A WEIBULL-NORMAL DISTRIBUTION: ITS PROPERTIES AND APPLICATIONS

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BY

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DECLARATION

I declare that the work in this dissertation titled "AWeibull-Normal Distribution: Its Properties and Applications" has been carried out by me in the Department of Statistics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other institution.

_____ _____

Terna Godfrey IEREN Date

CERTIFICATION

This dissertation titled A WEIBULL-NORMAL DISTRIBUTION: ITS PROPERTIES AND APPLICATIONS by Terna Godfrey IEREN meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This work is dedicated to GodAlmighty for His wisdom, provision and protection over meandI also dedicate this research to my beloved family, my parents Ieren FredrickShima and Felicia AshidooIeren, and to my late sisters, may their souls rest in perfect peace, Amen.

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ABSTRACT

TheNormal distribution is a very important and well known probability distribution for dealing with problems in several areas of life, however there are numerous situations when the assumption of normality is not validated by the data. In this work, we propose a new model calledWeibull-Normal distribution, an extension of the normal distribution by adding two skewness parameters to the normal distribution using the Weibull generator proposed by Bourguignon *et al.*, (2014). This study has derived explicit expressions for some of its basic statistical properties such as moments, moment generating function, the characteristics function, reliability analysis and the distribution of order statistics. The implications of the plots for the survival and hazard functions indicate that the Weibull-Normal distribution would be appropriate in modeling time or age-dependent events, where survival and failure rate decreases with time or age. Also, the plots for the pdf of the distribution showed that it is negatively skewed. The method of maximum likelihood estimation is used to estimate the parameters of our proposed model. The usefulness of the Weibull-normal distribution has been illustrated by some applications to two real data sets. The results showed that new distribution (Weibull-Normal distribution) performs better (provides better fits) than the generalizations of the normal distribution such as Kumaraswamy-Normal, Beta-Normal, Gamma-Normal, Kummer Beta-Normal and the normal distributions when the data set is negatively skewed however, the results from the second data confirmed that this distribution is more flexible and appropriate for modeling negatively skewed data sets.

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CHAPTER ONE: INTRODUCTION

1.1 Background of the study

The celebrated Normal (Gaussian) distribution has been known for centuries. Its popularity has been driven byits analytical simplicity and the associated Central Limit Theorem. The multivariate extensionis straightforward because the marginal and conditionals are both normal,a property rarely found in most of the other multivariate distributions. Yet there have been doubts, reservations, and criticisms about the unqualified use of normality. There are numerous situations when the assumption of normality is not validated by the data. In fact Geary (1947) remarked, "Normality is a myth; there never was and never will be a normal distribution." As an alternative, many near normal distributions have been proposed. Some families of such near normal distributions, which include the normal distribution and to some extent share its desirable properties, have played a crucial role in data analysis. For description of some such families of distributions, see Mudholkar and Hutson (2000). See also Azzalini (1985), Turner (1960) and Prentice (1975).Many of the near norma1 distributions mentioned above deal with effects of asymmetry. These families of asymmetrical distributions are analytically tractable, accommodate practical values of skewness and kurtosis, and strictly include the normal distribution. These distributions can be quite useful for data modeling and statistical analysis.

Distribution functions, their properties and interrelationships play a significant role in modeling naturally occurring phenomena. For this reason, a large number of distribution functions which are found applicable to many events in real life have been proposed and defined in literature. Various methods exist in defining statistical distributions. Many of these arose from the need to model naturally occurring events. For example, the Normal distribution addresses real-valued

variables that tend to cluster at a single mean value, while the Poisson distribution models discrete rare events. Yet few other distributions are functions of one or more distributions.

Numerous standard distributions have been extensively used over the past decades for modeling data in several fields such as Engineering, Economics, Finance, Biological, Environmental and Medical Sciences etc. However, generalizing these standard distributions has produced several compound distributions that are more flexible compared to the baseline distributions. For this reason, several methods for generating new families of distributions have been studied.

1.2 Probability Distribution and Estimation Theory

In real life there is no certainty about what will happen in the future, but decisions still have to be made. Therefore, decision processes must be able to deal with the problems of uncertainty. Events that cannot be predicted precisely are often called random events. Many, if not most, of the inputs to, and processes that occur in, our systems are to some extent random. Hence, so too are the outputs or predicted impacts, and even people's reactions to those outputs or impacts. To ignore this randomness or uncertainty is to ignore reality. One of the commonly used tools for dealing with uncertainty in planning and management is probability. Probability is a branch of mathematical statistics that is used for quantitative modeling of random variables. The probability of an event represents the proportion of times under certain conditions that the outcome can be expected to occur. A probability density functionis a mathematical description that approximately agrees with the frequencies or probabilities of possible events of a random variable.

Maximum Likelihood is a popular estimation technique for many distributions because it picks the values of the distribution's parameters that make the data more likely" than any other valueof

the parameter. This is accomplished by maximizing the likelihood function of the parameters given the data. Some appealing features of Maximum Likelihood estimators include their asymptoticunbiasedness, in that the bias tends to zero as the sample size *n* increases; they are asymptotically efficient, in that they achieve the Cramer-Rao lower bound as *n* approaches 1; and they are asymptotically normal.

1.3 The Weibull and the Normal Distributions

The WeibullDistribution is a very popular continuous probability distribution named after a Swedish Engineer, Scientist and Mathematician, WaloddiWeibull (1887 – 1979). He proposed and applied this distribution in 1939 to analyze the breaking strength of materials. Since then, it has been widely used for analyzing lifetime data in reliability engineering. It is a versatile distribution that can take on the characteristics of other types of distributions, based on the value of the shape parameter. The Weibull distribution is a widely used statistical model for studying fatigue and endurance life in engineering devices and materials.

If a random variable X has the Weibull distribution with scale parameter $\alpha>0$ and shape parameter $\beta>0$, then its cdf and pdf are, respectively, given by

$$F(x) = 1 - e^{-\alpha x^{\beta}} \quad (1.1)$$

and

$$f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} \quad (1.2)$$

For $x > 0, \alpha > 0, \beta > 0$ where α *and* β are the scale and shape parameters respectively.

The Normal distribution is also called Gaussian distribution, after a German Mathematician Carl Freidrich Gauss (1777 -1855), who introduced it in connection with the theory of error. It is an extremely important probability distribution in many fields. The *pdf* and the *cdf* of the normal distribution with location parameter $-\infty \le \mu \le \infty$ and scale parameter $\sigma > 0$ are respectively given by

$$g(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) \qquad (1.3)$$

where $\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$

and

$$F(x;\mu,\sigma) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(v-\mu)^2}{2\sigma^2}\right\} dv \qquad (1.4)$$

1.4 Aim and Objectives of the Study

The aim of this research is to propose a Weibull-Normal distribution, propose some of its properties and evaluate its performance using a real life data. The stated aim is expected to be achieved through the following objectives. By

i. defining and expressing the *pdf* of the proposed distribution as a mixture of exponentiated-G density functions.
ii. deriving some statistical properties of the proposed distribution such as the moments (mean and variance), moment generating function, characteristics function, reliability functions and the distribution of order statistics.

iii. estimating the parameters of the proposed distribution using the method of Maximum Likelihood Estimation (MLE) technique.

iv. evaluating the performance of the proposed model compared to other generalizations of the normal distribution.

1.5 Limitation

The study is focused only on extending the normal distribution and deriving mathematical expressions for some selected properties of the proposed distribution such as mean, variance, moment generating function, characteristics function, density functions for the minimum and maximum order statistics and estimating the model parameters by using only the method of MLE.

1.6 Motivation

Bourguignon *et al*. (2014) highlighted some special distributions that can be obtained by their Weibull generator. These distributions include the Uniform, Exponential, Weibull, Normal, Frechet, Half-logistic, Power function, Pareto, Burr XII, Log-Logistic, Lomax, Gumbel and Kumaraswamy distribution.

Also, Johnson *et al*. (1994) stated that, the use of four-parameter distributions should be sufficient for most practical purposes. According to them, at least three parameters are needed but they doubted any noticeable improvement arising from including a fifth or sixth parameter.

Statistically, modeling of real life scenario help us to better understand and explain random events when they occur, thereby enabling us to reproduce such a scenario either on a large and/ or on a simplified scale aimed at describing only critical parts of the phenomenon. These real life phenomena are captured by means of statistical distribution models, which are extracted or

learned directly from data gathered about them. Every distribution model has a set of parameters that needs to be estimated. These parameters specify the model and provide a mechanism for efficient and accurate use of data.

Besides these facts, the normal distribution is a well-known and commonly used standard continuous distribution with many areas of application. However, there has been no study on the hybrid (Weibull-Normal) distribution since the introduction of the Weibull generator by Bourguignon *et al*. (2014). Therefore we proposed the Weibull-Normal distribution, study some of its properties and its application to a real life data.

1.7 Significance of the Study

The study of the proposed distribution, its properties and the parameter estimates has increased the flexibility of the normal distribution and itwill make it model more easily and appropriatelydata sets that do not properly fit the normaldistribution. This study compared the proposed model to some existing generalizations of the normal distribution using three information criteria (*AIC*, *CAIC* and *BIC*) to identify the most fitted model under two real life data sets.

1.8 Statement of the Problem

The quality of the procedures used in statistical analysis depends heavily on the assumed probability model or distribution that the random variable follows. Many lifetime data used for statistical analysis follow a particular probability distribution and therefore knowledge of the appropriate distribution that any phenomenon follows greatly improves the sensitivity, reliability and efficiency of the statistical analysis associated with it.

Furthermore, it is true that several probability distributions exist for modeling lifetime data; however, some of these lifetime data do not follow any of the existing and well known standard probability distributions (models) or at least are inappropriately described by them. This, therefore, creates room for developing new distributions which could better describe some of these phenomena and therefore provide greater flexibility and wider acceptability in the modeling of lifetime data.

1.9 Definition of Terms

1.9.1 Probability distribution

The probability density function of a random variable X depends on whether X is a discrete or a continuous random variable. A random variable is a variable whose value changes from one subject to the other. Probability is used to describe the likelihood or the chances that these random variables will equal specific values or be within a given range of specific values. A probability density function is a mathematical expression that approximately agrees with the frequencies of possible events of a random variable. A random variable X is said to be discrete if it has a finite or countablyinfinite range. Discrete random variables are used to model experiments which have to do with count data such as the number of defectives, the number of boys in a family of n children, the sex of a baby etc. Similarly, a random variable is said to be continuous if its range contains an interval of real numbers. They represent measured data such as weight, height, temperature, time, blood pressure etc.

For a continuous random variable X, the cumulative distribution function (cdf) is defined as;

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du \qquad (1.5)$$

whilethe probability density function (*pdf*) is defined as;

$$f(x) = \frac{dF(x)}{dx} \tag{1.6}$$

with the following properties:

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P(a < X < b) = \int_{a}^{b} f(x)dx$

1.9.2Moments

Moments are used to study some of the most important features and characteristics of a random variable such as mean, variance, skewness and kurtosis.Let X be a continuous random variable, the n^{th} moment of X about the origin can be defined as;

$$\mu_n^{'} = E\left[X^n\right] = \int_{-\infty}^{\infty} x^n f(x)dx \tag{1.7}$$

Also The n^{th} central moment of X, say μ_n, or moment about the mean can be obtained as

$$\mu_n = E\left[X - \mu_1^{'}\right]^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \mu_1^{'i} \mu_{n-i}^{'} \tag{1.8}$$

Hence, mean, μ_1', is the 1st moment about the origin while the variance (σ^2) is the 2^{nd}central moment or moment about the mean.

Skewness is a measure of the degree of asymmetry or lack of symmetry of a distribution. The coefficient of skewness is the standardized third central moment of X or moment about the mean and can be obtained using the expression;

$$Sk = E\left(\frac{x-\mu_1^{'}}{\sigma}\right)^3 = \frac{\mu_3}{(\sigma)^3} \qquad (1.9)$$

Kurtosis is a measure of the degree of peakness or flatness of a density near the center.Whereas the coefficient of kurtosis is the standardized fourth central moment of *X* or moment about the mean and is given by;

$$Ku = E\left(\frac{x-\mu_1^{'}}{\sigma}\right)^4 = \frac{\mu_4}{(\sigma)^4} \qquad (1.10)$$

Where σ can be obtained using equation (1.8) for *n*=2 and μ_3 and μ_4 using (1.8) for *n*=3 and *n*=4 respectively.

1.9.3Moment Generating Function

This is a clever way of organizing all the moments into one mathematical quantity, and that object is called the moment generating function. The moment generating function of a random variable *X* can be obtained as;

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)dx \qquad (1.11)$$

In other words, the moment generating functiongenerates the moments ofof a random variable*X* by differentiation i.e., for any real number say *k*, the k^{th} derivative of $M_X(t)$evaluated at $t = 0$is the k^{th}moment μ_k' of *X*.

1.9.4Characteristics Function

The characteristics function has many useful and important properties which give it a central role in statistical theory. Its approach is particularly useful for generating moments, characterization of distributions and in analysis of linear combination of independent random variables.

A representation for the characteristics function is given by

$$\varphi_x(t) = E\left[e^{itx}\right] = E\left[\cos(tx) + i\sin(tx)\right] = E\left[\cos(tx)\right] + E\left[i\sin(tx)\right] \qquad (1.12)$$

1.9.5Reliability Analysis

Survival Function

Survival function is the probability function that a system or an individual will survive beyond a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \quad (1.13)$$

where $F(x)$ is *cdf* of a baseline distribution

Hazard Function

Hazard function is also called the failure or risk function and is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{S(x)} \quad (1.14)$$

where*F(x)* and *f(x)* are the *cdf and pdf* of a baseline distribution.

1.9.6 Order Statistics

Order statistics are used in a wide range of problems including robust statistical estimation and detection of outliers, characterization of probability distributions and goodness of fit tests,

entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials.Suppose $X_1, X_2, ..., X_n$ is a random sample from a distribution with *pdf, f(x),* and let $X_{1:n}, X_{2:n}, ..., X_{i:n}$ denote the corresponding order statistic obtained from this sample. The pdf, $f_{i:n}(x)$ of the i^{th} order statistic can be defined as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i} \qquad (1.15)$$

1.9.7Maximum Likelihood Method

Let $x_1, x_2, ..., x_n$ be a random sample from a population *X* with probability density function $f(x;\theta)$, where θ is an unknown parameter. The likelihood function, $L(\theta)$, is defined to be the joint density of the random variables $x_1, x_2, ..., x_n$. That is,

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) \qquad (1.16)$$

The sample statistic that maximizes the likelihood functions, $L(\theta)$, is called the maximum likelihood estimator of θ and is denoted by $\hat{\theta}$.

