A STUDY OF FUNDAMENTAL COMBINATORIAL PRINCIPLES, THEIR APPLICATIONS AND SOME OF THEIR EXTENSIONS TO MULTISETS.

BY

MUTARI HARUNA DUNARI B.SC(UDUSOK 1995)

MSC/SCI/45006/2004-2005

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
AHMADU BELLO UNIVERSITY, ZARIA

A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL, AHMADU BELLO UNIVERSITY ZARIA-NIGERIA, IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF MASTERS OF SCIENCE DEGREE IN MATHEMATICS

October, 2009

DECLARATION

I declare that this research work entitled " A study of fundamental combinatorial principles, their applications and some of their extensions to multisets" is the hand work of myself under the supervision of Prof. D. Singh, Department of mathematics. This work has not been presented for the award of any degree in any higher institution of learning. All sources of information used in this work are duly acknowledged by means of reference.

Mutari Haruna Dunari

Sign ----------

Date:----------

CERTIFICATION

This thesis entitled " A STUDY OF FUNDAMENTAL COMBINATORIAL PRINCIPLES, THEIR APPLICATIONS AND SOME OF THEIR EXTENSIONS TO MULTISETS" by Mutari Haruna Dunari, meets the requirement of the regulation governing the award of the degree of masters of science in mathematics of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

---------- ----------

Professor D. Singh Date
Major supervisor

---------- ----------

Dr. A. A. Tijjani Date
Minor supervisor

---------- ----------

Professor S. U. Momoh Date
External Examiner

---------- ----------

Dr. B. Sani Date
Head, Department of Mathematics

---------- ----------

Professor S. A. Nkom Date
Dean, Postgraduate School

DEDICATION

I dedicate this work of mine to my parents, my children and the entire members of my family.

ACKNOWLEDGEMENT

All praise is to Almighty Allah, His blessing be upon His Prophet and his immaculate progeny. I must thank Almighty Allah who created me and sustained my life up to this time of great achievement of M.sc degree in Mathematics.

My sincere gratitude goes to my supervisor, Professor D. Singh, for his tireless effort in making this work completed. His support in providing materials that are necessary and his criticims in putting all this material in the right order is duly acknowledged and I must be grateful for all the effort.

I am indebted to acknowledge the contribution of Professor Jai N. Singh of the Department of Mathematics and Computer Science, Barry University, USA, for his effort in sending some relevant material used in my work.

I must be indebted to thank my second supervisor, Dr. A. A. Tijjani, for putting me through towards achieving this target. My thank goes to all my Lecturers for the various courses I was taught by them during my course work. I also extend my gratitude to all the Lecturers in the Department of Mathematics, Ahmadu Bello University, Zaria. My special thank goes to Mr. Ibrahim Adeku Musa, a senior colleague in the Department of Mathematics, ABU Zaria for his diligent assistance in making this work a success.

My appreciation goes to members of my kith and kin for their moral support and assistance which they rendered that led me to this level of achievement. I must express my admiration to my wife, Khadija and all my children for all their goodness, thoughtfulness, endurance, understanding and love which they constantly showed on me all the time.

Worth mentioning are my course mates like, Umar Shehu, Ishaq Abdulkarim and Awwal Bichi for their support and advice in one way or the other. I thank them all and wish all of them the best of achievement. Amen!

My colleagues in my place of work, particularly from the Department of Mathematics /Statistics and Computer science have helped me in various ways in this work. May Allah reward them all abundantly.

Finally, I pray Almighty Allah to reward and bless those who helped me in this work, those mentioned and those not mentioned for one reason or the other.

ABSTRACT

Traditionally, combinatorial identities along side with their applications in graph and design theory are seen to constitute the ingredients of combinatorics. The properties of finite set must lie at the heart of any study of combinatorial theory.

Considering all these and the enormous applications of multisets in mathematics, computer science and linguistics, it has become pertinent to conduct studies in the area most especially from the combinatorial point of view. Combinatorics of multisets provide explicit solutions to a host of mathematical problems and programming problems in computer science.

The aim of this thesis is to conduct a comprehensive study of the fundamental principles of combinatorics, their applications and some of their extensions to multisets.

Accordingly, the following chapters are presented:

- **Chapter one** gives a general introduction and discusses some mathematical preliminaries required in dealing with combinatorial problems. A novel approach (Wikipedia) for introducing relations and functions in their combinatorial form is discussed. Finally, a comprehensive method to define specifically the "closure" of a transitve relation is outlined.
- **Chapter two** provides a comprehensive literature review.
- **Chapter three** discusses the basic combinatorial principles with some few illustrations of their applications. In this section we have raised a new problem and provided a partial solution.
- **Chapter four** deals with essentials of multisets and multiset theory. Essentially, this section deals with the basics of multisets and emphasizes the related issues. In particular, a method to compute the power multiset of a given multiset is outlined.

- **Chapter five** considers some combinatorial applications of multisets. In this section we have closely studied the work done in[SS 03] and found that the results are quite innovative.
- **Chapter six** gives the conclusion and some future directions.

TABLE OF CONTENTS

TITLE PAGE ---------- i

DELARATION ---------- ii

CERTIFICATION ---------- iii

DEDICATION ---------- iv

AKNOWLEDGEMENT ---------- v

ABSTRACT ---------- vii

TABLE OF CONTENTS ---------- ix

CHAPTER ONE ---------- 1

MATHEMATICAL PRELIMINARIES---------- 1

1.0 INTRODUCTION ---------- 1

1.1 RELATIONS ---------- 2

1.1.1 DEFINITION ---------- 2

1.1.2 PROPERTIES OF RELATIONS ---------- 7

1.1.3 COMPOSITIONS OF RELATIONS ---------- 8

1.1.4 POWERS OF RELATIONS ---------- 9

1.1.5 CLOSURES OF RELATIONS ---------- 10

1.2 FUNCTIONS AND SEQUENCES ---------- 17

1.2.1 FUNCTIONS ---------- 17

1.2.2 INJECTIONS, SURJECTIONS, BIJECTIONS ---------- 18

1.2.3 FUNCTIONAL INVERSES ---------- 18

1.2.4 SEQUENCES ---------- 18

1.2.5 NOTATIONS ---------- 19

1.3 COVERING AND PARTITION OF SETS ---------- 20

CHAPTER TWO ---------- 21

LITERATURE REVIEW ---------- 21

CHAPTER THREE ---------- 25

BASIC COMBINATORICS ---------- 25

3.0 INTRODUCTION ---------- 25

3.1 COMBINATORIAL PRINCIPLES(Counting Techniques)---------- 27

3.1.1 ADDITION PRINCIPLE---------- 27

3.1.2 MULTIPLICATION PRINCIPLE---------- 27

3.1.3 INDUCTION RULE ---------- 28

3.1.4 INCLUSION EXCLUSION PRINCIPLE ---------- 29

3.1.5 BIJECTION PRINCIPLE - ---------- 35

3.1.6 RECURRENCE RELATION ----- ---------- 36

3.1.7 PIGEON HOLE PRINCIPLE(PHP)---------- 37

3.2 APPLICATIONS ---------- 39

3.2.1 APPLICATION OF PHP ---------- 39

3.2.2 COUNTING NUMBER OF RELATIONS ---------- 40

3.2.3 COUNTING NUMBER OF COVERINGS ---------- 41

CHAPTER FOUR ---------- 48

FUNDAMENTALS OF MULTISETS ---------- 48

4.1 INTRODUCTION ---------- 48

4.2 MULTISETS ---------- 48

4.2.1 STRUCTURE OF MULTISETS---------- 49

4.3 SUBSET OF MULTISETS ---------- 52

4.4 EQUALITY OF MULTISETS ---------- 54

4.5 SIMILAR MULTISETS ---------- 54

4.6 MULTISETS OPERATION ---------- 54

4.7 DIRECT PRODUCT OF MULTISETS ---------- 57

4.8 COMPLEMENT OF MULTISET ---------- 57

4.9 FUNCTIONS BETWEEN MSETS ---------- 58

4.10 POWER MULTISETS ---------- 59

CHAPTER FIVE ---------- 61

SOME COMBINATORIAL RESULTS ON MULTISETS --------- 61

5.1 INTRODUCTION ---------- 61

5.2 PERMUTATIONS OF MULTISETS ---------- 61

5.2.1 r-PERMUTATIONS ---------- 62

5.3 COMBINATIONS OF MULTISETS ---------- 63

5.3.1 r- COMBINATIONS ---------- 66

5.4 BINOMIAL COEFFICIENT ---------- 69

5.5 PASCAL IDENTITY ---------- 69

5.6 VANDERMONDE'S IDENTITY ---------- 71

5.7 PIGEON HOLE PRINCIPLES ---------- 72

5.8 SOME USEFUL RESULTS ---------- 73

CHAPTER SIX ---------- 80

CONCLUSION AND FUTURE DIRECTIONS ---------- 80

6.1 CONCLUSION ---------- 80

6.2 FUTURE DIRECTIONS ---------- 80

REFERENCES ---------- 81

CHAPTER ONE

MATHEMATICAL PRELIMINARIES

1.0 Introduction:

In set theoretic notion, a **relation** is a property that assigns **truth values** to combinations (k-tuples) of *k* individuals. Typically, the property describes a possible connection between the components of a *k*-tuple. For a given set of *k*-tuples, a truth value is assigned to each *k*-tuple according to whether the property does or does not hold.

The variable *k* giving the number of "*places*" in the relation, which is a non-negative integer 0, 1, 2,… called the relation's *arity*, *adicity*, or *dimension*. A relation with *k* places is variously called a *k-ary*, a *k-adic*, or a *k-dimensional* relation. Relations with a finite number of places are called *finite-place* or *finitary* relations. It is possible to generalize the concept to include *infinitary* relations between infinitudes of individuals. For example, infinite sequences. However, we will consider finitary relations in this work in its more combinatorial view, which will be simply called relations.

Since there is only one 0-tuple, the so-called empty tuple ( ), there are only two zero-place relations, one for the property "is a 0-tuple", and one for its negation ("is not a 0-tuple"). A zero place relation is called *nullary* or *medadic*. One-place relations are called unary or monadic relations. Two-place relations are called binary or *dyadic.* The latter term has historic priority. Binary relations are very common, For example:

- **Equality** and **inequality**, denoted by signs such as "=" and "<" in statements like "$a = b$"and "5 < 12" respectively.
- Being a **divisor** of, denoted by the sign "|" in statement like "5 | 15";

- **Set membership**, denoted by the sign "∈" in statement like "1 ∈ **N**" (set of Natural numbers).

A *k-ary* relation, $k \neq 2$, is a straightforward generalization of a binary relation.

A **relation**, as conceived in the combinatorial theory of relations, is a mathematical object that in general can have a very complex type, the complexity of which is best approached in several stages, as will be indicated.

In order to approach the combinatorial definition of a relation, it helps to introduce a few preliminary notions that can serve as stepping stones to the general idea.

A relation in mathematics is defined as an object that has its existence as such within a definite context or setting. It is literally the case that to change this setting is to change the relation that is being defined. The particular type of context that is needed here is formalized as a collection of elements from which are chosen the elements of the relation in question. [Wikipedia]

1.1 Relations

Definition 1.1.1

A **relation** R is defined by specifying two mathematical objects as its constituent parts: the *figure* of R, $F(\mathrm{R})$ and the *ground* of R, $G(\mathrm{R})$.

In the case of a **finitary relation** i.e. ***k*-place relation** for concreteness, the concepts of figure and ground are defined as follows:

The **ground** of R is a sequence of k nonempty sets, $X_1, \ldots, X_k$, called the *domains* of the relation R.

The **figure** of R is a subset of the Cartesian product taken over the domains of R, that is, $F(\mathrm{R}) \subseteq G(\mathrm{R}) = X_1 \times \ldots \times X_k$.

Hence, the relation R is represented as R = (F(R), G(R)). However, it is customary to use the single name R for either the couple R = (F(R), G(R)) or the figure F(R). In fact, customarily R is said to be determined by its figure F(R) or the graph of R.

Definition 1.1.2

*A **k-ary relation** R over the nonempty sets X_1, …, X_k is a (1+k) - tuple R = (F(R), X_1, …, X_k) where F(R) is a subset of the cartesian product $X_1 \times \ldots \times X_k$. If all of the X_j for j = 1 to k are the same set X, then R is called a **k-ary relation over X**.*

A **local incidence property** (LIP) of a relation R is a property that depends in turn on the properties of special subsets of R that are known as its *local flags*.

Definition 1.1.3

Let R be a k-place relation $R \subseteq X_1 \times \ldots \times X_k$. Let X_j denote a relational domain and x one of its elements. Then $R_{x,j}$ is a subset of R, called the *flag* of R with x at j, or the x,j-flag of R. In other words, $R_{x,j} = \{(x_1,....,x_j,.....,x_k)\ R: x_j=x\}$.

Any property c of the local flag $R_{x,j} \subseteq R$ is said to be a *local incidence property* of R with respect to the locus x at j.

Definition 1.1.4

A k-adic relation $R \subseteq X_1 \times \ldots \times X_k$ is said to be c-regular at j if and only if every flag of R with x at j has the property c, where x is taken to vary over the fixed domain X_j. Equivalently, R is c-regular at j if and only if $|R_{x,j}| = c$ for all x in X_j.

*A **numerical incidence property** (NIP) of a relation is a local incidence property that depends on the cardinalities of its local flags.*

Example 1.1.1

A k-adic relation R is said to be *c*-regular at *j* if and only if the cardinality of the local flag $L_{x,j}$ is *c* for all *x* in X_j, or, if and only if $|R_{x,j}| = c$ for all *x* in X_j. In a similar fashion, the definition of NIPs (< *c*)-regular at *j*, (> *c*)-regular at *j*, and so on.

R is *c*-regular at *j* if and only if $| R_{x,j} | = c$ for all *x* in X_j.

R is (< *c*)-regular at *j* if and only if $| R_{x,j} | < c$ for all *x* in X_j.

R is (> *c*)-regular at *j* if and only if $| R_{x,j} | > c$ for all *x* in X_j.

The definition of a local flag can be broadened from a point *x* in X_j to a subset *M* of X_j, arriving at the definition of a *regional flag* in the following way:

Suppose that $R \subseteq X_1 \times \ldots \times X_k$, and choose a subset $M \subseteq X_j$. Then $R_{M,j}$ is a subset of R that is said to be the *flag* of R with *M* at *j*, or the *M,j*-flag of R, an object which has the following definition:

$$R_{M,j} = \{ (x_1, \ldots, x_j, \ldots, x_k) \in R : x_j \in M \}.$$

Returning to 2-adic relations, it is useful to describe some familiar classes of objects in terms of their local and their numerical incidence properties.

Let $R \subseteq S \times T$ be an arbitrary 2-adic relation. The following properties of R can be defined:

R is *Total* at *S* if and only if R is (≥1)-regular at *S*.

R is *Total* at *T* if and only if R is (≥1)-regular at *T*.

R is *Tubular* at S if and only if R is ($\leq$1)-regular at S.

R is *Tubular* at T if and only if R is ($\leq$1)-regular at T.

If R $\subseteq S \times T$ is tubular at S, then R is called a *partial function* or a *pre-function* from S to T, sometimes indicated by giving R an alternate name, say, "p", and writing R = $p : S \multimap T$. Just by way of formalizing the definition:

R = $p : S \multimap T$ if and only if R is tubular at S.

If R is a prefunction p: $S \multimap T$ that happens to be total at S, then R is called a *function* from S to T, indicated by writing R = $f: S \to T$. To say that a relation R $\subseteq S \times T$ is *totally tubular* at S is to say that it is 1-regular at S. Thus, we may formalize the following definition:

R = $f : S \to T$ if and only if R is 1-regular at S.

In the case of a function $f : S \to T$, one has the following additional definitions:

f is *surjective* if and only if f is Total at T.

f is *Injective* if and only if f is Tubular at T.

f is *Bijective* if and only if f is 1-regular at T.

Definition 1.1.5

Let R be a k-place relation $R \subseteq X_1 \times \ldots \times X_k$. *A subset* $R_{x.\,j}$ *of R is defined as* $R_{x.\,j} = \{ (x_1, \ldots, x_j, \ldots, x_k) \in R : x_j = x \}$.

K-Tuples

A collection of form (a_1, a_2, ……. ,a_k) or < a_1, a_2, ……. ,a_k> with a_1 as the first element, a_2 as the second element …….a_k as the k^{th} element is called the *K-Tuples*.

Definition 1.1.6

The ordered k-tuple $(a_1, a_2 ...., a_k)$ *is the ordered collection that has* a_1 *as its first element,* a_2 *as its second element… and* a_k *as its* k^{th} *element.*

We say that two ordered k-tuples are equal if and only if each corresponding pair of their elements is equal. In other words $(a_{1,} a_2, ..., a_k) = (b_1, b_2 ..., b_k)$ if and only if $a_i = b_i$, for i =1,2,…,k. In particular, 2-tuples are called ordered pairs. The ordered pairs (a, b) and (c, d) are equal if and only if a = c and b = d. Note that (a, b) and (b, a) are not equal unless a = b.

Cartesian products.

The order of elements in a collection is often important. Since sets are unordered, a different structure is needed to represent ordered collections. This is provided by ordered k-tuples.

Definition 1.1.7

Let A and B be sets. The Cartesian product of A and B, denoted by A x *B, is the set of all ordered pairs (a, b) where* $a \in A$ *and* $b \in B$. *Hence,*

$A \text{ x } B = \{(a,b) / a \in A \Lambda b \in B\}$

***Illustration*1.1.1**

Let A represent the set of all students of Mathematics Dept; and let B represent the set of all courses offered in the Department. What is the Cartesian product A x B?

