EXPONENTIATED-EXPONENTIAL WEIBULL DISTRIBUTION: ITS PROPERTIES AND APPLICATIONS

By

AMIRU Faisal Muhammad

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NIGERIA.

DECLARATION

I declare that the work in this dissertation entitled "EXPONENTIATED-EXPONENTIAL WEIBULL DISTRIBUTIONS: ITS PROPERTIES AND APPLICATIONS " has been carried out by me in the Department of Statistics. The information derived from the literature has been well acknowledged in the text and a list of references provided and I assured that no part of this dissertation was previously presented for another degree or any form of research at this or any other institution.

AMIRU Faisal Muhammad — Date

CERTIFICATION

This dissertation titled "EXPONENTIATED-EXPONENTIAL WEIBULL DISTRIBUTION: ITS PROPERTIES AND APPLICATIONS" by AMIRU Faisal Muhammad meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and presentations carried out.

_____ _____

Dr. H. G. Dikko Date

Chairman, Supervisory Committee

_____ _____

Prof. O. E. Asiribo Date

Member, Supervisory Committee

_____ _____

Date

External Examiner

_____ _____

Dr. H. G. Dikko Date

Head of Department

_____ _____

Prof. Sadiq. Z. Abubakar Date

Dean Postgraduate School

DEDICATION

This dissertation is dedicated to my late father Muhammad Amiru Isah.

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ABSTRACT

A four-parameter distribution called Exponentiated-Exponential Weibull (EEW) distribution was proposed using a generator introduced in earlier research. Statistical properties of the distribution were derived along with its reliability functions and asymptotic behaviour. The distribution parameters were estimated using the method of Maximum Likelihood Estimation (MLE). The distribution generalizes some well-known special lifetime models such as Exponential, Exponential Weibull and Generalized Exponential distribution. The probability density of the distribution indicate that it is skewed to the right (Positively skewed). We compared the performance of EEW distribution with some related distributions from the literature using the AIC and CAIC method of comparison. Applicability of the distribution was also observed using two real datasets and it was found that the distribution fitted the dataset. Finally it is recommended that the distribution can be used for modelling real life data especially if it is positively skewed.

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CHAPTER ONE

INTRODUCTION

1.1 Background to the Study

Several classical distributions have been widely used over the past decades for modelling lifetime data in many areas such as reliability, engineering, economics, biological studies, environmental actuarial, environmental and medical sciences, demography, and insurance. However, in many applied areas such as lifetime analysis, finance, and insurance, there is a clear need for extended forms of these distributions. This is because there still remain many important problems where the real data does not follow any of the classical or standard probability models. For that reason, numerous methods for generating new families of distributions have been considered (Bourguignon *et al.,* 2014). To handle this, there is a strong need to propose useful models for the better study of the real-life marvel. Introducing new probability models or their classes is an old practice and has ever been considered as valuable as many other practical problems in statistics. According to Tahir and Cordeiro (2016), the idea simply started with defining different mathematical functional forms, and then adding of location, scale or shape parameter(s).

Inducing of a new shape parameter(s) introduces a model into greater family of distributions and can give significantly skewed and heavy-tailed distributions and also provides greater flexibility in the form of new distribution. Gupta *et al.,* (1998) proposed a generator for defining a new univariate continuous family of distributions by adding one shape parameter to the baseline (parent) distribution. Gupta and Kundu (1999) pioneered the study of two-parameter generalization, in which they studied two-parameter Generalized-Exponential (GE) distribution also called Exponentiated-Exponential (EE) distribution, after which several other authors worked on GE distribution due to its attractive features, among which are Gupta and Kundu (2001a, 2001b, 2002 and 2003), Kundu *et al.,* (2005) among others. The

authors present two real life data sets, where it is observed that in one data set Exponentiated exponential distribution has a better fit compared to Weibull or gamma distribution and in the other data set Weibull has a better fit than Exponentiated exponential or gamma distribution. The induction of parameter(s) has been proved useful in discovering tail properties and also for improving the goodness-of-fit of the proposed distribution (Saboor *et al.,* 2015).

The Generalized-Exponential (GE) distribution also known as Exponentiated Exponential (EE) is a continuous probability distribution belonging to the Exponentiated Family (EF) of distributions. Al-hussaini and Ahsanullah (2015) stated that the history of adding a parameter by exponentiation can be traced back to Gompertz (1825) and Verhulst (1838, 1845, 1847) where they used the CDF to compare known human mortality tables and represent mortality growth. The CDF is given as $G(t) = (1 - \rho e^{-t\lambda})^{\alpha}$ for $t > \frac{1}{\lambda}\ln\rho$, where ρ, α and λ are all positive. So also, in twentieth century, Ahuja and Nash (1967) considered this model and some further generalization. The Gompertz-Verhulst's CDF happened to be the first member of the EF of distributions. The GE distribution is a particular case of this family for $\rho = 1$.

The CDF and the pdf of the GE distribution is given by equation (1.3.1) and (1.3.2) respectively.

$$G(x) = (1 - e^{-\lambda x})^{\alpha} \tag{1.1.1}$$

$$g(x) = \alpha\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1} \tag{1.1.2}$$

where x, α and λ are greater than zero.

The Weibull distribution is a continuous probability distribution. It was named after Swedish mathematician Waloddi Weibull, who describe it in detail in 1951, although it was first recognized by Frechet (1927) and first applied by Rosin and Rammler (1933) to described a unit size of distribution. Weibull distribution has the scale and shape parameters. This distribution has become very popular in analysing lifetime data and for many applications where a skewed distribution is required. It

also has increasing and decreasing failure rates depending on the shape parameter. The Weibull distribution is often preferred for analysing lifetime data because, in presence of censoring, it is much easier to handle than the gamma distribution. Also in many positive datasets it is observed that the Weibull distribution fits very well. As one of the disadvantages of the distribution it maximum likelihood estimators has a very low asymptotic convergence to normality (Bain, 1976).

1.2 Statement of the Problem

Most of the lifetime data does not follow any of the classical distribution due to its uniqueness. Therefore, the need to provide univariate distributions that lifetime data follows is of great importance to address this problem. Adding new shape parameter(s) to the baseline distribution increases the flexibility and provide goodness-of-fit to new univariate continuous distributions. Induction of parameter(s) has been proved useful in discovering tail properties and also for improving the goodness-of-fit of the proposed distribution (Saboor *et al.,* 2015). Hence, developing new distributions that would enhance in addressing this problem is of paramount importance. Our interest here is to propose a new probability distribution function which can serve as an extension of the Exponential Weibull (EW) called Exponentiated-Exponential Weibull (EEW) distribution using Gupta *et al.,* (1998) generator so that it will address this problem appropriately to some extent.

1.3 Aim and Objectives

This research is aimed at developing a new probability density function called Exponentiated-Exponential Weibull (EEW) distribution using the generator proposed by Gupta *et al.,* (1998).

The specific objectives through which the stated aim will be achieved are to:

i. derive and establish a new Exponentiated-Exponential Weibull distribution;

ii. study some of its statistical properties such as Survival function, Hazard function, Asymptotic behaviour, Quantile function, Median, Moment, Moment Generating Function and Order Statistics;

iii. estimate the parameters of the distribution using the method of Maximum Likelihood Estimation (MLE) and;

iv. compare the fitness of the proposed probability distribution with other related distribution using a real-life data.

1.4 Scope of the Study

The study focused on extending the Exponential Weibull distribution and deriving mathematical expressions for some selected properties of the proposed distribution such as Survival function, Hazard function, Asymptotic behaviour, Quantile function and Median, Moment, Moment Generating Function and Order Statistics and estimating the model parameters by using the method of Maximum Likelihood.

1.5 Significance of the Study

The significance of any research is to deliver a new way of applying the acquired knowledge to solve problems not necessarily a real-life problem. So, the proposed distribution can be used in modelling lifetime data, since induction of new shape parameter(s) has been proved useful in exposing tail properties and also improve the goodness-of-fit of the proposed generator family (Saboor *et al.,* 2015). The proposed distribution will be compared with some existing distributions to identify the distribution that will provide better fit using a real dataset.

1.6 Motivation

Several standard theoretical distributions have been found to be useful in the field of reliability, insurance, engineering, medicine, economics, finance and so on. Yet, generalizing these standard distributions have produced several compound distributions that are more flexible compared to the baseline distributions. Hence, we intend to develop a new probability density function called Exponentiated-Exponential Weibull (EEW) distribution using Exponential Weibull distribution as our baseline and apply it to a real-life dataset.

1.7 Definition of Terms

1.7.1 Probability

Probability is a branch of mathematics that deals with calculating the likelihood of a given event's occurrence, which is expressed as a number between 1 and 0. An event is considered as "certain" if it has a probability of 1 while an event with the probability of 0 is considered as an impossible or uncertain event.

1.7.2 Probability Distribution

A probability distribution is a statistical function that describes the frequencies or possible values and likelihoods that a random variable can take.

1.7.3 Continuous Random Variable

A non-discrete random variable X is said to be continuous random variable if its distribution function can be represented as

$$F(X) = P(X \leq x) = \int_{-\infty}^{x} f(u)du \tag{1.7.1}$$

for every real number x and evaluated from $-\infty$ to a real number x. while the probability density function (pdf) is defined as;

$$f(x) = \frac{dF(x)}{dx} \tag{1.7.2}$$

and it has the following properties:

1. $f(x) \geq 0, \forall x$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P(a < X < b) = \int_{a}^{b} f(x)dx$

1.7.4 Moments

Moment is one of the statistical properties of any distribution that is used to study some of the characteristics of a random variable such as mean, variance, skewness (Sk) and kurtosis (Ku). Mathematically, it can be defined as: let X be a continuous random variable, then the r^{th} moment of X about the origin is given in equation (1.8.3)

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} X^r f(x)dx \tag{1.7.3}$$

where $f(x)$ is the proposed pdf.

1.7.5 Quantile Function

Quantile function is used for calculating the median, skewness and kurtosis and for simulation of random numbers.

1.7.6 Skewness

It is a measure of the degree of asymmetric of a probability distribution. The skewness is either to the right (positively skewed) or to the left (negatively skewed) on the normal distribution plot.

1.7.7 Kurtosis

Is a statistical technique that measures the peak of distribution. The kurtosis of a normal distribution is zero and if the kurtosis vary from zero, it means that the distribution is peaked more than normal or flattered.

1.7.8 Moment Generating Function

The Moment Generating Function provides an alternative way to analytical result rather than working directly with the pdf. The MGF of a random variable X can be obtained using equation (1.8.7)

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \tag{1.7.4}$$

1.7.9 Reliability Analysis

Reliability analysis can be defined as the study and modelling of observed product lives. Life data can be lifetime of products in the marketplace such as the time product operated effectively or the time the product operated before it collapsed.

