# A STUDY OF THEORY OF PARTIALLY ORDERED MULTI SETS

By

**Funmilola BALOGUN**

BSc (B.U.K 2003), MSc (A.B.U 2010)

PHD/SCI/ 45360/ 2012 - 2013

A THESIS SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES, AHMADU BELLO UNIVERSITY, ZARIA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF DOCTOR OF PHILOSOPHY IN MATHEMATICS

DEPARTMENT OF MATHEMATICS,

FACULTY OF PHYSICAL SCIENCES,

AHMADU BELLO UNIVERSITY,

ZARIA

AUGUST, 2017

## ABSTRACT

During the last three decades, researchers have made significant contributions to the theory of partially ordered sets (posets). In this thesis, partially ordered multisets (pomsets) are studied. These structures generalize sets, bags, list, trees, and other ordered types and therefore provide a uniform representation for them. A study of multiset orderings on the class of finite multisets defined over a set $S$ is presented. In order to introduce hierarchies between the points of a finite multiset $M$, a new partial multiset ordering, $\preccurlyeq\leq$, is defined via the ordering induced by the partially ordered base set. Properties of the structure $\mathcal{M} = (M, \preccurlyeq\leq)$, called a partially ordered multiset, are investigated. By restricting the ordering $\preccurlyeq\leq$ to submsets of $M$, substructures of $\mathcal{M}$ are constructed. Notions and results on chains and antichains are extended from set theoretical context to multisets. In particular, by exploiting set-based partitioning, a set of necessary and sufficient conditions is proved for characterizing the *width* and *height* of a partially ordered multiset, which are extensions of the classical theorem of Dilworth and its dual on partially ordered sets. Also, combinatorial parameters are studied on the ordered multiset structure using a partially ordered base set. The concepts of *linear extensions*, *realizers* and *dimension* of a partially ordered set are extended to the case where the ordered structure is a multiset. It is shown that if $m_i x_i, m_j x_j$ are incomparable points in $\mathcal{M}$ then, there exists a pomset $\mathcal{N}$ extending $\mathcal{M}$ such that $m_i x_i \preccurlyeq\leq_{\mathcal{N}} m_j x_j$. This result generalizes Szpilrajn's extension theorem for partially ordered sets. In the sequel, a heuristic algorithm for generating mset linear extensions is constructed and implemented on a randomly generated ordered multiset. It is also shown that the notion of realizers is well-defined on the ordered multiset structure and $\mathcal{M} = \bigcap \ell_i$, where each $\ell_i$ is an mset linear order on $\mathcal{M}$.

Finally, the concept of *dimension* for ordered multisets is defined using the ordering $\preccurlyeq\leq$. The relationship between the dimension of a partially ordered multiset and that of the underlying generic set is investigated. Among other results, it is shown that the dimension of an ordered multiset is monotonic and bounded in terms of the dimension of the base set. A multiset can be viewed as an extended notion of a set, hence, establishing these concepts and results on the new partially ordered multiset structure leads to generalizations in the theory of partially ordered sets.

# CHAPTER ONE

# INTRODUCTION

## 1.1 Background of the Study

Mathematical structures are the means to model the way certain mathematical objects are related to each other in a given system. Ordered structures are present throughout mathematics. In view of wide practical applications of orders, numerous special kinds of ordered structures have been developed, some of which have grown into mathematical fields of their own. The origin of partially ordered sets can be traced back to the work: *An Investigation of the Laws of Thought* of George Boole in 1854 on which are founded the mathematical theories of logic and probabilities. Georg Cantor in 1883 proposed that it was a valid law of thought that every set could be well ordered. However, most mathematicians at the time found it difficult to visualize this fundamental concept. In 1904, Ernst Zermelo proved the well ordering theorem (also known as Zermelo's theorem) using the axiom of choice. Today, it has come to occupy a central position in developing various other useful ordered structures in order to explicate complex application problems arising in probability theory, combinatorial theory of algorithms, and many algebraic theories. The well-ordering principle makes every set susceptible to the powerful technique of transfinite induction.

### 1.1.1 Partially ordered sets

Interest in finite partially ordered structures has been heightened in recent years by a steady stream of theorems combining brilliant ad hoc arguments with powerful techniques from other areas of Mathematics (Trotter, 1995).

Partial orders generalize the more familiar total orders. Most application problems are presented such that no relationship exists between some elements of a given system (*incomparable elements*), hence investigation via partial orders is a more practical approach in explicating real life application problems.

A *partially ordered set* (or *ordered set* or *poset*) is a structure $(P, \preccurlyeq)$ consisting of a set $P$ and a binary relation $\preccurlyeq$ contained in $P \times P$, called the *order* (or the *partial order*) on $P$ such that, $\preccurlyeq$ is *reflexive*, *antisymmetric* and *transitive*..

The antisymmetry of the order relation ensures that there are no two-way ties ($p \prec q$ and $q \prec p$ for distinct $p$ and $q$) in the relation. The transitivity (in conjunction with antisymmetry) ensures that no cyclic ties ($p_1 \prec p_2 \prec \cdots \prec p_n \prec p_1$ for distinct $p_1, p_2, \ldots, p_n$) exist.

