This complete the proof. □ We now

prove the following theorem.

**Theorem 4.4.3** *Let $C$ be a nonempty, closed and convex subset of a real reflexive Banach space $E$ and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of $E$. For each $k = 1, 2, \cdots, m$, let $g_k$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\{T_{i=1}^N\}$ be a finite family of $L_i-$Lipschitzian, $i = 1, 2, 3, \cdots, N$ closed and quasi-Bregman strictly pseudocontractive self mapping of $C$ such that $F := \left(\cap_{k=1}^m EP(g_k)\right) \bigcap \left(\cap_{i=1}^N F(T_i)\right) \neq \emptyset$. Let $\{x_n\}_{n=1}^\infty$ be a sequence generated by $x_1 = x \in C, C_1 = C$ and*

$$\begin{cases} x_1 \in C \\ y_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T_n x_n) \\ u_{j,n} = Res_{g_j}^f y_n, \ j = 1, 2, 3, \cdots, m \\ w_n = \nabla f^*(\sum_{j=1}^m \beta_{j,n} \nabla f(u_{j,n})) \\ C_{n+1} = \{w \in C_n : D_f(x_n, w_n) \leq \frac{1}{1-k}\langle x_n - w, \nabla f(x_n) \\ \quad - \nabla f(T_n x_n)\rangle + \langle x_n - w, \nabla f(T_n x_n) - \nabla f(w_n)\rangle\} \\ x_{n+1} = P_{C_{n+1}}^f(x), \ n \in \mathbb{N}, \end{cases} \tag{4.4.6}$$

*where $T_n = T_{n(mod\ N)}$, and $k \in [0, 1)$, for each $i = 1, 2, \cdots, N$, $T_i$ is uniformly continuous. Suppose $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_{j,n}\}_{n=1}^\infty, j = 1, 2, \cdots, m$ are sequences in (0,1) such that (i) $\liminf_{n\to\infty}(1-\alpha_n) > 0$, (ii) $\sum_{j=1}^m \beta_{j,n} = 1, n \geq 1$. Then $\{x_n\}_{n=1}^\infty$ converges strongly to $P_F^f(x)$, where $P_F^f$ is the Bregman projection of $E$ onto $F$.*

**Proof.** The proof is divided into six steps.

**Step I.** Show that $F = \left(\cap_{j=1}^m EP(g_j)\right) \bigcap \left(\cap_{i=1}^N F(T_i)\right)$ is closed and convex. From Lemma 4.4.2, $\cap_{i=1}^N F(T_i)$ is closed and convex and from (iv) of Lemma 3.7.9 $\cap_{j=1}^m EP(g_j)$ is closed and convex. So, $F = \left(\cap_{j=1}^m EP(g_j)\right) \bigcap \left(\cap_{i=1}^N F(T_i)\right)$ is closed and convex.

**Step II.** Show that $C_n$ is closed and convex for all $n \geq 1$. For $n = 1, C_1 = C$ is closed and convex. Assume that $C_h$ is closed and convex for some $h > 1$.

For $w \in C_{h+1}$, one obtains

$$\begin{aligned} D_f(x_h, w_h) &\leq \frac{1}{1-k}\langle x_h - w, \nabla f(x_h) - \nabla f(T_h x_h)\rangle \\ &\quad + \langle x_h - w, \nabla f(T_h x_h) - \nabla f(w_h)\rangle. \end{aligned}$$

Using the fact that $\langle \nabla f(x_h) - \nabla f(T_h x_h), \cdot\rangle$ and $\langle \nabla f(T_h x_h) - \nabla f(w_h), \cdot\rangle$ are continuous and linear in $E$, then for $h \geq 1$, $C_{h+1}$ is closed and convex.

**Step III.** Show that $F \subset C_n$ for every $n \geq 1$. Note that $F \subset C_1 = C$. Suppose $F \subset C_h$, then for $h \geq 1$, and for all $w \in F \subset C_h$, since $u_{j,h} = Res^f_{g_j}(y_h)$ for each $j = 1, 2, \cdots, m$, from (3.7.4) and Lemma 3.7.9, we have

$$\begin{aligned} D_f(w, w_h) &= D_f\Big(w, \nabla f^*(\sum_{j=1}^{m} \beta_{j,n} \nabla f(u_{j,n}))\Big) \\ &\leq \sum_{j=1}^{m} \beta_{jh} D_f(w, u_{jh}) \\ &\leq \sum_{j=1}^{m} \beta_{jh} D_f(w, y_h) \\ &= D_f(w, y_h) \end{aligned} \tag{4.4.7}$$