1.7.10 Survival Function:

The survival function, also known as the reliability function in engineering, is the distinctive of an independent variable that maps a set of events, frequently related with mortality of some system onto time. It is the probability that the system will survive beyond a specified period of time. Mathematically it is given by

$$S(x) = P(X > x) = \int_x^{\infty} f(u)du = 1 - F(x) \tag{1.7.5}$$

where $F(x)$ is the CDF of any distribution.

1.7.11 Hazard Function

The hazard function is defined as the probability per unit time that a case which has survived to the beginning of the respective interval will fail in that interval. It is computed as the number of failures per unit time in the respective interval, divided by the average number of surviving cases at the mid-point of the interval. Mathematically its given by

$$H(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \tag{1.7.6}$$

where $f(x)$ is pdf of any distribution.

1.7.12 Order Statistic

Order statistics is widely used in many areas of statistical theory and practice, for instance, detection of outlier in statistical quality control processes.

Suppose $X_1, X_2, \cdots, X_n$ is a random sample from a distribution with pdf $f(x)$ and let $X_{1:n}, X_{2:n}, \cdots, X_{i:n}$ denote the corresponding order statistic obtained from this sample. The pdf $f_{i:n}(x)$ of the i^{th} order statistic can be express as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} [1 - F(x)]^{n-i} \tag{1.7.7}$$

1.7.13 Maximum Likelihood Estimation

The method of maximum likelihood estimation is used for estimating parameter of statistical model given observations, by finding parameter value that makes the known likelihood distribution a maximum. It is considered as robust of the parameter estimation techniques. Suppose $X_1, X_2, \cdots, X_n$ is a random sample from a population w ith pdf $f(x; \theta)$, where θ is an unknown parameter to be estimated. The likelihood function, L(θ), is defined to be the joint density of the random variables $X_1, X_2, \cdots, X_n$. That is,

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) \tag{1.7.8}$$

The sample statistic that maximizes the likelihood function $L(\theta)$ is known as maximum likelihood estimator of θ and is denoted as $\hat{\theta}$.

CHAPTER TWO

LITERATURE REVIEW

In the modern era of distribution theory, practitioners and applied researchers faced challenges in modelling lifetime data with classical probability distributions. Many researchers have worked and some are still working to see that they come up with new probability distributions through the use of link functions known as generators to add one or more parameter(s) to the existing probability distribution with the hope that the new proposed distribution will be more flexible. The proposed distribution function serves as a generalization of the existing probability distributions for better study and modelling of lifetime dataset. Marshall and Olkin (1997) were the first to introduce the approach of probability distributions' generalization suggested by adding one parameter to the survival function $F(x) = 1 - G(x)$ where $G(x)$ is the cumulative distribution function (CDF) of the baseline distribution. The generator we use in this research has its root from Generalized Exponential (GE) distribution also known as Exponentiated Exponential (EE) distribution which is one of the members of Exponentiated Family (EF) of distributions. The generator was proposed by Gupta *et al.,* (1998).

We reviewed some literature related to GE family of distributions as follows: Following Gupta *et al.,* (1998), Gupta and Kundu (1999) proposed the study of two-parameter generalization of the GE distribution. The distribution has received increased attention due to its attractive nature and many researchers have investigated its properties and also proposed comparisons with other distributions. Some of these distributions include: Gupta and Kundu (2001a) studied the properties of the EE family of distribution and it was observed that the two-parameter EE family are quite similar in nature to the other two-parameter family like Weibull family or gamma family, which can be used as an alternative to a Weibull distribution or a gamma distribution. Here, the two parameters of the distribution are shape and scale parameter respectively. When the shape parameter says α = 1, it represents

the exponential family. The Generalized-Exponential distribution: different method of Estimation by Gupta and Kundu (2001b). The authors considered five estimation procedures (Method of maximum likelihood estimators, method of moment, estimators base on percentiles, least squares and weighted least square estimators and L-moment estimators) and compare their performances through numerical simulations.

Gupta and Kundu (2002), used the ratio of the maximized likelihoods (RML) in discriminating between Weibull and GE distributions. Asymptotic distributions of the logarithm of the RML under null hypotheses are obtained and they are used to determine the minimum sample size required in discriminating between two overlapping families of distributions for a user specified probability of correct selection and tolerance limit. Gupta and Kundu (2003), proposed a probability distribution called "Closeness of Gamma and Generalized Exponential distribution". It was observed that for a given gamma distribution there exists a generalized exponential distribution so that the two distribution functions are almost identical. Since the gamma distribution function does not have a compact form, efficiently generating gamma random numbers is known to be problematic. The authors observed that for all practical purposes it is possible to generate approximate gamma random numbers using generalized exponential distribution and the random samples thus obtained can not be differentiated using any statistical tests. However, if there is a skewed data set where gamma distribution fits very well, the generalized exponential distribution also can be used. Adeyemi and Adebanji (2004) introduced a three-parameter generalized Pareto distribution, the exponentiated generalized Pareto distribution (EGP) and present a comprehensive treatment of the mathematical properties of this new distribution.

Gupta and Kundu (2004)proposed another probability distribution called the Discriminating between the Gamma and Generalized Exponential distribution were they use the ratio of the maximized likelihoods in choosing between a generalized

exponential distribution and a gamma distribution, and obtain asymptotic distributions of the logarithm of the ratio of the maximized likelihoods under null hypotheses and use them to determine the sample size needed to discriminate between two overlapping families of distributions for a user specified probability of correct selection and a tolerance limit. Gupta and Kundu (2007) developed a distribution called the Generalized Exponential distribution: Existing results and some recent development. The distribution has a right skewed unimodal density function and monotone hazard function similar to the density functions and hazard functions of the gamma and Weibull distributions. It is observed that it can be used quite effectively to analyse lifetime data in place of gamma, Weibull and log-normal distributions. Gupta and Kundu (2008) introduced the Generalized Exponential distribution: Bayesian inference. The distribution consider the Bayes estimators of the unknown parameters under the assumptions of gamma priors on both the shape and scale parameters. The Bayes estimators cannot be obtained in explicit forms and approximate Bayes estimators were computed using the idea of Lindley.

Carrasco *et al.,* (2008), proposed a four parameter generalization of the Weibull distribution capable of modelling a bathtub-shaped hazard rate function.The beauty and importance of this distribution lies in its ability to model monotone as well as non-monotone failure rates, which are quite common in lifetime problems and reliability. The new distribution has a number of well-known lifetime special sub-models, such as the Weibull, extreme value, exponentiated Weibull, generalized Rayleigh and modified Weibull distributions, among others. Nadarajah (2011) proposed another continuous probability distribution called Exponentiated Exponential distribution in which he studied its mathematical properties and estimate the parameters using the method of maximum likelihood estimate including the case of censoring, and provide simpler expressions for the Fisher information matrix than those given by Gupta and Kundu. Lemonte and Cordeiro (2011) proposed the Exponentiated Generalized Inverse Gaussian distribution which extends the exponentiated standard gamma distribution (Nadarajah and Kotz, 2006). Various structural properties of

the new distribution were derived, including expansions for its moments, moment generating function, moments of the order statistics, and so forth. El-Gohary (2012) proposed the generalized Gompertz distribution(GGD). The main advantage of this new distribution is that it has increasing or constant or decreasing or bathtub curve failure rate depending upon the shape parameter. This property makes GGD very useful in survival analysis.

Sarhan *et al.,* (2013) provide a continuous probability distribution called the Exponentiated Generalized linear exponential distribution. The new distribution is important since it contains as special sub-models some widely well-known distributions such as the exponentiated Weibull distribution among many others and it also provides more flexibility to analyse complex real data sets. Cordeiro *et al.,* (2013) proposed a new density function called the Beta Exponentiated Weibull distribution. The new distribution is an important competitive model to the Weibull, exponentiated exponential, exponentiated Weibull, beta exponential and beta Weibull distributions since it contains all these models as special cases. The method of maximum likelihood was employed in estimation the parameters of the distribution.

Pal *et al.,* (2006), proposed the Exponentiated Weibull distribution and it was opined that the distribution fitted the data better than the exponentiated exponential probability distribution used in the literature. If (α, γ) denotes the shape parameters and λ is the scale parameter: for $\gamma = 1$, it represents the EE family and for $\alpha = 1$, it represents the Weibull family. Ayal and Tsokos (2011) proposed a generalization of the Weibull probability distribution called Transmuted Weibull (TW) distribution. The distribution fits the subject data better than the two-parameter Weibull and Exponentiated Weibull distribution. Cordeiro *et al.,* (2013), proposed a new method of adding two parameters to a continuous distribution. This method leads to a new class of Exponentiated generalized distributions. The distribution fitted the subject data better when compared with other distributions in the literature. The Exponentiated-G family of distribution allows flexibility of its tails and can be widely applied in many areas of engineering and biology. Cordeiro *et al.,* (2013),

proposed three parameter probability distribution for modelling lifetime data called Exponential Weibull distribution. The mathematical properties of the EW distribution were investigated.

Tibor *et al.,* (2015) proposed a new four-parameter distribution called the Marshall-Olkin exponential Weibull distribution. The distribution is quite flexible for analysing positive data and in-terms of applicability, it provides a better fit than related distributions used. Cordeiro *et al.,* (2015), studied some mathematical properties of a new generalized Weibull family of distributions with two extra positive parameters and concluded that the new distribution provides consistently better fits than any other competitive distributions. He also introduced a new distribution named Gamma extended-Weibull distribution which generalizes some important distributions by taking the pdf and CDF of the three-parameter extended-Weibull distribution with scale parameter $\lambda > 0$ and shape parameters $\alpha, \beta > 0$.

Saboor *et al.,* (2015) proposed the Transmuted Exponential Weibull (TEW) distribution with applications. The distribution fits the subject data better than any other distributions used in the literature. Cordeiro *et al.,* (2016) provided some properties of the new Kumaraswamy exponential-Weibull (KwEW) distribution. The distribution generalizes a number of well-known special cases of lifetime distributions which include Weibull, Exponential, Rayleigh, Modified Rayleigh, Modified exponential and Exponential-Weibull (EW) distributions among others. The distribution fits the dataset better than the Kumaraswamy Weibull, Marshall-Olkin Exponential Weibull, Extended Weibull, Exponential Weibull and Weibull models. Tahir and Cordeiro (2016) conducted an investigation on compound univariate distribution, their extensions and classes and also proposed some new generalized classes of distribution such as Exponentiated Kumaraswamy-G-geometric class, McDonald-G-geometric class, Beta-G-geometric class, Exponentiated Kumaraswamy-G-Poisson class and Exponentiated-G-logarithmic class among others. Inclusion, our interest here is to propose new distribution function which will serve as an extension of Exponential-Weibull distribution named Exponentiated-Exponential Weibull

(EEW) distribution using Gupta *et al.,* (1998) generator called Exponentiated Exponential.