1.9.8 Lifetime data

Real life data sets are data sets obtained based on real happenings or normal life occurrences. Real life data are not simulated in any way but based on what has been observed and recorded in our day to day activities. Meanwhile, lifetime data are data collected on living subjects that has to do with life. Loosely speaking we can say that lifetime data are mostly found in the clinic or hospital e.g data on cancer, tuberculosis, HIV/AIDS e.t.c. Hence, lifetime data is also part of real life data sets.

CHAPTER TWO: LITERATURE REVIEW

In recent times, researchers have worked tirelessly to produce compound probability distributions which have been evaluated and considered to have performed much better than the well known classical or standard probability distributions. Most of these studies are meant purposely to introduce a higher level of skewness in the existing probability distributions by generalizing on a well-known distribution under certain facts and assumptions. A review of some of these generalizations is as listed: the ßeta generalized family (Beta-G) by Eugene *et al*. (2002) which generalized the beta distribution under some assumptions, the Kumaraswamy-G by Cordeiro and de Castro (2011) using the Kumaraswamy distribution as a baseline distribution, Transmuted family of distributions by Shaw and Buckley (2007), Gamma-G (type 1) by Zografos and Balakrishnan (2009) using the Gamma distribution, McDonald-G by Alexander *et al*. (2012), Gamma-G (type 2) by Ristic*et al*. (2012), Gamma-G (type 3) by Torabi and Montazari (2012), Log-gamma-G by Amini*et al*. (2012), Exponentiated T-X by Alzaghal*et al*. (2013), Exponentiated-G (EG) by Cordeiro*et al*. (2013), Logistic-G by Torabi and Montazari (2014), Gamma-X by Alzaatreh*et al*., (2013), Logistic-X by Tahir*et al*. (2015), Weibull-X by Alzaatreh*et al*. (2013), Weibull-G by Bourguignon *et al*. (2014) and Beta Marshall-Olkin family of distributions by Alizadeh*et al*. (2015) and several other studies of the same kind. Further work on some of these studies above have led to the development of some compound probability distributions such as skew normal distribution by Azzalini (1985), the generalized Weibull distribution by Mudholkar and Kollia (1994), the exponentiatedWeibull distribution by Mudholkar*et al.* (1995), the beta normal distribution by Eugene *et al*. (2002), the beta-Weibull distribution by Famoye*et al.* (2005), the generalized normal distribution by Nadarajah*et al*. (2005) and the kumaraswamy normal distribution by Cordeiro and De'Castro (2011).

In an attempt to introduce skewness (reduce the level of symmetry in the distribution) into the well known normal distribution, many researchers have proposed many generalizations of this distribution by employing several methodologies. Details of some of the recent extensions of the normal distribution in addition to the aforementioned ones above in the literature include the following:

Cordeiro*et al*. (2012) presented a five parameter life time distribution, referred to as the McDonald normal (*MCN*) distribution, which includes as special cases most of the commonly used distribution in the lifetime literature such as the normal, skew-normal, exponentiated normal, beta normal and kumaraswamy normal distributions, among others. They obtained their ordinary moments, moment generating function and mean deviation. They also derived the ordinary moments of the order statistics. They used the method of maximum likelihood to fit the new distribution and illustrated its potentiality with three applications to real data. Their results proved that the new model is much more flexible than the exponentiated normal, beta normal, skew normal and kumaraswamy normal models proposed recently.

Alzaatreh*et al*. (2014) proposed a new family of distribution called the gamma – X family where X represents any continuous distribution. They discussed some properties of the gamma – x family and studied a member of the family; the gamma – normal distribution. They provided the limiting behaviours, moments, mean deviations, dispersions, and Shannon entropy for the gamma-normal distribution. They also obtained bounds for the non-central moments. The method of maximum likelihood estimation was used for estimating the parameters of the gamma-normal distribution. Two real data sets were used to illustrate the application of the gamma-normal distribution. The result showed that the gamma-normal distribution provide a good fit to each data set compared to other existing generalized forms of the normal distribution.

Nadarajah*et al*. (2014) also proposed a new family of skewed distribution referred to as modified beta distributions. They studied some properties of the new family including estimation procedures. They also studied a member of this family called the modified beta normal distribution. A real data application and simulation studies were used to show superior performance of this new model versus other known models.

Lima *et al*. (2015) proposed a new three parameter distribution called the gamma normal distribution which extends the normal distribution. They defined and studied some of its structural properties. The study provided explicit expressions for the ordinary and incomplete moments, Quantile and generating functions, mean deviations, Renyi entropy, Shannon entropy, order statistics and their moments. They estimated the model parameters using the method of maximum likelihood estimation. This study also determined the observed information matrix of the proposed distribution. The potentiality of the proposed model was evaluated based on three criteria and the results showed that the new distribution provides a better fit than the beta normal distribution, kumaraswamy-normal and the standard normal distribution.

Pescim*et al*. (2015) introduced a new extension of the normal distribution called the Kummer beta normal distribution. They defined and expressed the pdf of the new distribution as a linear combination of exponentiated normal pdfs which allowed them to derive some of its mathematical properties like its ordinary and incomplete moments, mean deviations and order statistics. The estimation of parameters of the proposed model was done by the method of maximum likelihood and Bayesian analysis. The usefulness of the model was illustrated by an application to a real data set. Their results revealed that the fit of the Kummer beta normal distribution outperforms all the distributions considered in their study based on the data set used. So they concluded that the proposed distribution can yield better fits than the normal, beta

normal, gamma normal, McDonald normal and the modified beta normal distribution and therefore may be an interesting alternative to these distributions for modeling skewed data sets.

Bourguignon *et al*. (2014) introduced a new wider family of distributions called the Weibull generalized (Weibull-G) family of distributions. According to these authors, this generator can extend several widely known distributions such as the Uniform, Exponential, Weibull, Frechet, Half-logistic, Power Function, Pareto, Burr XII, Logistic, Lomax, Normal, Gumbel and Kumaraswamy distributions. They derived general mathematical properties of the new wider Weibull family of distributions. They also showed that the Weibull-G density function can be expressed as a mixture of exponentiated-G density functions. This mixture representation is important to derive several structural properties of this family in full generality. They provided some of this structural properties such as the ordinary and incomplete moments, Quantile function and order statistics. The study also proposed that for any Weibull-G family distributions, the estimation of model parameters should be done by method of maximum likelihood estimation. These authors also recommended that for each baseline distribution G, their methods and results should be adopted.

Merovci*et al*. (2015) used the Weibull generalized family of distributions proposed by Bourguignon *et al*. (2014) to study the Weibull Rayleigh (WR) distribution. They derived expressions for the moments and the moments generating functions. Estimation of parameters of their model was done by the method of maximum likelihood estimation. Their model performance was compared to some baseline distributions such as beta Weibull,ExponentiatedWeibull and the standard Weibull distribution. The results of this comparison using the likelihood ratio statistics showed that the Weibull Rayleigh distribution can be used quite effectively to provide better fit than the Weibull distribution.

Oguntunde*et al*. (2015) also used the Weibull-G family of distribution introduced by Bourguignon *et al*., (2014) to study the Weibull Exponential distribution (*WED*). They also derived explicit expression for some of its basic mathematical properties like moments, moment generating function, reliability analysis, limiting behavior and order statistics. The method of maximum likelihood estimation was proposed for estimating its parameters and real life applications were provided to illustrate the flexibility of the new distribution over the standard exponential distribution.

Afify*et al*. (2016) proposed a four parameter distribution named the WeibullFrechet distribution using the Weibull-G family by Bourguignon *et al*. (2014). They studied some of its mathematical and statistical properties. The applications of the new distribution with some baseline distributions showed that it is more fitted compared to kumaraswamyFrechet (*KFr*), exponentiatedFrechet (*EFr*), beta Frechet (*BFr*), gamma extended Frechet (*GEFr*), transmitted marshallOlkinFrechet (*TMOFr*) and Frechet (*Fr*) distributions.

With this understanding, this research seeks to increase the flexibility of the normal distribution using the weibull-G family of distribution proposed by Bourguignon *etal*. (2014). Hence, this study proposed "Weibull-Normal distribution: its properties and applications".

CHAPTER THREE: METHODOLOGY

3.1 The Definition of the Weibull-Normal Distribution*(WND)*

For any continuous distribution with cdf $G(x;\xi)$ and *pdf*, $g(x;\xi)$ Bourguignon *et al*. (2014) proposed the Weibull generalized (denoted "Weibull-G) family of distributions that provides greater flexibility in modeling of real data sets. The cumulative distribution function (*cdf*) of the Weibull generalized family of distributions according to Bourguignon *et al*. (2014) is defined as

$$F_{WG}(x;\alpha,\beta,\xi) = \int_{-\infty}^{\frac{G(x;\xi)}{G'(x;\xi)}} \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}} dt \quad (3.1)$$

Where $G(x;\xi)$ is the cdf of any continuous distribution which depends on the parameter vector ξ, $G'(x;\xi) = 1 - G(x;\xi)$ and $\alpha > 0$ and $\beta > 0$ are the scale and shape parameters respectively.

The interpretation of the above Weibull generator is given as follows: Let *Y* be a lifetime random variable having a certain continuous $G(x;\xi)$ distribution. The odds ratio that an individual (or component) following the lifetime *Y* will die (failure) at time *x* is $G(x;\xi)/G'(x;\xi)$. If the variability of these odds of death is represented by the random variable *X* and assumed it follows a Weibull distribution with *pdf* $f(x) = \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}$ where α is the scale parameter and β is the shape parameter. Hence, we can represent the above interpretation as

$$P(Y \le x) = P\left(X \le \frac{G(x;\xi)}{G'(x;\xi)}\right) = F(x;\alpha,\beta,\xi) = \int_{-\infty}^{\frac{G(x;\xi)}{G'(x;\xi)}} \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}} dt \quad (3.2)$$

Using integration by substitution in equation (3.2), we perform the following operations:

let

$u = e^{-\alpha t^{\beta}} = e^{-v}$ and

$v = \alpha t^{\beta}$

Then,

$$F_{WG}(x;\alpha,\beta,\xi) = \int_{-\infty}^{\frac{G(x;\xi)}{G'(x;\xi)}} \alpha\beta t^{\beta-1} U dt \quad (3.3)$$

$\frac{dU}{dt} = \frac{dU}{dv} \times \frac{dv}{dt}$ (function of a function rule or the chain rule)

Where

$$\frac{du}{dv} = -e^{-v} \text{ and } \frac{dv}{dt} = \alpha\beta t^{\beta-1}$$

$$\frac{du}{dt} = \frac{du}{dv} \times \frac{dv}{dt} = -\alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}$$

$$dt = -\frac{du}{\alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}} = -\frac{du}{\alpha\beta t^{\beta-1} u} \qquad (3.4)$$

Therefore, substituting for *dt* in equation (3.3) and simplifying, we obtain

$$F_{WG}(x;\alpha,\beta,\xi) = -\int_{-\infty}^{\frac{G(x;\xi)}{G'(x;\xi)}} dU \quad (3.5)$$

$$F_{WG}(x;\alpha,\beta,\xi) = -\left[U\right]_{-\infty}^{\frac{G(x;\xi)}{G'(x;\xi)}} = -\left[e^{-\alpha t^{\beta}}\right]_{-\infty}^{\frac{G(x;\xi)}{G'(x;\xi)}} \quad (3.6)$$

The odd ratio that an individual following the lifetime *Y* will die at time *x* is $G(x;\xi)/G'(x;\xi)$ where $G'(x;\xi)=1-G(x;\xi)$. If *x*assumes a Weibull distribution as in (3.2), then as $x \to -\infty$, $t=G(x;\xi)/G'(x;\xi)$ will tend to $t=G(-\infty;\xi)/G'(-\infty;\xi)=0$

Evaluating the integrand in equation (3.6) yields

$$F_{WG}(x;\alpha,\beta,\xi)=1-\exp\left\{-\alpha\left[\frac{G(x;\xi)}{G'(x;\xi)}\right]^{\beta}\right\}, -\infty<x<\infty, \alpha,\beta>0 \quad (3.7)$$

Therefore, equation (3.7) is the cumulative distribution function (*cdf*) of the Weibull-G family of distributions proposed by Bouguignon*et al*. (2014) and the corresponding *pdf* of the Weibull-G family can be obtained from equation (3.7) by taking the derivative of the *cdf* with respect to *x* as follows:

Recallthat

$$f(x)= f_{WG}(x;\alpha,\beta,\xi)=\frac{d(F_{WG}(x;\alpha,\beta,\xi))}{dx} \quad (3.8)$$

where

$$F_{WG}(x;\alpha,\beta,\xi)=1-\exp\left\{-\alpha\left[\frac{G(x;\xi)}{G'(x;\xi)}\right]^{\beta}\right\}=1-y$$

Let

$$y=\exp\left\{-\alpha\left[\frac{G(x;\xi)}{G'(x;\xi)}\right]^{\beta}\right\}=e^{-u}$$

$u = \alpha t^{\beta}$ and

$t = \frac{G(x;\xi)}{G'(x;\xi)}$

Such that $y = e^{-\alpha t^{\beta}} = e^{-U}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}$ (function of a function rule or the chain rule)

then,

$$\frac{dy}{du} = -e^{-u}, \frac{du}{dt} = \alpha\beta t^{\beta-1} and \frac{dt}{dx} = \frac{g(x;\xi)}{\left[G'(x;\xi)\right]},$$

Note that $t = \frac{G(x;\xi)}{G'(x;\xi)} = \frac{u}{v}$ and by quotient rule,

$$\frac{dt}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2} = \frac{\left[1-G(x;\xi)\right]g(x;\xi) - G(x;\xi)\left[-g(x;\xi)\right]}{\left[G'(x;\xi)\right]^2} = \frac{g(x;\xi)}{\left[G'(x;\xi)\right]^2}$$

But recall that

$$f(x) = f_{WG}(x;\alpha,\beta,\xi) = \frac{d(F(x;\alpha,\beta,\xi)}{dx} = -\frac{dy}{dx}$$

Where

$$-\frac{dy}{dx} = -\left\{-\alpha\beta t^{\beta-1}\frac{g(x;\xi)}{\left[G'(x;\xi)\right]^2}\mathrm{e}^{-u}\right\} = \alpha\beta\frac{g(x;\xi)}{\left[G'(x;\xi)\right]^2}\left(\frac{G(x;\xi)}{G'(x;\xi)}\right)^{\beta-1}\exp\left\{-\alpha\left[\frac{G(x;\xi)}{G'(x;\xi)}\right]^{\beta}\right\} \quad (3.9)$$

Hence,

$$f(x) = f_{WG}(x;\alpha,\beta,\xi) = \alpha\beta g(x;\xi)\frac{(G(x;\xi))^{\beta-1}}{(G'(x;\xi))^{\beta+1}}\exp\left\{-\alpha\left[\frac{G(x;\xi)}{G'(x;\xi)}\right]^{\beta}\right\} \qquad (3.10)$$

Finally, (3.10) is the probability density function (*pdf*) of the Weibull-G family of distributions where $g(x;\xi)$ and $G(x;\xi)$ are the pdf and the cdf of any baseline continuous distribution respectively which depends on the parameter vector $\xi = (\mu,\sigma)$ where $\alpha>0$ and $\beta>0$ are the scale and shape parameters respectively.