Solution:

The Cartesian product A x B consist of all the ordered pairs of the form (a, b), where a is a student at the department and b is a course offered at the department. The set A x B can be used to represent all possible enrollments of the students in courses at the department.

***Illustration*1.1.2**

If $A = \{1,2\}$ and $B = \{a, b, c\}$, show that the Cartesian product B x A is not equal to the Cartesian product A x B

Solution: $A \times B = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$

$B \times A = \{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$.

Hence $A \times B \neq B \times A$

Definition 1.1.8

The Cartesian product of the set $A_1, A_2, ..., A_k$, *denoted by* $A_1 \times A_2 \times ... \times A_k$ *is the set of ordered k –tuples* $(a_1, a_2, ..., a_k)$ *where* a_i *belongs to* A_i *for* $i = 1,2,,...,k$. *in other words.* $A_1 \times A_2 \times,...,\times A_k = \{(a_1, a_2, .., a_k) \setminus a_i \in A_i$ for $i = 1,2,..k\}$

***Illustration* 1.1.3**

What is the Cartesian product A x B x C, where $A = \{0,1\}$, $B = \{1,2\}$ and

C={0,1,2}?

Solution:

The Cartesian product A x B x C consists of all ordered triples *(a, b, c),* where

$a \in A, b \in B$, and $c \in C$. Hence,

A x B x C = {0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),(0,2,2),(1,1,0),(1,1,1),(1,1,2),

(1,2,0),(1,2,1),(1,2,2)}

1.1.2 Properties of Relations

1.1.2.1 Reflexive Property

In some relations an element is always related to itself. For instance, let R be the relation on the set of all people consisting of pairs (x, y)where x and y have the same mother and the same father. Then xRx. We can define such relation as follow:

Definition:1.1.2.1

A relation R on a set A is called reflexive if (a, a) $\in R$ *for every element* $a \in A$.

1.1.2.2 Symmetric Property

Some relations have the property that an element is related to a second element if and only if the second element is also related to the first element. A relation consisting of pairs (a, b) where a and b are students of Maths Department, with at least one common level, has this property. Other relations have the property that if an element is related to a second element, then this second element is not related to the first. A typical example is the relation consisting of the pairs (a, b) where a

and *b* are students of Maths Department, where *a* has a higher grade point than *b*. This type of relation is called Antisymmetric.[Ros 99]

Definition: 1.1.2.2

A relation R on a set A is called symmetric if $(b, a) \in R$ *whenever* $(a, b) \in R$, *for* $a,b \in A$. *A relation R on a set A such that* $(a, b) \in R$ *and* $(b, a) \in R$ *only if* $a=b$, *for* $a,b \in A$, *is called antisymmetric.*

1.1.2.3 Transitive Property

Let R be a relation consisting of all pairs (*a*, *b*) of students in Maths Department, where *a* has taken more courses than *b*. Suppose that *a*R*b* and *b*R*c*. This means that *a* has taken more courses than *b* and *b* has taken more courses than *c*. We conclude here that *a* has taken more courses than *c*, so that *a*R*c*. This illustrates that R is transitive.

Definition: 1.1.2.3

A relation R on a set A is called transitive if whenever $(a, b) \in R$ *and* $(b, c) \in R$, *then* $(a, c) \in R$, *for* $a,b,c \in A$.

The above properties can be summarized as below. Given a set A with n elements, n = 1,2,3,..... a relation R on A is called:-

i) ***Reflexive*** if, for all $a \in A$. *a*R*a* (follows that R is reflexive iff $\Delta_A \subset R$.)

ii) ***Irreflexive*** if, for all $a \in A$, $\neg aRa$.

iii) ***Symmetric*** if, for all $a,b \in A, aRb \Rightarrow bRa$

iv) ***asymmetric*** if, for all $a,b \in A, aRb \Rightarrow \neg bRa$

v) ***antisymmetric*** if, for all $a,b \in A, aRb \wedge bRa \Rightarrow a = b$

vi) ***transitive*** if, for all $a,b,c \in A, aRb \wedge bRc \Rightarrow aRc$

vii) ***total*** if, for all $a,b \in A, a \neq b \Rightarrow aRb$ or bRa

viii) ***Circular*** if, for all $a,b,c \in A, aRb \wedge bRc \Rightarrow cRa$

1.1.2.4 Equivalence relation

Definition:1.1.2.4

A relation that is both reflexive, symmetric and transitive is called an equivalence relation.

1.1.3 Composition of relations

Definition: 1.1.3.1

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that (a, b) $\in R$ and (b, c) $\in S$. The composition of relations R and S is denoted by S $\circ$ R.

Illustration: 1.1.3.1

What is the composite of the relations R= {(1,1),(1,4),(2,3),(3,1),(3,4)}, from sets {1,2,3} to {1,2,3,4} and the relation S={(1,0),(2,0),(3,1),(3,2),(4,1)}, from sets {1,2,3,4} to {0,1,2}?

Solution

S $\circ$ R is found by using all the ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of

the ordered pair in S. Computing all the ordered pairs in the composite, gives $S \circ R = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$.

1.1.4 Powers of relations

Let R be a relation on A. The powers R^n, n= 1,2,3,.... are defined inductively by

$R^1 = R$ and $R^{n+1} = R^n \circ R$.

Hence $R^3 = R^2 \circ R$ and $R^2 = R \circ R$.

Illustration:1.1.4.1

Let $A=\{a,b,c\}$ *and* $R= \{(a,b),(b,c),(c,a)\}$. *Then* R^2, R^3 *and* R^4 *can be given as follows:-*

$$R^2 = R \circ R = \{(a,c),(b,a),(c,b)\}$$

$$R^3 = R^2 \circ R = \{(a,a),(b,b),(c,c)\}$$

$$R^4 = R^3 \circ R = \{(a,b),(b,c),(c,a)\}$$

Proposition: ***1.1.4.1***

A relation R on a set A is transitive if and only if $R^n \subseteq R$ for n =1,2,3,.....

Proof:

Suppose that $R^n \subseteq R$ *for n = 1,2,3,....In particular* $R^2 \subseteq R$. *This clearly implies the transitvity of R. Note that if* $(a,b) \in R$ *and* $(b,c) \in R$, *then* $(a,c) \in R^2$ *by definition of composition. Since* $R^2 \subseteq R$, *means that* $(a,c) \in R$. *Hence R is transitive.*

Conversely, by mathematical induction, we see that $R^n \subseteq R$ *is trivially true for n=1. Assume that* $R^n \subseteq R$ *where n is a positive integer. Inductively we see also that* $R^{n+1} \subseteq R$. *To show this, assume that* $(a,b) \in R^{n+1}$. *Then, since* $R^{n+1} = R^n \circ R$, *there is an element x with* $x \in A$ *such that* $(a,x) \in R$ *and* $(x,b) \in R^n$. *The inductive*

hypothesis, namely, that $R^n \subseteq R$, *implies that* $(x,b) \in R$. *Furthermore, since R is transitve, and* $(a,x) \in R$ *and* $(x,b) \in R$, *it follows that* $(a,b) \in R$. *This shows that* $R^{n+1} \subseteq R$. *Hence the proof is complete.*[see Ros 99 for details]

1.1.5 Closure of Relations:

Closure of relation has the property that when ever a certain property does not hold for a relation and to minimally augment the relation for the property to hold.

It is easy to follow this procedure to construct the closure of Reflexive and Symmetric relations where as a little more complex task is required for Transitive relation.

1.1.5.1 Reflexive Closure:

Illustration: 1.1.5.1

Consider the relation R = *{(a, a), (a, b), (b, a), (c, b)} on {a, b, c}*

R is not reflexive; the pairs missing are: *(b, b), (c, c).*

Now adding the missing pairs, we obtain the new relation:

$$\{(a, a), (a, b), (b, a), (b, b), (c, b), (c, c)\}.$$

The new relation formed is reflexive. This is a reflexive closure of the relation *R.* If Δ is the **diagonal relation** on *A* defined as: $\Delta = \{(a, a) \mid a \in A\}$.

The reflexive closure of *R* is then: $\boldsymbol{R \cup \Delta}$**.**

The following Properties hold:

- $R \subseteq (R \cup \Delta)$;
- $R \cup \Delta$ is reflexive;
- $\forall S \quad (R \subseteq S \ \wedge \ S$ is reflexive$) \ \rightarrow \ (R \cup \Delta) \subseteq S.$

Remarks: In zero-one matrix notation: $\mathbf{M}_R \vee \mathbf{M}_\mathbf{A}$

Example 1.1.5.1

Consider $R = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a < b\}$

The reflexive closure of relation R is:

$R \cup \Delta$ and $\{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a < b\} \cup \{(a, a) \mid a \in \mathbf{Z}\}$

and $\{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a \leq b\}$

1.1.5.2 Symmetric Closure

Illustration1.1.5.2

Consider also the relation $R = \{(a, a), (a, b), (b, b), (b, c), (c, a), (c, b)\}$

R is not symmetric because the pairs (b, a), (a, c) are missing:

If we add the pairs, we obtain the new relation which is symmetric:

Therefore the resulting symmetric relation form the symmetric closure of R

Example 1.1.5.2

Consider $R = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a < b\}$. The symmetric closure of relation R is:

- $R \cup R^{-1}$
- $\{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a < b\} \cup \{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid b < a\}$
- $\{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a \neq b\}$

1.1.5.3 Paths in Directed Graphs

Definition 1.1.5.3

A *path* from a to b in a digraph G is a sequence of 1 or more adjacent edges (a,x_1), (x_1,x_2), (x_2,x_3), …, (x_{n-1}, b).

- Denoted: $a, x_1, x_2, x_3, \ldots, x_{n-1}, b$
- Has *length n.*
- If $a = b$, the path is called a *circuit* or a *cycle*.

Definition 1.1.5.4

There is a **path** from a to b in R if there is a sequence of elements: $x_1, x_2, \ldots, x_{n-1}$ with $(a,x_1) \in R, (x_1,x_2) \in R, \ldots, (x_{n-1}, b) \in R$.

Example 1.1.5.3

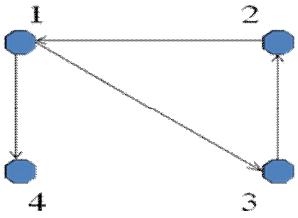
$R = \{(1,3), (1,4), (2,1), (3,2)\}$

- $\exists$ a path in R (length 2) from 1 to 2: $(1,3), (3,2) \in R$.
- $\exists$ a path in R (length 3) from 3 to 4: $(3,2), (2,1), (1,4) \in R$.
- There is no path in R from 4 to 1, or to 2, or to 3.

Example 1.1.5.4

$R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$

- $\exists$ a path of length 2 from 1 to 2: edges (1,3), (3,2).
- $\exists$ a path of length 3 from 3 to 4: (3,2), (2,1), (1,4).
  - There are no paths from 4 to 1, 4 to 2, 4 to 3



Paths in Relations

Theorem 1.1.5.1

Let R be a relation on a set A.

$\exists$ a path in R of length n from a to $b \Leftrightarrow (a, b) \in R^n$. [see Ros 99 pp.399 for the proof]

1.1.5.4 Transitive Closures

Illustration1.1.5.4

Consider the $R = \{(a, c), (a, d), (b, a), (c, b)\}$.

The relation R is not transitive; the following pairs $(a, b), (b, c), (b, d), (c, a)$ are missing.

Adding these pairs, we obtain the new relation: $\{(a, b), (a, c), (a, d), (b, a), (b, c), (b, d), (c, a), (c, b)\}$.

The new relation is still not transitive. The transitive closure is more complicated to build. We now show that finding transitive closure of a relation is equivalent to determining which pairs of vertices in the associated digraph are connected by a path. With this in mind, we define a new relation as follows:

Definition 1.1.5.6

Let R be a relation on a set A.

The *connectivity relation* (*Transitive Closure*) is the relation R* defined as:

$R^* = \{(a, b) \mid \exists$ a path in R from a to $b\}$.

- From the above definition of R^* and the previous theorem, R^* can be written as: $R^* = \cup^{\infty}_{k=1} R^k$.

Theorem 1.1.5.2

The transitive closure of R is R^*.

Proof

1. $R \subset R^*$, because if there is an edge from *a* to b then there is a path from a to b.
2. R^* is transitive, because if ∃ a path from *a* to *b* and ∃ a path from *b* to *c* then ∃ a path from *a* to *c*.
3. $\forall S\ (R \subseteq S \wedge S \text{ is transitive}) \rightarrow R^* \subseteq S$. [see Ros 99 for details]

Lemma 1.1.5.1

Let *A* be a set with *n* elements, and *R* be a relation on *A*. If there is a path in *R* from *a* to *b* (with $a \neq b$), then there is such a path with length ≤ *n*-1. i.e.

$R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$. [see Ros 99 for the proof]

Theorem 1.1.5.3

Let M_R be the zero-one matrix of relation *R* on a set with *n* elements. The zero-one matrix of the transitive closure *R** is: $M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee .. \vee M_R^{[n]}$. [see Ros 99 p. 401 for details]

Illustration1.1.5.4

Find the matrix $\mathbf{M}_R^*$ of the transitive closure of *R* where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution: From Theorem 1.1.5.3, it follows that $M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$.

Since

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

it follows that

$$\mathbf{M}_R^{*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The theorem can be used as a basis for an algorithm for computing $\mathbf{M}_R^{*}$. The procedure is displayed as an algorithm below:

Procedure for computing <u>transitive closure</u>:

$\mathbf{A} = \mathbf{M}_R$;

$\mathbf{B} = \mathbf{A}$;

For i=2 to n do the following:

$\mathbf{A} = \mathbf{A}$ o $\mathbf{M}_R$; // Boolean product

$\mathbf{B} = \mathbf{B} \vee \mathbf{A}$; // Join operation

Solution:

$$\mathbf{B}_1 = \mathbf{A}_1 = \mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$\mathbf{A}_k = \mathbf{A}_{k-1}$ o $\mathbf{M}_R$ $\quad$ $\mathbf{B}_k = \mathbf{B}_{k-1} \vee \mathbf{A}_k$

$$\mathbf{A}_2=\mathbf{M}_R^{[2]}=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 1\end{bmatrix} \qquad \mathbf{B}_2=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}$$

$$\mathbf{A}_3=\mathbf{M}_R^{[3]}=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 1\end{bmatrix} \qquad \mathbf{M}_R^{*}=\mathbf{B}_3=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}$$

Definition 1.1.5.7

Let R be a relation on $S=\{v_1,v_2,\ldots, v_n\}$. The *interior vertices* of a path of length m from a to b: $a, x_1, x_2, x_3 \ldots x_{m-1}, b$ are: $x_1, x_2, x_3, \ldots, x_{m-1}$.

Warshall's algorithm iteratively constructs 0-1 matrices

$\mathbf{W}_0 = \mathbf{M}_R$;

$\mathbf{W}_1= [w^{[1]}_{ij}]$, where $w^{[1]}_{ij}=1 \leftrightarrow \exists$ a path from v_i to v_j

with interior vertices in $\{v_1\}$;

$\mathbf{W}_2=[w^{[2]}_{ij}]$, where $w^{[2]}_{ij}=1 \leftrightarrow \exists$ a path from v_i to v_j

with interior vertices in $\{v_1,v_2\}\ldots$

$\mathbf{W}_k=[w^{[k]}_{ij}]$, where $w^{[k]}_{ij}=1 \leftrightarrow \exists$ a path from v_i to v_j

with interior vertices in $\{v_1,v_2, \ldots, v_k\}\ldots$

$\mathbf{M}_{R^*} = \mathbf{W}_n$. [see Ros 99, p. 403 for details].

Solution: $\mathbf{W}_0 = \mathbf{M}_R$ $\mathbf{W}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & \mathbf{1} \end{bmatrix}$ and $\mathbf{W}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & \mathbf{1} & 1 \end{bmatrix}$

$$\mathbf{M}_R^* = \mathbf{W}_3 = \begin{bmatrix} 1 & \mathbf{1} & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Lemma 1.1.5.2

Let R be a relation on $S=\{v_1,v_2,\ldots,v_n\}$, and let $\mathbf{W}_k=[w^{[k]}_{ij}]$ be the 0-1 matrix $w^{[k]}_{ij}=1 \leftrightarrow \exists$ a path from v_i to v_j with interior vertices in $\{v_1, v_2, \ldots, v_k\}$.

Then

$$\forall i,j,k \leq n \quad w^{[k]}_{ij} = w^{[k-1]}_{ij} \vee (w^{[k-1]}_{ik} \wedge w^{[k-1]}_{kj}).$$

- Procedure for computing <u>transitive closure</u>: <u>Warshall's algorithm:</u>

  $\mathbf{W} = \mathbf{M}_R$

  For k=1 to n, begin

    for i=1 to n, begin

      for j=1 to n do the following:

        $w_{ij} = w_{ij} \vee (w_{ik} \wedge w_{kj})$;

    end { $\mathbf{W} = [w_{ij}]$ is $\mathbf{M}_R^*$}

// That is, at step k: add row k to all other rows which have 1 as intersection with kth column

Solution: $\mathbf{W}_0 = \mathbf{M}_R$

Add row 1 to row 3: $\mathbf{W}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & \mathbf{1} \end{bmatrix}$

Add row 2 to row 3: $\mathbf{W}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Add row 3 to row 1: $\mathbf{M}_R^* = \mathbf{W}_3 = \begin{bmatrix} 1 & \mathbf{1} & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

1.2 Functions and Sequences

1.2.1 Functions:

Functions play a fundamental role in nearly all of mathematics. Combinatorics is no exception. In the next section we review the basic terminology and notation for functions. Permutations are special functions that arise in a variety of ways in Combinatorics. Functions form the mathematical basis of most computer logic.