A tool used in describing the covering relation between elements of an ordered set is the *order diagram* also known as the *Hasse diagram* (named after the German algebraic number theorist Helmut Hasse, who used diagrams to picture the ordered sets of subfields or field extensions). Order diagrams provide good visual tools for translating real life problems into Mathematics. Such diagrams can represent many structures from everyday life, like public transportation, supply chain, road maps, computer networks, water supply systems, etc. Order diagrams are not canonical, the same ordered set can be drawn in different ways according to the investigator's preferences or needs as shown in figure 1.1.

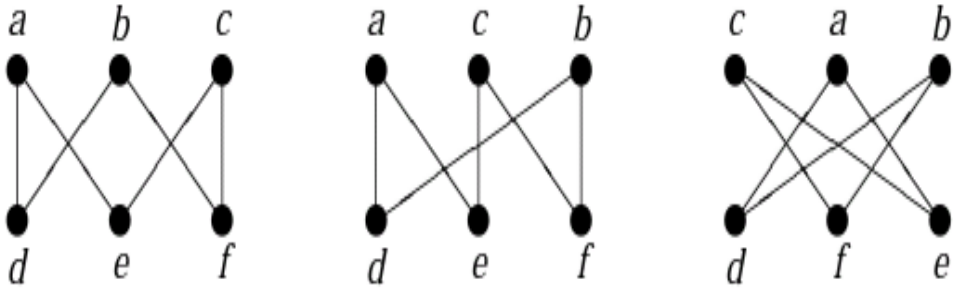



Figure 1.1 Three diagrams of the same ordered set (Schroder, 2003)

It is known that various characterizations of subsets of a finite set are the central tools to study combinatorial theory. Much of the combinatorial interest in ordered sets is inextricably linked to the combinatorial features of the diagrams associated with them. Aspects of combinatorics include counting the structures of a given kind and size, deciding when certain criteria can be met, constructing and analyzing objects that meet the criteria (as in combinatorial designs and matroid theory), finding largest, smallest, or optimal objects (extremal combinatorics and combinatorial optimization), and studying combinatorial structures that arise in an algebraic context, or applying algebraic techniques to combinatorial problems (algebraic combinatorics). Significant notions of combinatorics are frequently used in many areas of computer science such as scheduling, systems controls, data mining and networking (Rival, 1983; Faigle and Schrader, 1984). A basic problem of combinatorics is to determine the number of possible configurations (for example, graphs, designs, and arrays) of a given type. Though the rules specifying the configuration may be relatively simple, enumeration on the other hand may present formidable difficulties. Finding a good upper and lower bound for a given problem usually gives the best possible solution.

There has been rapid growth in research activity centered on combinatorial problems for partially ordered sets, an evidence of this is the introduction of the AMS subject classification 06A07: Combinatorics of Partially Ordered Sets (Trotter, 1996).

Any partially ordered set can be sorted into a linear order maintaining all preexistent relations. Linear orders correspond to our intuition of *sorted list* of elements.

A *linear extension* of a finite partially ordered set $P$ is an order-preserving bijection $\alpha: P \rightarrow \{1,2,\dots,|P|\}$ such that $x \prec y$ in $P$ implies $\alpha(x) \prec \alpha(y)$. The set of linear extensions of P can be viewed as a set of permutations of the elements of P. Many combinatorial objects can be represented by permutations subject to various restrictions. Following the fundamental results of Szpilrajn in *Sur l'extension de l'ordre partiel*, numerous works have appeared on linear extensions (also known as *topological sorts*) of partially ordered sets (Bouchitte and Habib, 1987; Bertet *et al*., 2003; Fernandez *et al*., 2013; Tella *et al*., 2014B). The concept of linear extensions of a partially ordered set has been employed in solving a wide class of application problems which includes the *classical matroid secretary* problem. This concept is also found useful in solving problems like the *scheduling problem* (or *jump number* problem) (Rival 1983; Faigle and Schrader, 1984; Ehrenborg *et al*., 2011) by providing an embedding of the partial order into a linear order. For optimization, different classes of linear extensions have been studied in the literature, this include, *greedy* linear extensions (Cogis and Habib, 1979; Rival and Zaguia, 1987; Syslo, 1995) and *super greedy* linear extensions (Kierstead *et al*., 1987; Kierstead and Trotter, 1989). This is achieved mainly by placing certain restrictions on the methods by which the linear extensions can be constructed. Figure 1.1 is an order diagram of an ordered set and some of its linear extensions. There has been a good deal of interest in the maximal sized antichains of a partially ordered set (Freese, 1974; Behrendt, 1988). Besides its theoretical

interest, the abstraction of maximal antichains of an ordered set has structural properties which gives it interesting practical behaviors. These substructures play a significant role in characterizing ordered sets.