Equation (3.10) is the *pdf* of any Weibull-G family of distributions and is most tractable when both *cdf* and *pdf* have simple analytic expressions. The major benefit of (3.10) is to offer more flexibility to extremes of the *pdfs* and therefore it becomes suitable for analyzing data with high degree of asymmetry.

Henceforth, let *G* be a continuous baseline distribution, for each *G* distribution, we define the Weibull-G distribution with two extra parameters α and β by following equation (3.10).

A random variable *X* with *pdf* (3.10) is denoted by X~Weibull-*G*(α,β,ξ). The additional parameters induced by the Weibull generator above are sought as a vehicle to furnish a more flexible distribution. This class of distribution becomes a special case of the exponential generator if $\beta=1$.

The *cdf* and *pdf* of the normal distribution with location parameter $-\infty<\mu<\infty$ and scale parameter $\sigma>0$ is given by

$$G(x;\mu,\sigma) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}}\exp\left\{-\tfrac{(v-\mu)^2}{2\sigma^2}\right\}dv \quad (3.11)$$

and

$$g\left(x;\mu,\sigma\right)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}=\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)(3.12)$$

respectively.

Taking the *cdf* (3.11) and the *pdf* (3.12) of the normal distribution with location parameter, $-\infty<\mu<\infty$ and dispersion parameter $\sigma>0$.

The *cdf* and *pdf* of the*WND* are obtained from equation (3.7) and (3.10) as

$$F(x)=F(x;\alpha,\beta,\mu,\sigma)=1-\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]}\right]^{\beta}\right\},-\infty<x<\infty,\alpha,\beta>0\ (3.13)$$

and

$$f(x)=f(x;\alpha,\beta,\mu,\sigma)=\frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}(3.14)$$

respectively. Where $-\infty<X<\infty,\sigma>0$ is the dispersion parameter, $-\infty<\mu<\infty$ is the location parameter and $\alpha>0$ and $\beta>0$ are the scale and shape parameters respectively

Model Validity Check

The above model is valid if and only if $\int_{-\infty}^{\infty}f(x)dx=1$

Proof:

$$f(x)=\frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}$$

$$\int_{-\infty}^{\infty}f(x)dx=\int_{-\infty}^{\infty}\frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}dx \quad (3.15)$$

let

$$u=\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}=\alpha t^{\beta}\text{ and }t=\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}$$

then

$$\int_{-\infty}^{\infty}f(x)dx=\int_{-\infty}^{\infty}\frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{t^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{2}}\exp\{-u\}dx \quad (3.16)$$

$\frac{du}{dx}=\frac{du}{dt}\times\frac{dt}{dx}$ (function of a function rule or the chain rule)

$$\frac{du}{dt}=\alpha\beta t^{\beta-1}\text{ and }\frac{dt}{dx}=\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{\sigma\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{2}}$$

hence

$$\frac{du}{dx} = \alpha\beta t^{\beta-1} \frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{\sigma\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^2}$$

and

$$dx = \alpha\beta t^{\beta-1} \frac{\sigma\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^2 du}{\alpha\beta t^{\beta-1}\Phi\left(\frac{x-\mu}{\sigma}\right)} \qquad (3.17)$$

Therefore, substituting for *dx* in equation (3.16) and simplifying, we obtain

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \exp\{-u\}\, du = -\left[e^{-u}\right]_{-\infty}^{\infty} \quad (3.18)$$

$$\int_{-\infty}^{\infty} f(x)dx = -\left[e^{-\infty} - e^{\infty}\right] = -[0-1] = 0+1 = 1 \quad (3.19)$$

Hence, equation (3.19) is a proof that the model in equation (3.14) is a valid probability density function.

The plots of the *pdf* and *cdf* of the *WND* for selected parameter values are displayed in **figures 3.1** and **3.2** below respectively.

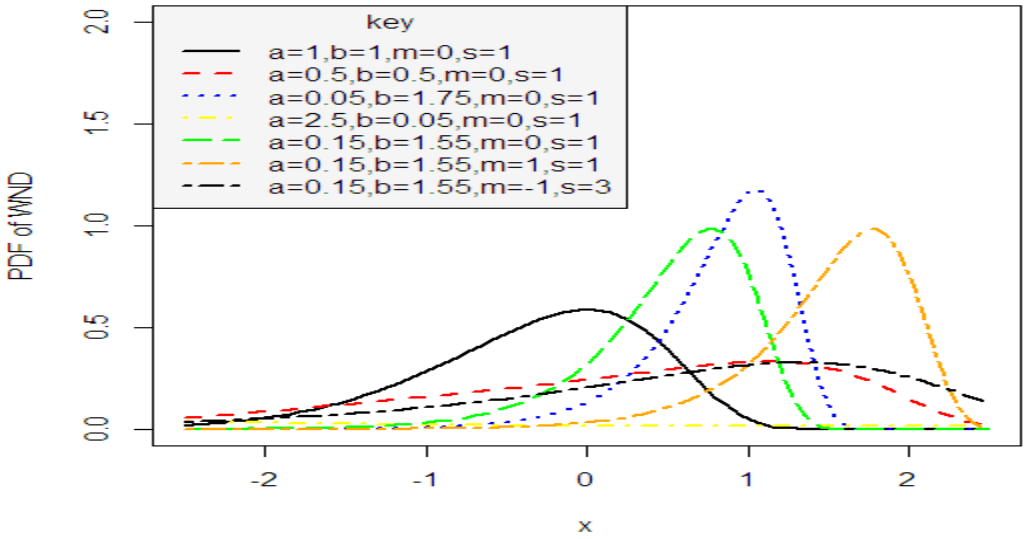

Figure 3.1: The graph of *pdf* of the *WND* at different parameter values where $a = \alpha, b = \beta, m = \mu and s = \sigma.$

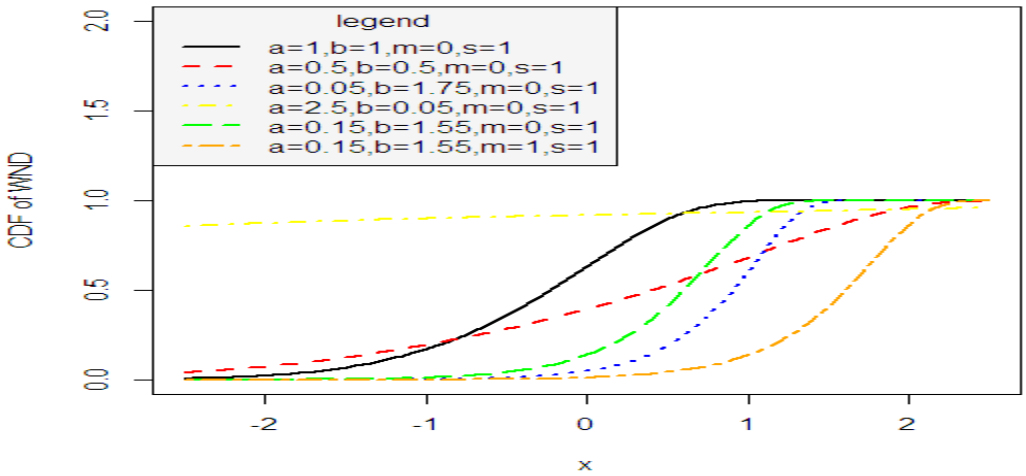

Figure 3.2: The graph of *cdf* of the *WND*at different parameter values where $a = \alpha, b = \beta, m = \mu and s = \sigma.$

The plot for the *pdf* shows that the *WND* is skewed to the left (negatively skewed) and therefore will be an appropriate candidate or model for modeling skewed data sets unlike the standard

normal distribution. Also, the *cdf* has an increasing upward trend. It is evident that the shapes of the new distribution are much more flexible than its standard counterpart.

Expansions for (3.13) and (3.14) can be derived using the concept of exponentiated distributions as follows:

Consider the exponentiated normal (*EN*)distribution by Nadarajah*et al*., 2006 with power parameter *a>0*and distribution coefficient *b*>0 defined by *Y~EN(a,μ,σ),* with *cdf* and *pdf* given by

$$H_a(y) = H(y,a) = \Phi\left(\frac{x-\mu}{\sigma}\right)^a \quad (3.20)$$

and

$$h_a(y,a) = h(y,a) = \frac{a}{b}\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\frac{x-\mu}{\sigma}\right)^{a-1} \quad (3.21)$$

respectively.

By expanding the exponential term in (3.14) using power series, we obtain:

$$\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\} = \sum_{k=0}^{\infty}\frac{(-1)^k(\alpha)^k}{k!}\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{k\beta} \quad (3.22)$$

Making use of the result in (3.22) above, equation (3.14) becomes

$$f(x) = \sum_{k=0}^{\infty}\frac{(-1)^k(\alpha)^{k+1}\beta}{\sigma k!}\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\frac{x-\mu}{\sigma}\right)^{\beta(k+1)-1}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{-[\beta(k+1)+1]} \quad (3.23)$$

Also, using the generalized binomial theorem, we can write the last term from the above result as:

$$\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\left[\beta(k+1)+1\right]}=\sum_{j=0}^{\infty}\frac{\Gamma\left(\beta\left(k+1\right)+j+1\right)}{j!\Gamma\left(\beta\left(k+1\right)+1\right)}\Phi\left(\frac{x-\mu}{\sigma}\right)^{j} \quad (3.24)$$

Again making use of the expansion in (3.24) above, equation (3.23) can be written as:

$$f(x)=\sum_{j=0,k=0}^{\infty}\frac{(-1)^{k}(\alpha)^{k+1}\beta}{\sigma k!}\frac{\Gamma\left(\beta\left(k+1\right)+j+1\right)}{j!\Gamma\left(\beta\left(k+1\right)+1\right)}\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\frac{x-\mu}{\sigma}\right)^{\beta(k+1)+j-1} \quad (3.25)$$

Using the result in (3.21), we can now express the *pdf* of the Weibull normal distribution as an infinite linear combination of exponentiated generalized (exponentiated-G) density functions as:

$$f(x)=\sum_{j=0,k=0}^{\infty}w_{j,k}h_{\beta(k+1)+j}(x) \quad (3.26)$$

where

$$h_{\beta(k+1)+j}(x)=h(x;\beta\left(k+1\right)+j,\mu,\sigma)=\left[\beta\left(k+1\right)+j\right]\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\frac{x-\mu}{\sigma}\right)^{\beta(k+1)+j-1} \quad (3.27)$$

whichCorresponds to equation (3.21) denotes the exponentiated normal *pdf*which is given by

$$h_{a}(x)=h(x;a,\mu,\sigma)=a\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\frac{x-\mu}{\sigma}\right)^{a-1}$$

while$w_{j,k}$ represents a constant or the coefficient of the exponentiatedWeibull-Normal distribution and is given by

$$w_{j,k} = \frac{(-1)^k (\alpha)^{k+1} \beta \Gamma(\beta(k+1)+j+1)}{\sigma k!\, j! \Gamma(\beta(k+1)+1)}$$

Now, by integrating the *pdf* in (3.26) with respect to *x*, we obtain the corresponding *cdf* as:

$$F(x) = \sum_{j,k=0}^{\infty} w_{j,k} H(x; \beta(k+1)+j, \mu, \sigma) = \sum_{j,k=0}^{\infty} w_{j,k} \Phi\left(\frac{x-\mu}{\sigma}\right)^{\beta(k+1)+j} \quad (3.28)$$

Now if β is a positive non-integer, we can expand the last term in (3.28) as

$$\Phi\left(\frac{x-\mu}{\sigma}\right)^{\beta(k+1)+j} = \left\{1-\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]\right\}^{\beta(k+1)+j} = \sum_{p=0}^{\infty} (-1)^p \binom{\beta(k+1)+j}{p} \left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^p$$

Using the binomial expansion in the above result, we have

$$\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^p = \sum_{r=0}^{p} (-1)^r \binom{p}{r} \Phi\left(\frac{x-\mu}{\sigma}\right)^r$$

and then,

$$\Phi\left(\frac{x-\mu}{\sigma}\right)^{\beta(k+1)+j} = \sum_{p=0}^{\infty} \sum_{r=0}^{p} (-1)^{p+r} \binom{\beta(k+1)+j}{p} \binom{p}{r} \Phi\left(\frac{x-\mu}{\sigma}\right)^r$$