CHAPTER THREE

METHODOLOGY

3.1 Definition of the Proposed Distribution

Gupta *et al.,* (1998), studied the generalized exponential family of distribution. Let X denote a random variable from an arbitrary parent distribution G(x), the CDF of the resulting Exponentiated family of distribution is given by :

$$F(x) = (G(x))^{\gamma} \tag{3.1.1}$$

where $G(x)$ is the CDF of the baseline (parent) distribution. The corresponding probability density function (pdf) is obtained by differentiating equation (3.1.1) with the respect to x as follows;

$$f(x) = \frac{dF(x)}{dx} = \frac{d\,(G(x))^{\gamma}}{dx} \tag{3.1.2}$$

Letting $u = G(x)$ then

$$f(x) = \frac{du^{\gamma}}{du} \times \frac{du}{dx} \tag{3.1.3}$$

therefore,

$$f(x) = \gamma\,(G(x))^{\gamma-1}\,g(x) \tag{3.1.4}$$

Cordeiro *et al.,* (2013) defines the CDF of Exponential and Weibull distributions as follows; Let Z and T be independent random variables with exponential and Weibull distributions describing the sub-system lifetimes. Their cumulative distribution functions are

$$G_z(z) = 1 - e^{\alpha z} \tag{3.1.5}$$

and

$$G_t(t) = 1 - e^{-\beta t^{\gamma}} \tag{3.1.6}$$

where $\alpha > 0$ and $\gamma > 0$ are shape parameters and $\beta > 0$ is the scale parameter respectively.

The random variable $x = min(Z, T)$ will define the time to failure of the system with cumulative distribution function. The CDF of the Exponential-Weibull distribution is given by

$$F(x) = F(x; \alpha, \beta, \lambda) = 1 - e^{-(\alpha x + \beta x^{\lambda})}, x > 0 \tag{3.1.7}$$

and

$$f(x) = (\alpha + \beta\lambda x^{\lambda-1})e^{-(\alpha x + \beta x^{\lambda})} \tag{3.1.8}$$

where $\alpha > 0$ and $\lambda > 0$ are the shape parameters and $\beta > 0$ is the scale parameter. Now, taking the CDF and the pdf in equation (3.1.7) and (3.1.8) of the EW distribution with the scale parameter $\beta > 0$ into equation (3.1.1) and equation (3.1.4), we define new four-parameter distribution named Exponentiated-Exponential Weibull (EEW) distribution having parameters α, β, λ and γ denoted as EEW(Θ), where Θ is a parameter of vector define by $\Theta = (\alpha, \beta, \lambda, \gamma)'$.

Definition

Let X be a random variable that follows EEW distribution with the parameters α, β, λ and γ, the cumulative distribution function (cdf) and probability density function (pdf) from equation (3.1.1) and (3.1.4) are given as

$$F(x) = \left(1 - e^{-(\alpha x + \beta x^{\lambda})}\right)^{\gamma} \tag{3.1.9}$$

and

$$f(x) = \gamma\left(1 - e^{-(\alpha x + \beta x^{\lambda})}\right)^{\gamma-1}(\alpha + \beta\lambda x^{\lambda-1})e^{-(\alpha x + \beta x^{\lambda})}, x > 0 \tag{3.1.10}$$

where $\gamma > 0$ is the shape parameter and α, β and λ all greater than zero are the parameters defined by our baseline distribution. The main reason why we used the GE family of distribution is to make the proposed distribution more flexible compared to the baseline distribution and also to make it possible to construct heavy-tailed distributions that are not long-tailed for modelling real datasets.

3.1.1 Model Validity Checking

Let X be the random variable that follows EEW distribution, then the probability model $f(x)$ in equation (3.1.10) can be valid probability density function (pdf) if and only if

$$\int_0^\infty f(x)dx = 1 \tag{3.1.11}$$

Proof: Substituting equation (3.1.10) into equation (3.1.11), we have

$$\int_0^\infty f(x)dx = \gamma \int_0^\infty \left(1 - e^{-(\alpha x+\beta x^\lambda)}\right)^{\gamma-1} (\alpha + \beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x^\lambda)}dx \tag{3.1.12}$$

Let

$$u = 1 - e^{-(\alpha x+\beta x^\lambda)} \tag{3.1.13}$$

As $x \to 0$, $u \to 0$ and as $x \to \infty$, $u \to 1$

$$\frac{du}{dx} = -e^{-(\alpha x+\beta x^\lambda)} \times \frac{d(-(\alpha x + \beta x^\lambda))}{dx} \tag{3.1.14}$$

$$\frac{du}{dx} = (\alpha + \beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x^\lambda)} \tag{3.1.15}$$

making dx the subject, equation (3.1.15) becomes

$$dx = \frac{du}{(\alpha + \beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x^\lambda)}} \tag{3.1.16}$$

substituting equation (3.1.16) in equation (3.1.12), we get

$$\int_0^1 f(x)dx = \int_0^1 \gamma u^{\gamma-1}(\alpha + \beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x^\lambda)} \frac{du}{(\alpha + \beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x^\lambda)})} \tag{3.1.17}$$

$$= \gamma \int_0^1 u^{\gamma-1}du$$

$$= u^\gamma|_0^1 = 1 - 0$$

$$\int_0^\infty f(x)dx = 1$$

Hence, it is a valid pdf.

To understand the behaviour of Exponentiated-Exponential Weibull distribution graphically, figure (3.1.1) and (3.1.2) in appendix **A** and **B** displays the CDF and pdf of the EEW distribution for some selected parameter values respectively.

where $a = \alpha, b = \beta, c = \lambda$ and $d = \gamma$. The CDF plot indicates that the distribution has an increasing upward trend that approaches value (1) as the random variable X increases and also as required for any CDF plot. Furthermore, the pdf plot shows that the distribution is positively skewed (i.e skewed to the right) implying that the new distribution can be used for modelling positively skewed dataset and also ensured the validity of the pdf.

Special Case:

For $\gamma = 1$, $\beta = 0$ in equation (3.1.9), the proposed distribution reduce to Exponential distribution. For $\beta = 0$ in equation (3.1.9), it is reduced to Generalized Exponential (GE) distribution. For $\gamma = 1$ in equation (3.1.9), it is reduced to Exponential Weibull (EW) distribution.

3.2 Reliability Analysis

In this section, we present the survival (also known as reliability) function and hazard function of our proposed distribution named EEW distribution.

3.2.1 Survival function of the EEW distribution

Survival function simply means the probability that the system will survive (or does not fall) beyond a specified period of time. It is mathematically given as

$$S(x) = 1 - F(x) \tag{3.2.1}$$

The survival function of the EEW distribution can be useful in characterizing the failure rate of a given system because of the analytical structure. The reliability function $S(x)$ of the EEW distribution is given as

$$S_{EEW}(x) = 1 - \left(1 - e^{-(\alpha x + \beta x^{\lambda})}\right)^{\gamma} \tag{3.2.2}$$

The plot of the survival function is presented in figure (3.2.1) in appendix **C**. The figure indicated that the value of the survival function equals 1 at the initial time or at the origin and decreases as x increases and also tends to zero as x increases.

3.2.2 Hazard function of the EEW distribution

The Hazard function, also known as failure rate or risk function or force of mortality is the probability that a component which has survived to the beginning of the respective interval will fail in that interval of time. The hazard function is defined as;

$$H(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \tag{3.2.3}$$

The Hazard function of the EEW distribution is defined as;

$$H(x) = \frac{\gamma\left(1 - e^{-(\alpha x + \beta x^{\lambda})}\right)^{\gamma-1}(\alpha + \beta\lambda x^{\lambda-1})e^{-(\alpha x + \beta x^{\lambda})}}{\left(1 - e^{-(\alpha x + \beta x^{\lambda})}\right)^{\gamma}} \tag{3.2.4}$$

The plot of the hazard function of EEW distribution for some selected values of the parameters is given in figure (3.2.2) in appendix **D**. It can be observed that the

hazard function is flexible and hence, the proposed distribution should be useful in modelling increasing, decreasing and constant failure rates behaviour.

3.3 Statistical Properties

3.3.1 Asymptotic Behaviour of the EEW Distribution

Here, we investigate the behaviour of the proposed probability model given in equation (3.1.10). The limit of equation (3.1.10) is $x \to 0$ and as $x \to \infty$ i.e.

$$\lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x) = 0$$

this implies that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \gamma\left(1 - e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma-1} (\alpha + \beta\lambda x^{\lambda-1}) e^{-(\alpha x+\beta x^{\lambda})}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \gamma\left(1 - e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma-1} \times \lim_{x\to 0} (\alpha + \beta\lambda x^{\lambda-1}) e^{-(\alpha x+\beta x^{\lambda})} \tag{3.3.1}$$

$$\lim_{x\to 0} f(x) = 0 \times \lim_{x\to 0} (\alpha + \beta\lambda x^{\lambda-1}) e^{-(\alpha x+\beta x^{\lambda})} = 0 \tag{3.3.2}$$

So also,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \gamma\left(1 - e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma-1} (\alpha + \beta\lambda x^{\lambda-1}) e^{-(\alpha x+\beta x^{\lambda})}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \gamma\left(1 - e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma-1} \times \lim_{x\to\infty} (\alpha + \beta\lambda x^{\lambda-1}) e^{-(\alpha x+\beta x^{\lambda})} \tag{3.3.3}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \gamma\left(1 - e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma-1} (\alpha + \beta\lambda x^{\lambda-1}) \times 0 = 0 \tag{3.3.4}$$

Hence, equation (3.3.2) and equation (3.3.4) shows that the EEW distribution has a uni-modal.