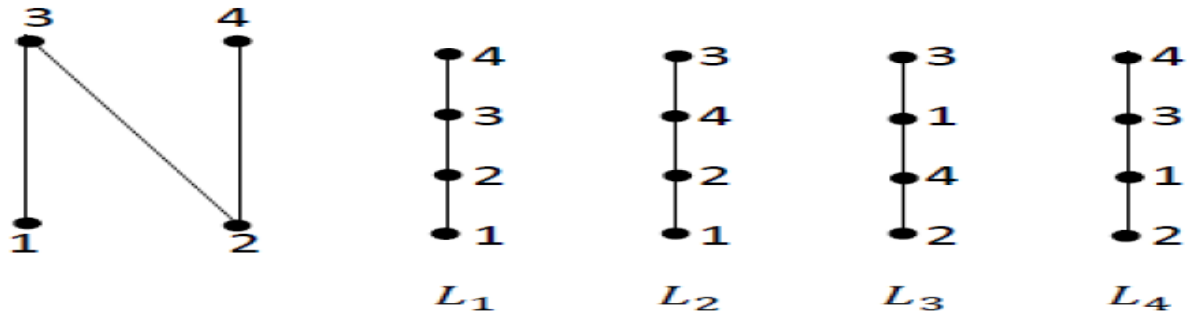


Figure 1.2 A partially ordered set and its linear extensions.

Various algorithms have been developed for generating the linear extensions of partially ordered sets (Bianco *et al*., 1997; Korsh and Lafollete, 2002; Felsner *et al*., 2003; Chen, 2006). Figure 1.2 is an order diagram of a partially ordered set and some of its linear extensions. Algorithmically, every linear extension $L = \{x_1 \prec x_2 \prec \cdots\}$ of a partially ordered set $P$ can be constructed through the following steps:

1. Choose a minimal element $x_1$ in $P$.
2. Given $x_1, x_2, \ldots, x_i$, choose a minimal element from $P\backslash\{x_1, \ldots, x_i\}$ and call this element $x_{i+1}$.

For any sequence of choices of the points $x_{i+1}$, the above algorithm produces a linear extension of the partially ordered set $P$. In his work, Szpilrajn (1930) also showed that *every ordered set P on a set S is the intersection of a set of linear orders on S*. A family of such linear orders is called a *realizer* of $P$. In figure 1.3, the linear orders on the right form a realizer for the order diagram on the left.

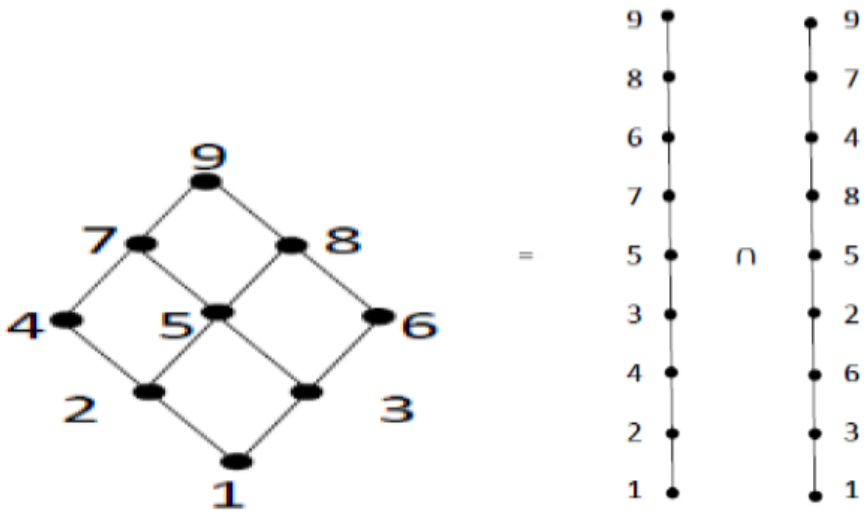



Figure 1.3 A partially ordered set and its realizer.

An interesting parameter of an ordered set is the minimum number of linear extensions needed to characterize it i.e., the *dimension* of an ordered set. Dushnik and Miller (1941) defined the dimension of an ordered set $P$ as the least positive integer $d$ for which $P$ has a realizer of size $d$. As observed by Trotter (2017), it is arguably true that dimension is the single most widely studied combinatorial parameter for ordered sets. Dimension for partially ordered sets plays a role which in many instances is analogous to chromatic number for graphs. A natural extension of this concept arises if the linear orders which form a realizer of $P$ are required to have certain additional properties. The dimension of a partially ordered set is a comparability invariant i.e., two partially ordered sets with the same comparability graph have the same dimension. Several characterizations are known for the dimension of an ordered set (Kelly and Trotter, 1982; Kierstead and Trotter, 1985; Trotter and Wang 2015; Joret et al., 2016A, 2016B).