Replacing $\sum_{p=0}^{\infty} \sum_{r=0}^{p}$ by $\sum_{r=0}^{\infty} \sum_{p=r}^{\infty}$, we obtain

$$\Phi\left(\frac{x-\mu}{\sigma}\right)^{\beta(k+1)+j} = \sum_{r=0}^{\infty} \sum_{p=r}^{\infty} (-1)^{p+r} \binom{\beta(k+1)+j}{p} \binom{p}{r} \Phi\left(\frac{x-\mu}{\sigma}\right)^r$$

and

$$\Phi\left(\frac{x-\mu}{\sigma}\right)^{\beta(k+1)+j} = \sum_{r=0}^{\infty} S_r\left(\beta(k+1)+j\right)\Phi\left(\frac{x-\mu}{\sigma}\right)^r \quad (3.29)$$

where the coefficients are

$$S_r\left(\beta(k+1)+j\right) = \sum_{r=0}^{\infty}\sum_{p=r}^{\infty}(-1)^{p+r}\binom{\beta(k+1)+j}{p}\binom{p}{r}$$

Because $S_r(m) = \sum_{p=r}^{\infty}(-1)^{p+r}\binom{m}{p}\binom{p}{r}$

Combining equations (3.28) and (3.29), the Weibull-Normal *cdf* can be expressed from (3.28) as

$$F(x) = \sum_{j,k=0}^{\infty} w_{j,k}\sum_{r=0}^{\infty} S_{r\left(\beta(k+1)+j\right)}\Phi\left(\frac{x-\mu}{\sigma}\right)^r$$

$$F(x) = \sum_{j,k=0}^{\infty}\sum_{r=0}^{\infty} w_{j,k}s_{r\left(\beta(k+1)+j\right)}\Phi\left(\frac{x-\mu}{\sigma}\right)^r$$

$$F(x) = \sum_{r=0}^{\infty}\eta_r\Phi\left(\frac{x-\mu}{\sigma}\right)^r = \sum_{r=0}^{\infty}\eta_r H\left(x;r,\mu,\sigma\right) \quad (3.30)$$

where $\eta_r = \sum_{j,k=0}^{\infty} w_{j,k}s_{r\left(\beta(k+1)+j\right)}$

By differentiating equation (3.30) and changing indices, we can obtain the *pdf* of the Weibull-Normal distribution as

$$f(x) = \frac{dF(x)}{dx} = \sum_{r=0}^{\infty}\eta_r h\left(x;r,\mu,\sigma\right)$$

$$f(x)=\sum_{r=0}^{\infty}\eta_r\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\frac{x-\mu}{\sigma}\right)^{r-1} \quad (3.31)$$

where $\eta_r = \sum_{j,k=0}^{\infty} w_{j,k} s_{r(\beta(k+1)+j)}$ and $\sum_{r=0}^{\infty}\eta_r = 1$

Equation (3.31) is the *pdf* of the Weibull-Normal distribution (*WND*) defined as a linear combination of *ENpdf*. So, several properties of the *WND* can be obtained by following those properties of *EN* distribution.

3.2 Moments

Moments are used to study some of the most important features and characteristics of a random variable such as mean, variance,skewness and kurtosis.

3.2.1 Ordinary Moments

Here, we obtain the ordinary moments of *X* as follows:

Let $X_1, X_2, \ldots\ldots\ldots, X_n$ denote a random sample from the standard Weibull-Normal distribution given by the *pdf*

$$f(x)=\sum_{r=0}^{\infty} r\eta_r\phi(x)\Phi(x)^{r-1}$$

which is obtained from equation (3.31) for $\mu=0$ and $\sigma=1$, that is $X\sim WN(\alpha,\beta,0,1)$.

The n^{th} moment of *X* can be obtained as

$$\mu_n^{'} = E\left[X^n\right] = \int_{-\infty}^{\infty} x^n f(x)dx \quad (3.32)$$

$$\mu_n^{'} = E\left[X^n\right] = \sum_{r=0}^{\infty} r\eta_r \int_{-\infty}^{\infty} x^n h(x;r,0,1)\,dx$$

$$\mu_n^{'} = E\left[X^n \right] = \sum_{r=0}^{\infty} r\eta_r \int_{-\infty}^{\infty} x^n \phi(x) \Phi(x)^{r-1} dx \quad (3.33)$$

Recall that

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2} \right)$$

and

$$\Phi(x) = \left\{ 1 + erf\left(\frac{x}{\sqrt{\pi}} \right) \right\}$$

Where $-\infty < x < \infty$ and *erf(.)* denotes the error function defined by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2 \right) dt$$

Now, substituting for $\phi(x)$and $\Phi(x)$ in equation (3.33) and simplifying, we have:

$$\mu_n^{'} = \sum_{r=0}^{\infty} r\eta_r \frac{2^{-[r-1]}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n \exp\left(-\frac{x^2}{2} \right) \left[1 + erf\left(\frac{x}{\sqrt{2}} \right) \right]^{r-1} dx \quad (3.34)$$

Using binomial expansion on the middle term in the integral in (3.34) and simplifying, we obtain:

$$\mu_n^{'} = \sum_{r=0}^{\infty} \sum_{p=0}^{r-1} r\eta_r \frac{2^{-[r-1]}}{\sqrt{2\pi}} \binom{r-1}{p} \int_{-\infty}^{\infty} x^n \left[erf\left(\frac{x}{\sqrt{2}} \right) \right]^p \exp\left(-\frac{x^2}{2} \right) dx \quad (3.35)$$

$$\mu_n^{'} = \sum_{r=0}^{\infty} \sum_{p=0}^{r-1} r\eta_r \frac{2^{-[r-1]}}{\sqrt{2\pi}} \binom{r-1}{p} I(n, p) \quad (3.36)$$

Where *I(n,p)* represents the $(n,p)^{th}$ probability weighted moment (*PWM*) for any *n* and *p* positive integers of the standard normal distribution and is found as follows;Nadarajah (2008).

Using the power series expansion of the error function;

$$\exp(x)=\frac{2}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)m!}$$

This implies that

$$\exp\left(\frac{x}{\sqrt{2}}\right)=\frac{2}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{m+1/2}(2m+1)m!} \quad (3.37)$$

Therefore the integral *I(n,p)* in (3.36) can be expressed as:

$$I(n,p)=\int_{-\infty}^{\infty}x^n\left[\frac{2}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{m+1/2}(2m+1)m!}\right]^p\exp\left(-\frac{x^2}{2}\right)dx \quad (3.38)$$

$$I(n,p)=\left(\frac{2}{\sqrt{\pi}}\right)^p\sum_{m_1=0}^{\infty}\cdots\sum_{m_p=0}^{\infty}\frac{(-1)^{m_1+\ldots+m_p}}{2^{m_1+\ldots+m_p+p/2}(2m_1+1)\ldots(2m_p+1)m_1!..m_p!}\int_{-\infty}^{\infty}x^{2(m_1+\ldots+m_p)+p+n}\exp\left(-\frac{x^2}{2}\right)dx \quad (3.39)$$

Elementary integration shows that

$$\int_{-\infty}^{\infty}x^{2(m_1+\ldots+m_p)+p+n}\exp\left(-\frac{x^2}{2}\right)dx=\begin{cases}2^{m_1+\ldots+m_p+\frac{(p+n-1)}{2}}\Gamma\left(m_1+\ldots+m_p+\frac{p+n+1}{2}\right), if: p+n=even\\ 0, if; p+n=odd\end{cases} \quad (3.40)$$

Hence equation (3.39) reduces to

$$I(n,p)=\pi^{-\frac{p}{2}}2^{\left(\frac{n-1}{2}\right)+p}\sum_{m_1=0}^{\infty}\cdots\sum_{m_p=0}^{\infty}\frac{(-1)^{m_1+\ldots+m_p}}{2^{m_1+\ldots+m_p+p/2}(2m_1+1)\ldots(2m_p+1)m_1!..m_p!}\Gamma\left(m_1+\ldots+m_p+\frac{p+n+1}{2}\right) \quad (3.41)$$

Now, according to Nadarajah (2008),

$$(f)_n = \frac{\Gamma(f+n)}{(f)} \quad (3.42)$$

and

$$F_{(A)}^{(n)}(a;b_1,...,b_n;c_1,...,c_n;x_1,...,x_n) = \sum_{m_1=0}^{\infty}...\sum_{m_n=0}^{\infty} \frac{a_{m_1+...+m_n}(b_1)m_1...(b_n)m_n x_1^{m_1}...x_n^{m_n}}{(c_1)m_1...(c_n)m_n!..m_n!} (3.43)$$

Using these definitions in (3.42) and (3.43) we can express (3.41) as

$$I(n,p) = \pi^{-\frac{p}{2}} 2^{\left(\frac{n-1}{2}\right)+p} F_{(A)}^{(p)}\left(\tfrac{p+n+1}{2};\tfrac{1}{2},...,\tfrac{1}{2};\tfrac{3}{2},...,\tfrac{3}{2};-1,...,-1\right), if: p+n = even \leq (3.44)$$

Combining (3.36) and (3.44), we can express the n^{th} moment of the standard Weibull-Normal distribution in terms of the Lauricella function of type A (Exton, 1979) demonstrated by Nadarajah (2008), Nadarajah and Kotz (2006), Pescim*et al*. (2015), Nadarajah (2005), Lima *et al*. (2015), e.t.c as

$$\mu_n^{'} = \sum_{r=0}^{\infty}\sum_{p=0}^{r-1} r\eta_r \binom{r-1}{p} \pi^{-\frac{(p+1)}{2}} 2^{\left(\frac{n}{2}\right)-r+p} \Gamma\left(\tfrac{p+n+1}{2}\right) F_{(A)}^{(p)}\left(\tfrac{p+n+1}{2};\tfrac{1}{2},...,\tfrac{1}{2};\tfrac{3}{2},...,\tfrac{3}{2};-1,...,-1\right), if: p+n = even (3.45)$$

3.2.2 The Central Moments

The n^{th} central moment or moment about the mean of *X*, say μ_n, can be obtained as

$$\boldsymbol{\mu}_n = E\left[X - \mu_1^{'}\right]^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \mu_1^{'i} \mu_{n-i}^{'} (3.46)$$

The variance of *X* is the central moment about the mean of order two (*n*=2) and is given by

$$\text{var}(X) = \sigma^2 = \mu_2^{'} - \mu_1^{'2} (3.47)$$

where $\mu_2'$and $\mu_1'^2$ are the second ordinary moment about the origin and first ordinary moment about the origin squared respectively which could be obtained using equation (3.45) for *n*=2 and 1 respectively.

The coefficient of skewness is the standardized third central moment of *X* about the mean and can be obtained using the expression;

$$Sk = E\left(\frac{x-\mu_1^{'}}{\sigma}\right)^3 = \frac{\mu_3}{(\sigma)^3} \quad (3.48)$$

Whereas the coefficient of kurtosis is the standardized fourth central moment of *X* about the mean and is given by;

$$Ku = E\left(\frac{x-\mu_1^{'}}{\sigma}\right)^4 = \frac{\mu_4}{(\sigma)^4} \quad (3.49)$$

Where σ can be obtained using equation (3.47) and μ_3 and μ_4 using (3.46) respectively.

3.2.3Incomplete Moments

We derive an expression for the n^{th} incomplete moment of *X* as given below:

Consider the case of a standard Weibull-Normal distribution based equation (3.31), we denote the n^{th} incomplete moment of *X* as

$$I_x(y) = \int_{-\infty}^{y} x^n f(x)dx \qquad (3.50)$$

$$I_x(y) = \sum_{j,k=0}^{\infty} r\eta_r \int_{-\infty}^{y} x^n \phi(x)\Phi(x)^{r-1} dx \quad (3.51)$$

Note that we can write $\Phi(x)$ using power series as

$$\Phi(x) = \sum_{j=0}^{\infty} a_j x^j \qquad (3.52)$$

Where $a_0 = \frac{\left(1+\sqrt{\frac{2}{\pi}}\right)^{-1}}{2}, a_{2j+1} = \frac{(-1)^j}{\left[\sqrt{2\pi}\,2^j\,(2j+1)\,j!\right]}$, $for: j = 0,1,2...and, a_{2j} = 0; for: j = 1,2,..,$

Using the identity given by Gradshteyn and Ryzhik (2007) for power series raised to a positive integer *i*, we have

$$\left(\sum_{j=0}^{\infty} c_j x^j\right)^i = \sum_{j=0}^{\infty} c_{i,j} x^j \quad (3.53)$$

Where the coefficient $C_{i,j}$t (for i=1,2,….) are easily obtained from the recurrence equation as

$$c_{i,j} = (ja_0)^{-1} \sum_{m=0}^{\infty} \left[m(i+1) - j\right] a_m c_{i,j-m}; and, c_{i,0} = a_0^i \qquad (3.54)$$

Hence, using the expressions in (3.52) and (3.53), it implies that

$$\Phi(x)^{r-1} = \sum_{j=0}^{\infty} c_{r-1,j} x^j \qquad (3.55)$$

wherethe coefficients$C_{r-1,j}$ can be determined from the recurrence equation (3.54). Thus using (3.55) and simplifying the integral, it follows from (3.51) that the n^{th} incomplete moment of *X* can be obtained as

$$I_x(y) = \sum_{j,r=0}^{\infty} r\eta_r c_{r-1,j} \int_{-\infty}^{y} x^{j+n} \exp\left(-\tfrac{x^2}{2}\right) dx \qquad (3.56)$$

Elementary integration and simplification of terms gives;

$$I_x(y) = \frac{1}{\sqrt{2\pi}} \sum_{j,r=0}^{\infty} r\eta_r c_{r-1,j} 2^{\frac{1}{2}(j+n-1)} \Gamma\left(\frac{j+n+1}{2}, \frac{y^2}{2}\right)$$

$$I_x(y) = \pi^{-\frac{1}{2}} \sum_{j,r=0}^{\infty} r\eta_r c_{r-1,j} 2^{\frac{1}{2}(j+n)-1} \Gamma\left(\frac{j+n+1}{2}, \frac{y^2}{2}\right) (3.57)$$

Where $\Gamma(s,x) = \int_x^{\infty} t^{s-1} e^{-t} dt$ denotes the complementary incomplete gamma function.