3.3.2 Quantile Function and Median

In this subsection, we determine the explicit formula for quantile function and the median of the EEW distribution. The quantile function is obtained using the relation

$$F(x_u) = u \tag{3.3.5}$$

where $F(x_u)$ is the CDF of EEW in (3.1.9) and $0 < u < 1$.
Let

$$u = F(x) = \left(1 - e^{-(\alpha x + \beta x^{\lambda})}\right)^{\gamma}$$

multiplying the power of both side by $1/\gamma$, we have

$$u^{1/\gamma} = 1 - e^{-(\alpha x + \beta x^{\lambda})} \tag{3.3.6}$$

equation (3.3.6) lead us to equation (3.3.7)

$$1 - u^{1/\gamma} = e^{-(\alpha x + \beta x^{\lambda})} \tag{3.3.7}$$

taking the natural log of both side, it leads to

$$\ln(1 - u^{1/\gamma}) = -(\alpha x + \beta x^{\lambda}) \tag{3.3.8}$$

$$-\ln(1 - u^{1/\gamma}) = (\alpha x + \beta x^{\lambda}) \tag{3.3.9}$$

by taking natural log, it implies that

$$\ln\left(-\ln(1 - u^{1/\gamma})\right) = \ln\alpha + \ln x + \ln\beta + \lambda \ln x \tag{3.3.10}$$

$$\ln\left(-\ln(1 - u^{1/\gamma})\right) - \ln\alpha\beta = \ln x(1 + \lambda) \tag{3.3.11}$$

divide both side by $(1+\lambda)$, it leads to equation (3.3.12)

$$\frac{\ln\left(-\ln(1-u^{1/\gamma})\right)-\ln\alpha\beta}{(1+\lambda)}=\ln x \tag{3.3.12}$$

$$e^{\left(\frac{\ln\left(-\ln(1-u^{1/\gamma})\right)-\ln\alpha\beta}{(1+\lambda)}\right)}=x_u \tag{3.3.13}$$

we can re-write equation (3.3.13)as

$$x_u=e^{\left(\frac{\ln\left(-\ln(1-u^{1/\gamma})\right)-\ln\alpha\beta}{(1+\lambda)}\right)} \tag{3.3.14}$$

Hence, equation (3.3.14) is the quantile function of the EEW distribution and the median is determined by substituting for $u=0.5$ at x_u thus

$$x_{0.5}=e^{\left(\frac{\ln\left(-\ln(1-0.5^{1/\gamma})\right)-\ln\alpha\beta}{(1+\lambda)}\right)} \tag{3.3.15}$$

3.3.3 Moments

Moment is one of the statistical properties of any distribution that is used to study some of the characteristics of a random variable such as mean, variance, skewness (Sk) and kurtosis (Ku). Also moments of a probability distributions is a collection of descriptive constants that can be used for measuring its properties. The moment of the EEW distribution can be mathematically defined as follows:
Let X be a continuous random variable, then the r^{th} moment of X about the origin is given in equation (3.3.16)

$$\mu_r'=E(X^r)=\int_{-\infty}^{\infty}X^r f(x)dx \tag{3.3.16}$$

where $f(x)$ is the pdf of EEW distribution. Substituting (3.1.10) into (3.3.16), we get

$$\mu'_r = E(x^r) = \int_0^\infty \gamma x^r \left(1 - e^{-(\alpha x + \beta x^\lambda)}\right)^{\gamma-1} (\alpha + \beta\lambda x^{\lambda-1}) e^{-(\alpha x + \beta x^\lambda)} \tag{3.3.17}$$

Theorem: The r^{th} moment of the random variable $X \sim$ EEW $(\alpha, \beta, \lambda, \gamma)$ is given by

$$u'_r = E(X^r) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\gamma)}{j!k!l!\Gamma(\gamma-j)} \left(\frac{\gamma\alpha^{l+1}\Gamma(\frac{r+k+l+1}{\lambda})}{\lambda\left(\beta(1+j)\right)^{\frac{r+k+l}{\lambda}}} + \frac{\beta\alpha^{l}\Gamma(\frac{r+k+l}{\lambda})}{\left(\beta(1+j)\right)^{\frac{r+k+l+\lambda-l}{\lambda}}} \right)$$

Proof:

$$E(X^r) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\gamma)}{j!k!l!\Gamma(\gamma-j)} \left(\frac{\gamma\alpha^{l+1}\Gamma(\frac{r+k+l+1}{\lambda})}{\lambda\left(\beta(1+j)\right)^{\frac{r+k+l}{\lambda}}} + \frac{\beta\alpha^{l}\Gamma(\frac{r+k+l}{\lambda})}{\left(\beta(1+j)\right)^{\frac{r+k+l+\lambda-l}{\lambda}}} \right)$$

$$\int_0^\infty \gamma\alpha x^r e^{-(\alpha x+\beta x^\lambda)} \left(1 - e^{-(\alpha x+\beta x^\lambda)}\right)^{\gamma-1} dx + \int_0^\infty \beta\lambda x^{r+\lambda-1} e^{-(\alpha x+\beta x^\lambda)} \left(1 - e^{-(\alpha x+\beta x^\lambda)}\right)^{\gamma-1} dx \tag{3.3.18}$$

factoring the constants out

$$\gamma\alpha \int_0^\infty x^r e^{-(\alpha x+\beta x^\lambda)} \left(1 - e^{-(\alpha x+\beta x^\lambda)}\right)^{\gamma-1} dx + \beta\lambda \int_0^\infty x^{r+\lambda-1} e^{-(\alpha x+\beta x^\lambda)} \left(1 - e^{-(\alpha x+\beta x^\lambda)}\right)^{\gamma-1} dx \tag{3.3.19}$$

using binomial expansion for

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} (-1)^j \begin{pmatrix} b-1 \\ j \end{pmatrix} z^j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j!\Gamma(b-j)} z^j \tag{3.3.20}$$

and the series expansion for

$$e^x = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!} \tag{3.3.21}$$

then,

$$\left(1 - e^{-(\alpha x+\beta x^\lambda)}\right)^{\gamma-1} = \sum_{j=0}^{\infty} \begin{pmatrix} \gamma-1 \\ j \end{pmatrix} e^{-(\alpha x+\beta x^\lambda)^j}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\gamma)}{j!\Gamma(\gamma - j)} e^{-j(\alpha x + \beta x^{\lambda})} \tag{3.3.22}$$

but

$$e^{-j(\alpha x + \beta x^{\lambda})} = e^{-j\alpha x} \times e^{-j\beta x^{\lambda}}$$

considering

$$e^{-j\alpha x} = \sum_{k=0}^{\infty} (-1)^k \frac{(\alpha j)^k x^k}{k!} \tag{3.3.23}$$

this implies that

$$e^{-j(\alpha x + \beta x^{\lambda})} = e^{-j\beta x^{\lambda}} \sum_{k=0}^{\infty} (-1)^k \frac{(\alpha j)^k x^k}{k!} \tag{3.3.24}$$

substituting equation (3.3.22) and (3.3.24) into equation (3.3.19) we obtain equation (3.3.25)

$$\alpha\gamma \int_0^{\infty} x^r e^{-(\alpha x + \beta x^{\lambda})} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\gamma)}{j!\Gamma(\gamma - j)} \sum_{k=0}^{\infty} (-1)^k \frac{(\alpha j)^k x^k}{k!} e^{-j\beta x^{\lambda}} dx +$$

$$\beta\lambda \int_0^{\infty} x^{r+\lambda-1} e^{-(\alpha x + \beta x^{\lambda})} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\gamma)}{j!\Gamma(\gamma - j)} \sum_{k=0}^{\infty} (-1)^k \frac{(\alpha j)^k x^k}{k!} e^{-j\beta x^{\lambda}} dx \tag{3.3.25}$$

$$\alpha\gamma \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} (\alpha j)^k \Gamma(\gamma)}{j!k!\Gamma(\gamma - j)} \int_0^{\infty} x^{r+k} e^{-(\alpha x + \beta x^{\lambda})} e^{-j\beta x^{\lambda}} dx +$$

$$\beta\lambda \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} (\alpha j)^k \Gamma(\gamma)}{j!k!\Gamma(\gamma - j)} \int_0^{\infty} x^{k+r+\lambda-1} e^{-(\alpha x + \beta x^{\lambda})} e^{-j\beta x^{\lambda}} dx \tag{3.3.26}$$

again

$$e^{-(\alpha x + \beta x^{\lambda})} = e^{-\alpha x} \times e^{-\beta x}$$

but

$$e^{-\alpha x} = \sum_{l=0}^{\infty} (-1)^l \frac{\alpha^l x^l}{l!} \tag{3.3.27}$$

substituting equation (3.3.27) into equation (3.3.26), it leads to equation (3.3.28)

$$\alpha\gamma\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}(\alpha j)^k\Gamma(\gamma)}{j!k!\Gamma(\gamma-j)}\sum_{l=0}^{\infty}(-1)^l\frac{\alpha^l x^l}{l!}\int_0^{\infty}x^{r+k}e^{-\beta x^{\lambda}(1+j)}dx+$$

$$\beta\lambda\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}(\alpha j)^k\Gamma(\gamma)}{j!k!\Gamma(\gamma-j)}\sum_{l=0}^{\infty}(-1)^l\frac{\alpha^l x^l}{l!}\int_0^{\infty}x^{r++k+\lambda-1}e^{-\beta x^{\lambda}(1+j)}dx \tag{3.3.28}$$

$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+k+l}\Gamma(\gamma)}{j!k!l!\Gamma(\gamma-j)}\left(\ \gamma\alpha^{l+1}\int_0^{\infty}x^{r+k+l}e^{-\beta x^{\lambda}(1+j)}dx+\beta\lambda\alpha^l\int_0^{\infty}x^{r+k+l+\lambda-1}e^{-\beta x^{\lambda}(1+j)}dx\ \right) \tag{3.3.29}$$

let

$$w=\beta x^{\lambda}(1+j) \tag{3.3.30}$$

this implies that

$$\frac{w}{\beta(1+j)}=x^{\lambda}$$

therefore,

$$x=\frac{w^{\frac{1}{\lambda}}}{(\beta(1+j))^{\frac{1}{\lambda}}} \tag{3.3.31}$$

$$\frac{dx}{dw}=\frac{\frac{1}{\lambda}w^{\frac{1-\lambda}{\lambda}}}{(\beta(1+j))^{\frac{1}{\lambda}}} \tag{3.3.32}$$

this implies that

$$dx=\frac{w^{\frac{1-\lambda}{\lambda}}}{\lambda\,(\beta(1-j))^{\frac{1}{\lambda}}}dw \tag{3.3.33}$$

putting equation (3.3.30), (3.3.31) and (3.3.33) into (3.3.29) we have

$$E(X^r)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+k+l}\Gamma(\gamma)}{j!k!l!\Gamma(\gamma-j)}\times$$

$$\left(\gamma\alpha^{l+1}\int_0^{\infty}\frac{w^{\frac{r+k+l}{\lambda}}}{(\beta(1+j))^{\frac{r+k+l}{\lambda}}}e^{-w}\frac{w^{\frac{1-\lambda}{\lambda}}}{\lambda\,(\beta(1-j))^{\frac{1}{\lambda}}}dw+\beta\lambda\alpha^l\int_0^{\infty}\frac{w^{\frac{r+k+l-1}{\lambda}}}{(\beta(1+j))^{\frac{r+\lambda+l-1}{\lambda}}}e^{-w}\frac{w^{\frac{1-\lambda}{\lambda}}}{\lambda\,(\beta(1-j))^{\frac{1}{\lambda}}}dw\right) \tag{3.3.34}$$

but

$$\int_0^{\infty}w^{\frac{r+k+l}{\lambda}-1}e^{-w}dw=\Gamma\left(\frac{r+k+l+1}{\lambda}\right) \tag{3.3.35}$$

also,

$$\int_0^{\infty} w^{\frac{r+k+l}{\lambda}-1} e^{-w} dw = \Gamma\left(\frac{r+k+l}{\lambda}\right) \tag{3.3.36}$$

taking equation (3.3.35) and (3.3.36) into (3.3.34), it leads to

$$E(X^r) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\gamma)}{j!k!l!\Gamma(\gamma-j)} \left(\frac{\gamma\alpha^{l+1}\Gamma\left(\frac{r+k+l+1}{\lambda}\right)}{\lambda\left(\beta(1+j)\right)^{\frac{r+k+l}{\lambda}}} + \frac{\beta\lambda\alpha^{l}\Gamma\left(\frac{r+k+l}{\lambda}\right)}{\lambda\left(\beta(1+j)\right)^{\frac{r+k+l+\lambda-l}{\lambda}}} \right) \tag{3.3.37}$$

after further simplification, equation (3.3.36) becomes

$$E(X^r) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\gamma)}{j!k!l!\Gamma(\gamma-j)} \left(\frac{\gamma\alpha^{l+1}\Gamma\left(\frac{r+k+l+1}{\lambda}\right)}{\lambda\left(\beta(1+j)\right)^{\frac{r+k+l}{\lambda}}} + \frac{\beta\alpha^{l}\Gamma\left(\frac{r+k+l}{\lambda}\right)}{\left(\beta(1+j)\right)^{\frac{r+k+l+\lambda-l}{\lambda}}} \right) \tag{3.3.38}$$

Hence proved.