Definition 1.2.1

If A and B are sets, a function from A to B is a rule that tells us how to find a unique $b \in B$ *for each* $a \in A$. *We write f:* $A \mapsto B$ *to indicate that f is a function from A to B. The set A is called the domain of f and the set B is the co domain of f.*

Or

Let E and F be sets. With each element x of E, let there be associated a unique element f(x) of F. Then f is called a function from E into F, and f is said to map E into F. We write f : $E \mapsto F$ *to indicate it.*

Let f be a function from E into F. For x in E, the point $f(x)$ in F is called the *image* of x or the value of f at x. Similarly, for $A \subset E$, the set $\{y \in F : y = f(x)$ for some x

$\in A\}$ is called the *image* of A. In particular, the image of E is called the *range* of f. Moving in the opposite direction, for $B \subset F$,

$$f^{-1}(B) = \{x \in E : f(x) \in B\}$$

is called the *inverse image* of B under f. Obviously, the inverse of F is E.

Terms like mapping, operator, transformation are synonyms for the term "function" with varying shades of meaning depending on the context and on the sets E and F. Sometimes, $x \mapsto f(x)$ is used to indicate the mapping f; for instance, the mapping $x \mapsto x^3 + 5$ from R into R is the function $f : \mathrm{R} \mapsto \mathrm{R}$ defined by $f(x) = x^3 + 5$.

1.2.2 Injections, Surjections, Bijections.

Let f be a function from E into F. It is called an *injection*, or is said to be *injective*, or is said to be *one-to-one*, if distinct points have distinct images (that is, if $x \neq y$ implies $f(x) \neq f(y)$). It is called a *surjection*, or is said to be *surjective*, if its range is F, in which case f is said to be from E *onto* F. It is called a *bijection*, or is said to be *bijective*, if it is both injective and surjective.

These terms are relative to sets E and F. For example, $x \mapsto \mathrm{e}^{\mathrm{x}}$ is an injection from R into R, but it is not a bijection from R into (0,1). The function $x \mapsto \sin x$ from R into R is neither injective nor surjective, but it is a *surjection* from R onto $[-1, 1]$.

1.2.3 Functional Inverse:

Let f be a bijection from E onto F. Then, for each y in F there is a unique x in E such that $f(x) = y$. In other words, in the notation, $f^{-1}(\{y\}) = \{\mathrm{x}\}$ for each y in F and some unique x in E. In this case, the brackets is dropped and write $f^{-1}(y) = x$. The resulting function f^{-1} is a bijection from F onto E; it is called the *functional inverse* of f. This particular usage should not be confused with the general notation of f^{-1}. (Note that $f^{-1}(B) = \{x \in E : f(x) \in B\}$) defines a function f^{-1} form F

into E, where $\mathcal{F}$ is the collection of all subsets of F and $\mathcal{E}$ is the collection of all subsets of E.) [wikipedia, Ros99]

1.2.4 Sequences

A *sequence* is a function from N into some set. If f is a sequence, it is customary to denote $f(n)$ by something like x_n and write (x_n) or $(x_1, x_2, \ldots)$ for the sequence (instead of f). Then, the x_n are called the *terms* of the sequence. For instance, (1, 3, 4, 7, 11, . . .) is a sequence whose first, second, etc. terms are $x_1 = 1$, $x_2 = 3$, ... If A is a set and every term of the sequence (x_n) belongs to A, then (x_n) is said to be a sequence in A or a sequence of elements of A, and we write $(x_n) \subset$ A to indicate this.

A sequence (x_n) is said to be a *subsequence* of (y_n) if there exist integers $1 \leq k_1 < k_2 < k_3 < \cdots$ such that $x_n = y_{kn}$ for each n.

For instance, the sequence (1, 1/2, 1/4, 1/8, . .) is a subsequence of (1, 1/2, 1/3, 1/4, 1/5, . .).

Definition 1.2.4.1 *Let $x_1, x_2, \ldots x_N$ be a sequence of real numbers. A subsequence of this sequence is a sequence of the form* $x_{i_1}, x_{i_2}, \ldots\ldots x_{i_m}$ *, where* $1 \leq i_1 < i_2 < \ldots\ldots < i_m \leq N$.

Definition 1.2.4.2 *A sequence is called* ***strictly increasing*** *if each term is larger than the one that precedes it, and it is called* ***strictly decreasing*** *if each term is smaller than the one that precedes it.*

1.2.5 Notations:

Let f: A $\mapsto$ B. If A $\subset$ Z, f is called a sequence in B. Usually, A = P, N, or [k], for some k $\in$ P. Sequences are usually denoted using ordered lists, enclosed in parentheses, instead of using functional notation. Thus, with $t_i = f(i)$, a sequence f :

P $\mapsto$ B is usually denoted (t_1, t_2.....,), a sequence f : N $\mapsto$ B is usually denoted (t_0, t_1, t_2,.....), and a sequence f : [k] $\mapsto$ B is usually denoted (t_1, t_2, ... , t_k). The latter is called a sequence of length k in B.

A sequence of length k in B is also called a word of length k in the "alphabet" B. When a sequence is construed as a word, it is typically written without enclosing parentheses and with no commas. Thus, the sequence (t_1; t_2; : : ; t_k) is written t_1t_2 - --t_k when construed as a word. Also, whereas one refers to t_i as the *i*th term in the sequence (t_1, t_2,.....t_k), it is called the i^{th} letter of the word t_1t_2--- t_k.

The function f : [k] $\mapsto$ B associated with the sequence (t_1, t_2, -----t_k) by the rule $f(i) = t_i$ is injective if and only if all the terms in the sequence are distinct. Similarly, injectivity of the function associated with the word t_1t_2 ---- t_k is equivalent to there being no repeated letters in the word. Sequences and words associated with injective functions from [k] to B are also called permutations of length k in B, or permutations of the members of B taken k at a time.

The function f : [k] $\mapsto$ B associated with the sequence (t_1, ----, t_k) by the rule $f(i)$ = t_i is surjective if and only if each element of B occurs at least once as a term in the sequence. Similarly, surjectivity of the function associated with the word t_1t_2 ----- t_k is equivalent to the fact that each element of B occurs at least once as a letter of the word.[Wikipedia, Ros 99]

1.3 Covering and partition of a finite set

Definition: 1.3.1 Let *S* be a nonempty finite set. A decomposition of S of the form $S = A_1 \cup A_2 \cup .....\cup A_k$, where $A_i \neq \phi$ for all $i = 1,2,....,k$ is called a *covering* of *S*. If, in addition, $A_i \cap A_j = \phi$ for $i \neq j$, then $\{A_1, A_2,...., A_k\}$ is called a *partition* of *S* and the sets $A_1, A_2,...., A_k$ are called the *blocks* of the partition.

CHAPTER TWO

LITERATURE REVIEW

The theory of relations treats the subject matter of relations in its combinatorial aspect, as distinguished from, though related to, its more properly logical study on one side and its more generally mathematical study on another. (www.wikipedia.org).

Since relations play essential role in the theory of combinatorics, we have studied the theory of relations as a support to this work in chapter one. In the same place, the emphasis is given on determining the closures of relations. In particular, the *compatibility* relation is studied.

On our own part, we have developed intensively the closures of the transitive relations [wik, Ros99] and studied the Warshall's Algorithm in dealing with the formation of the transitive closures. This has enormous applications in Mathematics, in particular, relational combinatorics (counting problems) and Computer Science.

Fundamentally, as in [Wikipedia, Wag, Ros 99]; COMBINATIORICS deals with counting problems which are usually associated with *selection* of several objects from a given collection of objects and also their arrangements in a certain situation. Mathematical term for arrangement is *Permutation* and that for selection (choice or grouping) is *Combination* in which arrangement or ordering of objects is not taken into consideration. We may say that permutation is an *ordered set* whereas combination is an *unordered set* formed from elements of a given set. In general, the solution of problem on Combinatorics needs ad-hoc methods and deep insight into the situation of the problem but the treatment and techniques of such solutions are based on certain basic principles called combinatorial principles or counting techniques which we extensively discussed. Combinatorics was

studied as long ago as the seventeenth century, when combinatorial questions arose in the study of gambling games. In order to study combinatorics on multisets, which principally formed the last chapter, the theory of Multisets was studied.

Multiset, otherwise known as set with multiplicities, was first suggested by N. G. de Bruijn in private correspondence with Knuth [Bli 89]. The term has now replaced the terms; bag, bunch, heap, fireset, weighted set, and so on. N. G. de Bruijn's interest in multiset was as a result of his investigation into the combinatorial properties of the set of divisors of a number.

It is difficult to categorically state the origin of multiset but it is believed that the concept is as old as the concept of number. In ancient time, any number n was often represented by a collection of n strokes and tally marks. For example, the number seven was identified as a collection of objects like /////// ([Bli 91]).

The very origin of numbers involves the use of collections of repeated units, or multisets of "ones".

Enumeration of permutations of a set was known in ancient times and historically the first known document is the Hebrew Book of Creation (C. 100 A. D.), followed by the Indian Classic *Anuyogadvar-sutra* (C. 500 A. D.); The corresponding results for multiset seems to have appeared first in another Indian Classic *Lilávati of Bháscara Achárya* (C. 1150). (see[Knu 81], p. 23, for details).

In [Singh 06] it was observed that an early reference to multiset is found in the work of Marius Nizolius (1498 – 1576), cf. [Ang 65] and [Sin 94]. In the same work [Singh 06], observed that Hailperin [Hai 86] has justified (by introducing signed heaps) in stating that Boole's Laws of Thought may be interpreted as a lecture dealing with multisets.

Weierstrass, defines real numbers as certain multisets of rational numbers in which finitely many repetitions are allowed (see [Bli 91] for details). Mathematics of multisets was considerably treated in the work of [Whi 33], particularly in terms of algebra of characteristic functions of sets and subsequently, that of 'generalized' sets. He cites "chains in analysis situs" in which "each element is counted any number of times" ([Whi 33], p. 412).

Rado [Rad 75] defines an mset to be any cardinal-valued function whose nontrivial domain (the collection of elements not mapped to zero) is a set. The class of msets is called the cardinal module, like "a module over the semi-group of all cardinals" (p.135), An mset f represents a family of sets $\alpha = (x_i)$, $i \in I$ just in case $f(x_i)$ equals the number of times x_i occurs in α (that is, $f(x) = |\{i \in I : x_i = x\}|$ for all x). Rado also makes use of signed msets (functions which may have negative cardinal values). Parker-Rhodes ([Par 81], p. xiii) observes that "… there exists no branch of mathematics in which a third parity-relation, besides equality and inequality, is admitted". Parker-Rhodes [Par 81] develops a theory of "sorts" (collections of indistinguishables), an elaborate mathematical system, and applies it to explicate some fundamental problems of physics. "Copies of elements behave as identicals when they appear as elements of different classes, but as a plurality (each of them contributes to cardinality) when they are elements of the same class" (p.7). Elements of multisets conform to the Parker-Rhodes principle of indistinguishables (see [Singh 06] for details).

Meyer and MacRobie [MMc 82] make use of multisets in the study of *relevant implication* by way of reckoning how often a premise is repeated in the course of derivation.

Anderson [And 87] investigates properties of collections of subsets of a finite set and generalizations of these properties to collections of multisubsets of a finite multiset. This area of combinatorics is sometimes called "extremal set theory".

Clement [Cle 88] generalized multiset lattice of divisors of a number to that of an abstract multiset. Also, Bender [Ben 74] and [BDR 84] investigate partitions of multisets. A partition of multiset M is a collection of multisets (called blocks) whose multiset union (union with repetitions) is M. This is an area we studied and extended to finding the cardinality of partitions and coverings of sets.

Blizard [Bli 89] defines a first-order theory MST (Multiset Theory) for multisets (finite multiplicities). The theory MST is a generalization of classical ZFC (Zermelo-Fraenkel plus the axiom of Choice) set theory. MST contains an exact copy ZFC' of ZFC. In this work of ours, an overview of it is presented.

Wildberger ([Wil 03], p.3), brought out some notably examples from the real world, such as "….to a shopkeeper, any two dollar coins are equal even if their years of minting are different, whereas a coin collector would regard them essentially different, etc". He proposed some very innovative conclusions like "the notion of equality is often a relative one", "traditional mathematics works with the implicit assumption that all objects exist uniquely in some ideal sense" and "relies heavily on equivalence relation", etc. Wildberger argues that two physical objects are either different or they are the same (or equal) but separate or they are coinciding (and identical). He further went to consider water molecule with two hydrogen atoms and one oxygen atom as an example. He concluded that one hydrogen atom say H^1 and O (oxygen atom) are obviously different (and, for that matter, H^2 and O are different), however H^1 and H^2 are the same but separate, while H^1 and H^1 are coinciding and identical. Many interesting ideas on multisets have been proposed in [Wil 03]. In this work of ours, results on combinations of multisets are obtained keeping Wildberger's approach into view.

CHAPTER THREE

BASIC COMBINATORICS

3.0 Introduction

Combinatorics is the study of arrangements of objects; it is an important part of discrete mathematics. Combinatorics was studied as long ago as the seventeenth century, when combinatorial question arose in the study of gambling games. Enumeration, the counting of objects with certain properties, is an important part of combinatorics. We must count objects to solve many different types of problems. For instance, counting is used to determine the complexity of algorithms. Counting is also required to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand. Furthermore, counting techniques are used extensively when probabilities of events are computed.

The basic rules of counting can solve tremendous variety of problems. For instance we can use these rules to enumerate the different phone numbers possible in a particular country say Nigeria, the allowable password in a computer system, and the different orders in which the runners in a race can finish. Another important combinatorial tool is the pigeonhole principle, which states that when objects are placed in boxes and there are more objects than boxes, then there is a box containing at least two objects. For instance, we can use this principle to show that among a set of 15 or more students, at least 3 were born on the same day of the week.

Many counting problems in terms of ordered or unordered arrangements of a set can be phrased in what is called permutations and combinations. For instance, suppose that 100 top finishers on a competitive exam taken by 2000 students are

invited to a banquet. We can enumerate the possible sets of 100 students that will be invited, as well as the top 10 prizes that can be awarded.

We can analyze gambling games, such as poker, using counting techniques. We can also use these techniques to determine the probabilities of winning lotteries, such as the probability that a person will win a lottery where 6 numbers are chosen from the first 48 positive integers.

Another problem in combinatorics involves generating all the arrangements of a specified kind. This is often important in computer simulations.

3.1 Combinatorial Principles (Counting Techniques)

The entire structure of Combinatorics is built upon the two basic counting principles viz. Addition Principle (AP) and Multiplication Principle (MP) which appear most trivial in nature and form, but are so powerful that none of combinatorial problems can be solved without their application. Other counting principles are: [see And 97, Ros 99]

Induction Principle (IP), Inclusion-Exclusion Principle (IEP), Bijection Principle (BP) and Pigeon hole Principle (PHP).

3.1.1 Addition Principle (AP) (Sum Rule)

Statement 3.1.1

If one work can be done in n ways and other independent work in m ways, then either work can be done in n + m ways.

More generally, *Sum Rule* may be stated as:

"If there are k different works w_1, w_2....w_k which can respectively be done in n_1, n_2…….n_k ways separately, then w_1 or w_2 or ..... or w_k can be done in $n_1 + n_2 +$ ……$+ n_k$ ways." AP may also be stated in the following form "If there are k pair

wise disjoint events $E_1, E_2, \ldots\ldots, E_k$ which can occur respectively in $n_1, n_2, \ldots, n_k$ ways , then the number of ways for at least one of these k events to occur is $n_1 + n_2 + \ldots\ldots + n_k$.

Illustration: 3.1.1

If one can go from Kano to Kaduna in 4 ways by rail-route, R_1, R_2, R_3, R_4, in 2 ways by air-routes A_1, A_2 and in 3 ways by bus-route B_1, B_2, B_3, then the total number of ways in which he can go from Kano to Kaduna by either route is 4 + 2 + 3 = 9. Note that these 9 routes are R_1, R_2, R_3, R_4, A_1, A_2, B_1, B_2, B_3.

3.1.2 The product Rule (Multiplication Principles "MP")

Statement: **3.1.2**

If two works W_1 and W_2 are to be done such that W_1 can be done in p different ways and corresponding to each such ways W_2 can be done in q different ways, then both works W_1 and W_2 can be done together in p.q different ways.

More generally, MP can be stated as:

"If there are k different works $W_1, W_2, \ldots\ldots, W_k$ such that W_1 can be done in n_1 ways and W_i $(1 < i \leq k)$ can be done in n_1 ways corresponding to each way of doing the works $W_1, W_2, \ldots\ldots W_{i-1}$ together, then $W_1, W_2 \ldots\ldots W_k$ can be done together in $n_1, n_2, \ldots, n_k$ ways."

Illustration: 3.1.2

There are 3 rail routes R_1, R_2, R_3, and 2 bus routes B_1, B_2 between Zaria and Kano. If one goes from Zaria to Kano (W_1) by rail route and comes back from

Zaria (W_2) by bus route, then he can perform both way journey (W_1 and W_2 together) in 3x2=6 ways. Note that these 6 routes are R_1B_1, R_1B_2, R_2B_1, R_2B_2, R_3B_1 and R_3B_2 where R_iB_j denotes onward journey by rail route R_i and return journey by bus route B_j.