The standard example $S_d$ for $d \geq 2$ is the height 2 partially ordered set consisting of $d$ minimal elements $\{a_1, a_2, \ldots, a_d\}$ and $d$ maximal elements $\{b_1, b_2, \ldots, b_d\}$ with $a_i < b_j$ in $S_d$ if and only if

$i \neq j$. It is well known that dimension of $S_d = d$ for all $d \geq 2$. Furthermore, for $d \geq 3$, $S_d$ is irreducible, i.e., the removal of any point leaves a subposet whose dimension is $d - 1$ (Trotter and Wang, 2015). The standard example plays a prominent role in many characterization problems for ordered sets. The presence of a large standard example as a subposet is enough to force the dimension to be large, but the converse need not be true. The standard example also shows that in general, a partially ordered set with large height does not necessarily have large dimension. Figure 1.4 is a diagram of the standard example $S_d$.

In recent years, researchers have shown renewed interest in combinatorial properties of ordered sets determined by geometric properties of its order diagram and topological properties of its cover graph (Trotter and Wang, 2015). The roots for these problems can be traced back to the 1970's (Trotter and Moore, 1977) and sometimes even earlier.

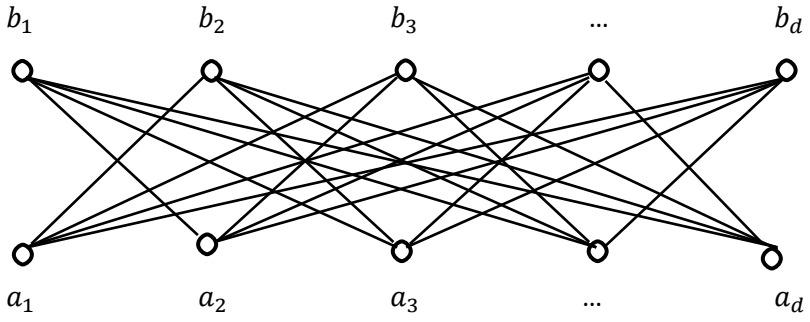



Figure 1.4 The standard example $S_d$ (Kelly and Trotter, 1982)

1.1.2 Partially ordered multisets

In a classical (or Cantorian) set, two mathematical objects are *either equal or different*. Howbeit, in real life applications mathematical objects are not necessarily distinct. For instance, repeated roots of a polynomial equation, repeated observations in a statistical sample and repeated

hydrogen atoms in a water molecule to mention a few. If repeated occurrences of any object is allowed in a set, then a mathematical structure known as a multiset (mset, for short) is obtained. This type of structure plays an important role in the area of applications. A multiset is considered to be a generalization of a set. Formally, a multiset $A$ over $S$ is a *cardinal-valued* function i.e., $A: S \to \mathbb{N} = \{0,1,2, \dots\}$, also called a *generalized characteristic* function, where $S$ is a set called the *ground* or *generic* set of the class of all multisets containing objects from $S$ (Singh and Isah, 2016). In view of various applications of multisets in areas such as logic, linguistics, computer science, just to mention a few, it became a part of research problem to extend results already obtained in set theoretical context to multisets. The problem of extending various mathematical notions and results related to ordered sets to multisets has attracted serious attention during the last couple of decades (Singh and Singh, 2003; Kilibarda and Jovovic, 2004; Conder and Slinko, 2007; Singh *et al*., 2012; Thanh, 2013). A relatively early contribution in this line of research is the work of Dershowitz and Manna (1979). For two multisets $M\ and\ N$ in $M(S)$ i.e., the class of all finite msets over a set $S$, the Dershowitz-Manna multiset ordering is presented as follows:

$M \ll N$ if there exist two multisets $X\ and\ Y$ in $M(S)$ such that,

i. $\emptyset \neq X \subseteq N$

ii. $M = (N \backslash X) \uplus Y$, and

iii. $X$ dominates $Y$, that is, for all $y \in Y$, there is some $x \in X$ such that $y < x$.

The Dershowitz-Manna ordering otherwise known as the *standard multiset ordering* seems to form the basis of all other multiset orderings. Several definitions of multiset orderings have been proposed building on this ordering, each embodying certain limitations and advantages over others (Huet and Oppen, 1980; Jouannaud and Lescanne 1982; Martin 1989). Well-founded

partial orderings on the set of all finite multisets are employed in proving program termination (Bachmair *et al.*, 1986; Dershowitz and Manna, 1979), also in the theory of partitions (Brandt, 1982).

The term *partially ordered multisets* (*pomsets*, for short) was proposed by Pratt (1984). A partially ordered multiset relation is a multiset relation that is *reflexive*, *antisymmetric*, and *transitive*. Intuitively, a partially ordered multiset can be viewed as a string in which the total order has been relaxed to be a partial order. As proposed by a number of researchers, partially ordered multisets can be used as mathematical representations of runs of a concurrent system (Pratt, 1986; Gischer, 1988).

### 1.2 Statement of the Problem

The theory of partially ordered structures is relatively well developed in the classical setting. Owing to applications of multisets in both hard and soft sciences, many researches have appeared on extending notions and results related to orders obtained on sets to multisets. This problem has attracted the attention of a large number of researchers during the last couple of decades (Dershowitz and Manna, 1979; Prat, 1986; Kilibarda and Jovovi, 2004; Girish and Sunil, 2009; Singh *et al*., 2012).