3.3.4 Moment Generating Function

The moment generating function of the random variable X that follows EEW distribution having a probability density function $f(x)$ as given in equation (3.1.10) can be defined as

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \tag{3.3.39}$$

Theorem: The MGF of the random variable X $EEW(\alpha, \beta, \lambda, \gamma)$ is given by

$$E(e^{tx}) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{j+k+l}(t)^m\Gamma(\gamma)}{j!k!l!m!\Gamma(\gamma-j)} \alpha^l \left\{ \frac{\gamma\alpha\Gamma\left(\frac{k+l+m+1}{\lambda}\right)}{\lambda\left(\beta(1+j)\right)^{\frac{k+l+m+1}{\lambda}}} + \frac{\beta\Gamma\left(\frac{k+l+m}{\lambda}\right)}{\left(\beta(1+j)\right)^{\frac{k+l+m+\lambda-1}{\lambda}}} \right\}$$

Proof: Replacing X^r with e^{tx} in equation (3.3.29), we obtain

$$E(e^{tx}) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\gamma)}{j!k!l!\Gamma(\gamma-j)} \times$$

$$\left(\gamma\alpha^{l+1}\int_0^\infty e^{tx}x^{k+l}e^{-\beta x^\lambda(1+j)}dx+\beta\lambda\alpha^l\int_0^\infty e^{tx}x^{k+l+\lambda-1}e^{-\beta x^\lambda(1+j)}dx\right) \tag{3.3.40}$$

but

$$e^{tx}=\sum_{m=0}^{\infty}\frac{(tx)^m}{m!} \tag{3.3.41}$$

putting equation (3.3.42) in equation (3.3.41), it leads

$$E(e^{tx})=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+k+l}\Gamma(\gamma)}{j!k!l!r!\Gamma(\gamma-j)}\sum_{m=0}^{\infty}\frac{(tx)^m}{m!}\times$$

$$\left(\gamma\alpha^{l+1}\int_0^\infty x^{k+l+m}e^{-\beta x^\lambda(1+j)}dx+\beta\lambda\alpha^l\int_0^\infty x^{k+l+\lambda-1}e^{-\beta x^\lambda(1+j)}dx\right) \tag{3.3.42}$$

Letting

$$\begin{aligned}&w=\beta x^\lambda(1+j)\ \Rightarrow\ \tfrac{w}{(\beta(1+j))}=x^\lambda\\&\Rightarrow\tfrac{w^{1/\lambda}}{(\beta(1+j))^{1/\lambda}}=x\\&\tfrac{dx}{dw}=\tfrac{1/\lambda w^{\frac{1-\lambda}{\lambda}}}{(\beta(1+j))^{1/\lambda}}\end{aligned} \tag{3.3.43}$$

$$dx=\frac{w^{\frac{1-\lambda}{\lambda}}}{\lambda\left(\beta(1+j)\right)^{1/\lambda}}dw \tag{3.3.44}$$

$$E(e^{tx})=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{j+k+l}(t)^m\Gamma(\gamma)}{j!k!l!m!\Gamma(\gamma-j)}(\gamma\alpha^{l+1}\int_0^\infty\frac{w^{\frac{k+l+m}{\lambda}}}{(\beta(1+j))^{\frac{k+l+m}{\lambda}}}e^{-w}\frac{w^{\frac{1-\lambda}{\lambda}}}{\lambda\left(\beta(1+j)\right)^{1/\lambda}}dw+$$

$$\beta\lambda\alpha^l\int_0^\infty\frac{w^{\frac{k+l+m+\lambda-1}{\lambda}}}{(\beta(1+j))^{k+l+m+\lambda-1}}e^{-w}\frac{w^{\frac{1-\lambda}{\lambda}}}{\lambda\left(\beta(1+j)\right)^{\frac{1}{\lambda}}}dw \tag{3.3.45}$$

but

$$\int_0^\infty w^{\frac{r+k+l}{\lambda}-1}e^{-w}dw=\Gamma(\frac{r+k+l+1}{\lambda}) \tag{3.3.46}$$

also,

$$\int_0^{\infty} w^{\frac{r+k+l}{\lambda}-1} e^{-w} dw = \Gamma(\frac{r+k+l}{\lambda}) \tag{3.3.47}$$

putting equation (3.3.47) and (3.3.48) into equation (3.3.45), it gives

$$E(e^{tx}) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{j+k+l}(t)^m \Gamma(\gamma)}{j!k!l!m!\Gamma(\gamma-j)} \alpha^l \{ \frac{\gamma\alpha\Gamma\left(\frac{k+l+m+1}{\lambda}\right)}{\lambda\left(\beta(1+j)\right)^{\frac{k+l+m+1}{\lambda}}} + \frac{\beta\Gamma\left(\frac{k+l+m}{\lambda}\right)}{\left(\beta(1+j)\right)^{\frac{k+l+m+\lambda-1}{\lambda}}} \} \tag{3.3.48}$$

Hence proved.

3.4 Order Statistic

Order statistics is widely used in solving statistical problems including robust statistical estimation, detection of outliers, analyses of censored samples, reliability analysis, characterization of probability distributions and goodness of fit tests, among others. In this section, we derived closed form expressions for the pdf of the i^{th} order statistic of the Exponentiated-Exponential Weibull distribution.

Suppose $X_1, X_2, \cdots, X_n$ is a random sample from a distribution with pdf $f(x)$ and let $X_{1:n}, X_{2:n}, \cdots, X_{i:n}$ denote the corresponding order statistic obtained from this sample. The pdf $f_{i:n}(x)$ of the i^{th} order statistic can be expressed as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \left(1 - F(x)\right)^{n-i} \tag{3.4.1}$$

where $F(x)$ and $f(x)$ are the CDF and pdf of EEW distribution.

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!} f(x) F(x)^{1-1} \left(1 - F(x)\right)^{n-1} \\ &= n f(x) \left(1 - F(x)\right)^{n-1} \end{aligned} \tag{3.4.2}$$

applying the binomial expansion on $(1-F(x))^{n-1}$ and putting equation (3.1.9) and (3.1.10), we obtain

$$f_{1:n} = n\gamma\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma-1}(\alpha+\beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x^{\lambda})}\sum_{j=1}^{n-1}\begin{pmatrix} n-1 \\ j \end{pmatrix}(-1)^j\left(\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma}\right)^j \tag{3.4.3}$$

$$= \sum_{j=1}^{n-1}\frac{n(n-1)!}{(n-1-j)!j!}\gamma\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma-1}(\alpha+\beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x^{\lambda})}((1-e^{-(\alpha x+\beta x^{\lambda})^{\gamma})^j}$$

$$f_{1:n} = \sum_{j=1}^{n}\frac{n!}{(n-1-j)!j!}\gamma\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma-1}(\alpha+\beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x^{\lambda})}\left(\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma}\right)^j \tag{3.4.4}$$

$$f_{1:n} = \sum_{j=1}^{n}\frac{n!}{(n-1-j)!j!}\gamma\left(\alpha+\beta\lambda x^{\lambda-1}\right)e^{-(\alpha x+\beta x^{\lambda})}\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma(j+1)-1} \tag{3.4.5}$$

also,

$$f_{n:n}(x) = \frac{n!}{(n-1)!(n-n)!}f(x)F(x)^{n-1}(1-F(x))^{n-n} \tag{3.4.6}$$

$$= \frac{n(n-1)!}{(n-1)!}\gamma\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma-1}(\alpha+\beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x^{\lambda})}\left(\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma}\right)^{n-1}$$

this imply that

$$f_{n:n}(x) = n\gamma\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma-1}(\alpha+\beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x^{\lambda})}\left(\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma}\right)^{n-1} \tag{3.4.7}$$

$$f_{n:n}(x) = n\gamma(\alpha+\beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x^{\lambda})}\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma n-1} \tag{3.4.8}$$

Finally, the pdf of the minimum order statistic X_1 and maximum order statistic X_n of the EEW distribution are given by

$$f_{1:n} = \sum_{j=1}^{n}\frac{n!}{(n-1-j)!j!}\gamma\left(\alpha+\beta\lambda x^{\lambda-1}\right)e^{-(\alpha x+\beta x^{\lambda})}\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma(j+1)-1} \tag{3.4.9}$$

and

$$f_{n:n}(x) = n\gamma(\alpha+\beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x^{\lambda})}\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma n-1} \tag{3.4.10}$$

respectively.