There are 5 men, 3 women, 4 boys, and 2 girls. Number of ways in which any person (either a man, or a woman or a boy or a girl) can be chosen is 5+3+4+2=14 ways vide 'AP'. But the number of ways in which 4 persons – a man, a woman, a boy, a girl (together) can be chosen in 5x3x4x2=120 vide Multiplication principle.

3.1.3 Induction Principle (IP)

Statement: 3.1.3

If a property or statement P(n) holds for n=1 and whenever it holds for some number $m \in N$*, it also holds for m+1, then the property holds for all* $n \in N$.

In other words

" $[P(1) and P(m) \Rightarrow P(m+1)] \Rightarrow P(n) \forall n \in N$ "

Illustration: 3.1.3

Prove that 1+2+3+……+n=½n (n+1) $\forall n \in N$

Solution:

Let P (n) = 1+2+3+……+n=½n(n+1)

(1st step). To examine whether p(1) is true, we put n=1 on both sides of the statement P(n) and see that

LHS = 1and RHS = ½ x 1 x 2 = 1= LHS

Which implies that p(1) is true …….(i)

(2^{nd} step). To examine whether $P(m) \Rightarrow P(m+1)$ for some $m \in N$

Suppose that p(m) is true i.e. suppose that

1+2+3+……+m = ½ m(m+1) for some $m \in N$

Then 1+2+3+……+m + (m+1) = ½ m(m+1) + (m+1), by supposition

= ½ (m+1)(m+2)

$\therefore P(m) \Rightarrow P(m+1)$ ……(ii)

From IP (i) and (ii), p(n) is true $\forall n \in N$ Hence

1+2+3+……+n=½n(n+1) $\forall n \in N$.[And 97]

Illustration: 3.1.3

Prove that $n^{n+1} > (n+1)$ $\forall n \in N$ such that $n \geq 3$

Solution:

The given statement can be rewritten as

$P(n) \equiv (1 + 1/n)^n < n \forall n \geq 3$

(1^{st} step). Putting n=3, we see that the LHS of the inequality = $(1+⅓)^3$

$=(4/3)^3 = \frac{64}{27} = 2\frac{10}{27} < 3$ (=RHS)

Hence P(3) is true ----------(i)

(2^{nd} step). Suppose that for some m(>3) $\in N$

$(1 + 1/m)^m < m$

Then $(1+\frac{1}{m+1})^{m+1} < (1+\frac{1}{m})^{m+1} = (1+\frac{1}{m})^{m} \times (1+\frac{1}{m}) < m \times (1+\frac{1}{m})$ by supposition = m+1

$\therefore (1+{}^{1}\!/_{m+1})^{m+1} < m+1$. If $(1+{}^{1}\!/_{m})^{m} < m$ for some m>3

i.e. $P(m) \Rightarrow P(m+1)$ .....(ii)

From IP (i) and (ii) $\Rightarrow P(n)$ is true $\forall n \geq 3$, Hence proved.

3.1.4 Inclusion-Exclusion Principles (IEP)

This is a set theoretic counting principle. When two tasks can be done at the same time, we cannot use the sum rule to count the number of ways to do one of the two tasks. Adding the number of ways to do each task leads to over count, since the ways to do both tasks are counted twice. To correctly count the number of ways to do one of the two tasks, we add the number of ways to do each of the two tasks and then subtract the number of ways to do both tasks. This technique is called the **principle of inclusion-exclusion.**

Statement 3.1.4

If $A_1, A_2, ..., A_n$ are finite sets, then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n}|A_i| - \sum_{i,j:i<j}|A_i \cap A_j| + \sum_{i,j,k:i<j<k}|A_i \cap A_j \cap A_k| - .....\pm|A_1 \cap .....\cap A_n| \quad \textit{where} \quad |A|$$

denotes the cardinality of the set A .

The Principle of Inclusion and Exclusion (PIE) is a very useful tool to count sets that can be expressed as a union or an intersection of simpler sets. The idea is to generalize the well-known formula to compute the cardinality of a union:

$|A \cup B| = |A| + |B| - |A \cap B|$.

For the case of three sets it is also easy to find a formula by inspection:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

As we see, to compute the size of the union we sum the sizes of the components, subtract the intersections of pairs, and sum back the size of the triple intersection. The Principle of Inclusion and Exclusion is a generalization of this idea.

3.1.4 Theorem *(PIE)*

Let $A_1, A_2, \ldots, A_n$ be subsets of a set X. Then

$$\left|A_1 \cup A_2 \cup \ldots\ldots \cup A_n\right| = \sum_{i=1}^{n}\left|A_i\right| - \sum_{1\le i<j\le n}\left|A_i \cap A_j\right| + \sum_{1\le i<j<k\le n}\left|A_i \cap A_j \cap A_k\right| + \ldots. + (-1)^{n-1}\left|A_1 \cap \ldots. \cap A_n\right|$$

Remarks: The PIE is stated in terms of unions, but can also be used to count intersections. Indeed,

$$\left|A_1 \cap A_2 \cap \ldots\ldots\ldots \cap A_n\right| = \left|\left(A_1^c \cup A_2^c \cup \ldots. \cup A_n^c\right)^c\right| = \left|X\right| - \left|A_1^c \cup A_2^c \cup \ldots \cup A_n^c\right|,$$

where B^c stands for the complement of the set B in X. Now, using PIE,

$$\left|A_1 \cap \ldots. \cap A_n\right| = \left|X\right| - \sum_{i=1}^{n}\left|A_i^c\right| + \sum_{1\le i<j\le n}\left|A_i^c \cap A_j^c\right| + \ldots.. + (-1)^n\left|A_1^c \cap \ldots.... \cap A_n^c\right| .$$

One of the main tricks in using PIE is to choose the sets A_i suitably so that computing the intersections $\left|A_{i1} \cap \ldots.. \cap A_{ik}\right|$ is feasible. It is often the case that $\left|A_{i1} \cap \ldots.. \cap A_{ik}\right|$ does not depend on the A_{ij} but only on k, the number of sets in the intersection. In this case, the PIE has a simpler form. Let A^k denote the size of the intersection of any k of the A_i.

Then, $\left|A_1 \cup A_2 \cup \ldots.. \cup A_n\right| = nA^1 - \binom{n}{2}A^2 + \binom{n}{3}A^3 + \ldots. + (-1)^{n-1}A^n$.

Proof.

There are several proofs of the PIE. The one with more combinatorial flavour is proved by induction method. Now to prove the theorem, we check that the formula counts each element in $A_1 \cup A_2 \cup .... \cup A_n$ just once. Let x be in $A_1 \cup A_2 \cup .... \cup A_n$; by relabeling the sets if necessary, we can assume that x belongs to A_1, A_2,…., A_p but does not belong to A_{p+1},…, A_n. Then in the RHS of the above formula, x contributes with

$$p - \binom{p}{2} + \binom{p}{3} + ... + (-1)^{p-1}\binom{p}{p}$$ This equals

$$\sum_{i=1}^{p}(-1)^{i-1}\binom{p}{i} = \sum_{i=0}^{p}(-1)^{i-1}\binom{p}{i} + 1 = 0 + 1 = 1,$$

since we know that the sum of signed binomials is zero.

Illustration: 3.1.4a

Suppose *n* people leave their coats at the cloakroom of a theater. At the end of the play, the attendant gives the coats back without looking at the tickets. Which is the probability that nobody gets his/ her own coat?

Let us identify the coats with the integers from 1 to *n*. Each way of giving back the coats is a permutation of [*n*]. For instance, the permutation 123…..*n* represents the case that each person gets back their own coat. Let π be a permutation of [*n*]. If for all *i* we have $\pi(i) \neq i$, then nobody gets their own coat back. We call such permutations *derangements.*

Then the probability asked is

$$\frac{\text{number of derangements of } [n]}{n!}$$

Hence the goal is to compute the number of derangements of [*n*]; let us denote the set of derangements by $\mathcal{D}_n$ and its cardinality by $\mathbf{d}_n$.

Let S_n be the set of the $n!$ Permutations of n elements and for each i with $1 \le i \le n$ let A_i be the subset of all permutations π such that $\pi(i) = i$. By the remarks above, we have that

$$\mathcal{D}_n = A_1^c \cap A_2^c \cap \ldots \cap A_n^c = S_n - (A_1 \cup A_2 \cup \ldots \cup A_n).$$

Hence,

$$\mathbf{d}_n = n! - |A_1 \cup A_2 \cup \ldots \cup A_n| ,$$

and by **PIE**

$$\mathbf{d}_n = n! - \sum_{i=1}^{n} |A_i| + \sum_{1 \le i < j \le n} |A_i \cap A_j| - \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^n |A_1 \cap \ldots \cap A_n| .$$

The only thing left now is to count the sizes of the intersections.

- $|A_i| = |\{\pi \in S_n / \pi(i) = i\}| = (n-1)!$, since the element i is fixed and we can permute the remaining n-1 in any way. Note that this is independent of the element i.

- $|A_i \cap A_j| = |\{\pi \in S_n / \pi(i) = i, \pi(j) = j\}| = (n-2)!$, since two elements are fixed and the remaining n- 2 can be permuted arbitrarily. Again, the result is the same for all pairs i, j.

And in general,

- $|A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}| = (n-k)!, for 1 \le i_1 < i_2 < \ldots \ldots < i_k \le n.$

Therefore,

$$\mathbf{d}_n = n! - \sum_{i=1}^{n} (-1)^i \binom{n}{i} (n-i)! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

Hence, the probability asked is $\sum_{i=0}^{n} \frac{(-1)^i}{i!}$.

1. Euler's ϕ function.

Recall from elementary number theory the definition of the function ϕ of Euler. Given a positive integer *n*, ϕ (*n*) is the number of integers smaller than *n* that are relatively prime to *n* (including 1). For instance,

$$\phi(2) = |\{1\}| = 1, \quad \phi(3) = |\{1,2\}| = 2, \quad \phi(4) = |\{1,3\}| = 2, \quad \phi(5) = |\{1,2,3,4\}| = 4.$$

Note that if *n* is prime, then $\phi(n) = n-1$, since all integers smaller than *n* are relatively prime to *n*. We need to find a formula for ϕ (*n*) for any integer *n*. We assume that we have the decomposition of *n* into prime factors,

$$n = \prod_{i=1}^{r} p_i^{\alpha_i},$$

where *r* is the number of distinct prime factors of *n*, the p_i are the distinct prime factors, and α_i stands for their multiplicities.

The integers that are relatively prime with *n* are those that do not contain any of the p_i as a factor. This suggests to define $B_i = \{m : m < n, p_i / m\}$, that is, the set of integers smaller than *n* that are divisible by p_i. Hence,

$$\phi(n) = \left|B_1^c \cap B_2^c \cap ..... \cap B_n^c\right|.$$

By **PIE**,

$$\phi(n) = n - \sum_{i=1}^{r} |B_i| + \sum_{1 \le i < j \le r} |B_i \cap B_j| - \sum_{1 \le i < j < k \le r} |B_i \cap B_j \cap B_k| + ... + (-1)^r |B_1 \cap B_2 \cap .... \cap B_r|.$$

Again, the problem reduces to computing the intersections of B_i's. It is not difficult to show that

$$|B_{i1} \cap B_{i2} \cap ..... \cap B_{ik}| = |\{m : m < n, p_{i1} ..... p_{ik} / m\}| = \frac{n}{p_{i1} .... p_{ik}}.$$

Note that in this case, the size of $B_{i1} \cap B_{i2} \cap .....\cap B_{ik}$ not only depends on *k* but also on the specific sets we intersect. Putting this into the formula given by **PIE**, we have

$$\phi(n) = n - n\left(\sum_{1\le i\le r}\frac{1}{p_i}\right) + n\left(\sum_{1\le i<j\le r}\frac{1}{p_i p_j}\right) + ...+ (-1)^r n\frac{1}{p_1.....p_r},$$

which can be written more compactly as

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right).....\left(1-\frac{1}{p_r}\right).$$

Inclusion-exclusion principle is also a general form of Sum rule for the purpose of counting number of subsets of a general set. Sets under consideration for counting may not necessarily be disjoint. They may overlap. This form of IEP may be stated as:-

Statement: ***3.1.4***

If U be a set of m objects and A, B be two subsets of U, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

and ----------(i)

$$n(A^c \cap B^c) = m - \{n(A) + n(B)\} + n(A \cap B)$$

Where $n(X)$ *denotes the number of objects in X and* X^c *denotes the complement of X in U i.e. U-X.* More generally,

"If U be a finite set of objects and A_i (i=1,2,3,….,k) be subsets of U, then

$$n(\bigcup_{i=1}^{k} A_i) = S_1 - S_2 + S_3.....+ (-1)^{k-1} S_k \quad \text{---------- (ii)}$$

Where S_r denotes the sum of elements common to all r-subsets of U; and complementing this equation from De-Morgan's Rule, we have

$$n(A_1^c \cap A_2^c \cap .....\cap A_k^c) = m - S_1 + S_2 - S_3 + ...+ (-1)^k S_k \text{ ---------- (iii)}$$

To understand the general IEP, let k = 3 and A, B, C be 3 subsets of *U*, where n(*U*) = *m*, then the statement becomes

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - \{n(A \cap B) + n(A \cap C) + n(B \cap C)\} + n(A \cap B \cap C)$$

and

$$n(A^c \cap B^c \cap C^c) = m - \{n(A) + n(B) + n(C)\} + n(A \cap B) + n(A \cap C) + n(B \cap C) - n(A \cap B \cap C) - -(iv)$$

Illustration: 3.1.4b

1. Find the number of integers between 1 and 100 which are (a) divisible by 2 or 5; (b) neither divisible by 2 nor 5.

Solution:

Let U= {1,2,3……,100} $\Rightarrow$ m=100;

A = set of all elements of U divisible by 2

and ={2,4,6,…….,98,100} $\Rightarrow$ n(A) = 50;

B = set of all elements U divisible by 5

={5,10,15,……95,100} $\Rightarrow$ n(B) = 20.

which $\Rightarrow (A \cap B) =$ set of all elements of U divisible by 2x5=10

$\Rightarrow n(A \cap B) = 10.$

a) Number of elements of U divisible by 2 or 5

$= n(A \cup B) = n(A) + n(B) - n(A \cap B)$.....(*i*)

= 50+20-10=60 and

b) Number of elements of U divisible by neither 2 nor 5

$= n(A^c \cap B^c) = m - [n(A) + n(B) - n(A \cap B)]$

$=100 - 60 = 40$

In general, IEP enables one to find out the number of elements in some subsets satisfying certain property and also to find number of elements of a set which do not have such property.

3.1.5 Bijection Principle (BP)

Sometimes, it is not quite easy to count the actual number of a very big collection say A. In such case, one can find *n*(A), without counting, with the help of another set B, which is equivalent with A (i.e. B ~ A) provided that the number of elements of B i.e. *n*(B) is known. BP states that:

Statement: 3.1.5

If A and Bare two finite sets such that there is a bijection (i.e. one-one-onto map), from A to B, then n(A) = n(B).

Illustration 3.1.5

The sitting capacity of a lecture theatre is known to be 1000. In a certain lecture, the hall is full. Then without actually counting we can say that the total number of students present in that lecture is also 1000.

Since all the seats are occupied by the students when the hall is full and there is no seat vacant, it implies that there is a bijection from A (set of students) to B (set of chairs of the lecture hall) $\Rightarrow$ n(A) = n(B) = 1000.

But if the theater is not full and some seats remain vacant, $\exists$ Injection (1-1-into) map from A to B but not bijection, which implies that n(A) < n(B) i.e. the number

of students is less than 1000. But actual number of students cannot be obtained without counting in such a case.

3.1.6 Recurrence Relation (RR)

Some counting problems cannot be solved by the above stated techniques, but can be solved by finding relationships called recurrence relations, between the terms of the sequence. Variety of counting problems can be modeled using recurrence relations. Sequences can be defined recursively, i.e. sequences specify one or more initial terms and a rule for determining subsequent terms from those that precede them. This recursive definition is used in counting problems and the rule for finding terms from those that precede them is called recurrence relation.

Statement: 3.1.6

A recurrence relation for the sequence $\{a_n\}$is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1,..,a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

Also recurrence relation can be stated as:

Given a sequence $< a_1, a_2, a_n,.....>$, or simply $<a_n>$, of positive numbers, an equation relating a_n to its predecessors, valid for all suffixes greater than some integer m is called a recurrence relation.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

Recurrence relation provides efficient means for computing the terms of sequence which satisfy that relation. Sometimes it is not easy to compute the general term, but the recurrence relation provides a powerful computational tool.

Illustration: 3.1.6

The following recurrence relation is given for a sequence $<a_n>$ such that $a_n = 3a_{n-1}$ (where $a_0 = 1$).

Then by putting 1,2,3,…..n for n, we get,

$a_1 = 3a_0$

$a_2 = 3a_1$

$a_3 = 3a_2$

$a_n = 3a_{n-1}$

Multiplying both sides, we get,

$(a_1\ a_2\ a_3\ ….a_{n-1}\ a_n) = 3^n(a_0\ a_1\ a_2\ …..\ a_{n-1})$

Cancelling common factors, we get,

$a_n = a_0(3^n) = 1\times3^n\ (\because a_0 = 1)$

$\therefore\ a_n = 3^n$

gives the solution of the recurrence relation. Terms of this sequence are 1, 3, 3^2, …. 3^n, ….. which are in geometric progression.

3.1.7 Pigeon-Hole Principle (PHP)

Suppose that a flock of pigeon flies into a set of pigeonholes to roost. The pigeonhole principle states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it. This principle, of course applies to other objects besides pigeons and pigeonholes. The pigeonhole principle is also called Dirichlet drawer principle.