Nevertheless, a number of problems related to extending many applicable notions and results established on ordered sets to multisets, require further serious attention; for example, developing rich ordered multiset structures and identifying their characteristic properties, including their computational aspects for application, hence the need for a partially ordered multiset structure that is suitable for establishing these concepts.

## 1.3 Justification of the Study

The Cantorian set theory has provided a firm foundation for bivalent logic based mathematics in which all variables as well as relations between variables are crisp. However, this demand of the Cantorian set embodies certain limitations. Repetition of elements is significant in a multiset structure, hence a multiset can be viewed as an extended notion of the Cantorian set. The significance of allowing duplicates in a collection cannot be overemphasized. The concept of multiple-membership collections has found applications in various areas such as logic, linguistics, computer science, just to mention a few. Researchers have turned to partial orders in order to understand the semantics of parallel programs. Partially ordered structures have usages in ranking scenarios where certain pairs of elements are incomparable, such as ranking conference submissions, also, the combined use of duplicates with partial order is natural in practical data. Notably, partially ordered multisets have various applications in computer science ranging from database systems to distributed computing. Numerous advanced applications such as CAD/CAM, software engineering and text retrieval require such data types. Partially ordered multisets are also common in geographical information systems and linguistics.

## 1.4 Aim and Objectives of the Study

The aim of this research is to study the theory of partially ordered structures and develop an ordered multiset structure with the intent of extending existing notions and results particularly on certain combinatorial parameters studied for sets, to multisets.

In order to achieve this aim, our main objectives are to:

i. Develop a partially ordered multiset structure that admits hierarchy between the points of a multiset and establish properties satisfied by this structure.

ii. Construct substructures of the ordered multiset and investigate their properties.

iii. Extend combinatorial parameters of linear extensions, realizers, and dimension to multisets using the structure constructed in i above.

iv. Extend results on these parameters from the set theoretical context to multisets and obtain some new results that are peculiar to ordered multisets.

v. Investigate algorithms for constructing linear extensions of a partially ordered set and develop a heuristic algorithm for generating multiset linear extensions to be implemented on a partially ordered multiset structure.

**1.5 Methodology**

To develop the partially ordered multiset structure $\mathcal{M} = (M, \preccurlyeq\leq)$, the multiset $M \in M(S)$ i.e., the class of finite multisets over a set $S$, is defined over a partially ordered base set $P = (S, \preccurlyeq)$ such that the ordering on the base set induces an ordering on $M$.

Substructures of $\mathcal{M}$ are constructed by restricting the ordering $\preccurlyeq\leq$ to submsets of $M$. Properties of these substructures are investigated through the defined suborder. Set-based partitioning studied in Jouannaud and Lescanne (1982) is exploited in establishing conditions for characterizing the width and height of the ordered multiset and hence partitioning $\mathcal{M}$ into

minimum number of mset chains and mset antichains. Due to the well-ordering principle, the proofs follow using the method of transfinite induction.

Combinatorial parameters of mset linear extensions, mset realizers and dimension are established on the ordered multiset structure via the ordering induced by the partially ordered base set. Characterizations for these parameters are obtained by bounding them in terms of their corresponding classical parameters and other comparability invariants. Mset linear extensions are generated by sorting the ordered multiset into a linear order while maintaining all preexistent relations.

### 1.6 Organization of the Thesis

The thesis consists of six chapters organized as follows:

Besides chapter one, a comprehensive and up-to-date literature review of related works is presented in chapter two. In chapter three, fundamentals of partially ordered sets and multisets are discussed. A comprehensive review on multiset orderings is also presented in this chapter. In chapter four, a partially ordered multiset structure is constructed using a partially ordered base set and exploiting the ordering induced by the underlying generic set. Notions of comparability and incomparability are defined, also, properties satisfied by this ordered structure are presented. In the sequel, substructures are introduced, and some related results on ordered sets are established on the ordered multiset structure. In chapter five, combinatorial notions on partially ordered sets are extended to ordered multisets. A new heuristic algorithm and an exact algorithm for generating mset linear extensions are constructed and implemented on a randomly generated ordered multiset. In chapter six, summary, conclusions and recommendations for further study are presented.

# CHAPTER TWO

# LITERATURE REVIEW

## 2.1 Partially Ordered Sets and Combinatorial Theory

The theory of ordered sets lies at the confluence of several branches of Mathematics which includes in particular, set theory, lattice theory, combinatorial theory and some aspects of modern operational research. Over the last thirty years, researchers have made considerable efforts towards developing the theory of ordered sets. Partially ordered sets can be said to have

their origin in the work of George Boole (1854), *An investigation of the laws of thought*, while developing an axiomatic theory of propositional logic. In 1883, George Cantor, founder of set theory, envisaged that every set could be well ordered. This well ordering principle for which Cantor could not provide a proof, remained at the heart of the theory of ordinal and cardinal numbers. In 1904, Ernst Zermelo using the axiom of choice, provided a proof of the well ordering theorem.