3.5 Estimation of Parameters

In this section, we employ the method of maximum likelihood to estimate the parameters of the EEW distribution. Let $X_1, ..., X_n$ be a random sample from EEW distribution with unknown parameters $(\alpha, \beta, \lambda, \gamma)$. The likelihood function $\boldsymbol{L}$ is given by

$$L(\alpha,\beta,\lambda,\gamma) = L(X_1, X_2, \cdots, X_n/(\alpha,\beta,\lambda,\gamma)) = \prod_{i=1}^{n} f(x_i/(\alpha,\beta,\lambda,\gamma)) \tag{3.5.1}$$

$$= \prod_{i=1}^{n} \{\gamma\left(1 - e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma-1} (\alpha + \beta\lambda x^{\lambda-1}) e^{-(\alpha x+\beta x^{\lambda})}\}$$

$$L(\alpha,\beta,\lambda,\gamma) = \ln \prod_{i=1}^{n} \gamma(\alpha + \beta\lambda x^{\lambda-1}) e^{-(\alpha x+\beta x^{\lambda})} \left(1 - e^{-(\alpha x+\beta x^{\lambda})}\right)^{\gamma-1} \tag{3.5.2}$$

$$L(\alpha,\beta,\lambda,\gamma) = n\ln\gamma + n\ln\alpha + n\ln\beta + n\ln\lambda + (\lambda-1)\sum_{i=1}^{n}\ln x - \sum_{i=1}^{n}(\alpha x + \beta x^{\lambda}) +$$

$$(\gamma-1)\sum_{i=1}^{n}\ln(1 - e^{-(\alpha x+\beta x^{\lambda})}) \tag{3.5.3}$$

The estimate of the parameters where obtained by taking a partial derivative of $L(\alpha, \beta, \lambda, \gamma)$ with respect to each parameter as given below

$$\frac{\partial L(\alpha,\beta,\lambda,\gamma)}{\partial\gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n}\ln\left(1 - e^{-(\alpha x+\beta x^{\lambda})}\right)$$

$$\frac{\partial L(\alpha,\beta,\lambda,\gamma)}{\partial\beta} = \frac{n}{\alpha} - \sum_{i=1}^{n} x + (\gamma-1)\sum_{i=1}^{n}\frac{(\alpha + \beta\lambda x^{\lambda-1}) e^{-(\alpha x+\beta x^{\lambda})}}{1 - e^{-(\alpha x+\beta x^{\lambda}}}$$

$$\frac{\partial L(\alpha,\beta,\lambda,\gamma)}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x^{\lambda} + (\gamma-1)\sum_{i=1}^{n}\frac{x^{\lambda}(\alpha + \beta\lambda x^{\lambda-1})}{1 - e^{-(\alpha x+\beta x^{\lambda}}}$$

$$\frac{\partial L(\alpha,\beta,\lambda,\gamma)}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\ln x - \beta\sum_{i=1}^{n} x^{\lambda}\ln x + (\gamma-1)\sum_{i=1}^{n}\frac{\beta x^{\lambda}\ln x e^{-(\alpha x+\beta x^{\lambda})}}{1 - e^{-(\alpha x+\beta x^{\lambda}}}$$

The estimate of the unknown parameters can be obtained by setting $\partial L(\alpha, \beta, \lambda, \gamma)$ to zero and solving them simultaneously. These equations cannot be solved analytically but statistical software (e.g. R package) can be used to solve the equations numerically by means of iterative technique such as the Newton-Raphson algorithm.

CHAPTER FOUR

ANALYSIS AND DISCUSSION

4.1 Application

In this section, we provide an application of the developed EEW distribution to show its usefulness to fit real data set. The EEW distribution is compared with two related distributions, namely Exponential-Weibull (EW) and Kumaraswamy Exponential-Weibull (KwEW) distribution whose pdf's are given as follows:

1. The density function of the EW distribution is given by

$$f(x) = (\alpha + \beta\lambda x^{\lambda-1})e^{-(\alpha x+\beta x)}$$

   where $\alpha > 0, \beta > 0$ and $\lambda > 0$

2. The density function of the KwEW distribution is given by

$$f(x) = \alpha\gamma(\lambda + k\beta x^{k-1})e^{-\lambda x-\beta x^k}(1 - e^{-\lambda x-\beta x^k})^{-1+\alpha} \times \{1 - (1 - e^{-\lambda x-\beta x^k})\}$$

   where $\lambda > 0, \beta > 0, k > 0, \alpha > 0$ and $\gamma > 0$.

We carried out the model selection using the AIC (Akaike Information Criteria) and the CAIC (Consistent Akaike Information Criteria). The smaller the values of goodness-of-fit measured, better the fit of the data. These measures are given as

$$AIC = -2\ell(\widehat{\Theta}) + 2q \tag{4.1.1}$$

$$CAIC = 2\ell(\widehat{\Theta}) + \frac{2qn}{n-q-1} \tag{4.1.2}$$

respectively, where $\ell(\widehat{\Theta})$ denotes the log-likelihood function evaluated at the likelihood estimates, q is the number of parameters and n is the sample size. Table 3

and 6 provides the MLEs of the model parameters. Table 4 and 8 lists the numerical values of the $-2\ell(\widehat{\Theta})$, AIC and CAIC respectively. These numerical results are obtained using the R-Package. Note that the Θ denote the parameters.

Table 1: The uncensored real dataset on breaking stress of carbon fibres (in Gba) reported by Cordeiro and Lemonte (2011).

3.78, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.25, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53

Table 2: Summary of the dataset.

Parameters	N	Q1	Median	Q3	Mean	Variance	Skewness	Kurtosis
Value	68	2.102	2.835	3.288	2.769	0.863	0.006	3.114

Table 3: MLE's of the parameters

Distribution	MLE's of the parameter(s)
$EW\,(\alpha,\beta,\lambda)$	$\widehat{\alpha}$ = 0.36113, $\widehat{\beta}$ = 0.02490, $\widehat{\lambda}$ = 3.28146
$EEW\,(\alpha,\beta,\lambda,\gamma)$	$\widehat{\alpha}$ = 0.82264, $\widehat{\beta}$ = 0.09651, $\widehat{\lambda}$ = 2.43514, $\widehat{\gamma}$ = 10.77656
$KwEW\,(\alpha,\beta,\lambda,k,\gamma)$	$\widehat{\alpha}$ = 0.865898, $\widehat{\beta}$ = −0.252877, $\widehat{\lambda}$ = 1.688675, $\widehat{k}$ = 0.080005, $\widehat{\gamma}$ = 0.008235

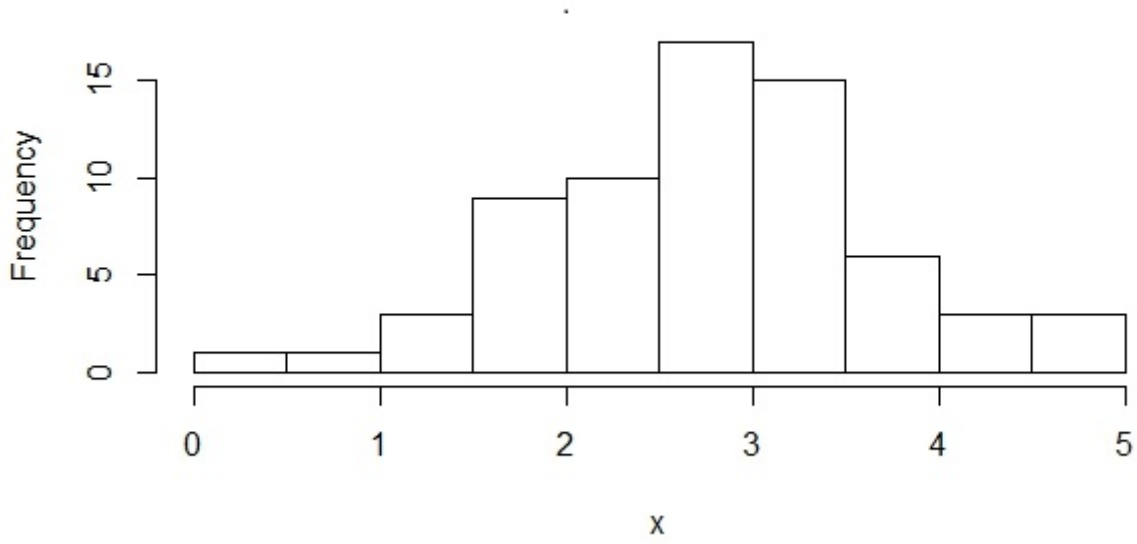


Figure 4.1.1: Histogram of the uncensored real data set on the breaking stress of the carbon fibres (in Gba) reported by Cordeiro and Lemonte (2011)

Table 4: Model Performance Comparisons

Distribution	**Measures**		
	$-2\ell(\widehat{\Theta})$	AIC	$CAIC$
$EW\,(\alpha,\beta,\lambda)$	457.291	463.291	464.621
$EEW\,(\alpha,\beta,\lambda,\gamma)$	335.4498	343.291	342.7798
$KwEW\,(\alpha,\beta,\lambda,k,\gamma)$	487.104	497.104	494.434

Table 5: Remission times (in month) of 128 patients of Bladder Cancer (Lee and Wang, 2003).

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69

Table 6: Summary of the dataset.

Parameters	N	Q1	Median	Q3	Mean	Variance	Skewness	Kurtosis
Value	128	3.348	6.395	11.840	9.366	110.425	3.287	18.483

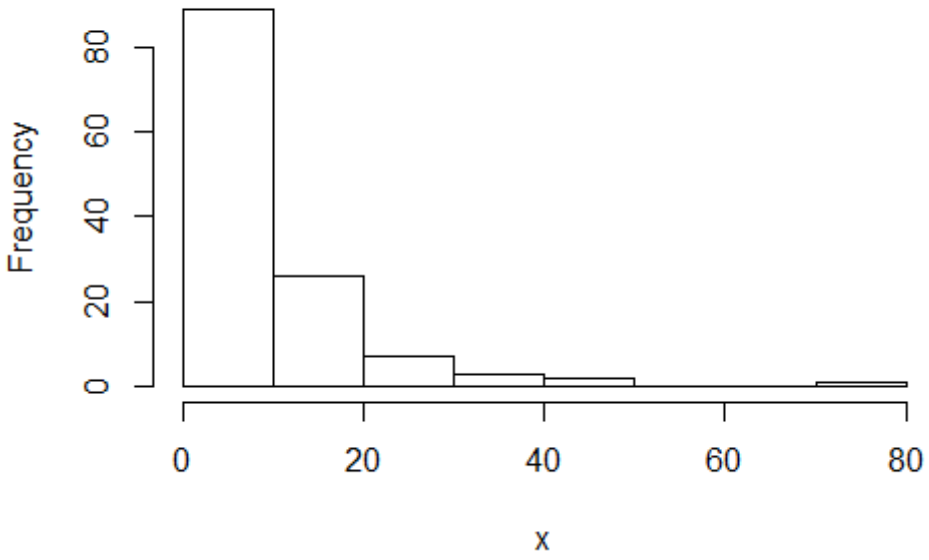


Figure 4.1.2: Histogram of the Remission times (in month) of 128 patients of Bladder Cancer (Lee and Wang, 2003)

From figure (3.1.2), table 6 and figure (4.1.2) above, it is evident that the dataset is skewed to the right (positively skewed). Hence, the two dataset can be applied on the distribution and also the distribution can be applied to positively skewed data.