also from (3.7.4) and (4.4.1), we obtain

$$\begin{aligned} D_f(w, y_h) &= D_f(w, \nabla f^*(\alpha_h \nabla f(x_h) + (1-\alpha_h)\nabla f(T_h x_h))) \\ &\leq \alpha_h D_f(w, x_h) + (1-\alpha_h) D_f(w, T_h x_h) \\ &\leq \alpha_h D_f(w, x_h) + (1-\alpha_h)[D_f(w, x_h) + k D_f(x_h, T_h x_h) \\ &\leq D_f(w, x_h) + k D_f(x_h, T_h x_h) \\ &\leq D_f(w, x_h) + \frac{k}{1-k}\langle x_h - w, \nabla f(x_h) - \nabla f(T_h x_h)\rangle. \end{aligned} \tag{4.4.8}$$

But, from (3.7.3)

$$D_f(w, w_h) = D_f(w, x_h) + D_f(x_h, w_h) + \langle w - x_h, \nabla f(x_h) - \nabla f(w_h)\rangle. \tag{4.4.9}$$

From (4.4.7), (4.4.8) and (4.4.9), we obtain

$$\begin{aligned} D_f(x_h, w_h) &\leq \frac{k}{1-k}\langle x_h - w, \nabla f(x_h) - \nabla f(T_h x_h)\rangle \\ &\quad +\langle x_h - w, \nabla f(x_h) - \nabla f(w_h)\rangle \\ &= \frac{k}{1-k}\langle x_h - w, \nabla f(x_h) - \nabla f(T_h x_h)\rangle \\ &\quad +\langle x_h - w, \nabla f(x_h) - \nabla f(T_h x_h)\rangle \\ &\quad +\langle x_h - w, \nabla f(T_h x_h) - \nabla f(w_h)\rangle \\ &= \Big(\frac{k}{1-k} + 1\Big)\langle x_h - w, \nabla f(x_h) - \nabla f(T_h x_h)\rangle \\ &\quad +\langle x_h - w, \nabla f(T_h x_h) - \nabla f(w_h)\rangle \\ &= \frac{1}{1-k}\langle x_h - w, \nabla f(x_h) - \nabla f(T_h x_h)\rangle \\ &\quad +\langle x_h - w, \nabla f(T_h x_h) - \nabla f(w_h)\rangle \end{aligned} \tag{4.4.10}$$

This shows that $w \in C_{h+1}$ which implies $F \subset C_n$ for every $n \geq 1$.

**Step IV.** Show that $\lim_{n\to\infty} D_f(x_n, x)$ exists. From (4.4.6), $x_n = P^f_{C_n} x$, which from (a) of Lemma 3.7.3 implies

$$\langle y - x_n, \nabla f(x) - \nabla f(x_n)\rangle \leq 0, \ \forall y \in C_n.$$

Since $F \subset C_n$, we have

$$\langle w - x_n, \nabla f(x) - \nabla f(x_n)\rangle \leq 0, \ \forall \ w \in F. \tag{4.4.11}$$

From (b) of Lemma 3.7.3 we have

$$\begin{aligned} D_f(x_n, x) &= D_f(P^f_{C_n} x, x) \leq D_f(w, x) - D_f(w, P^f_{C_n} x) \\ &\leq D_f(w, x), \ \forall n \geq 1, w \in F. \end{aligned} \tag{4.4.12}$$

This implies that $\{D_f(x_n, x)\}$ is bounded, and from Lemma 3.7.6, $\{x_n\}$ is bounded. By the construction of $C_n$, we have $x_m \in C_m \subset C_n$, and $x_n = P^f_{C_n} x$,

for any positive integer $m \geq n$. Then, we obtain

$$\begin{aligned} D_f(x_m, x_n) &= D_f(x_m, P^f_{C_n} x) \\ &\leq D_f(x_m, x) - D_f(P^f_{C_n} x, x) \\ &= D_f(x_m, x) - D_f(x_n, x). \end{aligned} \tag{4.4.13}$$

In particular,

$$D_f(x_{n+1}, x_n) \leq D_f(x_{n+1}, x) - D_f(x_n, x).$$

Since $x_n = P^f_{C_n} x$ and $x_{n+1} = P^f_{C_{n+1}} x \in C_{n+1} \subset C_n$, we obtain $D_f(x_n, x) \leq D_f(x_{n+1}, x)$ $\forall n \geq 1$. This show that $\{D_f(x_n, x)\}$ is nondecreasing and hence the limit $\lim_{n\to\infty} D_f(x_n, x)$ exists. Thus from (4.4.13), taking the limit as $m, n \to \infty$, we obtain $\lim_{n\to\infty} D_f(x_m, x_n) = 0$. Since $f$ is totally convex on bounded subsets of $E$, $f$ is sequentially consistent (see, Censor and Lent (1981)). It follows that $||x_m - x_n|| \to 0$ as $m, n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence in $C$. As $\{x_n\}$ is Cauchy in a complete space $E$, there exists $p \in E$ such that $x_n \to p$ as $n \to \infty$. Clearly $p \in C$.