In 1930, Szpilrajn established a fundamental result in ordered set theory: *Every ordering on a set can be extended to a linear ordering*. He proved that any order $\leq$ on an ordered set P has at least one linear extension. He also proved a stronger result: *Let* $\leq$ *be an order on P and let* $x, y$ *be elements of P such that* $x||y$ *i.e.,* $x$ *and* $y$ *are incomparable, then there exist two linear extensions* $\leq_1, \leq_2$ *of* $\leq$ *such that* $x \leq_1 y$ *and* $y \leq_2 x$.

Building on Szpilrajn's results, Dushnik and Miller (1941) showed that every partial order is the intersection of a collection of linear orders. They also defined the *dimension* of an ordered set P as the minimum cardinality of a realizer of P. Their results showed that there exist partially ordered sets of arbitrarily large dimensions.

Dilworth (1950) established that in a finite ordered set P, the minimum number of chains whose set union is P equals the maximum number of pairwise incomparable elements of P. This classical theorem of Dilworth has played a significant role in motivating research in ordered set theory. Like a number of other results in combinatorics, Dilworth's theorem is equivalent to *Hall's Marriage theorem*, *Konig-Egervary theorem*, *Menger's theorem* and the *maxflow-mincut theorem* for network flows. Dilworth's theorem has been extended and modified through deep and detailed investigations by a number of researchers.

Hiraguchi (1951) used Dilworth's theorem to prove that the dimension of an order never exceeds its width. This results showed that if P is a poset on $n$ points with $n \geq 4$, then $\dim(P) \leq \lfloor n/2 \rfloor$. An analogue of Dilworth's theorem in the infinite setting was presented by Perles (1963). However, the theorem does not extend so simply to ordered sets with infinite width. In this case the size of the largest antichain and the minimum number of chains needed to cover the partial order may be different from each other. In particular, for every infinite cardinal number $\kappa$ there is an infinite ordered set with width $\aleph_0$ whose partition into the fewest chains has $\kappa$ chains (Harzheim, 2005).

Hansson (1968), proved an analogue of Szpilrajn's theorem for ordering extensions of quasi orderings.

In 1971, Mirsky inspired by Dilworth's theorem, proved that for every partially ordered set, the height also equals the minimum number of antichains into which the set may be partitioned. For sets of order dimension 2, the two theorems coincide but for more general partial orders the theorems differ. Dilworth's theorem and Mirsky's theorem are also related through the theory of perfect graphs. Mirsky's theorem states that comparability graphs are perfect. Analogously, Dilworth's theorem states that every complement graph of a comparability graph is perfect. The perfect graph theorem of Lovasz (1972) states that the complements of perfect graphs are always perfect, and can be used to deduce Dilworth's theorem from Mirsky's theorem and vice versa (Golumbic 1980).

Fishburn (1973) showed that every quasi-ordering has an ordering extension. While Hansson's proof proceeds directly without making use of Szpilrajn's extension theorem, Fishburn's proof utilizes Szpilrajn's results (Hansson, 1968).

Dynamic programming has been successfully used in the past to solve a large class of precedence constrained sequencing problems. The main limitations of this method has been the amount of storage required, which is directly related to the number of ideals in the precedence structure. Knuth (1973) presented a fast algorithm for computing *linear extensions* (or *topological sorts*) of an ordered set together with its implementation.

Maximum antichains also called *maximum Sperner families*, named after Sperner who first characterized them, in the case where P is a finite Boolean lattice. Sperner showed that the maximum size of an antichain containing n elements is

$$W_{max(n)} = \binom{n}{\lfloor n / 2 \rfloor}.$$

Freese (1974) showed that under the natural ordering, the maximal sized antichains of an ordered set form a distributive lattice, a theorem attributed to Dilworth on the decomposition of partially ordered sets.

In 1976, Suzumura in his work: *Remarks on the theory of collective choice* extended Hansson's theorem. He showed that consistency was a necessary and sufficient condition for a binary relation to have an ordering extension.

The notion of *greedy linear extensions* of an ordered set was introduced in Cogis and Habib (1979). It was shown that every greedy linear extension is optimal for the jump number on the class of series-parallel partially ordered sets.

In line with this, Bouchitte *et al.* (1985) introduced a new concept of dimension for partially ordered sets; the *greedy dimension*. Several inequalities bounding the greedy dimension as a function of other parameters of the ordered set were developed in Kierstead and Trotter (1985).

The authors showed that the greedy dimension of P does not exceed the width of P and if A is an antichain in P with $|P - A| \geq 2$ then the greedy dimension of P does not exceed $|P - A|$. Consequently, the greedy dimension of P does not exceed $|P|/2$ when $|P| \geq 4$. If the width of $P - A$ is $n$ and $n \geq 2$, then the greedy dimension of P does not exceed $n^2 + n$. If A is the set of minimal elements of P, then this inequality can be strengthened to $2n - 1$ and if A is the set of maximal elements, then the inequality can be further strengthened to $n + 1$.