Table 7: MLE's of the parameters

Distribution	MLE's of the parameter(s)
$EW\,(\alpha,\beta,\lambda)$	$\widehat{\alpha}$ = 0.106773, $\widehat{\beta}$ = 0.0.093888, $\widehat{\lambda}$ = 1.047835
$EEW\,(\alpha,\beta,\lambda,\gamma)$	$\widehat{\alpha}$ = 0.14187, $\widehat{\beta}$ = 0.90801, $\widehat{\lambda}$ = 0.44624, $\widehat{\gamma}$ = 8.32639
$KwEW\,(\alpha,\beta,\lambda,k,\gamma)$	$\widehat{\alpha}$ = −0.010053, $\widehat{\beta}$ = 0.891136, $\widehat{\lambda}$ = 0.024100, $\widehat{k}$ = 0.005000, $\widehat{\gamma}$ = 0.021816

Table 8: Model Performance Comparisons

Distribution	**Measures**		
	$-2\ell(\widehat{\Theta})$	AIC	$CAIC$
$EW\,(\alpha,\beta,\lambda)$	1656.8576	1662.858	1665.1828
$EEW\,(\alpha,\beta,\lambda,\gamma)$	1566.5542	1574.554	1574.8794
$KwEW\,(\alpha,\beta,\lambda,k,\gamma)$	1769.388	1779.388	1777.7132

Form Table 4 and 8 above, it is evident that the EEW distribution has lowest values for the AIC and CAIC compared to EW and KwEW which means that the EEW distribution has better performance and could be used for fitting the subject data than EW and KwEW.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 Summary

In this research, we proposed a new four-parameter distribution named the Exponentiated-Exponential Weibull distribution which extends the Exponential Weibull distribution proposed by Cordeiro *et al.,* (2013) using a generator introduced by Gupta *et al.,* (1998) with the expectation that it will add more flexibility to the distribution. Some statistical properties of the distribution including Asymptotic behaviour, Reliability analysis, quantile function, median, moments, moment generating function and order statistics have been studied accordingly. The parameters of the distribution were obtained using the method of maximum likelihood. We applied the EEW distribution on the dataset to compare its usefulness with some distributions named Exponential Weibull (EW) and Kumaraswamy Exponential Weibull(KwEW) distribution. The new distribution provides better fits compared to EW and KwEW distributions. This indicates that the EEW distribution should be used better for fitting the subject dataset.

5.2 Conclusion

A four-parameter model called the Exponentiated-Exponential Weibull distribution was proposed. We provide some of its mathematical and statistical properties. The pdf plot shows that the distribution is skewed to the right (positively skewed) implying that the distribution can be used for modelling positively skewed dataset. We compared the new distribution with some related distributions from the literature to obtain the goodness-of-fit of the new distribution using AIC and CAIC criteria. The proposed distribution is applied to a real-life data set. The EEW

distribution provides a better fit than EW and KwEW distribution. We hope that the new model will bring the wider application in several areas such as engineering, medicine, survival lifetime data among others.

5.3 Recommendation

Based on the above results of this research , we recommend that the proposed model should be applied for modelling real datasets that are positively skewed.

5.4 Contribution to Knowledge

We derived a four-parameter lifetime distribution name "Exponentiated-Exponential Weibull" (EEW) distribution which is found to be useful in modelling real dataset especially when the data is positively skewed. Some of its statistical properties were obtained and we also estimated the parameters of the new distribution.

5.5 Areas for Further Research

In view of this research work, interested researcher(s) can derive the estimation of the confidence interval of the proposed distribution parameters to obtain its characteristics function, estimate the parameters using the method of moment and also use the Anderson Darling test for the purpose of theoretical comparison and validation of methodology.

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APPENDIX A

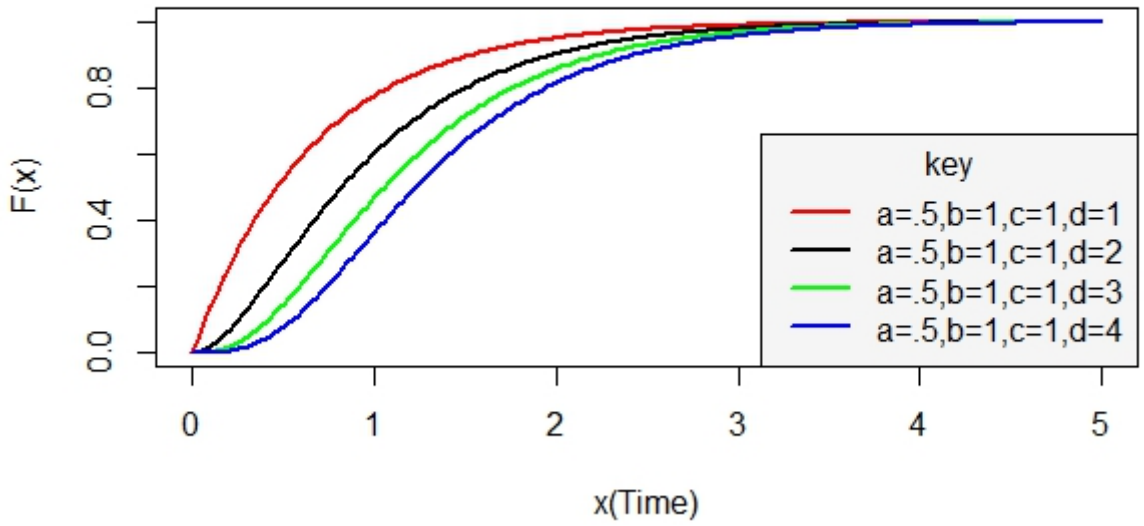


Figure 5.5.1: CDF of EEW Distribution

where $a = \alpha, b = \beta, c = \lambda$ and $d = \gamma$.

APPENDIX B

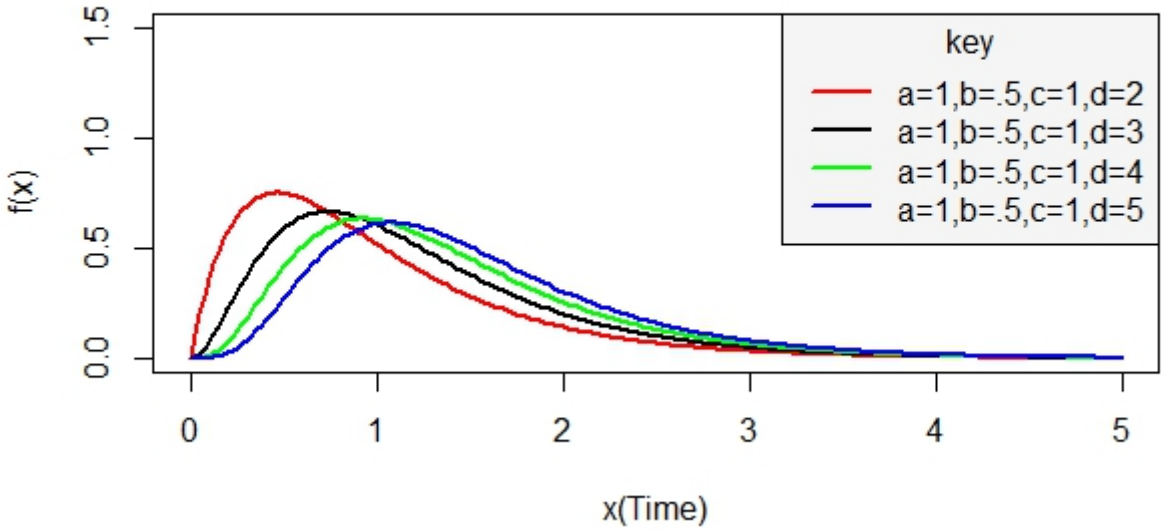


Figure 5.5.2: PDF of EEW Distribution

where $a = \alpha, b = \beta, c = \lambda$ and $d = \gamma$.

APPENDIX C

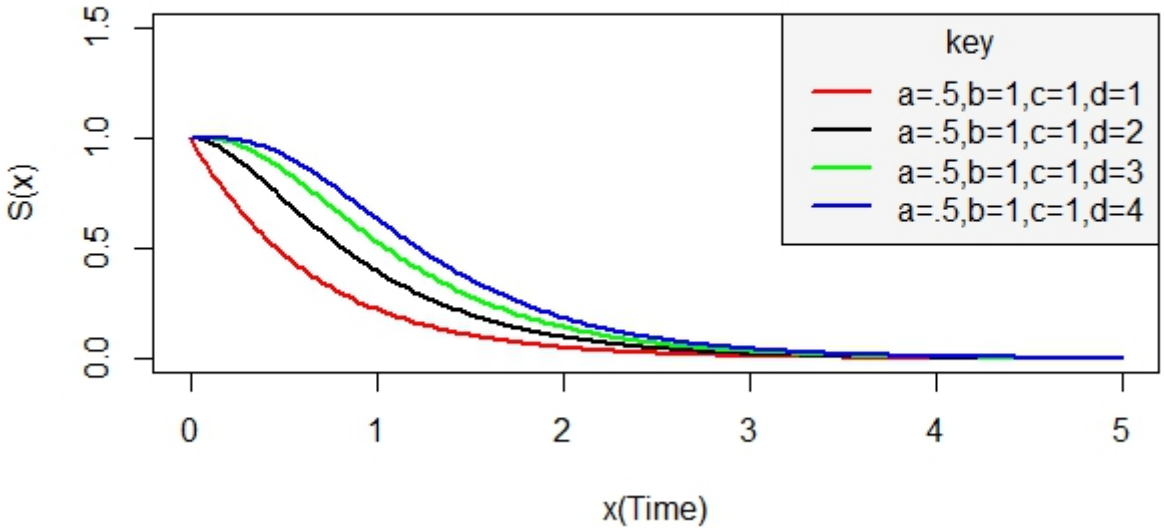


Figure 5.5.3: Survival function of EEW Distribution

where $a = \alpha, b = \beta, c = \lambda$ and $d = \gamma$.

APPENDIX D

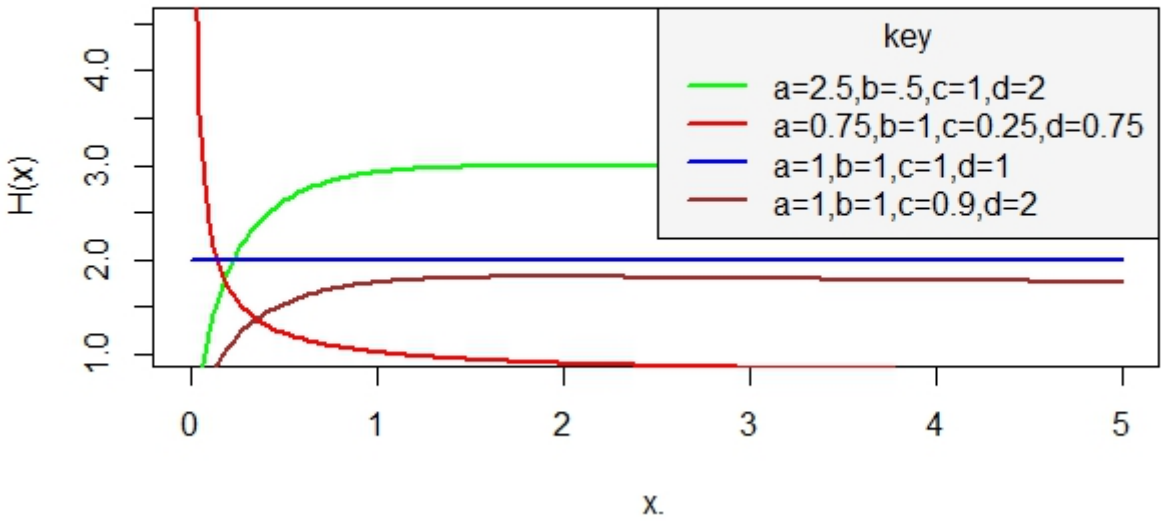


Figure 5.5.4: Hazard function of EEW Distribution

where $a = \alpha, b = \beta, c = \lambda$ and $d = \gamma$.