Statement: 3.1.7

If at least (n + 1) objects (pigeons) are to be distributed among n boxes (pigeon holes) then one of the boxes must contain 2 objects (pigeons).

The pigeonhole principle in a generalized form states that there must be at least two objects in the same box when there are more objects than boxes. However, even more can be said when the number of object exceeds a multiple of the number of boxes. For instance, among a set of 21 decimal digits there must be 3 that are the same. This follows because when 21 objects are distributed into 10 boxes, one box must have more than 2 objects.

More generally, PHP may be stated as:

"For positive integers k, n and r if $m = kn + r(1 \le r \le n-1)$ *objects are distributed among n boxes then one of the boxes must contain at least k + 1 objects."*

Illustration: 3.1.7a

1. If 10960 children were born in Zaria in 1995, then prove that at least 31 children were born on a particular day of the year.

Solution:

In this problem we find that m = 10960 is the number of objects and since there are 365 days in 1995, the number of boxes = n = 365. Since 10960 = 30(k) × 365(n) + 10(r) = kn + r, it follows that in this case. k = 30. Hence on a particular day of 1995 at least k + 1 = 31 children were born.

Theorem: 3.1.7

If N objects are placed into k boxes, then there is at least one box containing at least [N/k] objects.

Proof:

Suppose that none of the boxes contains more than [N/k] – 1 objects. Then the total number of objects is at most k([N/k] – 1) < k(((N/k) + 1) – 1) = N where the inequality [N/k] < (N/k) + 1 has been used. This is a contradiction since there are a total of N objects.

Illustration: 3.1.7b

In a mathematics class, what is the minimum number of students required to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D and F?

Solution

The minimum number of students needed to guarantee that at least six students receive the same grade is the smallest integer N such that [N/5] = 6. The smallest such integer is N = 5.5 + 1 = 26. Thus 26 is the minimum number of students needed to be sure that at least 6 students will receive the same grade.

3.2 Applications

3.2.1 Application of Pigeonhole Principle

The objects to be placed in boxes (pigeonholes) of PHP in many interesting application must be chosen in a clever way. A few of such application are illustrated below:

Illustration: 3.2.1a

During a month with 30 days a base ball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution:

Let a_j be the number of games played on or before the j^{th} day of the month. Then $a_1, a_2 \ldots a_{30}$ is an increasing sequence of distinct positive integers, with $1 \leq a_j \leq 45$. Moreover, $a_1+14, a_2+14,\ldots.., a_{30}+14$ is also an increasing sequence of distinct positive integers, with $15 \leq a_j +14 \leq 59$. The 60 positive integers, $a_1, a_2,\ldots.., a_{30}$, $a_1+14, a_2+14,\ldots, a_{30}+14$ are all less than or equal to 59. Hence, by the pigeonhole principle two of these integers are equal. Since the integers a_j, j = 1, 2, ….., 30 are all distinct and the integers a_j+14, j = 1, 2, …., 30 are all distinct, there must be indices *i* and *j* with $a_i = a_j+14$. This means that exactly 14 games were played from day *j + 1* to day *i*. [And 97, Ros 99].

Illustration: 3.2.1b

Show that among n + 1 positive integers not exceeding 2n there must be an integer that divides one of the other integers.

Solution

Write each of the n + 1 integers $a_1, a_2, \ldots., a_{n+1}$ as a power of 2 times an odd integer. In other words, let $a_j = 2^{k_j} q_j$ for j = 1, 2, …, n+1, where k_j is a nonnegative integer and q_j is odd. The integers $q_1, q_2, \ldots., q_{n+1}$ are all odd positive integers less than 2n. Since there are only n odd positive integers less than 2n, it follows from PHP that two of the integers $q_1, q_2, \ldots, q_{n+1}$ must be equal. Therefore, there are integers i and j such that $q_i = q_j$. Let q be the common value of q_i and q_j. Then $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$. It follows that if $k_i < k_j$, then a_i divides a_j; while if $k_i > k_j$, then a_j divides a_i.[see And 97]

3.2.2 Counting number of relations

Here, some important results are considered for counting number relations that are reflexive,symmetric and transitive.

Propositions 3.2.2

In a given set A with n elements, the number of relations that are:-

i) Reflexive $= 2^{n(n-1)}$

ii) Irreflexive $= 2^{n(n-1)}$

iii) Symmetric $= 2^{n(n+1)/2}$

iv) Antisymmetric $= 2^{n}\, 3^{n(n-)/2}$

v) Asymmetric $= 3^{n(n-)/2}$

vi) Reflexive and Symmetric $= 2^{n(n-1)/2}$

vii) Neither reflexive nor symmetric $= 2^{n2} - 2.2^{n(n-1)}$

Proof:

Since a relation is the subset of A x A, then the total number of elements in R is $2^{|A \times A|} = 2^{n^2}$. If R is reflexive, for example, each of n ordered pairs *(a,a)* for $a \in$ A must be in R, irrespective of the other *n(n-1)* ordered pairs of the form *(a,b)*, $a \neq b$, being present in R. Thus by the product rule of counting, the result in i) follows. This is seen from the number of permutations of the remaining *n(n-1)* ordered pairs, as described in (5.2) page 61-63. [see Ros 99 pp. 383] for more details.

3.2.3 Counting Number of Partitions and Coverings in a Set

3.2.3.1 Introduction

It may be recalled that counting the number of elements in a set is a broad mathematical problem and a large number of them yet remaining unsolved or partially answered.

However, in a typical combinatorial problem, as the one proposed in this paper, a relatively crisp description and some further structure are provided. Accordingly, such a problem may or may not have outright a closed formula solution even if it appears structurally feasible. In fact, many such problems do exist whereby either no solutions or partial solutions are available. For example, assume that a certain number *n* of objects is given. Is it possible to assign them sets so that each object

is in at least one set, each pair of objects is in exactly one set together, every two sets have exactly one object in common, and no set contains all or all but one of the objects? The answer depends on *n* and is only partially known to this day [Wik].

The next section is intended to discuss an extremal combinatorial problem concerning *partition* and *covering* of a finite set.

3.2.3.2 Definition of Covering and partition of a finite set

For the definition of covering and partition of sets refer to section 1.3 page 20.

From the definition, it follows that, {S} is both a covering and a partition on S.

Note also that an *n*-set can have atmost *n* blocks.

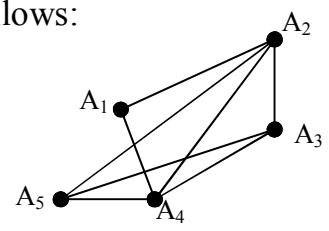
It is well-known that a relation on S which is reflexive, symmetric and transitive, called an *equivalence relation*, gives a partition of S. A related notion is that of compatibility relation. A relation on S which is reflexive and symmetric is called a *compatibility* relation, sometimes denoted by $\approx$.

Also, let R be a compatible relation on a set *X*, then $x, y \in X$ are called compatible if *x R y*. For example,

Let $X = \{A_1, A_2, A_3, A_4, A_5\}$ where $A_1 = \{1,2\}$, $A_2 = \{2,3,4\}$, $A_3 = \{3,4,5\}$, $A_4 = \{2,4\}$ $A_5 = \{4,5\}$ and let R be given by $R = \{(A_i, A_j) / A_i, A_j \in X \wedge A_i R A_j$ if A_i and A_j contain some common element$\}$.

Clearly, $A_1 R A_2 \wedge A_2 R A_3$, but $\neg(A_1 R A_3)$.

A simplified diagram is as follows:



Note that, in the case of a compatibility relation, it is not necessary to draw loops at each element nor is it necessary to draw both *x*R*y* and *y*R*x*.

Since a compatibility relation is not necessarily transitive, it does not necessarily define a partition; however, it does define a covering. A number of pleasing

properties of compatibility relation have been investigated (see [Ros 99] and [TM 00]) and used in solving certain minimization problems of switching theory, especially when the specification provided is incomplete.

From the diagram above, it is easy to observe that the elements in each of the sets $\{A_1, A_2, A_4\}$, $\{A_2, A_3, A_4\}$, $\{A_2, A_4, A_5\}$, $\{A_3, A_4, A_5\}$ and $\{A_2, A_3, A_4, A_5\}$ are mutually compatible and the sets are not mutually disjoint. Accordingly, it does define only a covering of X. Infact, there may exists more than one combination of these sets to yield the covering of X.

Further, in order to solve minimization problems, the following elaboration is in order.

A subset $A \subseteq X$ is said to be a maximal compatibility block if any element of A is compatible to all other elements of A and no element of $X - A$ is compatible to all the elements of A. Moreover, any element of the set which is related only to itself, and any two elements compatible to one another but to no other elements of the set are also the maximal compatibility blocks.

It follows that a maximal compatibility block is the largest complete polygon in which every vertex is related to its every other vertex. For example, in the diagram above, $\{A_1, A_2, A_3, A_4\}$ is not a complete polygon.

Clearly, $\{A_1, A_2, A_4\}$ and $\{A_2, A_3, A_4, A_5\}$ are the only maximal compatibility blocks. Note that no other subset of X forms a compatibility block (see [Ros 99] and [TM 00]) for more details).

Problem 3.2.3

Let S be a non-empty set with n elements, usually called an *n-set*. Find (a) the size of the family of all the partitions of S, denoted by $|par(s)|$, and (b) the size of the family of all the coverings of S, denoted by $|cov(s)|$.

Note that a relation between two partitions or two coverings or a partition and a covering will be understood by treating them as sets.

Results

Fact 1. $|cov(s)| > |par(s)|, for\ |s| \geq 2.$

Proof follows by definition.

Fact 2. In order to answer (a), one would wish to determine the number of all *possible* distinct equivalence relations on S, for which no general method is known to exist. Nevertheless, in view of the fact that there exists a one to one correspondence between the classes of distinct equivalence relations defined on a set and the partition of that set, a combinatorial formulation is known as follows (see [Mie 04], [Wag], and many others):

The number of partitions of an n-set into k blocks is given by $\left\{ {n \atop k} \right\}$ or $S(n,k)$, called a Starling number of the *second* kind. For $n \geq 1$,

$$\left\{ {n \atop 1} \right\} = 1, \qquad \left\{ {n \atop 2} \right\} = 2^{n-1} - 1, \qquad \left\{ {n \atop n-1} \right\} = \binom{n}{2}, \qquad \left\{ {n \atop n} \right\} = 1.$$

Also for $n \geq k \geq 1$,

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$ (see [Ann 04], p.18 and p.46 for the proof).

Note that this recurrence relation has a precedent in the theory of integer partition viz., for $n \geq k \geq 2$,

$$P_k(n) = P_{k-1}(n-1) + P_k(n-k)$$ (See [Ann 04], p.14 for the proof).

Moreover, in view of the existence of a correspondence between set partitions and surjective mappings, the following is an explicit formula for stirling number of the second kind viz.,

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} (k_j)^{j^n}$$

(see [Mie 04], p.19 and p.46 for the proof).

The total number of partitions of an n-set, given by $\sum_{k=1}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$,

is called a *Bell* number denoted by *B(n)*. Also, $B(n+1) = \sum_{k=0}^{n} \left(\begin{matrix} n \\ k \end{matrix} \right)$, *B(k)*.

Note, however, that no explicit formulation is known for *p(n)*, the number of partitions of the integer *n* ([Mie 04], p.14).

In order to compute p(*n*) of all partitions of any *n*-set, the following nice formula is constructed in [Coh 81]:

$$\frac{p(n) = n^{th} \text{ coefficient of the analytic function } f(x) = exp(exp(n)-1)}{n}$$

in its standard representation in the form of Taylor series.

Fact 3. No general formulation is known to explicitly determine $|cov(s)|$.

We put forward some motivating steps in this regard.

Step 1. It is known that the total number of nonempty subsets of an n-set

$S=2^n - 1$.

Step 2. As noted above, the total number of partitions of an *n*-set *S=B(n)*, the Bell number = the sets of all possible combinations of nonempty disjoint subsets of S which form a partition of S.

Step 3. Now, if one knows the number of all possible combinations of subsets of S which are not disjoint and form a covering of S, then the

sum of the two numbers determined in step 2 and step 3 would answer (b).

However, to our knowledge, no closed formula for this is known.

Nevertheless, the following is a further known reflection in this regard. The largest number (say ℓ) of subsets of an n-set S one can have, if no two of them are disjoint, is half the number of all subsets of S; that is, $\ell = 2^n \ 2 = 2^{n-1}$. The proof follows from the fact that between any subset T and its complement $S - T$, atmost one can be chosen. In order to attain the aforesaid half of the number, one element x of S is chosen and all the subsets of S that contain x are tracked. Clearly, the family of all such subsets of S would form a covering of S, for each element x of S. Accordingly, the number (say C) of all such coverings would be n. For example, if S={1,2,3}, then $\ell = 4$, and {{1},{1,2},{1,3},{1,2,3}} is a covering of S corresponding to the choice of the element 1 of S, and C = 4.

Note, however, that besides C = 4, there are many other coverings of S; for example, {{1}, {2},{1,2},{1,3},{1,2,3}}, etc.

At this end, we mention without proof a very nice result obtained in [BD 71] proving a best possible bound on the number of subsets of a given finite set that form its covering.

Let n, s, t be integers with $s>t>1$ and $n > (t+2)\ 2^{s-t-1}$. If n subsets of a set S with s elements have union S then some t of them have union S. The result is best possible ([BD 71], p.197).

Remarks

In view of the fact that a relation which is reflexive and symmetric (but not necessarily transitive) has been found characteristically useful in explicating a variety of problems e.g., minimization problems of switching theory (see [Ros 99] and [TM 00] for some details); it has increasingly become pertinent to closely investigate various issues in this regard.

In the first place, besides providing a sufficiently rich partition calculus, the paper puts forward some challenging problems of varying degrees specially from the pedagogical point of view; for example, constructing an algorithm to compute a maximal or minimal number of subsets of an n-set under varying conditions that would cover the set. In turn, for a given compatibility relation on an n-set, it is a good exercise to compute maximal compatibility blocks and coverings.

The paper also delineates an 'open' combinatorial problem to formulate a closed formula to determine the size of the family of all the coverings of an n-set and provides some motivating steps to solve it.

CHAPTER FOUR

FUNDAMENTALS OF MULTISETS

4.1 Introduction

The asumption of cantorian set theory dictates that no element shall be allowed to appear more than once in a set. That is, for this collection $\{1, 2, 2, 3\}$, it only becomes a set after deleting the repeated element to become $\{1, 2, 3\}$. This assumption is indeed handicapped for the requirement of various fields of human endeavours which requires mathematical formulation in solving some problems. In addition, the repeated nature of mathematical objects in this world is also a basis for the handicapped nature of this assumption.

Consider for example a water molecule, with repeated hydrogen atoms. Consider also coins of the same denomination and year of minting but which are physically distinct are essentially indistinguishable, like also repeated electrons, or grains of sand, or amino acids, in larger structures where their relative positions may allow us to distinguish them, like repeated roots of polynomial equations, repeated observation in statistical problems all these need to be counted for attaining adequacy and exactness. The only mathematics that solves this problem is the Multisets. By this concept, repeated elements are admitted in a set.

4.2 Multisets

Definition 4.2.1

Multiset (shorten as mset) is an unordered collection of objects (called elements) in which, unlike a standard (Cantorian) set, elements are allowed to repeat.[Singh 2006]

In other words, Multiset is a set in which elements may belong to more than once and hence it is a non-cantorian set [Knuth, ([Knu 81]p66)].

Multiset is a collection of objects, taken without regard to order, and with repetitions of the same object allowed. For example, M = {1, 1, 1, 2, 2, 2, 2, 3} is a multiset. Since the order in which objects in M are listed is immaterial, we could also write, among many other possibilities, M = {1, 3, 2, 2, 1, 2, 2, 1}. One sometimes encounters multisets written in "exponential" notation, where the "exponent" indicates the frequency of occurrence of an object in the multiset. With this notation, one would write M = $\{1^3, 2^4, 3^1\}$. The list of objects belonging to a multiset is always enclosed by a pair of curly brackets.

The cardinality (i.e., total number of elements) of a multiset takes account of repetitions (number of times an element belongs). So, for example, the multiset M above has cardinality 8. The number of times an element belongs to a multiset is the ***multiplicity*** of that member. [Wag]

The concept of multiset is a generalization of the concept of set. A set is simply a multiset in which there are no repetitions of the same object.

Multiset can also be defined as follows: Let X be a set. An mset M on X is a function $v : X \to N$ *such that* $v(x)$ *is finite for all* $x \in X$. *The number* $v(x)$ *is number of copies (or repetitions) of x, and* $\sum_{x \in X} v(x)$ *is the size of M. The set X is called the ground or generic set or root set of the class of all msets containing objects from X.*

Illustration: 4.2.1

Let X={a,b,c,d}. The multiset corresponding to the function v(a)=2 v(b)=0 v(c)=1 v(d)=3 can be represented as {a, a, c, d, d, d} with size of M=6.

4.2.1 Representation(structure) of Multisets

Since, multisets have multiple occurences but finite number of any element, one can define a multiset structure by employing the list method. However, in order to

avoid confusion, square brackets for multisets are used. In what follows we will employ the most suitable definition method for each case we encounter.