Since $D_f(x_m, x_n) \to 0$, as $m, n \to \infty$, we have in particular,

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0 \tag{4.4.14}$$

and this further implies that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{4.4.15}$$

**Step V.** Next we show that $x_n \to p \in F$.

Since $x_{n+1} = P^f_{C_{n+1}} x \in C_{n+1}$, we have from (4.4.6),

$$D_f(x_n, w_n) \leq \frac{1}{1-k} \langle x_n - x_{n+1}, \nabla f(x_n) - \nabla f(T_n x_n) \rangle \tag{4.4.16}$$

$$+ \langle x_n - x_{n+1}, \nabla f(T_n x_n) - \nabla f(w_n) \rangle \tag{4.4.17}$$

which implies that $\lim_{n\to\infty} D_f(x_n, w_n) = 0$. Since $f$ is totally convex on bounded subsets of $E$, $f$ is sequentially consistent (see, Censor and Lent (1981)). It follows that

$$\lim_{n\to\infty} ||x_n - w_n|| = 0. \tag{4.4.18}$$

From (4.4.15) and (4.4.18), we have

$$\lim_{n\to\infty} ||x_{n+1} - w_n|| = 0. \tag{4.4.19}$$

Since $f$ is uniformly Fréchet differentiable, it follows from Lemma 3.7.1 that $\nabla f$ is uniformly continuous and $f$ is uniformly continuous on bounded subsets of $E$. Hence

$$\lim_{n\to\infty} ||\nabla f(x_{n+1}) - \nabla f(w_n)|| = 0. \tag{4.4.20}$$

and

$$\lim_{n\to\infty} |f(x_{n+1}) - f(w_n)| = 0. \tag{4.4.21}$$

Since $x_{n+1} \in C_{n+1}$, it follows from (4.4.7), (4.4.8), that

$$\begin{aligned} &f(x_{n+1}) - f(w_n) - \langle x_{n+} - w_n, \nabla f(w_n)\rangle = D_f(x_{n+1}, w_n) \\ &\leq \quad D_f(x_{n+1}, y_n) \leq D_f(x_{n+1}, x_n) + \frac{k}{1-k}\langle x_n - x_{n+1}, \nabla f(x_n) - \nabla f(T_n x_n)\rangle. \end{aligned}$$

which implies from (4.4.21), (4.4.19), (4.4.14) and (4.4.15) that

$$\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0.$$

From the sequential consistency of $f$, we have

$$\lim_{n\to\infty} ||x_{n+1} - y_n|| = 0. \tag{4.4.22}$$

from (4.4.15) and (4.4.22), we obtain

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{4.4.23}$$

which implies $y_n \to p \in C$, since $x_n \to p \in C$. From the uniform continuity of $\nabla f$, we have

$$\lim_{n\to\infty} ||\nabla f(x_n) - \nabla f(y_n)|| = 0. \tag{4.4.24}$$

From (4.4.6), we have

$$||\nabla f(T_n x_n) - \nabla f(x_n)|| = \frac{1}{1-\alpha_n} ||\nabla f(x_n) - \nabla f(y_n)||$$

which implies from (4.4.24) that

$$\lim_{n\to\infty} ||\nabla f(T_n x_n) - \nabla f(x_n)|| = 0. \tag{4.4.25}$$

Since $f$ is strongly coercive and uniformly convex on bounded subsets of $E$, $f^*$ is uniformly Fréchet differentiable on bounded sets. Moreover, $f^*$ is bounded on bounded sets and from (4.4.25), we obtain

$$\lim_{n\to\infty} ||T_n x_n - x_n|| = 0. \tag{4.4.26}$$

On the other hand, we see that

$$\begin{aligned} ||x_n - T_{n+l}x_n|| &\leq ||x_n - x_{n+l}|| + ||x_{n+l} - T_{n+l}x_{n+l}|| \\ &\quad + ||T_{n+l}x_{n+l} - T_{n+l}x_n|| \\ &\leq (1+L)||x_n - x_{n+l}|| + ||x_{n+l} - T_{n+l}x_{n+l}|| \end{aligned}$$