3.3 Moment Generating Function

The moment generating function from a standard Weibull-Normal random variable *X(X~WN(α,β,0,1)* can be obtained as

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} f(x) dx \quad (3.58)$$

Recall that by power series expansion,

$$e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n (3.59)$$

Using the result in equation (3.59) and simplifying the integral in (3.58), therefore we have;

$$M_x(t) = E\left[e^{tx}\right] = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{\infty} x^n f(x) dx$$

$$M_x(t) = E\left[e^{tx}\right] = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mu_n^{'} (3.60)$$

where*n* and *t* are constants, *t* is a real number and μ_n' denotes the n^{th} ordinary moment of *X* and can be obtained from equation (3.45) as stated previously.

3.4 The Characteristics Function

The characteristics function has many useful and important properties which give it a central role in statistical theory. Its approach is particularly useful in analysis of linear combination of independent random variables.

A representation for the characteristics function is given by

$$\varphi_x(t) = E\left[e^{itx}\right] = E\left[\cos(tx) + i\sin(tx)\right] = E\left[\cos(tx)\right] + E\left[i\sin(tx)\right] \quad (3.61)$$

Recall from power series expansion that

$$\cos(tx) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} x^{2n}$$

$$E\left[\cos(tx)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \mu'_{2n}$$

And also that

$$\sin(tx) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} x^{2n+1}$$

$$E\left[\sin(tx)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \mu'_{2n+1}$$

Simple algebra and power series expansion proves that

$$\phi_x(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \mu'_{2n} + i\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \mu'_{2n+1} \quad (3.62)$$

Where μ'_{2n} and $\mu'_{2n+1}$are the moments of X for $n=2n$ and $n=2n+1$ respectively and can be obtained from μ'_n in equation (3.45)

3.5 Reliability Analysis of the WN Distribution

3.5.1 The survival function

The survival function, also known as the reliability function in engineering, is the characteristic of an explanatory variable that maps a set of events, usually associated with mortality or failure of some system onto time. It is the probability that the system will survive beyond a specified time.Mathematically, the survival function is given by:

$$S(x)=1-F(x)$$

where*F(x)* is *cdf* of the proposed distribution.

Therefore the survival function of a Weibull-Normal distribution is given by

$$S(X)=1-\left\{1-\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}\right\}$$

$$S(X)=\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\} \qquad (3.63)$$

A plot for the survival function of theWeibull-Normaldistribution at different parameter values is as shown in Figure **3.3** below;

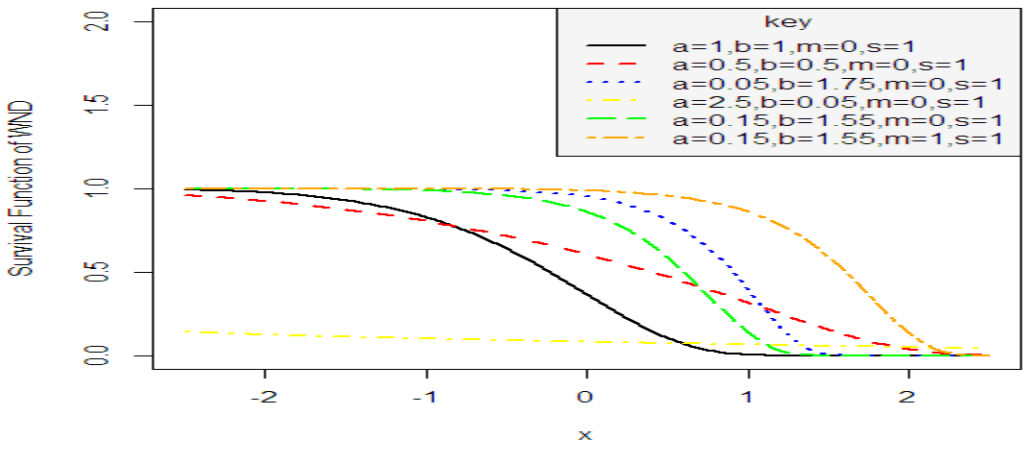

Figure.3.3: Survival function of theWeibull-Normaldistributionat different parameter values where $a = \alpha, b = \beta, m = \mu and s = \sigma$.

From the graph in figure **3.3**, we can see that the value of the survival function equals one (1) at initial time or early age and it decreases as$X$ increases and equals zero (0) as X becomes larger.

3.5.2 The Hazard Function

The hazard function is defined as the probability per unit time that a case which has survived to the beginning of the respective interval will fail in that interval. Specifically, it is computed as the number of failures per unit time in the respective interval, divided by the average number of surviving cases at the mid-point of the interval.

Mathematically, the hazard function for a random variable X is defined as

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \quad (3.64)$$

Where *f(x)* is the pdf and *F(x)* is the cdf of any continuous distribution respectively.

Hence the hazard function associated with the Weibull-Normal distribution from Equation (3.13) and Equation (3.14s) is

$$h(x)=\frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}\exp\left\{\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}$$

$$h(x)=\frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{-[\beta+1]} \qquad (3.65)$$

The following are some possible curves for the hazard rate at various values of themodel parameters

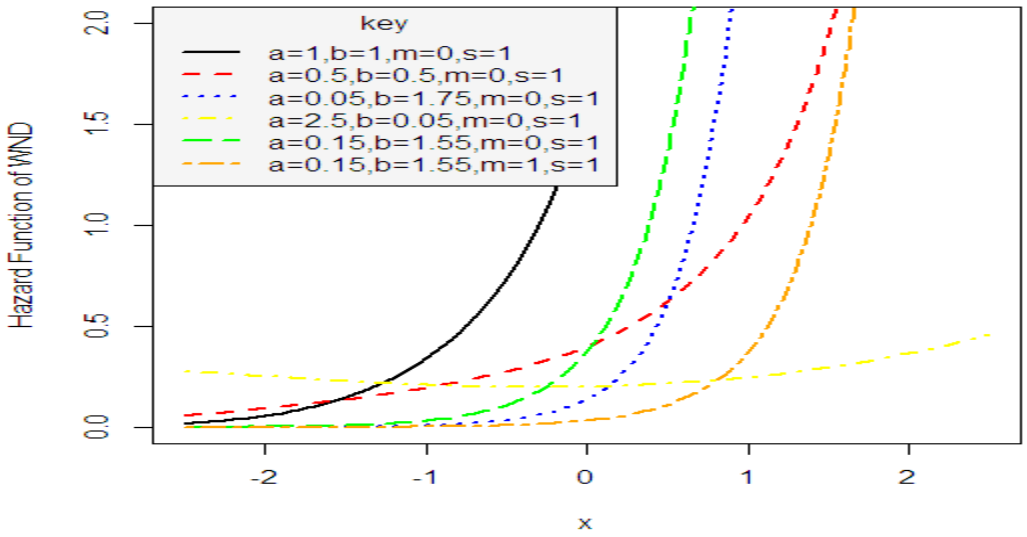

Figure3.4: Hazard function of theWeibull-Normal distributionat different parameter values where $a=\alpha, b=\beta, m=\mu and s=\sigma$.

From the graph, we can see that the value of the hazard function increases when *X* increases. It gets higher as the value of *X* increases. This means that the Weibull-Normaldistributionmay be appropriate in modeling time or age-dependent events, where risk or hazard increases with time or age. Many examples are found in products that fail as a result of the age of those components.

3.6Order Statistics

Order statistics have been used in a wide range of problems including robust statistical estimation and detection of outliers, characterization of probability distributions and goodness of

fit tests, entropy estimation, analyses of censored samples, reliability analysis, quality control and strength of materials. In this section, we derive closed form expressions for the *pdf* of the i^{th} order statistic of the Weibull normal distribution.

Suppose $X_1, X_2, ..., X_n$ is a random sample from the standard Weibull normal (WN) distribution and let $X_{1:n}, X_{2:n}, ..., X_{i:n}$ denote the corresponding order statistic obtained from this sample. The *pdf*, $f_{i:n}(x)$ of the ith order statistic can be obtained by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i} \quad (3.67)$$

Using binomial expansion, we have;

$$\left[1=F(x)\right]^{n-i} = \sum_{k=0}^{n-i}(-1)^k \binom{n-i}{k}\left[F(x)\right]^k \quad (3.68)$$

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k \binom{n-i}{k} f(x)F(x)^{k+i-1} \quad (3.69)$$

Using (3.13) and (3.14), the *pdf* of the i^{th} order statistic$X_{i:n}$, can be expressed from (3.69) as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k \binom{n-i}{k}\left[\frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}\right]\left[1-\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}\right]^{k+i-1} \quad (3.70)$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the Weibull normal distribution are given by

$$f_{1:n}(x)=n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[\frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}\right]\left[1-\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}\right]^{k} \quad (3.71)$$

and

$$f_{n:n}(x)=n\left[\frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}\right]\left[1-\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}\right]^{n-1} \quad (3.72)$$

respectively.

3.7Estimation of Parameters of the Weibull-Normal Distribution

In this section, the estimation of the parameters of the *WND* is done by using the method of maximum likelihood estimation. Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample from the *WND* with unknown parameter vector $\theta = (\alpha, \beta, \mu, \sigma)^T$. The total log-likelihood function for θ is obtained from *f(x)* as follows:

$$f(x)=\frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\} \quad (3.73)$$

The likelihood function is given by;

$$L\left(X_1, X_2, \ldots\ldots, X_n / \alpha, \beta, \mu, \sigma\right)=\left(\frac{\alpha\beta}{\sigma}\right)^n\sum_{i=1}^{n}\phi\left(\frac{x_i-\mu}{\sigma}\right)\frac{\sum_{i=1}^{n}\left[\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{\beta-1}}{\sum_{i=1}^{n}\left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\sum_{i=1}^{n}\left[\frac{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right]^{\beta}\right\} \quad (3.74)$$

Let the log-likelihood function, $l=\log L\left(X_1, X_2, \ldots\ldots, X_n / \alpha, \beta, \mu, \sigma\right)$, therefore

$$l = n\log\alpha + n\log\beta - n\log\sigma - \tfrac{n}{2}\log(2\pi) - \tfrac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 + (\beta-1)\sum_{i=1}^{n}\log\left[\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right] - (\beta+1)\sum_{i=1}^{n}\log\left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right] - \alpha\sum_{i=1}^{n}\left[\frac{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right]^{\beta}$$

(3.75)

Also let $Z_i = \Phi\left(\frac{x-\mu}{\sigma}\right)$

Hence

$$l = n\log\alpha + n\log\beta - n\log\sigma - \tfrac{n}{2}\log(2\pi) - \tfrac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 + (\beta-1)\sum_{i=1}^{n}\log[Z_i] - (\beta+1)\sum_{i=1}^{n}\log[1-Z_i] - \alpha\sum_{i=1}^{n}\left[\frac{Z_i}{1-Z_i}\right]^{\beta} \quad (3.76)$$

Recall that

$$Z_i = \Phi\left(\frac{x-\mu}{\sigma}\right) = F(x) = \int_{-\infty}^{x} f(s)ds \quad (3.77)$$

Such that

$$\frac{\partial Z_i}{\partial \mu} = \frac{\partial F(x)}{\partial \mu} = \int_{-\infty}^{x}\frac{\partial f(s)}{\partial \mu}ds$$

Also recall that

$$f(s) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\tfrac{(s-\mu)^2}{2\sigma^2}\right\} = \frac{1}{\sigma}\phi\left(\frac{s-\mu}{\sigma}\right)$$

Now, taking the partial derivative of *f(x)* with respect to the parameter, μ, we perform the following operations;

Let

$$u = \frac{1}{2}t^2, t = \frac{s-x}{\sigma}$$

Such that

$$f(x = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\tfrac{1}{2}t^2\right\} = \frac{1}{\sigma\sqrt{2\pi}} e^{-u}$$

$$\frac{\partial f(s)}{\partial \mu} = \frac{\partial f(s)}{\partial u} \times \frac{\partial u}{\partial t} \times \frac{\partial t}{\partial \mu}$$ (function of a function rule or the chain rule)

$$\frac{\partial f(s)}{\partial \mu} = \frac{(s-\mu)}{\sigma^3\sqrt{2\pi}} \exp\left\{-\tfrac{(s-\mu)^2}{2\sigma^2}\right\}$$ (3.78)

Hence

$$\frac{\partial F(x)}{\partial \mu} = \frac{\partial Z_i}{\partial \mu} = \int_{-\infty}^{x} \frac{(s-\mu)}{\sigma^3\sqrt{2\pi}} \exp\left\{-\tfrac{(s-\mu)^2}{2\sigma^2}\right\} ds$$ (3.79)

Using integration by substitution, we obtain:

Let

$$V = \frac{1}{2}t^2, t = \frac{s-\mu}{\sigma}$$

which implies that

$$\frac{\partial F(x)}{\partial \mu} = \frac{\partial Z_i}{\partial \mu} = \int_{-\infty}^{x} \frac{(s-\mu)}{\sigma^3\sqrt{2\pi}} \exp\{-V\} ds$$ (3.80)

Such that

$$\frac{\partial V}{\partial s} = \frac{\partial V}{\partial t} \times \frac{\partial t}{\partial s} = \frac{(s-\mu)}{\sigma^2}$$

$$\rightarrow ds = \frac{\sigma^2 dV}{(s-\mu)}$$

Substituting for *ds* in equation (3.80) and integrating, we have

$$\frac{\partial F(x)}{\partial \mu} = \frac{\partial Z_i}{\partial \mu} = \int_{-\infty}^{x} \frac{(s-\mu)}{\sigma^3\sqrt{2\pi}} \exp\{-V\} \frac{\sigma^2 dV}{(s-\mu)}$$

$$\frac{\partial F(x)}{\partial \mu} = \frac{\partial Z_i}{\partial \mu} = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp\{-V\} dV = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-V\} dV$$

Evaluating the integrand by taking the limits, we get:

$$\frac{\partial F(x)}{\partial \mu} = \frac{\partial Z_i}{\partial \mu} = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} (3.81)$$

To obtain the derivative of Z_i with respecttoσ, we will perform similar operations as follows:

Recall that

$$Z_i = \Phi\left(\frac{x-\mu}{\sigma}\right) = F(x) = \int_{-\infty}^{x} f(s) ds$$

Such that

$$\frac{\partial Z_i}{\partial \sigma} = \frac{\partial F(x)}{\partial \sigma} = \int_{-\infty}^{x} \frac{\partial f(s)}{\partial \sigma} ds$$