There are considerable number of ways or forms in which multisets are structured. We consider here some few such representations. [Singh 06]

4.2.1.1 Multiplicative Structure

It has almost become a standard form to represent an mset using a square bracket following the work of Meyer and McRobbie[1982], according to [Singh 2006]. Thus representing an mset containing one element(occurence) of a, two occurrences of b, and three occurrences of c is notationally written as [[a,b,b,c,c,c]] or [a,b,b,c,c,c] or $[a,b,c]_{1,2,3}$ or $[a^1,b^2,c^3]$ or [a1,b2,c3] the choice of which depends on individual conveience.

4.2.1.2 Linear structure:

This is yet another form of representing msets, linear form of representing mset is put forward by Wildberger[wil 03]. The linear notation looks quite innovative, especially when negative multiplicities are invoved(integral and rational). An example is representing the mset $M=[a,b,c]_{1,2,3}$ in linear form can be written as M= [a]+2[b]+3[c]. Similarly,for a rational mset,for example, $N=^2/_3[5]-^1/_2[1_8]$.

To be more precise and to enable accommodation of negative multiplicities round brackets are used for negative occurrences, e.g. (*a*) is an mset standing for a negative occurrence *a*.

For example: [3,5,(7),(7),5]=[3]+2[5]-2[7].

The above give the distinction between the terms “element” and ‘object’ occuring in an mset, this is explicitly explained below:

Each individual occurrence of an object X in an mset A is called an element of A.

Thus in the linear notation of M above, b is an object appearing twice, and every occurrence of b is an element of M. It follows that the distinct elements of an mset are the objects. An object is an element if its multiplicity is unity. The notation is clearly out in [wil 03 pp. 5-8].

A collection containing q,r,s,t,..... is denoted by {qrst.....} or {q_r_s_t.....} a set; {q,r,s,t,....} if it is an ordered set; [qrst.....]=[q_r_s_t_.....] if it is a multiset; and [q,r,s,t,....] if it is a list(which is an un ordered sequence of elements with repetition allowed).

4.2.1.3 Sequencial Structure:

A multiset can also be represented as sequence in which the multiplicity of an element equals the number of times the element occurs in the sequence; which is exactly Dedikind's frequency number. The idea is to construct an mset as a sequence (a function with the domain (N) a set of natural numbers) and ignore the odering of its elements, which can be done by taking all permutations of the domain of the sequence. (see permutations of multisets) chp. V p. 61-62.

4.2.1.4 Family of set structure

Multisets can also be represented as a family of sets, which is a generalization of the above structure (Sequencial Structure). Thus the family of sets F= $\{F_i\}$, $i \in I$, where $F_i = F_j$, if $i = j$, which identifies a repeated element, represent an mset. Clearly such a family F is a function: I $\{F_i / i \in I\}$, which in turn is a sequence if I = N.

4.2.1.5 Cardinal Valued Structure

Msets can also be represented as numeric-valued functions or cardinal valued functions, this especially come in an area of applications. (i.e. when multisets operations are involved). An mset is formally known as a mapping from some ground or generic or universal set into some set of numbers, for instance an mset

$\alpha = [q,r,s]_{1,2,3}$ is a mapping from a ground set S to N, the set of natural numbers with zero, defined by

$$\alpha(t) = \begin{cases} 1, & \text{if } t = q \\ 2, & \text{if } t = r \\ 3, & \text{if } t = s \\ 0, & \text{for all the remaining } t \in S \end{cases}$$

In general, for a given ground set S and a numeric set T, we call a mapping $\alpha : S \to T$, with the following definition:

$$\begin{cases} a \text{ set, if } T=\{0,1\} \\ a \text{ multiset if } T=N, \text{ the set of natural numbers;} \\ a \text{ signed multiset(or hybrid/shadow set) if } T = Z, \text{ the set of integers;} \end{cases}$$

a fuzzy(or hazy) set, if $T = [0,1] \subset R$, a two-valued boolean algebra.

4.2.1.6 Generalized Structure

Similar to the representation of a set by its characteristic function(function whose range is {0,1}, a multiset or hybrid set is determined by its generalized characteristics function (whose range is the set of integers, positive, negative or zero), see [whi 33] and for more details as in [Singh 2006,pp.12].

4.3 Multisubsets (or msubsets, for short)

Let A and B be two msets, A is called an msubset or a submultiset (submset) of B, written as $A \subseteq B$ or $B \supseteq A$, if $m_A(x) \leq m_B(x)$ for all $x \in D$, where D is the root set of B. Also, if $A \subseteq B$ and $A \neq B$, then A is called a ***proper submset*** of B. An mset is called the ***parent*** in relation to its msubsets. Clearly, ϕ is a submset of every mset. An msubset that contains all multiplicities of common is called a ***'whole' msubset*** while a ***'full' msubset*** contains all objects of the parent mset and accordingly, every mset contains a unique full msubset, its root set. Clearly, for any two msets A and B, if $A \subseteq B$ and Dom (A) = Dom (B), then A is a full msubset of B.

With the introduction of the notation $\in_+$ (dressed epsilon, is a binary predicate which means 'belongs to at least once') by [SS 07], A $\subseteq$ B is clearly understood as $\forall_z \forall_k (z \in^k A \rightarrow z \in_+^k B)$. With this definition $\subseteq$ is antisymmetric i.e., $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$, and it is a partial ordering on the class of msets defined on a given generic domain.

Definition 4.3.1

Suppose that $\mathcal{A} = \langle A, f \rangle$ is a multiset, then the subset B of A is called the support of $\mathcal{A}$ if for every x such that f(x) > 0 this implies that $x \in B$, and for every x such that f(x) = 0 this implies that $x \notin B$.

Illustration: 4.3.1

Given the multiset A = [a, a, a, b, c, c], then its support is the set {a, b, c}.

Definition 4.3.2

Let $\mathcal{A} = \langle A, f \rangle$ and $\mathcal{B} = \langle A, g \rangle$ be two multisets, we say that $\mathcal{A}$ is a sub-multiset of $\mathcal{B}$, denoted $\mathcal{A} \subseteq \mathcal{B}$ if for all $a \in A$

$$f(a) \leq g(a).$$

$\mathcal{A}$ is called a proper sub-multiset of $\mathcal{B}$, denoted $\mathcal{A} \subset \mathcal{B}$, if in addition for some $a \in A$

$$f(a) < g(a).$$

Obviously, it follows that for any two multisets $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$.

Definition 4.3.3

Let $\mathcal{A} = \langle A, f \rangle$ be a multiset, then $\mathcal{A}$ is the empty multiset if for all $x \in A$, $f(x) = 0$.

Definition 4.3.4

Suppose that $\mathcal{A} = 3$ is a multiset, then its cardinality, denoted card($\mathcal{A}$), is defined as $card(\mathcal{A}) = \sum_{a \in A} f(a)$

Assume that A is a set, then $\mathcal{P}^A$ is the set of all multisets which have A as their support set. Moreover, A is the smallest non-empty multiset in $\mathcal{P}^A$ in the sense that if $\mathcal{B} \in \mathcal{P}^A$, then card($\mathcal{B}$) $\geq$ card(A).

Remarks:

We observed that this definition does not completely capture the idea of submultiset. For example, given multisets A = [a, a, b, b, c] and B = [a, b, b, b, c, c], A $\not\subseteq$ B and B $\not\subseteq$ A, yet A and B have common root set, i.e. $A^*=B^*$. We say multisets A and B are ***similar***, the term which we will clearly define later.

4.4 Equality of Multiset

Two multisets A and B are said to be equal, written as A = B if and only if for $x \in D$, $m_A(x) = m_B(x)$. Also, $A = B \Rightarrow A^* = B^*$, but the converse need not hold.

4.5 Similar msets

Two msets A and B are said to be 'cognate' or 'similar' if $\forall x(x \in A \Leftrightarrow x \in B)$, where x is an object. Thus, similar msets have equal root sets but need not be equal themselves.

4.6 Multisets and their Operations

It is clear that if occurrences of any element is allowed to multiply finitely, we get a generalization of the notion of a set which is called a multiset. There are two different kinds of sets with a finite number of repeated elements: sets with distinguishable repeated element. For example, people sharing a common property, and sets with indistinguishable repeated element a "soup" of elementary particles. Monro [Mon87] calls the first kind of sets multisets and the second multinumbers. However, in order to avoid confusion, we will use the term multiset for Monro's multinumbers and the term real multiset for Monro's multisets.

Definition 4.3.1

A real multiset $\mathcal{X}$ *is a pair (X,* ρ*), where X is a set and* ρ *an equivalence relation on X. The set X is called the field of the real multiset. Elements of X in the same equivalence class will be said to be of the same sort; elements in different equivalence classes will be said to be of different sorts.*

Given two real multisets $\mathcal{X} = (X, \rho)$ and $\mathcal{Y} = (Y, \sigma)$, a morphism of real multisets is a function $f : X \rightarrow Y$ which respect sorts; that is, if $x, x^1 \in X$ and $x \ \rho \ x^1$, then $f(x) \ \sigma \ f(x^1)$.

4.6.1 Basic operations

[Knu 69] and [Knu 81] are the early references that gave intuitive sufficient operations under multiset. [Hai 86], [Bli 89], [Sin 94], [Pet 94], [Pet 97], [Wil 03] etc. are also notable works where operations of multisets were almost exhausted. The operations of addition, subtraction, union and intersection between multisets can be defined in the following order.

4.6.1.1 Sum of Multiset

Definition 4.6.1.1

Suppose that $\mathcal{A} = \langle A, f \rangle$ *and* $\mathcal{B} = \langle A, g \rangle$ *are two multisets, then their sum, denoted* $\mathcal{A} \uplus \mathcal{B}$*, is the multiset* $C = \langle A, h \rangle$ *where for all* $a \in A$*:*

$$h(a) = f(a) + g(a).$$

It can be easily shown that the multiset sum operation has the following properties:

i) Commutativity: $\mathcal{A} \overset{+}{\cup} \mathcal{B} = \mathcal{B} \overset{+}{\cup} \mathcal{A}$;

ii) Associativity: $(\mathcal{A} \overset{+}{\cup} \mathcal{B}) \overset{+}{\cup} C = \mathcal{A} \overset{+}{\cup} (\mathcal{B} \overset{+}{\cup} C)$;

iii) There exists a multiset, the null multiset ϕ, such that $A \cup \phi \overset{+}{=} A$.

It is important to note that there exists no inverse and multiset sum is not indempotent.

4.6.1.2 Difference of Multisets

Definition 4.6.1.2

Suppose that $\mathcal{A} = \langle A, f \rangle$ *and* $\mathcal{B} = \langle A, g \rangle$ *are two multisets, then the removal of multiset* $\mathcal{B}$ *from* $\mathcal{A}$*, denoted* $\mathcal{A} \ominus \mathcal{B}$*, is the multiset* $C = \langle A, h \rangle$ *where for all* $a \in A$*:*

$$h(a) = max(f(a) - g(a), 0).$$

4.6.1.3 Union of two Multisets

Definition 4.6.1.3

Suppose that $\mathcal{A} = \langle A, f \rangle$ *and* $\mathcal{B} = \langle A, g \rangle$ *are two multisets, then their union, denoted by* $A \cup B$*, is the multiset* $C = \langle A, h \rangle$*, where for all* $a \in A$*:*

$$h(a) = max(f(a), g(a)).$$

4.6.1.4 Intersection of Multisets

Definition 4.6.1.4

Suppose that $\mathcal{A} = \langle A, f \rangle$ *and* $\mathcal{B} = \langle A, g \rangle$ *are two multisets, then their intersection, denoted* $\mathcal{A} \cap \mathcal{B}$*, is the multiset* $C = \langle A, h \rangle$*, where for all* $a \in \mathcal{A}$*:*

$$h(a) = min(f(a), g(a)).$$

The following properties can easily be established as holding for union, intersection and sum of multisets:

i) Commutativity:

$$\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$$

$$\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A};$$

ii) Associativity:

$$\mathcal{A} \cup (\mathcal{B} \cup C) = (\mathcal{A} \cup \mathcal{B}) \cup C$$

$$\mathcal{A} \cap (\mathcal{B} \cap C) = (\mathcal{A} \cap \mathcal{B}) \cap C;$$

iii) Idempotency: (***Not exist in multiset sum***)

$$\mathcal{A} \cup \mathcal{A} = \mathcal{A}$$

$$\mathcal{A} \cap \mathcal{A} = \mathcal{A};$$

iv) Distributivity: $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$,

$$\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C});$$

v) $\mathcal{A} \overset{+}{\cup} (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \overset{+}{\cup} \mathcal{B}) \cup (\mathcal{A} \overset{+}{\cup} \mathcal{C})$

$$\mathcal{A} \overset{+}{\cup} (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \overset{+}{\cup} \mathcal{B}) \cap (\mathcal{A} \overset{+}{\cup} \mathcal{C});$$

vi) $\mathcal{A} \cap (\mathcal{A} \uplus \mathcal{B}) = \mathcal{A}$

$$\mathcal{A} \cup (\mathcal{A} \uplus \mathcal{B}) = \mathcal{A} \uplus \mathcal{B};$$

vii) $\mathcal{A} \overset{+}{\cup} \mathcal{B} = (\mathcal{A} \cup \mathcal{B}) \overset{+}{\cup} (\mathcal{A} \cap \mathcal{B})$.

The proof of all these identities follows from the interpretation of $\cup$, $\cap$ and $\overset{+}{\cup}$ of two natural numbers as maximum, minimum and (direct) sum respectively.

4.7 Direct Product of Multisets

Given two multisets A and B, the direct product of A and B denoted by A × B is the multiset consisting of all ordered pairs $\langle a, b \rangle$ where $a \in$ A and $b \in$ B. Clearly,

$$m_{A \times B}(\langle a, b \rangle) = m_A(a) m_B(b) \text{ or } |A \times B| = |A||B|$$

For example,

$$[1, 1, 3] \times [2, 2] = [\langle 1, 2 \rangle, \langle 1, 2 \rangle, \langle 1, 2 \rangle, \langle 1, 2 \rangle, \langle 3, 2 \rangle, \langle 3, 2 \rangle]$$
$$= [\langle 1, 2 \rangle, \langle 3, 2 \rangle]_{4, 2}$$

4.8 Complement of a multiset

Let $\Im = \{A_1, A_2, \ldots\}$ be a family of multisets composed of the elements of the generic set D. Then, the maximum multiset z is defined by $m_Z(x) = \max_{A \in \Im} m_A(x)$ for all $x \in D$ and all $A \in \Im$. Now, the complement of an mset A, denoted by $\overline{A}$, is defined as follows:

$$\overline{A} = Z - A = \left\{ m_{\overline{A}}(x) \,.\, x / m_{\overline{A}}(x) = m_Z(x) - m_A(x),\ for\ all\ x\ \varepsilon\ D \right\}$$

4.9 Functions between msets

The underlying assumption in defining a function between msets has been invariably not to allow mapping of identical elements to non-identical elements and hence, it amounts to defining the function between their root sets, which is just the classical definition of a function.

The function f: $A \rightarrow B$ is an injection if and only if

(i) f: $A^* \rightarrow B^*$ is an injection, and

(ii) $\forall z\ (z \in A^* \Rightarrow m_A(z) \leq m_B(f(z)))$

The function f: $A \rightarrow B$ is a surjection if and only if

(i) f: $A^* \rightarrow B^*$ is a surjection, and

(ii) $\forall z\ (z \in A^* \Rightarrow m_A(z) \geq m_B(f(z)))$.

The function f: $A \rightarrow B$ is a bijection if and only if

(i) f: $A^* \rightarrow B^*$ is a bijection and

(ii) $\forall z\ (z \in A^* \Rightarrow m_A(z) = m_B(f(z)))$.

For example, f: $[x]_3 \rightarrow [x]_{10}$ is an injection, f: $[x]_5 \rightarrow [y]_4$ is a surjection, and

f: $[x, y]_{6, 2} \rightarrow [x, y]_{3, 4}$ is neither an injection nor a surjection.

Note that some of the consequences of the aforesaid definitions are conflicting with some fundamental theorems of the classical set theory.

1. Having defined functions between msets as above, ([Hic 80]) p. 215) proved that Cantor's theorem does not hold and there is no injection from $A \rightarrow \wp(A)$.
2. Msets of equal cardinality need not have a bijection between them. For example, $[a, b]_{1, 3}$ and [a, b, c, c] both contain four elements, but there can be no bijection between them because the objects and their multiplicities are different in the domain and the range of any such function. In the other words, there is no bijection between their root sets viz; f: $\{a, b\} \rightarrow \{a, b, c\}$ cannot be a bijection.
3. Schröder–Bernstein theorem fails ([Hic 80]),p.215] and ([Bli 89], p.47).

4.10 Power multiset.

Power multiset of a given mset A, denoted by $\widetilde{\wp}(A)$ to distinguish it from the symbol $\wp(A)$ used for power set of A, is the multiset of all submultisets of A.

For example, let $A = [x, y]_{2, 1} = [x, x, y]$. Then,

$$\widetilde{\wp}(A) = [\phi, \{x\}, \{x\}, [x]_2, \{y\}, \{x, y\}, \{x, y\}, [x, y]_{2, 1}].$$

In this sense, $|\widetilde{\wp}(A)| = 2^{|A|}$, for any mset A.

However, as has been voiced by many researchers in the area of msets and their applications ([Hic 80], p. 213 and [Bli 89], p. 45, in particular), there is no 'good' reason for admitting repeated elements into a power multiset. Hence, a power multiset needs to be called a power set only and denoted by $\wp(A)$. Accordingly, for $A = [x, y]_{2, 1}$, $\wp(A) = [\phi, \{x\}, [x]_2, \{y\}, \{x, y\}, [x, y]_{2,1}]$, and hence $|\wp(A)| < 2^{|A|}$,

which implies that Cantor's power set theorem: $|A| < |\wp(A)|$ fails. However, for finite msets, Cantor's theorem holds for power mset.