Building on this, Pretzel (1985) introduced the notion of *super greedy linear extensions* at a meeting on ordered sets in Oberwolfach. This author proposed the definition of a super greedy linear extension (with a suggestion that it should be called an *imperial* linear extension in a lighter vein). Pretzel asked whether Kierstead and Trotter's work on greedy dimension could be extended to super greedy dimension, Kierstead *et al.*, (1987) in their work: *Representing an ordered set as the intersection of super greedy linear extensions*, answered this question. These authors studied super greedy linear extensions of a partially ordered sets P, super greedy realizers of P and the super greedy dimension of P. They derived the best possible upper bounds for the super greedy dimension of P in terms of $|P - A|$ and width$(P - A)$, where A is a maximal antichain.

Following the pioneering work of Kierstead; Bouchitte and Habib (1987) presented some complexity results on the construction of depth-first greedy linear extension. They proved that the recognition of Dilworth partially ordered sets of height 2 is $NP$-complete.

A finite ordered set is planar when it can be represented in the plane with the point $a$ lower than $b$ in the plane whenever $a < b$ in the ordered set, and with a straight-line segment from $a$ to $b$ whenever $a$ is covered by $b$. Kelly (1981) proved that it is equivalent to allow instead an arc

from $a$ to $b$ provided the arc does not go below the level of $a$ or above the level of $b$ (whenever $a$ is covered by $b$).

Caro *et al*. (1988) showed that every partially ordered set P on at least $\big(1 + 0(1)\big)n^3$ elements can be decomposed into subposets of size n that are *almost* chains or antichains. They showed that this lower bound on P is asymptotically best possible and presented similar results for other types of combinatorial structures.

Fishburn and Trotter (1992) considered the problem of characterizing which partially ordered sets on $n$ points with $k$ pairs in their ordering relations have the greatest number of linear extensions. Semiorders were found to be the ordered sets that maximize the number of linear extensions for each fixed $(n, k), 0 \leq k \leq \binom{n}{2}$. Except for special cases, it appears difficult to say precisely which semiorders solve the problem. A complete solution for $k \leq n$ and a near complete solution for $k = n + 1$ were given.

In Syslo (1995), *strongly* and *semi-strongly greedy chains* were introduced and their role in solving the jump number problem was presented. It was shown that every partially ordered set has an optimal linear extension which consist of strongly and semi-strongly greedy chains. Algorithmic aspect of finding linear extensions of this type was considered in Syslo (1987), where they provided a very efficient method for solving the jump number problem which is polynomial in the class of ordered sets whose arc representations contain a bounded number of dummy arcs.

Donaldson and Weymark (1998) extended the results of Dushnik and Miller (1941) to quasi orderings, their proof is based on the results of Suzumura, which showed that transitivity can be

weakened in Hasson's extension theorem. They proved that any quasi ordering is the intersection of orderings.

Bertet *et al.*, (2003) described a digraph representing all the weak-order extensions of a partially ordered set and algorithms for generating them. They also presented a digraph representing all of the minimal weak-order extensions of a partially ordered set with algorithms for obtaining them. It was shown that the number of weak-order extensions of an ordered set is a comparability invariant, whereas the number of minimal weak-order extensions of an ordered set is not a comparability invariant.

In Guenver and Rampon (2004), finite orders whose comparability graphs are split graphs are studied. A normalized representation of these orders was introduced using a family of open real intervals. Its repercussions are both structural and algorithmic. The authors showed that the dimension of split orders is at most 3 and a simple characterization was given for those of dimension 2. A linear time recognition algorithm for directed graphs whose transitive closures are split orders was presented.

Chen (2006) considered the problem of membership query for a given partially ordered set and devised a greedy algorithm which can produce *near-optimal* search strategies. The algorithm presented was shown to have fewer comparisons than the best known solution by at least a factor of 0.27 under random graph model as presented in Linial and Saks, (1985), and Carmo et al., (2004).

In Kloch (2007), the online dimension of upgrowing partial orders was investigated. The problem was treated as a *two-person game*. One person builds an order, one point at a time; the other person maintains its online realizer. The value of the game (i.e., the number of linear

extensions used) is compared with the offline width of the poset. It was shown that the online dimension of width 2 upgrowing order is 2.

In Piljs and Potharst (2011), a new proof for Dilworth's theorem was given. An algorithm for finding both the path cover and the antichain as considered in the theorem was presented.

In a second paper Piljs and Potharst (2013), a proof based on the minflow/maxcut property in flow networks was presented. In relation to the proof, a new method for finding both a Dilworth decomposition and a maximal antichain was presented.

Usually, dimension of an ordered set should be an integer valued parameter. Felsner and Trotter (2005) introduced a refined version of dimension for graphs which can assume a value $[t-1 \updownarrow t]$, thought to be between $t-1$ and $t$. They established that a graph is outer planar if and only if its dimension is at most $[2 \updownarrow 3]$. They also proved that the largest $n$ for which the dimension of the complete graph $k_n$ is at most $[t-1 \updownarrow t]$ is the number of antichains in the lattice of all subsets of a set of size $t$.