Also recall that

$$f(s)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(s-\mu)^2}{2\sigma^2}\right\}=\frac{1}{\sigma}\phi\left(\frac{s-\mu}{\sigma}\right)$$

Now, taking the partial derivative of $f(s)$ with respect to the parameter, σ, we perform the following operations;

Let

$$u=\frac{1}{2}t^2, t=\frac{s-x}{\sigma}$$

Such that

$$f(x=\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}t^2\right\}=\frac{1}{\sigma\sqrt{2\pi}}e^{-u}$$

$$\frac{\partial f(s)}{\partial\sigma}=\frac{\partial f(s)}{\partial u}\times\frac{\partial u}{\partial t}\times\frac{\partial t}{\partial\sigma}$$ (function of a function rule or the chain rule)

$$\frac{\partial f(s)}{\partial\sigma}=\frac{(s-\mu)^2}{\sigma^4\sqrt{2\pi}}\exp\left\{-\frac{(s-\mu)^2}{2\sigma^2}\right\}$$ (3.82)

Therefore,

$$\frac{\partial F(x)}{\partial\sigma}=\frac{\partial Z_i}{\partial\sigma}=\int_{-\infty}^{x}\frac{(s-\mu)^2}{\sigma^4\sqrt{2\pi}}\exp\left\{-\frac{(s-\mu)^2}{2\sigma^2}\right\}ds$$ (3.83)

Using integration by substitution, we perform these operations:

Let

$$V = \frac{1}{2}t^2, t = \frac{s-\mu}{\sigma}$$

which implies that

$$\frac{\partial F(x)}{\partial \sigma} = \frac{\partial Z_i}{\partial \sigma} = \int_{-\infty}^{x} \frac{(s-\mu)^2}{\sigma^4\sqrt{2\pi}} \exp\{-V\}\, ds \quad (3.84)$$

Such that

$$\frac{\partial V}{\partial s} = \frac{\partial V}{\partial t} * \frac{\partial t}{\partial s} = \frac{(s-\mu)}{\sigma^2}$$

$$\rightarrow ds = \frac{\sigma^2 dV}{(s-\mu)}$$

Substituting for *ds* in equation (3.84) and integrating, we have

$$\frac{\partial F(x)}{\partial \sigma} = \frac{\partial Z_i}{\partial \sigma} = \int_{-\infty}^{x} \frac{(s-\mu)^2}{\sigma^4\sqrt{2\pi}} \exp\{-V\} \frac{\sigma^2 dV}{(s-\mu)} \quad (3.85)$$

$$\frac{\partial F(x)}{\partial \sigma} = \frac{\partial Z_i}{\partial \sigma} = \int_{-\infty}^{x} \frac{(s-\mu)}{\sigma^2\sqrt{2\pi}} \exp\{-V\}\, dV = \frac{(s-\mu)}{\sigma^2\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-V\}\, dV$$

Evaluating the integrand by taking the limits, we get:

$$\frac{\partial F(x)}{\partial \sigma} = \frac{\partial Z_i}{\partial \sigma} = -\frac{(s-\mu)}{\sigma^2\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \quad (3.86)$$

Therefore, the partial derivative of $Z_i$with respect to μ and σ are given as:

$$\frac{\partial Z_i}{\partial \mu} = -\frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \quad (3.87)$$

and

$$\frac{\partial F(x)}{\partial \sigma} = \frac{\partial Z_i}{\partial \sigma} = -\frac{(s-\mu)}{\sigma^2\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \quad (3.88)$$

respectively.

Meanwhile, differentiating $l(\theta)$ partially with respect to each parameter $\alpha, \beta, \mu \; and \; \sigma$ and setting the results equal to zero gives the maximum likelihood estimates of the respective parameters. The partial derivatives of $l(\theta)$with respect to each parameter or the score function is given by:

$$U_n(\theta) = \left(\frac{\partial(\theta)}{\partial\alpha}, \frac{\partial(\theta)}{\partial\beta}, \frac{\partial(\theta)}{\partial\mu}, \frac{\partial(\theta)}{\partial\sigma}\right)$$

wherethe components of the score vector $U(\theta) = \left(U_\alpha, U_\beta, U_\mu, U_\sigma\right)$ are:

$$U_\alpha = \frac{\partial l(\theta)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^\beta \quad (3.89)$$

$$U_\beta = \frac{\partial l(\theta)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log\left[\frac{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right] - \alpha\sum_{i=1}^{n}\left[\frac{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right]^\beta \log\left[\frac{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right] \quad (3.90)$$

$$U_{\mu}=\frac{\partial l(\theta)}{\partial \mu}=\frac{n}{\sigma^{2}}\sum_{i=1}^{n}(x_i-\mu)-\frac{(\beta-1)}{\sigma\sqrt{2\pi}}\sum_{i=1}^{n}\left\{\frac{\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^{2}\right]}{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right\}-\frac{(\beta+1)}{\sigma\sqrt{2\pi}}\sum_{i=1}^{n}\left\{\frac{\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^{2}\right]}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right\}$$
$$+\frac{\alpha\beta}{\sigma\sqrt{2\pi}}\sum_{i=1}^{n}\left[\frac{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right]^{\beta-1}\left[\frac{\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^{2}\right]}{\left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{2}}\right] \quad (3.91)$$

$$U_{\sigma}=\frac{\partial l(\theta)}{\partial \sigma}=-\frac{n}{\sigma}+\frac{n}{\sigma^{3}}\sum_{i=1}^{n}(x_i-\mu)^{2}-\frac{(\beta-1)}{\sigma^{2}\sqrt{2\pi}}\sum_{i=1}^{n}\left\{\frac{(x-\mu)\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^{2}\right]}{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right\}$$
$$-\frac{(\beta+1)}{\sigma^{2}\sqrt{2\pi}}\sum_{i=1}^{n}\left\{\frac{(x-\mu)\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^{2}\right]}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right\}+\frac{\alpha\beta}{\sigma^{2}\sqrt{2\pi}}\sum_{i=1}^{n}\left[\frac{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right]^{\beta-1}\left[\frac{(x-\mu)\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^{2}\right]}{\left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{2}}\right] \quad (3.92)$$

Maximization of (3.75) can be performed by using well established routines like the nlm routine or optimize in the R statistical package. Setting these equations to zero, $U(\theta)=0$, and solving them simultaneously yields the maximum likelihood estimate (MLE) $\hat{\theta}$ of θ. These equations cannot be solved analytically and therefore statistical software can be used to solve them numerically by means of iterative techniques like the Newton-Raphson algorithm.

CHAPTER FOUR: ANALYSIS AND DISCUSSION

In this chapter, the potentiality of the Weibull-Normal distribution (model) is illustrated in two applications to real data. We have compared the fits of the Weibull-Normal (WN)distribution

with those of five models including the Beta-Normal distribution, the Kummaraswamy-Normal distribution, Gamma-Normal, Kummer Bata-Normal and the Normal distribution. The density functions for these distributions are given as follows;

The Beta-Normal Distribution

The *pdf* of the *BN* distribution is given as;

$$f(x;\alpha,\beta,\mu,\sigma)=\frac{\Gamma(\alpha+\beta)}{\sigma\Gamma(\alpha)\Gamma(\beta)}\phi\left(\frac{x-\mu}{\sigma}\right)\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha-1}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1} \qquad (4.1)$$

The Kummaraswamy-Normal Distribution

The *pdf* of the *KwN* is given as;

$$f(x;a,b,\mu,\sigma)=\frac{ab}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{a-1}\left[1-\Phi a\left(\frac{x-\mu}{\sigma}\right)\right]^{b-1} \qquad (4.2)$$

The Normal Distribution

The *pdf* of the normal distribution is given by;

$$f(x;\mu,\sigma)=\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) (4.3)$$

The Weibull-Normal Distribution

The pdf of the WN distribution is given as;

$$f(x;\alpha,\beta,\mu,\sigma)=\frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\} (4.4)$$

The Gamma-normal Distribution

The pdf of the Gamma-Normal distribution is given as:

$$f(x;a,\mu,\sigma)=\frac{1}{\sigma\Gamma(a)}\phi\left(\frac{x-\mu}{\sigma}\right)\left\{-\log\left[1-\Phi a\left(\frac{x-\mu}{\sigma}\right)\right]\right\}^{a-1} \quad (4.5)$$

The Kummer Beta-Normal distribution

Its pdf is given as follows:

$$f(x;a,b,c,\mu,\sigma)=\frac{K}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{a-1}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{b-1}\exp\left\{-c\Phi\left(\frac{x-\mu}{\sigma}\right)\right\} (4.6)$$

Where $a,b,c,\alpha,\beta,\sigma>0; and\ -\infty<\mu<\infty$ are the model parameters.

Data set I: This data is on the strength of 1.5cm glass fibers. The data was originally obtained by workers at the UK national physical laboratory and it has been used by smith and Naylor (1987),Barreto-Souza *et al*. (2011), Bourguignon *et al*. (2014),Oguntunde e*t al*. (2015) and Afify and Aryal (2016).This data can be found in **Table 4.1** as follows.

Table 4.1: Data set I

0.55	0.74	0.77	0.81	0.84	0.93	0.104	1.11	1.13	1.24	1.25	1.271.28	1.29		
1.30	1.36	1.39	1.42	1.48	1.51	1.52	1.53	1.54	1.55	1.55	1.58	1.61	1.62	
1.62	1.63	1.64	1.66	1.66	1.66	1.67	1.68	1.68	1.69	1.70	1.70	1.73	1.76	1.77
1.77	1.78	1.81	1.82	1.84	1.84	1.89	2.00	2.01	2.24					

Data set II: The second data set represents 66 observations of the breaking stress of carbon fibres of 50mm length (in GPa) given by Nicholas and Padgett (2006). This data set has been used by Cordeiro and Lemonte (2011), Al-Aqtash*et al*.,(2014), Afify*et al*., (2014), Oguntunde*et al*., (2015) and Afify*et al*.,(2016). This data can be found in **Table 4.2** as follows.

Table 4.2: Data set II

0.39, 1.87, 2.53, 2.85, 3.15, 3.56, 0.85, 1.89, 2.55, 2.87, 3.19, 3.60, 1.08, 2.03, 2.55, 2.88, 3.22, 3.65, 1.25, 2.03, 2.56, 2.93, 3.22, 3.68, 1.47, 2.05, 2.59, 2.95, 3.27, 3.70, 1.57, 2.12, 2.67, 2.96, 3.28, 3.75, 1.61, 2.35, 2.73, 2.97, 3.31, 4.20, 1.61, 2.41, 2.74, 3.09, 3.31, 4.38, 1.69, 2.43, 2.79, 3.11, 3.33, 4.42, 1.80, 2.48, 2.81, 3.11, 3.39, 4.70, 1.84, 2.50, 2.82, 3.15, 3.39, 4.90

The summary of the two data sets is also provided in **Table 4.3** as follows;

Table 4.3: Summary of the two data sets

Parameters	Data set I	Data set II
n	63	66
Minimum	0.550	0.390
Q_1	1.375	2.178
Median	1.590	2.835
Q_3	1.685	3.278

Mean	1.507	2.760
Maximum	2.240	4.900
Variance	0.105	0.795
Skewness	-0.8786	-0.1285
Kurtosis	3.9238	3.2230

We also provide some histograms for the two data sets as shown in **Figure 4.1** and **4.2**below respectively.