According to Blizard the powerset of an mset {x} can informally be defined as $(\tilde{\wp}(x))^* = \tilde{\wp}(x)$(so $y \in \tilde{\wp}(x) \Leftrightarrow y \subseteq x$). If $y = \phi$, then $y \in^1 \tilde{\wp}(x)$; and if $y \neq \phi$, then $y \in^k \tilde{\wp}(x)$ where $k = \prod_{z \in y^*} \begin{pmatrix} |x_z| \\ |y_z| \end{pmatrix}$ and $\begin{pmatrix} n \\ C \\ m \end{pmatrix} = \frac{n!}{m!(n-m)!}$ is the number of combination of n objects taken m at a time and where the product $\prod_{z \in y^*}$ is taken over distinct element z of the multiset y and where $|x_z| = n$ iff $z \in^n x$ and $|y_z| = m$ iff $z \in^m y$.

***Illustration**: 4.11.1*

If $x = [\phi, \{\phi\}]_{3,1}$ and $y = [\phi]_2$ then $k = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3$. W*ith the same mset x, if* $y = [\phi, \{\phi\}]_{2,1}$, then $k = \begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3$. I*f the mset x contains n elements, then* $\tilde{\wp}(x)$ *contains exactly* 2^n *elements.*

CHAPTER FIVE

SOME COMBINATORIAL RESULTS ON MULTISETS

5.1 Introduction

There are many interesting applications of Combinatorics as applied to counting numbers and their arrangements. Permutations and Combinations of multisets are some of the clear results on the combinatorics of multisets. When refering to permutations of a multiset we assume that the elements of the multiset are 1, 2,..t, that the number of occurences of i is n_i and that $N = n_1 + n_2 + n_t$. It is to be noted that, if $n_i = 1$ then we have our regular definition of permutation.

The study of permutations began in ancient times. The number of ways of arranging an *n* number of items was known to be *n*! for at least 2500 years. It is quite possible that permutations of a multiset were first considered when analyzing the probabilities of certain gambling outcomes, such as getting a full house in a poker hand.

Mathematics is the science of patterns and permutations are one of the most fundamental of all patterns. Therefore, permutations of both sets and multisets occur in much areas of mathematics.

5.2 PERMUTATION OF MULTISETS

In many counting problems, elements may be used repeatedly (Multisets). For instance, a letter or digit may appear more than once on a licence plate. Digit in our phone numbers may also be used repeatedly. When a dozen doughnuts are selected, each variety can be chosen repeatedly [Ros 99].

Also some counting problems involved indistinguishable elements. For instance, to count the number of ways the letters of the word SUCCESS can be arranged, the placement of identical letters must be considered.

Theorem: 5.2.1

The number of r-permutations of a multiset of n objects is n^r

Proof:

There are *n* ways to select an element of the mset for each of the r-positions in the r-permutation when there is repetition, since for each choice all *n* objects are available. Hence by the product rule, there are n^r r-permutations when the set is a multiset.

Illustration: 5**.2.1**

How many strings of length n can be formed from the English alphabets?

Solution:

By the product rule, since there are 26 letters, and since each letter can be used repeatedly, we see that there are 26^n strings of length n.

Illustration: 5**.2.2**

What is the probability of drawing three red balls in a row from an urn containing five red balls and seven blue balls if a ball is put back into the urn after it is drawn?

Solution:

By the product rule the number of successful outcomes-that is, the number of ways to draw three red balls is 5^3, since for each drawing there are five red balls in the urn. The total number of outcomes is 12^3, since for each drawing

there are 12 balls in the urn. Thus, the desired probability is $5^3/12^3 = 125/1728$. This is an example of sampling with replacement.

5.2.1 r-Permutations of sets with indistinguishable elements (Multisets)

Many atimes sets with some identical elements, which may be indistinguishable may as well fall into counting problems. When this is the case, care must be taken to avoid counting things more than once. The following solves the problem.

Theorem 5.2.1.1

The number of different permutations of n objects, where there are n_1 *indistinguishable objects of type 1.* n_2 *Indistinguishable objects of type 2… and* n_k *indistinguishable objects of type k is* $\dfrac{n!}{n_1!.n_2!...n_k!}$

Proof:

To determine the number of permutations, first take note that the n_1 objects of type one can be placed among the *n* positions in $C(n,n_1)$ ways, leaving $n-n_1$ positions free. Then the objects of type two can be placed in $C(n-n_1,n_2)$ ways, leaving $n-n_1-n_2$ positions free. Continue placing the objects of type three……., type k – 1 until at the last stage *nk* objects of type k can be placed in $C(n-n_1-n_2)$ ways, leaving $n-n_1-n_2$ positions free. Continue placing the objects of type three ….., type k – 1 until at the last stage *n*k objects of type k can be placed in $C(n-n_1-n_2-...-n_{k-1},n_k)$ ways. Hence by the product rule, the total number of different permutations is [see Ros 99]

$$C(n_1n_1)C(n-n_1,n_2).....C(n-n_1-...n_{k-1},n_k)$$

$$=\frac{n!}{n_1!(n-n_1)!}\frac{(n+n_1)!}{n_2!(n-n_1-n_2)}z.....z\frac{(n-n_1-...-n_{k-1})!}{n_k!O!}$$

$$= \frac{n!}{n_1!.n_2!..n_k!}$$

5.3 COMBINATION WITH REPETITION

Theorem 5.3.1

There are C (n + r -1, r) r – combinations from a set with n elements when repetition of elements is allowed.

Proof:

Each r – combination of a set with n elements when repetition is allowed can be represented by a list of n – 1 bar and r stars. Then n – 1 bars are used to mark off *n* different cells, with the i^{th} cell containing a star for each time the i^{th} element of set occurs in the combination. For instance, a 6 – combination of a set with four elements is represented with three bars and six stars. Here

$$**|*||***$$

represents the combination containing exactly two of the first elements one of the second elements, none of the third elements, and three of the fourth elements of the set. As we have seen, each different list containing n – 1 bars and r stars corresponds to an r-combination of the set with n element when repetition is allowed. The number of such list is $C(n-1+r,r)$, since each list corresponds to a choice of the r position to a place the r stars from the n – 1 + r positions that contain r stars and n – 1 bars.[As in Ros 99]

Illustration: **5.3.1**

Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen matter.

Solution:

The number of ways to choose six cookies is the number of 6 –combinations of a set with four elements. From the above theorem this equals

$$C(4+6-1,6) = C(9,6)$$

$$C(9,6) = C(9,3) = \frac{9.8.7}{1.2.3} = 84,$$

There are 84 different ways to choose the six cookies.

Illustration: **5.3.2**

How many solutions does the equation, $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2 and x_3 are non-negative integers?

Solution:

To count the number of solutions, we note that a solution corresponds to a way of selecting 11 items from a set with three elements, so that x_1 items of type one, x_2 items of type two and x_3 items of type three are chosen. Hence, the number of solutions is equal to the number of 11 – combinations with repetition allowed from a set with three elements. From theorem 5.3.1 above, it follows that there are:

$$C(3+11-1,11) = C(13,11) = C(13,2) = \frac{13.12}{1.2} = 78$$

Alternatively; the number of solutions of this equation can also be found when

the variables are subject to constraints. For instance, we can find the number of solutions where the variables are integers with $x_1 \geq 1, x_2 \geq 2$ and $x_3 \geq 3$. A solution to the equation subject to these constraints corresponds to a selection of 11 items with x_1 items of type one, x_2 items of type two, and x_3 items of type three where in addition, there is at least one item of type one, two items of type two, and three items of type three. So choose one item of type one, two of type two, and three of type three. Then select five additional items. By theorem 5.3.1, this can be done in

$$C(3+5-1,5) = C(7,5) = C(7,2) = 21 \text{ Ways.}$$

Thus, there are 21 solutions of the equation subject to the given constraints.

Also counting the number of combinations with repetition allowed arises in determining the value of a variable that is incremented each time a certain type of nested loop is traversed.

Illustration: 5.3.3

What is the value of *k* after the following pseudocode has been executed

k:=0

For $i_1 =$ 1 to n

For $i_2 =$ 1 to n

:

:

:

For i_m :1 to i_{m-1}

k:= k + 1

Solution:

Note that the initial value of k is o and that 1 is added to k each time the nested loop is traversed with a set of integers $i_1, i_2 \ldots\ldots i_m$ such that

$$1 \le i_m \le i_{m-1} \le \ldots\ldots \le i_1 \le n.$$

The number of such sets of integers is the number of ways to choose m integers from {1,2,…..n}, with repetition allowed. (to see this, note that once such a set has been selected, if we order the integers in the set in decreasing order this uniquely defines an assignment of $i_m, i_{m-1}, \ldots, i_1$. Conversely, every such assignment corresponds to a unique unordered set.) Hence from the theorem above, it allows that k = (n+m-1, m) after this code has been executed.

5.3.1 r-COMBINATIONS:

An r-combination of elements of a set is an unordered selection of r-elements from the set. Thus an r-combination is simply a subset of the set with r-elements.

Theorem: 5.3.1.1

The number of r-combinations of a set with n elements, where n is a positive Integer and r is an integer with $0 \le r \le n$, equals

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Proof:

The r-permutations of the set can be obtained by forming the $C(n,r)$ r combinations of the set, and then ordering the elements in each r-combination, which can be done in $P(r,r)$ ways.

Consequently,

$$P(n,r) = C(n,r) \bullet P(r,r)$$

This implies that

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

The following corollary, is useful and helpful in computing the number of r-combinations of a set.

Corollary: ***5.3.1.1***

Let n and r be nonnegative integers with $r \le n$. Then

$$C(n,r) = C(n,n-r)$$

Proof:

From the above theorem, it follows that

$$C\ (n\ ,\ r\ ) = \frac{n\,!}{r\,!\,(\ n\ -\ r\ )!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}$$

Hence

$$C(n,r) = C(n,n-r)$$

There is another common notation for the number of r-combinations from a set with n elements, namely, $\binom{n}{r}$.

Illustration: 5.3.1a

How many ways are there to select 5 players from a 10-member tennis team to make a trip to a match at another school?

Solution:

The answer is given by the number of 5-combinations of a set with 10 elements. By the above theorem, the number of such combination is

$$C\,(10\ ,5\,) = \frac{10\,!}{5!(10\ -\ 5)!} = \frac{10\,!}{5!.5!} = 252$$

Illustration: 5.3.1b

How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of 3 faculty members from the mathematics department and 4 from the computer science department, If

there are 9 faculty members of the mathematics department and 11 of the computer science department?

Solution:

By the product rule, the answer is the product of the number of 3-combinations of a set with an element and its number of 4 combinations of set with 11 elements. The above theorem, gives the number of ways to select the committee as:

$$C(9,3).C(11,4) = \frac{9!}{3!6!}.\frac{11!}{4!7!} = 84.330 = 27.720$$

5.4 BINOMIAL COEFFICIENTS

The number $\binom{n}{r}$ is also called a binomial coefficient. The name binomial coefficient is used because these numbers occur as coefficients in the expansion of powers of a binomial expression such as $(a+b)^n$.

There are some important properties of the binomial coefficients which involve counting problems, such properties include: Pascal's and Vandermonde's identities.

5.5 PASCAL'S IDENTITY

Theorem: 5.5.1

Let n and k be positive integers with $n \geq k$. *Then* $C(n+1,k) = C(n,k-1) + C(n,k).$

Proof:

Suppose that T is a set containing $n+1$ elements. Let a be an element in T, and let S=T $-\{a\}$. Note that there are $C(n+1,k)$ subsets of T containing k elements. However, a subset of T with k elements either contains a together with $k-1$ elements of S, or contains k elements of S and does not contain a. Since there are $C(n,k-1)$ subsets of $k-1$ elements of S, there are $C(n,k-1)$ subsets of k elements of T that contain a. And there are $C(n,k)$ subsets of k elements of S. Consequently.

$$C(n+1,k) = C(n,k-1) + C(n,k).$$

However, the proof above of the Pascal's identity is a combinatorial proof. It is possible to prove this identity by algebraic manipulation from the formula for $C(n, r)$[see Ros 99].

It follows that:

$$C(n,k-1) + C(n,k).$$

$$= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!}{k!(n-k+1)!} \bullet [k + (n-k+1)]$$

$$= \frac{(n+1)!}{k!(n+1-k)!} = C(n+1,k)$$

Pascal's identity is the basis for a geometric arrangement of the binomial coefficients in a triangle as below:-

The nth row in the triangle consists of the binomial coefficients $\binom{n}{k}, k = 0,1,...,n.$

By Pascal's identity;

$$\binom{6}{1}+\binom{6}{2}=\binom{7}{2}$$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
1 & & \mathbf{6} & & \mathbf{15} & & 20 & & 15 & & 6 & & 1 \\
1 & & 7 & & \mathbf{21} & & 35 & & 35 & & 21 & & 7 & & 1 \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}$$

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & \binom{0}{0} & & & & & & & & & \\
 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & & & \\
 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & & & \\
 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & & & \\
 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & & & \\
 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & & & \\
 & \binom{6}{0} & & \binom{\mathbf{6}}{\mathbf{1}} & & \binom{\mathbf{6}}{\mathbf{2}} & & \binom{6}{3} & & \binom{6}{4} & & \binom{\mathbf{6}}{\mathbf{5}} & & \binom{6}{6} & & \\
\binom{7}{0} & & \binom{7}{1} & & \binom{\mathbf{7}}{\mathbf{2}} & & \binom{7}{3} & & \binom{7}{4} & & \binom{\mathbf{7}}{\mathbf{5}} & & \binom{7}{6} & & \binom{7}{7} & \\
\binom{8}{0} & & \binom{8}{1} & & \binom{8}{2} & & \binom{8}{3} & & \binom{8}{4} & & \binom{8}{5} & & \binom{8}{6} & & \binom{8}{7} & & \binom{8}{8}
\end{array}$$

This triangle is known as Pascal's triangle. Pascal's identity shows that when two adjacent binomial coefficients in the triangle are added the binomial coefficient in the next row between these two coefficients is produced.

Theorem: 5.5.2

Let n be a positive integer. Then $\sum_{k=0}^{n} C(n,k) = 2^n$

Proof:

A set with n elements has a total of 2^n different subsets. Each subset has either zero elements, one element, two elements ….or *n* elements in it. There are C(*n*,0) subsets with zero elements, C(*n*,1) subsets with one element, C(*n*,2) subsets with two elements, ….C(*n*, k) subsets *n* elements. Therefore,

$$\sum_{k=0}^{n} C(n,k)$$

counts the total number of subsets of a set with *n* elements. This shows that

$$\sum_{k=0}^{n} C(n,k) = 2^n$$

5.6 VANDERMONDE'S IDENTITY

Theorem 5.6.1

Let m, n and r be non negative integers with r not exceeding either m or n. Then

$$C(m+n_1 r) = \sum_{k=0}^{r} C(m,r-k)C(n_1 k).$$

Proof:

Suppose that there are *n* items in one set and n items in a second set. Then the total number of ways to pick r elements from the union of these sets is $C(m+n,r)$. Another way to pick r elements from the union is to pick k elements from the first set and then r –k elements from the second set, where k is an integer with $0 \le k \le r$. This can be done in C(*m*, k) C(r – k) ways, using the product rule. Hence, the total number of ways to pick r elements from the union is

$C(m+n,r) = \sum_{k=0}^{r} C(m,r-k)C(n,k)$.

Hence the proof of the Identity [see Ros 99].

5.7 PIGEON HOLE PRINCIPLES (Dirichlet Drawing Principle)

The generalized pigeonhole principle states:

If N objects are placed into k boxes, then there is at least one box containing at least [N/k] objects.

Proof:

Suppose that none of the boxes contain more than [N/k] – 1 objects. Then the total number of objects is at most

$$k([N/k] - 1) < k(((N/k) + 1) - 1) = N$$

where the inequality [N/k] < (N/k) + 1 has been used. This is a contradiction since there are a total of N objects.

A clever application of the pigeonhole principle shows the existence of an increasing or a decreasing subsequence of a certain length in a sequence of distinct integers. Consider definitions 1.2.4.1 and 1.2.4.2 p. 19 for clarity, before this application is considered.

Theorem 5.7.1

Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length n+ 1 that is either strictly increasing or strictly decreasing.

Illustration 5.7.1

The sequence 8,11,9,1,4,6,12,10,5,7 contains 10 terms. Note that $10 = 3^2 + 1$. There are four increasing subsequences of length four, namely, 1,4,6,12; 1,4,6,7; 1,4,6,10; and 1,4,5,7. There is also a decreasing subsequence of length four, namely, 11,9,6,5.

For the proof of the theorem 5.7.1 and the aforesaid details, see ([Ros 99], p.245- 247).

5.8 SOME USEFUL RESULTS(mostly extracted from ([SS 03])

5.8.1 Multisubset (msubset, for short) relation $\subset$.

Definition: 5.8.1

For all mset terms u and v, $u \subseteq v$ (u is an msubset of v) iff

$\forall z \forall n(z \in^n u \rightarrow \exists m(n < m \ \wedge z \in^m v))$.

In relation to its msubsets, an mset is called the *parent* mset. For example, in the above definition v is the parent mset of u. The relation $\subseteq$ is reflexive, antisymmetric and transitive.

5.8.2 Part, whole, and full msubsets.

Definitions: 5.8.2

Msubset u is called a part of v iff $u \subseteq v \hat{} u \neq \phi \hat{} u \neq v$.

Msubset u is called a whole of v iff $\forall z \forall n(z \in^n u \rightarrow z \in^n v)$.