for all $l \in \{1, 2, \cdots, N\}$, where $L := \sup_{1 \leq i \leq N} L_i$. It follows from (4.4.15) and (4.4.26) that

$$\lim_{n\to\infty} ||x_n - T_{n+l}x_n|| = 0.$$

for all $l \in \{1, 2, \cdots, N\}$. Thus

$$\lim_{n\to\infty} ||x_n - T_l x_n|| = 0. \tag{4.4.27}$$

for all $l \in \{1, 2, \cdots, N\}$. Since $x_n \to p$ as $n \to \infty$, by the closedness of $T_l$ for each $l \in \{1, 2, \cdots, N\}$, we obtain $p \in \cap_{l=1}^{N} F(T_l)$.

Also, since $y_n \to p$ as $n \to \infty$, we have from Lemma 3.7.9, for each $j = 1, 2, \cdots, m$,

$$0 \leq D_f(p, u_{jn}) = D_f(p, Res^f_{g_j} y_n) \leq D_f(p, y_n) \to 0 \text{ as } n \to \infty.$$

Then, we have from Lemma 3.7.4, $\lim_{n\to\infty} ||p - u_{jn}|| = 0$, for each $j = 1, 2, \cdots, m$. Comsequently, we have

$$||u_{jn} - y_n|| \leq ||u_{jn} - p|| + ||p - y_n|| \to 0 \text{ as } n \to \infty. \tag{4.4.28}$$

From the uniform continuity of $\nabla f$, for each $j = 1, 2, \cdots, m$ we have

$$\lim_{n\to\infty} ||\nabla f(u_{jn}) - \nabla f(y_n)|| = 0. \tag{4.4.29}$$

From (3.7.5), we have, for $j = 1, 2, ..., m$

$$g_j(u_{jn}, y) + \langle y - u_{jn}, \nabla f(u_{jn}) - \nabla f(y_n)\rangle \geq 0, \ \forall y \in C.$$

Furthermore, using (A2) in the last inequality, we obtain

$$\langle y - u_{jn}, \nabla f(u_{jn}) - \nabla f(y_n)\rangle \geq g_j(y, u_{jn}) \ \forall y \in C.$$

By (A4), (4.4.29) and $u_{jn} \to p$ as $n \to \infty$, we have

$$g_j(y, p) \leq 0 \ \forall y \in C. \tag{4.4.30}$$

Let $z_t := ty + (1-t)p$ for $t \in (0, 1]$ and $y \in C$. This implies that $z_t \in C$. This yields $g_j(z_t, p) \leq 0$. It follows from (A1) and (A4) that

$$\begin{aligned} 0 &= g_j(z_t, z_t) \leq tg_j(z_t, y) + (1-t)g_j(z_t, p) \\ &\leq tg_j(z_t, y) \end{aligned}$$

and hence

$$0 \leq g_j(z_t, y).$$

From condition (A3), we obtain

$$g_j(p, y) \geq 0, \ \forall y \in C \text{ and } \forall j \in \{1, 2, 3, \cdots, m\}.$$

This implies that $p \in EP(g_j)$, for each $j = 1, 2, \cdots, m$. Thus, $p \in \cap_{j=1}^{m} EP(g_j)$. Hence we have $p \in F = \Big( \cap_{i=1}^{N} F(T_i) \Big) \bigcap \Big( \cap_{j=1}^{m} EP(g_j) \Big)$.

**Step VI.** Finally, we show that $p = P_F^f x$. Setting $n \to \infty$ in (4.4.11), we obtain

$$\langle w - p, \nabla f(x) - \nabla f(p)\rangle \leq 0, \ \forall w \in F.$$

By (a) of Lemma 3.7.3, we have $p = P_F^f x$. □

Next we give an example of a quasi-Bregman strictly peudocontractive mapping which is not quasi$-\phi-$ strictly pseudocontractive mapping, this shows that the former class is a generalization of the latter.