APPENDIX E

R-codes of the pdf of the Exponentiated-Exponential Weibull distribution

$p1 < -function(x, a, b, c, d)d * (1 - exp(-(a * x + b * x^c)))^(d - 1) * (a + b * c * x^(c - 1)) * exp(-(a *$

$c2 < -p1(x, a = 1, b = 0.5, c = 1, d = 1) > plot(x, c2, type = "n", xlab = "x(Time)", ylab =$

$"f(x)") > lines(x, p1(x, a = 0.5, b = 1, c = 1, d = 2), col = "red", lwd = 2, lty = 1)$

$> lines(x, p1(x, a = 0.5, b = 1, c = 1, d = 3), col = "black", lwd = 2, lty = 1)$

$> lines(x, p1(x, a = 0.5, b = 1, c = 1, d = 4), col = "green", lwd = 2, lty = 1)$

$> lines(x, p1(x, a = 0.5, b = 1, c = 1, d = 5), col = "blue", lwd = 2, lty = 1)$

$> legend("topright", cex = 1, title = "key",$

$c("a = 1, b = .5, c = 1, d = 2", "a = 1, b = .5, c = 1, d = 3", "a = 1, b = .5, c = 1, d = 4", "a =$

$1, b = .5, c = 1, d = 5"), horiz = F,$

$+ lty = c(1, 1, 1, 1), lwd = c(2, 2, 2, 2), col = c("red", "black", "green", "blue"), bg =$

$"grey96")$

R-codes of the CDF of the Exponentiated-Exponential Weibull distribution

$c7 < -function(x, a, b, c, d)$

$((1 - exp(-(a * x + b * x^c)))^d)$

$> c2 < -c7(x, a = 1, b = 0.5, c = 1, d = 1)$

$> plot(x, c2, type = "n", xlab =$

$"x(Time)", ylab = "F(x)")$

$> lines(x, c7(x, a = 0.5, b = 1, c = 1, d = 1), col = "red", lwd = 2, lty = 1)$

$> lines(x, c7(x, a = 0.5, b = 1, c = 1, d = 2), col = "black", lwd = 2, lty = 1)$

$> lines(x, c7(x, a = 0.5, b = 1, c = 1, d = 3), col = "green", lwd = 2, lty = 1)$

$> lines(x, c7(x, a = 0.5, b = 1, c = 1, d = 4), col = "blue", lwd = 2, lty = 1)$

$> legend("bottomright", cex = 1,$

$title = "key", c("a = .5, b = 1, c = 1, d = 1", "a = .5, b = 1, c = 1, d = 2", "a = .5, b = 1, c =$

$1, d = 3", "a = .5, b = 1, c = 1, d = 4"),$

$+ horiz = F, lty = c(1, 1, 1, 1), lwd = c(2, 2, 2, 2),$

$col = c(\text{"}red\text{"}, \text{"}black\text{"}, \text{"}green\text{"}, \text{"}blue\text{"}),$

$bg = \text{"}grey96\text{"})$

APPENDIX F

Datasets and R-codes of the Survival and Hazard function of the Exponentiated-Exponential Weibull distribution. $> x = c(0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.5$

$3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47,$

$14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62,$

$3.82, 5.32, 7.32, 10.06, 14.77, 32.152.64, 3.88, 5.32, 7.39, 10.34, 14.83,$

$34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69,$

$4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12,$

$46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87,$

$5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46,$

$4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02,$

$3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69)$

$> n = length(x)$

$> x = c(3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87,$

$1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28,$

$3.09, 1.87, 3.15, 4.90, 1.57, 4.90, 1.57, 2.67, 2.93,$

$3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85,$

$1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79,$

$4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43,$

$2.95, 2.97, 3.39, 2.96, 2.35, 2.25, 2.59, 2.03, 1.61,$

$2.12, 3.15, 1.08, 2.56, 1.80, 2.53)$

Survival function $c8 < -function(x, a, b, c, d)1 - ((1 - exp(-(a * x + b * x^c)))^d)$

$> c2 < -c8(x, a = 1, b = 0.5, c = 1, d = 1)$

$> plot(x, c2, type = "n", xlab = "x(Time)", ylab = "S(x)")$

$> lines(x, c8(x, a = 0.5, b = 1, c = 1, d = 1), col = "red", lwd = 2, lty = 1)$

$> lines(x, c8(x, a = 0.5, b = 1, c = 1, d = 2), col = "black", lwd = 2, lty = 1)$

$> lines(x, c8(x, a = 0.5, b = 1, c = 1, d = 3), col = "green", lwd = 2, lty = 1)$

$> lines(x, c8(x, a = 0.5, b = 1, c = 1, d = 4), col = "blue", lwd = 2, lty = 1)$

$> legend("topright", cex = 1, title = "key",$

$c(\text{"}a = .5, b = 1, c = 1, d = 1\text{"}, \text{"}a = .5, b = 1, c = 1, d = 2\text{"},$

$\text{"}a = .5, b = 1, c = 1, d = 3\text{"}, \text{"}a = .5, b = 1, c = 1, d = 4\text{"}), horiz = F, lty = c(1, 1, 1, 1), lwd =$

$c(2, 2, 2, 2),$

$col = c(\text{"}red\text{"}, \text{"}black\text{"}, \text{"}green\text{"}, \text{"}blue\text{"}), bg = \text{"}grey96\text{"})$ Hazard function plot $at <$

$-function(x, a, b, c, d)d * (1 - exp(-(a * x + b * x^c)))(d - 1) * (a + b * c * x(c - 1)) * exp(-(a * x + b$

$c1 < -at(x, a = 1, b = 1, c = .1, d = .5)$

$plot(x, c1, type = \text{"}n\text{"}, xlab = \text{"}x.\text{"}, ylab = \text{"}H(x)\text{"})$

$> lines(x, at(x, a = 1, b = 1, c = .25, d = .5), col = \text{"}red\text{"}, lwd = 2, lty = 1)$

$> lines(x, at(x, a = 1, b = 1, c = .25, d = .75), col = \text{"}blue\text{"}, lwd = 2, lty = 1)$

$> lines(x, at(x, a = 1, b = 1, c = .5, d = 1), col = \text{"}green\text{"}, lwd = 2, lty = 1)$

$> lines(x, at(x, a = 1, b = 1, c = .5, d = .95), col = \text{"}purple\text{"}, lwd = 2, lty = 1)$

$> lines(x, at(x, a = 1, b = 1, c = .5, d = 1.25), col = \text{"}black\text{"}, lwd = 2, lty = 1)$

$> lines(x, at(x, a = 1, b = 1, c = 1, d = 1.5), col = \text{"}magenta\text{"}, lwd = 2, lty = 1)$

$> legend(\text{"}topright\text{"}, cex = 1, title = \text{"}key\text{"},$

$c(\text{"}a = 1, b = 1, c = .25, d = .5\text{"}, \text{"}a = 1, b = 1, c = .25, d = .75\text{"}, \text{"}a = 1, b = 1, c = .5,$

$d = 1\text{"}, \text{"}a = 1, b = 1, c = .5, d = .95\text{"}, \text{"}a = 1, b = 1, c = .5, d = 1.25\text{"}), horiz = F,$

$lty = c(1, 1, 1, 1, 1), lwd = c(2, 2, 2, 2, 2),$

$col = c(\text{"}red\text{"}, \text{"}blue\text{"}, \text{"}green\text{"}, \text{"}purple\text{"}, \text{"}black\text{"}), bg = \text{"}grey96\text{"})$

$> EEWD < -function(p)n * log(p[1]) + n * log(p[2]) + n * log(p[3]) + n * log(p[4]) + (p[4] - 1) *$

$> a < -maxLik(EEWD, start = c(1, 1, 0.5, 0.95))$

$There were 50 or more warnings (use warnings() to see the first 50)$

$> summary(a)$

$- -$

$-Maximum Likelihood estimation\ Newton - Raphson maximisation, 18 iterations$

$Return code 2 : successive function$

$values within tolerance limit$

$Log - Likelihood : -783.2771$

$4 free parameters$

$Estimates :$

$Estimate Std.error t value Pr(>t)[1,]0.14187$

$0.0120911.739 < 2e-16 ***[2,]0.908010.363052.501$

$0.0124 *[3,]0.446240.100704.4319.37e-06 ***[4,]8.326393.906072.1320.0330*$

$> EWD <- function(p) n * log(p[3]) + n * log(p[1]) + n * log(p[2]) + (p[3] - 1) * sum(log(x)) - s$

$a <- maxLik(EWD, start = c(1, 1, 1))$

$There were 50 or more warnings (use warnings() to see the$

$first 50)$

$> summary(a)$

$--$

$Maximum Likelihood estimation$

$Newton-Raphson maximisation, 14 iterations$

$Return code 2: successive function values within tolerance$

$limit$

$Log-Likelihood: -828.4288$

$3 free parameters$

$Estimates:$

$Estimate Std.error t value Pr(>t)[1,]0.106773$

$0.00943811.314 < 2e-16 ***[2,]0.0938880.0190774.922$

$8.59e-07 ***[3,]1.047835$

$0.06756315.509 < 2e-16 *** --- Signif.codes: 0‘***’0.001‘**’0.01‘*’0.05‘.’0.1‘’1 -$

$--$

$> AIC(a)[1]1662.858$

KEWD $> KEWD <- function(p) n * log(p[4]) + log(p[5]) + sum(log(exp(-x^(p[3]) * (p[2]) - x *$

$> a <- maxLik(KEWD, start = c(0.05, 0.9, 0.025, 0.005, 0.025))$

$There were 50 or more warnings (use warnings() to see the$

$first 50)$

$> summary(a)----------$

$MaximumLikelihoodestimation$

$Newton - Raphsonmaximisation, 3iterations$

$Returncode3 : Laststepcouldnotfindavalueabovethe$

$current.$

$Boundaryofparameterspace?$

$Considerswitchingtoamorerobustoptimisationmethod$

$temporarily.$

$Log - Likelihood : -884.694$

$5freeparameters$

$Estimates :$

$EstimateStd.errortvaluePr(> t)[1,] - 0.010053$

$0.034035.6561.55e - 08 * * * [2,]0.891136$

$0.02224 - 1.1050.269[3,]0.024100$

$0.0030597.8773.34e - 15 * * * [4,]0.005000$

$0.00044211.311 < 2e - 16 * * * [5,]0.021816$

$0.0218161.0000.317 - - -$

$Signif.codes : 0` * * *' 0.001` * *' 0.01` *' 0.05`.' 0.1`' 1$

$- -$

$> AIC(a)[1]1779.388$