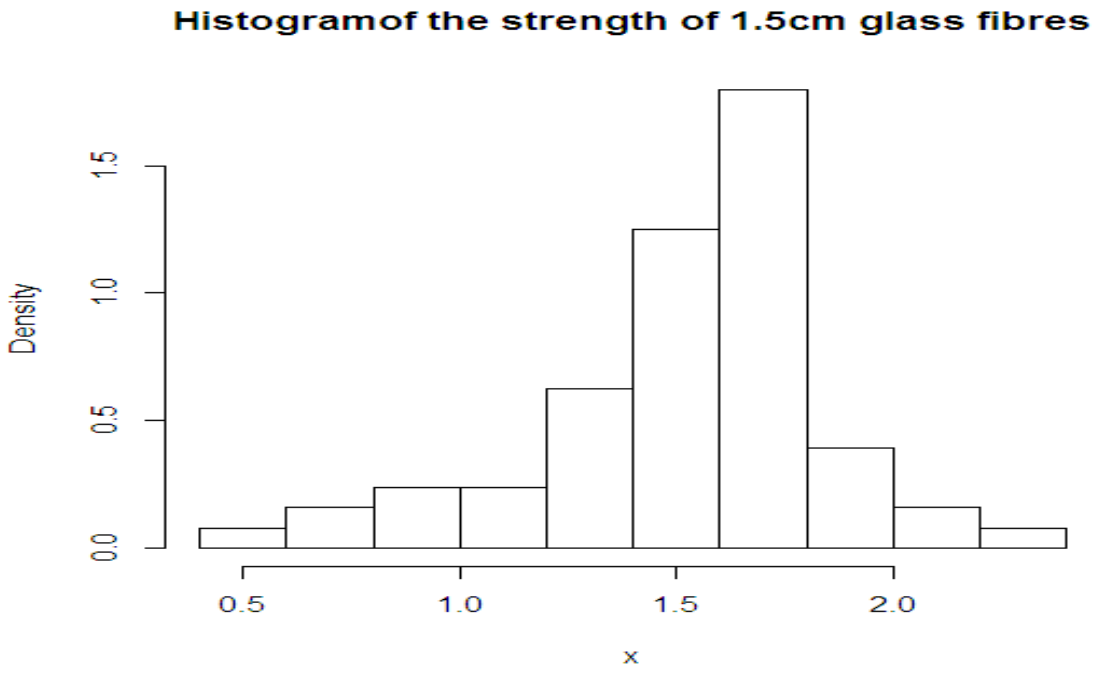

Figure4.1: A histogram for data set I

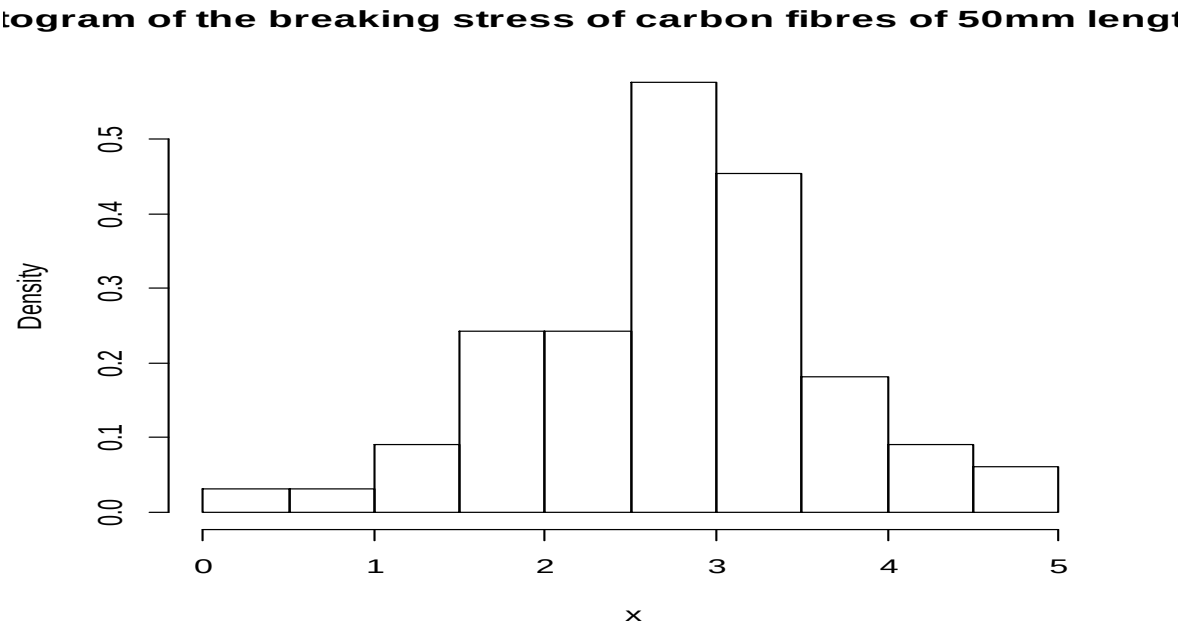

Figure4.2: A histogram for data set II

Both the summary and the histograms for the two data sets show that the first data set is skewed and therefore suitable for skewed or asymmetric distributions which the Weibull-Normal distribution is not an exception, while the second data set is approximately normal or symmetric which might not be analyzed adequately using asymmetric distributions like the Weibull-Normal distribution.

In order to compare these distributions, we will consider some criteria: the*AIC* (Akaike Information Criterion), *CAIC* (Consistent Akaike Information Criterion) and*BIC* (Bayesian Information Criterion). These statistics are given as:

$$AIC = -2ll + 2k$$
$$BIC = -2ll + k\log(n), and$$
$$CAIC = -2ll + \frac{2kn}{(n-k-1)}$$

Where $l\,l$ denotes the log-likelihood function evaluated at the *MLEs*, k is the number of model parameters and n is the sample size.

Note: The model with the lowest values for these statistics would be chosen as the best model to fit the data.

Table 4.4: Performance of the distribution using the *AIC*, *CAIC* and *BIC* values of the models based on data set I (strength of 1.5cm glass fibres).

Distributions	Parameter estimates	-$l\,l$ =(minus log-likelihood)	AIC	CAIC	BIC	Ranks of modelsperformance
Weibull-Normal	$\hat{\alpha}$=2.6945 $\hat{\beta}$=1.8183 $\hat{\mu}$=1.9253 $\hat{\sigma}$=0.8134	14.55	37.1	37.79	36.30	1
Kummaraswamy-Normal	$\hat{a}$=0.1959 $\hat{b}$=1.2525 $\hat{\mu}$=1.9219 $\hat{\sigma}$=0.1811	15.50	39.0	39.69	38.19	2
Normal	$\hat{\mu}$=1.5075 $\hat{\sigma}$=0.3191	17.70	39.4	39.60	38.99	3

Beta-Normal	$\hat{\alpha}$=1.0662 $\hat{\beta}$=2.5550 $\hat{\mu}$=1.7857 $\hat{\sigma}$=0.4019	15.93	39.86	40.55	39.06	4
Gamma-Normal	$\hat{a}$=5.8119 $\hat{\mu}$=4.0708 $\hat{\sigma}$=6.9595	43.2172	92.4344	91.8324	92.8412	6
Kummer Beta-Normal	$\hat{a}$=0.0122 $\hat{b}$=2.3949 $\hat{c}$=1.1020 $\hat{\mu}$=0.8118 $\hat{\sigma}$=0.7701	15.8533	41.7066	40.7033	42.7592	5

Table 4.4 shows parameter MLEs to each one of the six fitted distributions for the first data set (Data set I), the table alsoshows the corresponding values of -$\ell\ \ell$, *AIC*, *BIC* and *CAIC* for each model. The values in **Table 4.4** are evidence that the Weibull-Normal distribution has better performance and could be chosen as the best model compared to the other fivedistributions used here for fitting the same data set. Since this data is highly skewed to the left, we can equally say that this distribution is an appropriate model for analyzing asymmetric or non-normally distributed data sets most especially those skewed to the left. Hence, the Weibull-Normal distribution performs better than the Kumaraswamy-Normal, Beta-Normal, Gamma-Normal, Kummer Beta-Normal and the normal distributions when the data set is negatively skewed.

Table 4.5: Performance of the distribution using the *AIC*, *CAIC* and *BIC* values of the models based on data set II (breaking stress of carbon fibres of 50mm length in GPa).

Distributions	Parameter estimates	-l l =(minus loglikelihood)	AIC	AICC	BIC	Ranks based on modelsperformance
Weibull-Normal	$\hat{\alpha}$=4.7299 $\hat{\beta}$=0.3342 $\hat{\mu}$=4.8112 $\hat{\sigma}$=0.7494	85.1929	178.3858	179.0415	177.6639	4
Kummaraswamy-Normal	$\hat{a}$=0.5607 $\hat{b}$=0.0786 $\hat{\mu}$=0.6075 $\hat{\sigma}$=0.5299	99.1788	206.3576	207.0133	205.6356	6
Normal	$\hat{\mu}$=2.7595 $\hat{\sigma}$=0.8847	85.5625	175.1250	175.3155	174.7640	1

Beta-Normal	$\hat{\alpha}$=6.7049 $\hat{\beta}$=8.1327 $\hat{\mu}$=3.0991 $\hat{\sigma}$=2.6577	85.5276	179.0552	179.7109	178.3332	5
Gamma-Normal	$\hat{\alpha}$=1.9545 $\hat{\mu}$=1.8807 $\hat{\sigma}$=1.0054	85.7551	177.5102	176.9688	177.8973	2
Kummer Beta-Normal	$\hat{a}$=0.2096 $\hat{b}$=5.3899 $\hat{c}$=7.1186 $\hat{\mu}$=9.8267 $\hat{\sigma}$=2.0671	83.9363	177.8726	176.9703	178.8726	3

Table 4.5shows the parameter estimates to each one of the six selected and fitted distributions for the data set (data set II), the table alsoprovide the values of -ℓ ℓ , *AIC*, *BIC* and *CAIC* of the fitted models. The values in **Table 4.5**unlike those in the previous table (Table 4.4) indicate that theNormal distribution has better performance with the lowest values of *AIC*, *CAIC* and *BIC* followed by the Gamma-Normal, Kummer Beta-Normal, Weibull-Normal, Beta-Normal and the normal distributions. The reason behind this performance is simply because our second data set is approximately normal or symmetric and could be model best by the Normal distribution. This particular point also proved the fact that the Weibull-Normal distribution is a skewed model and can only be appropriate for modeling negatively skewed data sets as illustrated previously using the first data set.

CHAPTER FIVE: SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 SUMMARY

In this dissertation, we introduced a new four parameter model named the Weibull-Normal (WN) distribution, which extends the well-known Normal distribution by adding skewness to the classical Normal distribution. An obvious reason for generalizing a classical distribution is the fact that the generalization provides more flexibility to analyze real life data. The new distribution has proved to be versatile and analytically tractable during the generalization process. The Weibull-Normal distribution density function was expressed as a linear combination of exponentiated normal density function which allowed us to derive some of its mathematical properties like its ordinary and incomplete moments, moment generating function, characteristics function, reliability analysis and order statistics. The estimation of parameters has been done using the method of maximum likelihood estimation. The usefulness of the Weibull-normal distribution has been illustrated by some applications to two real data sets. The results

showed that thenew distribution (Weibull-Normal distribution) performs better (provides better fits) than the generalizations of the normal distribution such as Kumaraswamy-Normal, Beta-Normal, Gamma-Normal, Kummer Beta-Normal and the normal distributions when the data set is negatively skewed. However, the results from the second data confirmed that this distribution is more flexible and appropriate for modeling negativelyskewed data sets.

5.2CONCLUSION

In this research, a new four-parameter model called the Weibull-Normal distribution (*WND)* has been proposed. Some mathematical and statistical properties of the proposed distribution have been studied appropriately. The researcher has derived explicit expressions for its moments, moment generating function, characteristics function, survival function, hazard function, and ordered statistics. Some plots of the distribution revealed that it is a skewed distribution and has only one mode. The model parameters have been estimated using the method of maximum likelihood estimation.The study hasproposed reliability measures widely used in engineering using the Weibull-Normal distribution. This study has derived explicit expressions for the survival and hazard function of the distribution. The implications of the plots for the survival and hazard functions indicate that the Weibull-Normal distribution would be appropriate in modeling time or age-dependent events, where survival and failure rate decreases with time or age.

5.3 RECOMMENDATIONS

Based on the findings of this research, we recommend that the proposed distribution should be used for modeling if the data set in question is skewed most especially if it is negatively skewed. We also recommend that this distribution can be used in modeling time or age dependent events, systems, components or random variables.

5.4CONTRIBUTION TO KNOWLEDGE

We have proposed a four parameter distribution which is useful for modeling skewed and approximately symmetry data sets better than some existing compound normal based probability distributions.

We have also derived some of its properties which are useful for studying the shape characteristics of the distribution as well as the reliability functions which are useful in the field of engineering.

We have clearly estimated the parameters of the new distribution as a major step necessary for fitting/using any model both new and existing.

Finally, we have tested the performance of our model compared to some existing baseline distributions and therefore identify our model as the most appropriate when dealing with negatively skewed data sets or age-dependent random variables.

5.5 AREAS OF FURTHER RESEARCH

Based on our findings in this area of research, subsequent studies can look at estimation of confidence intervals for the parameters of the proposed distribution. Also, researchers can estimate the parameters of the new distribution using a non-classical approach for the purpose of theoretical comparison of methodology.

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APPENDIX

R COMMANDS USED IN THIS DISSERTATION

A-PLOTS

```
>a=1

> b=1

> m=0

> s=1

> x=rnorm(10,0,1)
```

PDF of the Weibull-Normal distribution=WN

```
> WN=function(x,a,b,m,s){a*b/s*dnorm((x-m)/s)*(((pnorm((x-m)/s)^(b-1))/((1-pnorm((x-m)/s))^(b+1)))*exp(-a*((pnorm((x-m)/s))/(1-pnorm((x-m)/s)))^b)}

> curve(WN(x,1,1,0,1),from=-2.5, to=2.5,lty=1,col="black",lwd=2,ylim=c(0,2),ylab="PDF of WND")

> curve(WN(x,0.5,0.5,0,1),from=-2.5, to=2.5,add=T,lty=2,col="red",lwd=2,ylim=c(0,2),ylab="PDF of WND")

> curve(WN(x,0.05,1.75,0,1),from=-2.5, to=2.5,add=T,lty=3,col="blue",lwd=2,ylim=c(0,2),ylab="PDF of WND")

> curve(WN(x,2.5,0.05,0,1),from=-2.5, to=2.5,add=T,lty=4,col="yellow",lwd=2,ylim=c(0,2),ylab="PDF of WND")
```

```
> curve(WN(x,0.15,1.55,0,1),from=-2.5, to=2.5,add=T,lty=5,col="green",lwd=2,ylim=c(0,2),ylab="PDF of
WND")

> curve(WN(x,0.15,1.55,1,1),from=-2.5, to=2.5,add=T,lty=6,col="orange",lwd=2,ylim=c(0,2),ylab="PDF of
WND")

>legend("topleft",cex=1,title="key",c("a=1,b=1,m=0,s=1","a=0.5,b=0.5,m=0,s=1","a=0.05,b=1.75,m=0,s=
1","a=2.5,b=0.05,m=0,s=1","a=0.15,b=1.55,m=0,s=1","a=0.15,b=1.55,m=1,s=1"),horiz=FALSE,lty=c(1,2,3,
4,5,6),lwd=c(2,2,2,2,2),col=c("black","red","blue","yellow","green","orange"),bg="grey96")
```

RELIABILITY ANALYSIS

Survival function=SFWN

```
> SFWN=function(x,a,b,m,s){exp(-a*((pnorm((x-m)/s))/(1-pnorm((x-m)/s)))^b)}

> curve(SFWN(x,1,1,0,1),from=-2.5, to=2.5,lty=1,col="black",lwd=2,ylim=c(0,2),ylab="Survival Function
of WND")

> curve(SFWN(x,0.5,0.5,0,1),from=-2.5, to=2.5,add=T,lty=2,col="red",lwd=2,ylim=c(0,2),ylab="Survival
Function of WND")

> curve(SFWN(x,0.05,1.75,0,1),from=-2.5,
to=2.5,add=T,lty=3,col="blue",lwd=2,ylim=c(0,2),ylab="Survival Function of WND")

> curve(SFWN(x,2.5,0.05,0,1),from=-2.5,
to=2.5,add=T,lty=4,col="yellow",lwd=2,ylim=c(0,2),ylab="Survival Function of WND")

> curve(SFWN(x,0.15,1.55,0,1),from=-2.5,
to=2.5,add=T,lty=5,col="green",lwd=2,ylim=c(0,2),ylab="Survival Function of WND")

> curve(SFWN(x,0.15,1.55,1,1),from=-2.5,
to=2.5,add=T,lty=6,col="orange",lwd=2,ylim=c(0,2),ylab="Survival Function of WND")

>legend("topright",cex=1,title="key",c("a=1,b=1,m=0,s=1","a=0.5,b=0.5,m=0,s=1","a=0.05,b=1.75,m=0,s
=1","a=2.5,b=0.05,m=0,s=1","a=0.15,b=1.55,m=0,s=1","a=0.15,b=1.55,m=1,s=1"),horiz=FALSE,lty=c(1,2,3
,4,5,6),lwd=c(2,2,2,2,2),col=c("black","red","blue","yellow","green","orange"),bg="grey96")
```