**Example 4.4.1** *Let $E = \mathbb{R}, C = [-1, 0]$ and define $T, f : [-1, 0] \to \mathbb{R}$ by $f(x) = x$ and $Tx = 2x$, for all $x \in [-1, 0]$. We want to show that $T$ is a quasi-Bregman strict pseudo-contractive but not quasi-phi strict pseudo-contractive.*

**Proof.** From the definition it is clear that $f$ is proper, lowersemi-continuous and convex, also $F(T) = \{0\}$. By definition of quasi-Bregman strict pseudo-contractive, we find $k \in [0, 1)$ such that $D_f(p, Tx) \leq D_f(p, x) + kD_f(x, Tx)$ for all $x \in C$ and $p \in F(T)$. Now,

$$\begin{aligned} D(0, Tx) &= f(0) - f(Tx) - \langle 0 - Tx, \nabla f(Tx)\rangle \\ &= 0 - 2x - \langle 0 - 2x, \nabla f(2x)\rangle \\ &= -2x - \langle -2x, 2\rangle \\ &= -2x + 4x = 2x. \end{aligned} \tag{4.4.31}$$

$$\begin{aligned} D(0, x) &= f(0) - f(x) - \langle 0 - x, \nabla f(x)\rangle \\ &= 0 - x - \langle -x, 1\rangle \\ &= -x + x = 0. \end{aligned} \tag{4.4.32}$$

and

$$\begin{aligned} D(x, Tx) &= f(x) - f(Tx) - \langle x - Tx, \nabla f(Tx)\rangle \\ &= x - 2x - \langle x - 2x, 2\rangle \\ &= -x + 2x = x. \end{aligned} \tag{4.4.33}$$

From (4.4.31), (4.4.32) and (4.4.33), we obtain

$$\begin{aligned} D(0,Tx) = 2x &\leq x \\ &\leq 0 + kx \quad \forall\ x \in [-1,0],\ k \in [0,1) \\ &\leq D(0,x) + kD(x,Tx) \quad \forall\ x \in [-1,0],\ k \in [0,1). \end{aligned}$$

Therefore

$$D(0,Tx) \leq D(0,x) + kD(x,Tx) \quad \forall x \in [-1,0],\ k \in [0,1).$$

Hence, $T$ is a quasi-Bregman strict pseudo-contractive map.

Further,

$$\begin{aligned} \phi(0,Tx) &= |0|^2 - 2\langle 0, J(Tx)\rangle + |Tx|^2 \\ &= 0 - 2\langle 0, J(2x)\rangle + |2x|^2 \\ &= 4|x|^2, \end{aligned} \tag{4.4.34}$$

$$\begin{aligned} \phi(0,x) &= |0|^2 - 2\langle 0, J(x)\rangle + |x|^2 \\ &= 0 - 2\langle 0, J(x)\rangle + |x|^2 \\ &= |x|^2 \end{aligned} \tag{4.4.35}$$

and

$$\begin{aligned}\phi(x,Tx) &= |x|^2 - 2\langle x, J(Tx)\rangle + |Tx|^2 \\ &= |x|^2 - 2\langle x, J(2x)\rangle + 4|x|^2 \\ &= |x|^2 - 4\langle x, J(x)\rangle + 4|x|^2 \\ &= |x|^2 - 4|x|^2 + 4|x|^2 \\ &= |x|^2. \end{aligned} \tag{4.4.36}$$

Since $4|x|^2 > |x|^2 + k|x|^2$, for all $k \in [0,1)$ and for all $x \in [-1,0]$, then

$$\phi(0,Tx) \leq \phi(0,x) + k\phi(x,Tx) \ \forall \ x \in [-1,0]$$

cannot hold for any $k \in [0,1)$. Hence, $T$ is not quasi-phi- strict pseudo-contractive map. □

### 4.4.1 Numerical example

In this section we discuss the direct application of Theorem 4.4.3 on a typical example on a real line. Consider the following:

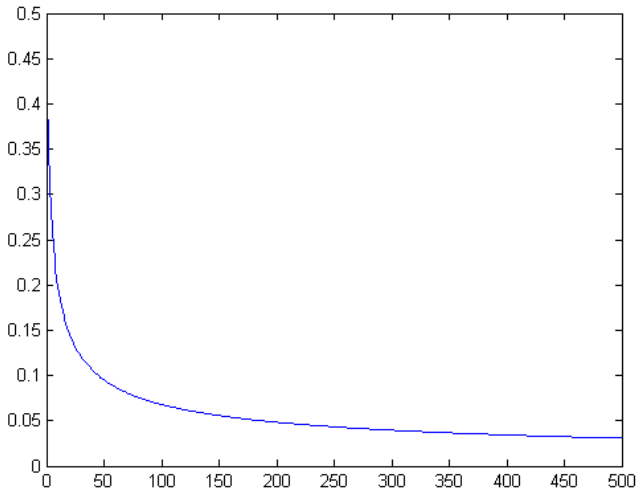
$$\begin{aligned}&\mathbb{E} = \mathbb{R}, C = [-1,\ 1],\ g(z,y) = y^2 + yz - 2z^2, \\ &f(x) = \frac{2}{3}x^2, \nabla f(x) = \frac{4}{3}x, Tx = -2x,\end{aligned}$$