Msubset u is called a full of v iff $u \subseteq v \hat{} \forall z(z \in v \rightarrow z \in u)$.

Propositions: 5.8.1

Let $\alpha = [x_1, x_2, ..., x_n]m_1, m_2, ..., m_n$ be any finite mset. Let us denote the size of α by $|\alpha|$. Thus, $|\alpha^*| = n$ and $|\alpha| = m_1 + m_2 + ... + m_n = s$ (say). We have the following:

i) Let $PW(\alpha)$ be the set of all whole msubsets of α. Then,

$$|PW(\alpha)| = 2^n, where |\alpha^*| = n.$$

This result is similar to the case for classical sets. Because, for constructing whole msubsets of α, every element of α with its full multiplicity behaves like an element in a classical set.

ii) Let $PF(\alpha)$ be the set of all full msubset of α.

Then, $|PF(\alpha)| = (m_1 \times m_2 \times ....m_n)$.

iii) Let $P(\alpha)$ be the power set of α, the set of all msubsets of α.

We note that $P(\alpha)$ in a reasonable context [Bli89], p.45 and [Hic80], p.214 needs to be a set and consequently $|P(\alpha)| < 2^s$, which shows that cantor's power set theorem fails for mset.

Proof:

The proof follows directly from combinatorial principles.

For example, Let $\alpha = [x, y, z]_{1,2,3}$. Thus, $|\alpha| = 6$ and $|\alpha^*| = 3$.

Now, $|P(\alpha)| = 24,$ (calculated by essentially enumerating all its elements) < 2^6, showing that cantor's powerset theorem fails for msets. Also $PW(\alpha) = 2^3 = 8$, and $|PF(\alpha)| = (1 \times 2 \times 3) = 6.$

Remarks: For a given finite mset α, one would wish to know if there exists a general method to find $|P(\alpha)|$, as against essentially enumerating all its elements,

and the answer is yes. In order to vindicate the claim, we mention below a couple of approaches exploited to deal with this problem, indicate their limitation and put forward a relatively efficient approach.

5.8.3 APPROACHES:

To find $|P(\alpha)|$, for a finite mset α. Let $P^{\sim}(\alpha)$ be the power multiset [Bli89], p.45] of a finite mset α. One can find $|P(\alpha)|$ by way of finding $|P^{\sim}(\alpha)|$ and, in turn, $|(P^{\sim}(\alpha))^{*}|$, which essentially amounts to enumerating term by term (see [Bli89], p.45 for details).

1. In various deliberations on combinatorics, one finds the following definitions: If $\alpha = [a_1, a_2, ...a_k]_{m1,m2,.....mk}$ and $m_1 + m_2 + ... + m_k = n$ then, α is called an *n*-mset. If one or more elements of an mset β occur infinitely, often written as $[a_1, a_2, ...]_{\infty,\infty,.....,}$ then β is called an infinite mset. Also, α described above is called an n-msubset of β. Accordingly, $[a_i]_{n,}\ i = 1,2,.....,$ are all n-msubsets of β for any *n*. It is well known that the number of combinations of *n* distinct objects taken r at a time when objects may occur with repetitions is ${}^{n}H_r = {}^{r+n-1}C_r$. It includes all the terms of the type $[a_i]_n$ which are coincidentally msubsets of an mset in which each distinct element occurs infinitely. For example, from a shop having an unlimited number of each of the three types of fruits, one can buy four fruits in ${}^{3}H_4 = {}^{6}C_4=15$ ways. Hence, the aforesaid formula straightforwardly works for finding the size of the collection of all n-msubsets of an mset in which each distinct element occurs infinitely.

However, if the mset is finite or contains an infinite number of distinct elements with finite multiplicities, the aforesaid formula does not seem to work efficiently. In other words, given an mset $\alpha = [x, y, z.....]_{m,p,t,.....}$

Along with the stipulations that in any msubsets of α the element x can occur at most m times, the element y can occur at most p times and so on; the aforesaid formula cannot efficiently determine $|P(\alpha)|$.

5.8.3.1 Application of Inclusion Exclusion Principles (IEP)

We consider the definitions in chapter 3, statement 3.1.4 page 29. The formula (iii) in page 34 is known as the Sieve formula. This well-known application is used to solve the aforesaid problem successfully though not efficiently.

Let $\alpha = [X, Y, Z]_{4,5,6,}$ be an mset, and the task is to determine all msubsets of α having cardinality 10. That is, we are required to determine all 10-combinations of (the elements of) U in which X can occur at most four times, Y can occur at most five times and Z can occur at most six times, where $U = [X, Y, Z]_{\infty,\infty,\infty}$.

Clearly from the formula to determine combinations in multisets, ${}^{n}H_r = {}^{r+n-1}C_r$ we have $n(U) = {}^{3}H_{10} = {}^{12}C_{10} = 66$, where U is supposed to have unlimited number of copies of X, Y and Z.

Let $A(B,C)$ respectively be the set of all 10-combinations of U having the property $P_1(P_2,P_3)$ satisfying that each combination contains more than four Xs(five Ys, six Zs). Then $A^c \cap B^c \cap C^c$ denotes the mset of all such 10-combinations of α which have neither more than four Xs nor more than five Ys nor more than six Zs.

Hence, by IEP, we have:

$$n(A^c \cap B^c \cap C^c) = n(U) - S_1 + S_2 - S_3$$

where

$$S_1 = n(A) + n(B) + n(C)$$

$$S_2 = n(A \cap B) + n(A \cap C) + n(B \cap C)$$

$$S_3 = n(A \cap B \cap C).$$

Now, since A consists of all 10-combinations of α in which X occurs at least five times, if five Xs are removed from any such combination we are left with a combination of α .

Hence, $n(A)$ = number of all 5-combinations of $\alpha = {}^3H_5 = {}^7C_5 = 21$.

Similarly, $n(B) = {}^3H_4 = {}^6C_4 = 15$ and $n(C) = {}^3H_3 = {}^5C_3 = 10$.

Again, $A \cap B$ consists of all 10-combinations of α in which there are at least five Xs and six Ys.

If we remove these 11 elements from any such 10-combinations, then we are left with (-1)-combination of α so that $n(A \cap B) = {}^3H_{-1} = 0$.

Similarly, $n(A \cap C) = {}^3H_{-2} = 0$, and $n(B \cap C) = {}^3H_{-3} = 0$.

Also, $n(A \cap B \cap C) = 0$.

Thus the desired number of 10-combinations of α = 66-(21+15+10)+(0+0+0)-0=20.

It is clear that when this approach is applied to determine the cardinality of the power set of a finite mset, will be awfully lengthy.

2. A more efficient approach in finding $|P(\alpha)|$ for a finite mset α is put forward here. This is a schematic PROGRAM for evaluating $|P(\alpha)|$.

Let α be a finite mset denoted by $\alpha = [k_1, k_2, \ldots, k_n]$, instead of $\alpha = [x_1, x_2, \ldots, x_n]k_1, k_2, \ldots, k_n$, for convenience. Here $|\alpha^*| = n$, $|\alpha| = k_1 + k_2 + \ldots + k_n$.

Let $\alpha = [k_1]$ and hence $|\alpha^*| = 1$, then $|P(\alpha)| = 1 + k_1$. Let $\alpha = [k_1, k_2]$, and hence $|\alpha^*| = 2$,

then $|P(\alpha)| = 1 + (k_1 + k_2) + k_1 k_2$. Let $\alpha = [k_1, k_2, k_3]$ and hence $|\alpha^*| = 3,$ then

$$|P(\alpha)| = 1 + (k_1 + k_2 + k_3) + k_1 k_2 + k_1 k_3 + k_2 k_3) + k_1 k_2 k_3, \text{etc.}$$

Let $\alpha = [k_1, k_2, ...k_n]$ and hence $|\alpha^*| = n,$ then

$$|P(\alpha)| = 1 + \left[\sum_{i=1}^{n} k_i\right] + \left[k_1 \sum_{i=2}^{n} k_i + k_2 \sum_{i=3}^{n} k_i + ... + k_{n-1} k_n\right]$$

$$+$$

$$\left[\left(k_1 k_2 \sum_{i=3}^{n} k_i + k_1 k_3 \sum_{i=4}^{n} k_i + .. + k_1 k_{n-1} k_n\right) + \left(k_2 k_3 \sum_{i=4}^{n} k_i + k_2 k_4 \sum_{i=5}^{n} k_i + ... + k_2 k_{n-1} k_n\right) + ... + \left(k_{n-2} k_{n-1} k_n\right)\right]$$

$$+$$

$$\left[\left(k_1 k_2 k_3 \sum_{i=4}^{n} k_i + k_1 k_2 k_4 \sum_{i=5}^{n} k_i + .. + k_1 k_{n-2} k_{n-1} k_n\right) + \left(k_2 k_3 k_4 \sum_{i=5}^{n} k_i + k_2 k_3 k_5 \sum_{i=6}^{n} k_i + . + k_2 k_{n-2} k_{n-1} k_n\right) + . + \left(k_{n-3} k_{n-2} k_{n-1} k_n\right)\right]$$

$$+ ..... + [(k_1 k_2 .... k_n)] \text{ ---------- (1)}$$

Example: 5.8.3.1

Let $\alpha = [x, y, z]_{m, p, t}$ and $|\alpha^*| = 3,$

Formula:

$$|p(\alpha)| = 1 + (m + p + t) + (m \times p + m \times t + p \times t) + (m \times p \times t).$$

3. An improved approach followed from the theorem below

Theorem 5.8.3.3

If ***P***(α) is the power set of an mset α = [x_1,x_2, x_3,...x_n] $_{k1}$, $_{k2}$,....,$_{kn}$ in which x_1 occurs k_1 times, x_2 occurs k_2 times, ..., x_n occurs k_n times, then

$$\left|P(\alpha)\right| = \prod_{i=1}^{n}(1+k_i).$$

Proof. We cosider the identity

$$(x+k_1)(x+k_2).....(x+k_n) \equiv A_o x^n + A_1 x^{n-1} + ...+ A_n..$$

(2)

Equating the coefficients of x^{β} (β=0,1,2,....,n) on both sides of equation (2), we get

A_o= 1

$$A_1 = \left[\sum_{i=1}^{n} k_i\right]; A_2 = \left[k_1\sum_{i=2}^{n} k_i + k_2\sum_{i=3}^{n} k_i + ... + k_{n-1}k_n\right],.....$$

$$A_n = [(k_1 k_2 .....k_n)]$$

In view of the above relation, equation (1) can be written as

$$\left|P(\alpha)\right| = A_o + A_1 + ... + A_n$$

(3)

Substituting x =1in equation (2) we get

$$(1+k_1)(1+k_2).....(1+k_n) = A_o + A_1 + ...+ A_n$$

(4)

Equations (1) and (3) imply

$$\left|P(\alpha)\right| = (1+k_1)(1+k_2)....(1+k_n)$$

Thus the cardinality of the power set ***P***(α) of an mset

$\alpha = [x_1, x_2, ..., x_n]k_1, k_2, ..., k_n$

Can be given by

$$\left|P(\alpha)\right| = (1+k_1)(1+k_2)....(1+k_n) = \prod_{i=1}^{n}(1+k_i)$$

(5)

In particular, an ordinary set $S = \{a_1, ...a_n\}$ can be thought of as a finite mset of the type $[a_1, ...a_n]_{1,1,1,...,1}$ in which each element occurs only once. In this case, $k_i = 1$, for all $i = 1,2,.....,n$ and the formula

$$\left|P(\alpha)\right| = \prod_{i=1}^{n}(1+k_i) = (1+k_1)(1+k_2)....(1+k_n)$$

reduces to

$$\left|P(\alpha)\right| = \prod_{i=1}^{n}(1+1) = (1+1)\cdot(1+1)....(1+1) = 2\cdot 2....n-times = 2^n$$

which is cosistent with the result obtained in ordinary set theory for cardinality of the power set of a finite set $S = \{a_1, ...a_n\}$ containing *n*-elements.

CHAPTER SIX

CONCLUSION AND FUTURE DIRECTIONS

6.1 CONCLUSION

We have found in the study that many interesting combinatorial applications exist in almost all areas of mathematical sciences. Also, many elegant applications of combinatorics were discussed which are very useful to the computer scientists, cryptographers, statisticians and more generally to combinatorialists. Computing the transitive closure of a digraph is an important problem in many computer science applications, which was also discussed. This helps the computer scientists in evaluation of recursive database queries, analysis of reachability (connectivity) of transition graphs in communication networks and in construction of parsing automata in compilers. Specially, for solving certain class of problems in circuit theory, the results on coverings of a set are found relevant. Also, a close study of the work in [SS03] revealed that some important innovative results on combinatorics of multiset are made available.

6.2 FUTURE DIRECTIONS

The content of this work, besides providing tools for combinatorialists,

cryptographers, probability and graph theorists, contain some relevant research for computer scientists,specially those grounded in their extension to multisets.

An

'open' combinatorial problem to formulate a closed formula to determine the size of the family of all the coverings of an n-set is put forward for further

vindication. Some motivating steps to this regard were provided.

REFERENCES

1. [And 74] Anderson I. (1974), A first course in Combinatorial Mathematics, Clarendon Press, Oxford.

2. [And 87] Anderson I. (1987), Combinatorics of finite Sets, *Clarendon Press, Oxford.*

3. [Ben 74] Bender E.A., (1974), Partitions Of Multisets. *Discrete Mathematics, Vol .9, pp.301 – 311.*

4. [BD 71] Brace A. and Daykin D.E., (1971). A finite set covering theorem, BULL AUSTRAL. MATH. SOC., vol. 5, *pp*. 197-202.

5. [BDR 84] Bender E.A., Devitt J.S., and Richmond. L.B., (1984), Partitions of multisets II *Discrete Mathematics, Vol. 50, pp. 1 – 8.*

6. [Bli 89] Blizard. W. (1989), Multiset theory. *Notre Dame Journal of formal logic, Vol. 30, pp. 36 – 66*.

7. [Bli 91] Blizard. W., (1991), The Development of Multiset Theory, *Modern Logic, Vol. 1, pp. 319 – 352.*

8. [Cle 88] Clements G. F.(1988), On Multisets k-families*, Discrete Mathematics, Vol. 69, pp. 153 – 164.*

9. [Cle 88] Clements G. F.(1988), Multisets Antichains having minimal downsets, *Journal of Combinatorial Theory, Series A, Vol. 48, pp. 255 – 258.*

10. [Coh 81] Cohn P. M., (1981) Universal Algebra, *Mathematics and Its Applications,* Vol. 6, D. Reidel Publishing Company.

11. [HL 06] Hegarty P. and Larson U.(2006), Permutations of the Natural Numbers with Prescribed Difference Multisets. *Integer: Electronic Journal of Combinatorial Number Theory 6, #A03.*

12. [Hai 86] Hailperin T. (1986), Boole's logic and probability, *Second edition, North Holland.*

13. [Hic 80] Hickman J.L.(1980), A Note on the concept of Multiset*, A bulletin of the Australian Mathematical society, Vol. 22, pp. 211 – 217.*

14. [Knu 73] Knuth D.E (1973), The Art of computer programming, *Vol. 3, (Sorting and Searching), Addison-Wesley.*

15. [Knu 81] Knuth D.E.(1981), The Art of computer programming,

Vol.2 (Seminumerical Algorithms) Second Edition, Addison-Wesley.

16. [Mie 04] Mier Anna de (2004), Lecture notes for Enumerative Combinatorics,

University of Oxford.

17. [MMc 82] Meyer R. K and McRobbie M. A., (1982), Multisets and Relevant Implication I and II, *Australasian Journal of Philosophy, Vol. 60, pp. 107 – 139 and pp. 265 – 281.*

18. [Mon 87] Monro G. P.(1987), The Concept of Multiset, *Zeitschrift fur Mathematische Logik und Grundlagen der Mathematik, Vol.3, pp. 171 – 178.*

19. [Par 81] Parker-Rhodes A.F., (1981), The theory of indistinguishables, *D. Reidel, Dordrecht.*

20. [Rad 75] Rado R., (1975), The Cardinal Module and some Theorems on Families of Sets, *Annali di Matematica pura ed applicata (4) Vol. 102, pp 135 – 154.*

21. [Ros 99] Rosen Kenneth H. (1999), Discrete Mathematics and Its Applications, McGraw-Hil, International Edition (4th print).

22. [Ser 05] Serguey N. Pozdnyakov (2005), Brilliant Ideas of Combinatorics, Math. Track. School of mathematical sciences. Vol. 1, No. 1, pp. 81-98, ISSN: 1817-3462.

23. [Sin 94] Singh D., (1994), A Note on the Development of Multiset Theory, *Modern Logic, Vol. 4, pp. 405 – 406.*

24. [SS 03] Singh D. and Singh J. N.(2003), Some Combinatorics of Multisets, *International Journal of Mathematical Education in Science and Technology, Vol. 34, No.4, pp. 489-499.*

25. [TM 00] Tremblay J.P. and Manohar R. (2000) Discrete Mathematical Structures with Applications to computer science, TATA McGraw-Hill Edition (11th Print).

26. [Wag] Carl G. Wagner, Basic Combinatorics, Department of mathematics, The University of Tennessee.

27. [Wil 03] Wildberger N.J. (2003), A new look at multiset. *School of Mathematics, UNSW Sydney 2052, Australia.*

28. [Wik] Wikipedia: User-contributed encyclopedia.

29. [Whi 33] Whitney H. (1933), Characteristic Functions and the Algebra of Logic, *Annals of Mathematics, Vol. 34, pp. 405 – 414*.