Hazard function=HFWN

```
>HFWN=function(x,a,b,m,s){a*b/s*dnorm((x-m)/s)*((pnorm((x-m)/s)^(b-1))/((1-pnorm((x-
m)/s))^(b+1)))}

> curve(HFWN(x,1,1,0,1),from=-2.5, to=2.5,lty=1,col="black",lwd=2,ylim=c(0,2),ylab="Hazard Function of
WND")

> curve(HFWN(x,0.5,0.5,0,1),from=-2.5, to=2.5,add=T,lty=2,col="red",lwd=2,ylim=c(0,2),ylab="Hazard
Function of WND")
```

```
> curve(HFWN(x,0.05,1.75,0,1),from=-2.5,
to=2.5,add=T,lty=3,col="blue",lwd=2,ylim=c(0,2),ylab="Hazard Function of WND")

> curve(HFWN(x,2.5,0.05,0,1),from=-2.5,
to=2.5,add=T,lty=4,col="yellow",lwd=2,ylim=c(0,2),ylab="Hazard Function of WND")

> curve(HFWN(x,0.15,1.55,0,1),from=-2.5,
to=2.5,add=T,lty=5,col="green",lwd=2,ylim=c(0,2),ylab="Hazard Function of WND")

> curve(HFWN(x,0.15,1.55,1,1),from=-2.5,
to=2.5,add=T,lty=6,col="orange",lwd=2,ylim=c(0,2),ylab="Hazard Function of WND")

>legend("topleft",cex=1,title="key",c("a=1,b=1,m=0,s=1","a=0.5,b=0.5,m=0,s=1","a=0.05,b=1.75,m=0,s=
1","a=2.5,b=0.05,m=0,s=1","a=0.15,b=1.55,m=0,s=1","a=0.15,b=1.55,m=1,s=1"),horiz=FALSE,lty=c(1,2,3,
4,5,6),lwd=c(2,2,2,2,2),col=c("black","red","blue","yellow","green","orange"),bg="grey96")
```

B-DATA ANALYSIS

Data Set I

```
x=c(0.55,1.28,1.51,1.61,1.70,2.00,0.74,1.29,1.52,1.62,1.7,2.01,0.77,1.3,1.53,1.62,1.73,2.24,0.81,1.36,1.5
4,1.63,1.76,0.84,1.39,1.55,1.55,1.64,1.76,0.93,1.42,1.55,1.66,1.77,1.04,1.48,1.58,1.66,1.78,1.11,1.48,1.5
9,1.66,1.81,1.13,1.49,1.6,1.67,1.82,1.24,1.49,1.61,1.68,1.84,1.25,1.5,1.61,1.68,1.84,1.27,1.50,1.61,1.69,
1.89)
```

Weibull-Normal distribution

```
> loglike1=function(p,x){-sum(log(p[1]*p[2]/p[4]*dnorm((x-p[3])/p[4])*(((pnorm((x-p[3])/p[4]))^(p[2]-
1))/((1-pnorm((x-p[3])/p[4]))^(p[2]+1)))*exp(-p[1]*((pnorm((x-p[3])/p[4]))/(1-pnorm((x-
p[3])/p[4])))^p[2]))}

> result1=optim(c(2.5,0.5,1.6,0.8),method="CG",fn=loglike1,hessian=T,x=x)

> result1$convergence

[1] 1

> result1

$par

[1] 2.6945420 1.8182798 1.9252568 0.8134207

$value

[1] 14.55321

$counts

function gradient

   314      101
```

```
$convergence
[1] 1
$message
NULL
$hessian
         [,1]      [,2]      [,3]      [,4]
[1,]   8.814753  -7.152704  -85.93153   16.40136
[2,]  -7.152704  41.231372   76.80800  -97.90449
[3,] -85.931533  76.808004  843.25698 -175.57286
[4,]  16.401355 -97.904486 -175.57286  234.48872
Normal distribution
>loglike3=function(p,x){-sum(log(1/p[2]*dnorm((x-p[1])/p[2])))}
> result3=optim(c(1.6,0.8),method="CG",fn=loglike3,hessian=T,x=x)
> result3$convergence
[1] 1
> result3
$par
[1] 1.5075016 0.3190663
$value
[1] 17.7012
$counts
function gradient
   1148      101
$convergence
[1] 1
$message
NULL
$hessian
```

```
          [,1]         [,2]
[1,] 628.66329689   -0.00635638
[2,]  -0.00635638 1257.43446033
```

Kumaraswamy-Normal distribution

```
> loglike4=function(p,x){-sum(log(p[1]*p[2]/p[4]*dnorm((x-p[3])/p[4])*((pnorm((x-p[3])/p[4]))^(p[1]-
1))*((1-(pnorm((x-p[3])/p[4]))^p[1])^(p[2]-1))))}
> result4=optim(c(0.8,0.4,0.7,0.9),method="CG",fn=loglike4,hessian=T,x=x)
> result4$convergence
[1] 1
> result4
$par
[1] 0.1958995 1.2525326 1.9218547 0.1810566
$value
[1] 15.50168
$counts
function gradient
    329      101
$convergence
[1] 1
$message
NULL
$hessian
         [,1]      [,2]      [,3]      [,4]
[1,]  2031.5240 -201.95410  1022.7743 -3313.4989
[2,]  -201.9541   40.79458  -168.0632   192.0334
[3,]  1022.7743 -168.06322   723.3113 -1258.0794
[4,] -3313.4989  192.03344 -1258.0794  6305.2513
```

Gamma-Normal distribution

```
> loglike6=function(p,x){-sum(log(1/(p[3]*gamma(p[1]))*dnorm((x-p[2])/p[3])*((-log(1-pnorm((x-
p[2])/p[3])))^(p[1]-1)))}

> result6=optim(c(0.8,0.5,0.7),method="CG",fn=loglike6,hessian=T,x=x)

> result6$convergence

[1] 1

> result6

$par

[1] 5.811970   4.070783  6.959472

$value

[1] 43.21716

$counts

function gradient

    356      101

$convergence

[1] 1

$message

NULL

$hessian

           [,1]      [,2]       [,3]

[1,] 2.035991e+03  0.4482117 0.02667184

[2,] 4.482117e-01 -0.1486733 0.42239299

[3,] 2.667184e-02  0.4223930 0.04832509

Kummer Beta Normal distribution

>  loglike7=function(p,x){-sum(log(1/(p[5])*dnorm((x-p[4])/p[5])*((pnorm((x-p[4])/p[5]))^(p[1]-1))*((1-
pnorm((x-p[4])/p[5]))^(p[2]-1))*exp(-p[3]*(pnorm((x-p[4])/p[5])))))}

> result7=optim(c(0.4,0.9,0.8,0.5,0.7),method="CG",fn=loglike7,hessian=T,x=x)

> result7$convergence

[1] 1
```

> result7

$par

[1] 0.01219683 2.39489581 1.10202536 0.81178116 0.77012166

$value

[1] 15.85333

$counts

function gradient

372 132

$convergence

[1] 1

$message

NULL

$hessian

[,1] [,2] [,3] [,4] [5,]

[1,] 382802.469 7772.60109 22100.0360 3166.58216 452.1123

[2,] 7772.601 175.56150 471.4569 69.53929 331.7655

[3,] 22100.036 471.45691 1319.1551 191.73637 211.1918

[4,] 3166.582 69.53929 191.7364 28.07536 101.0097

[5,] 452.1123 331.7655 211.191 101.0097 71.9844

Data set II

>x=c(0.39,1.87,2.53,2.85,3.15,3.56,0.85,1.89,2.55,2.87,3.19,3.60,1.08,2.03,2.55,2.88,3.22,3.65,1.25,2.03,2.56,2.93,3.22,3.68,1.47,2.05,2.59,2.95,3.27,3.70,1.57,2.12,2.67,2.96,3.28,3.75,1.61,2.35,2.73,2.97,3.31,4.20,1.61,2.41,2.74,3.09,3.31,4.38,1.69,2.43,2.79,3.11,3.33,4.42,1.80,2.48,2.81,3.11,3.39,4.70,1.84,2.50,2.82,3.15,3.39,4.90)

Weibull-Normal distribution

> loglike1=function(p,x){-sum(log(p[1]*p[2]/p[4]*dnorm((x-p[3])/p[4])*(((pnorm((x-p[3])/p[4]))^(p[2]-1))/((1-pnorm((x-p[3])/p[4]))^(p[2]+1)))*exp(-p[1]*((pnorm((x-p[3])/p[4]))/(1-pnorm((x-p[3])/p[4])))^p[2]))}

> result1=optim(c(4.0,3.5,3.8,3.2),method="CG",fn=loglike1,hessian=T,x=x)

> result1$convergence

```
[1] 1

> result1

$par

[1] 4.7299912 0.3341683 4.8111816 0.7493569

$value

[1] 85.19286

$counts

function gradient

    301      101

$convergence

[1] 1

$message

NULL

$hessian

          [,1]       [,2]      [,3]       [,4]

[1,]   2.950009   -47.42093  -14.79398    28.95241

[2,] -47.420931  1732.72138  348.55309 -1176.74284

[3,] -14.793983   348.55309   87.96808  -227.89538

[4,]  28.952414 -1176.74284 -227.89538   813.49476

Beta-Normal distribution

> loglike2=function(p,x){-sum(log(gamma(p[1]+p[2])/(p[4]*gamma(p[1])*gamma(p[2]))*dnorm((x-
p[3])/p[4])*((pnorm((x-p[3])/p[4]))^(p[1]-1))*((1-pnorm((x-p[3])/p[4]))^(p[2]-1))))}

> result2=optim(c(07,03,05,09),method="CG",fn=loglike2,hessian=T,x=x)

> result2$convergence

[1] 1

> result2

$par

[1] 6.704873 8.132701 3.099066 2.657730
```

```
$value
[1] 85.52761
$counts
function gradient
    282      101
$convergence
[1] 1
$message
NULL
$hessian
         [,1]       [,2]      [,3]       [,4]
[1,]  6.012557  -4.6014265  22.15479  -4.6500690
[2,] -4.601426   4.0332831 -18.16047   0.6790688
[3,] 22.154790 -18.1604743  84.50262 -10.2759925
[4,] -4.650069   0.6790688 -10.27599  19.7619687
Normal distribution
> loglike3=function(p,x){-sum(log(1/p[2]*dnorm((x-p[1])/p[2])))}
> result3=optim(c(1.0,0.2),method="CG",fn=loglike3,hessian=T,x=x)
> result3$convergence
[1] 1
> result3
$par
[1] 2.7595456 0.8846742
$value
[1] 85.56251
$counts
function gradient
    250      101
```

$convergence

[1] 1

$message

NULL

$hessian

[,1] [,2]

[1,] 8.432905e+01 -2.143352e-05

[2,] -2.143352e-05 1.686595e+02

Kumaraswamy-Normal distribution

```
> loglike4=function(p,x){-sum(log(p[1]*p[2]/p[4]*dnorm((x-p[3])/p[4])*((pnorm((x-p[3])/p[4]))^(p[1]-1))*((1-(pnorm((x-p[3])/p[4]))^p[1])^(p[2]-1)))))}

> result4=optim(c(0.4,0.2,0.5,0.8),method="CG",fn=loglike4,hessian=T,x=x)

> result4$convergence

[1] 1

> result4

$par

[1] 0.5607239 0.0785921 0.6074712 0.5299925

$value

[1] 99.17882

$counts

function gradient

   852      101

$convergence

[1] 1

$message

NULL

$hessian

        [,1]      [,2]       [,3]      [,4]
```

[1,] 29.54304 -113.3119 -185.6024 7085.799

[2,] -113.31189 10688.7484 -855.1604 -2732.789

[3,] -185.60243 -855.1604 159405.0916 -1652.387

[4,] 7085.79922 -2732.7890 -1652.3870 157968.057

Gamma-Normal distribution

```
> loglike6=function(p,x){-sum(log(1/(p[3]*gamma(p[1]))*dnorm((x-p[2])/p[3])*((-log(1-pnorm((x-
p[2])/p[3])))^(p[1]-1)))}
```

```
> result6=optim(c(0.8,0.5,0.7),method="CG",fn=loglike6,hessian=T,x=x)
```

```
> result6$convergence
```

[1] 1

```
> result6
```

$par

[1] 1.954542 1.880663 1.005429

$value

[1] 85.75514

$counts

function gradient

   328   101

$convergence

[1] 1

$message

NULL

$hessian

       [,1]   [,2]   [,3]

[1,] 43.81271 59.92221 37.09163

[2,] 59.92221 83.43363 66.50767

[3,] 37.09163 66.50767 185.10336

Kummer Beta Normal distribution

```
>  loglike7=function(p,x){-sum(log(1/(p[5])*dnorm((x-p[4])/p[5])*((pnorm((x-p[4])/p[5]))^(p[1]-1))*((1-
pnorm((x-p[4])/p[5]))^(p[2]-1))*exp(-p[3]*(pnorm((x-p[4])/p[5])))))}

> result7=optim(c(0.4,0.9,0.8,0.5,0.7),method="CG",fn=loglike7,hessian=T,x=x)

> result7$convergence

[1] 1

> result7

$par

[1] 0.2095611 5.3899577 7.1185744 9.8266979 2.06711443

$value

[1] 83.93633

$counts

function gradient

    407      271

$convergence

[1] 1

$message

NULL

$hessian

          [,1]      [,2]      [,3]      [,4]      [5,]

[1,] 17.044616 -3.807809 37.602259  5.804585 23.543200

[2,] -3.807809  1.130375 -9.283607 -3.962408 2.23112

[3,] 37.602259 -9.283607 85.839153 21.269225 19.22133

[4,]  5.804585 -3.962408 21.269225 27.384343 7.09011

[5,]  23.543200 2.23112 19.22133 7.09011 21.55342
```