$$\begin{aligned}&f^*(x^*) = \sup\{< x^*, x > -f(x) : x \in \mathbb{E}\}, \\ &f^*(z) = \frac{3}{8}z^2, \nabla f^*(z) = \frac{3}{4}z,\ \alpha_n = \frac{n+1}{4n}, \\ &\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n) = \frac{-(5n-3)}{3n}x_n, \\ &k = 1/2,\ x_1 = 1/2 \in C,\end{aligned}$$

then the scheme can be simplified as

$$
\begin{aligned}
&y_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)),\\
&\therefore y_n = \frac{-(5n-3)}{4n} x_n,\\
&u_n = \mathrm{Res}_g^f y_n = \frac{4}{13} y_n,\\
&w_n = \nabla f^*(\nabla f(u_n)) = u_n,\\
&C_{n+1} = \left\{ w \in C_n : w \leq x_n - \frac{(1-k)(x_n - w_n)^2}{2\left[(1+2k)x_n - (1-k)w_n\right]} \right\},\\
&x_{n+1} = \mathrm{P}^f_{C_{n+1}}(x_1) = x_n - \frac{(1-k)(x_n - w_n)^2}{2\left[(1+2k)x_n - (1-k)w_n\right]}.
\end{aligned}
$$

The values are generated from the above setting as in the following table, and the following graph illustrates the convergence.




### 4.4.2 Concluding remarks

In conclusion, the result in this chapter extends the result of Zhu et al. (2013), from Bregman strongly nonexpansive mapping and equilibrium problem to closed quasi-Bregman strictly pseudo-contraction mapping and system of equilibrium problem.

# CHAPTER FIVE

# SUMMARY, CONCLUSION AND RECOMMENDATIONS

## 5.1 Summary

In this thesis, we have considered four important class of operators, namely, asymptotically nonexpansive mapping, generalized asymptotically quasi-$\phi$-strict pseudo-contraction mapping, total quasi-$\phi$-asymptotically nonexpansive multi-valued mappings using $f$-projection and closed quasi-Bregman strictly pseudo-contraction mapping and system of equilibrium problem. We studied some new iterative methods for approximation of common fixed points of families of these operators. The common fixed point of these families approximated happen to be in some cases solutions of some variational inequality problems and in other cases solutions of system of equilibrium problems.

## 5.2 Conclusion

This work has been able to:

(i) extend the result of Ali (2009), from family of nonexpansive mappings to family of asymptotically nonexpansive mapping in real Banach space whose duality map is weakly sequentially continuous and its norm is uniformly Gâteaux differentiable.

(ii) extend the result of Osilike et al. (2013), from asymptotically nonexpansive mappings in Hilbert space to family of asymptotically nonexpansive mapping in Banach space with uniformly Gâteaux differentiable norm.

(iii) extend the result of Qin et al. (2012), from the class of asymptotically strict quasi-$\phi$-pseudo-contraction in the intermediate sense to family of generalized asymptotically quasi-$\phi$-strict pseudo-contraction mapping.

(iv) we extend the result of Zhang et al. (2013), from weak relatively multi-valued nonexpansive mapping to infinite family of total quasi-$\phi$-asymptotically nonexpansive multi-valued mappings using $f$-projection.

(v) extend the result of Zhang et al. (2013), from weak relatively multi-valued nonexpansive mapping to infinite family of total quasi-$\phi$-asymptotically nonexpansive multi-valued mappings using $f$-projection.

(vi) extend the result of Zhu et al. (2013), from Bregman strongly nonexpansive mapping and equilibrium problem to closed quasi-Bregman strictly pseudo-contraction mapping and system of equilibrium problem.

## 5.3 Recommendations

The increasing interest in the study of fixed point theory have been an important tools for solving problems arising in the fields of linear or nonlinear programming, variational inequalities, complementarity problems, optimization problems which widely applied to physics, structural analysis, management science, economics and so on.

We are recommending researchers to apply the results in this thesis to solve specific problems such as those mentioned above. It will also be of interest to carry out an intensive and extensive study of the result in this thesis with a view of possible improvement.

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