Fernandez *et al*., (2013) studied the problem of constructing sets of partially ordered sets that generate a given set of linear orders. This study reverses the problem of generating linear extensions of a given partially ordered set. Such a problem finds applications in computational neuroscience, systems biology, paleontology, and physical plant engineering. Two algorithms are presented for efficiently finding a single partially ordered set, if such an ordered set exits, whose linear extensions are exactly the same as the input set of linear orders. A variation of the problem where a minimum set of partially ordered sets that cover the input was also explored.

In Tella *et al*., (2014B), some of the known techniques and principal results for computing and counting topological sorts, realizers and dimension of a finite partially ordered set are outlined.

These authors observed that, removing one point from a partially ordered set does not increase any of the parameters of width, height, jump number and dimension. However, it can decrease each of them by at most one.

There has been a resurgence of interest in combinatorial properties of a partially ordered set that are associated with graph properties of its cover graph and order diagram. The following two theorems appearing in Trotter and Moore (1977) have played important roles in motivating this more modern research viz., (1) The dimension of a partially ordered set is at most 3 when its cover graph is a tree; (2) The dimension of a partially ordered set is at most 3 when the ordered set has a zero and its order diagram is planar. Although the underlying ideas lay dormant for more than thirty years, the first of these two results has become the base case for recent results bounding the dimension of a partially ordered set in terms of (a) the tree-width of its cover graph, and (b) the maximum dimension of its blocks. The second result is the base case for bounding the dimension of a planar ordered set in terms of the number of minimal elements.

Streib and Trotter (2014) showed that for each integer $h \geq 2$, there exists a least positive integer $c_h$ so that if $P$ is a partially ordered set having a planar cover graph and the height of $P$ is $h$, then the dimension of $P$ is at most $c_h$.

Felsner *et al*., (2015) showed that if $P$ is a partially ordered set with a planar comparability graph, then the dimension of $P$ is at most 4. Also, the dimension of $P$ is at most 4 if $P$ has an outerplanar cover graph. Finally, if $P$ has an outerplanar cover graph and the height of $P$ is 2, then the dimension of $P$ is at most 3. These three inequalities are best possible.

Trotter and Wang (2015) proved that the dimension of a planar poset with $t$ minimal element is at most $2t + 1$. They showed that this bound is tight for $t = 1$ and $t = 2$. For $t \geq 3$, they showed that there exist planar ordered sets with $t$ minimal elements having dimension $t + 3$.

Joret *et al*. (2016) investigated combinatorial problems involving finite graphs and partially ordered sets. They proved that the dimension of a finite partially ordered set is bounded in terms of its height and the tree-width of its cover graph. Cover graphs are useful in providing diagrams of partially ordered sets but they do not tell much about the combinatorial properties of the ordered sets associated with them, as ordered sets with the same cover graphs can vary in height, width and number of linear extensions among other parameters.

**2.2 Partially Ordered Multisets**

Multisets arise quite naturally in certain areas of mathematics, computer science, physics, and philosophy. Blizard (1989) noted that multisets can be traced back to the very origins of the concept of number. The application goes back to ancient times when numbers were represented as unordered collections of tally marks. However, they were first mentioned formally by Dedekind in 1901. As noted in Singh and Singh (2003), "many mathematicians have said that it is unfortunate for the development of mathematics that such a common concept, having so many applications in its disguised form (functions, permutations of sequences, collections of equivalence classes, families of sets, etc.), escaped our attention for such a long time before gaining some systematic treatment". According to Knuth (1969), the term multisets was first suggested by N. G. De Bruijn in a private communication to him. A deep survey and axiomatic introduction of multisets is presented in the work of Blizard (1989). Multisets have been used for

proving termination properties and in some search and sort algorithms, automata theory and software verification among others (Knuth, 1973; Dershowitz and Manna, 1979; Piskac and Kuncak, 2008).

Multiset orderings can be traced back to the work of Muirhead (1903) titled, 'some methods applicable to identities and inequalities of symmetric algebraic functions of n letters'. *Dominance multiset ordering* was used in a result, otherwise called *unusual application of multiset orderings*. An application of this result is its use in reaffirming that the arithmetic mean of any number of positive real numbers is greater than or equal to their geometric mean. Alternative formulations of the dominance ordering, particularly when the ground set is totally ordered, abound in mathematics. Floyd (1967) suggested that well-founded sets can be employed in proving that programs terminate. Knuth (1973) applied the dominance ordering to demonstrate termination of a sorting algorithm.

Dershowitz and Manna (1979) constructed a multiset ordering otherwise known as the *standard multiset ordering*. Using multisets defined over a *well-founded* set $S$, it was shown that the ordering on $S$ induces an ordering on the finite multisets over $S$. The defined multiset ordering was shown to be well-founded and enabled the use of relatively simple and intuitive termination functions in otherwise difficult termination proofs. The Dershowitz-Manna multiset ordering is used to prove the termination of production systems, programs defined in terms of sets of rewriting rules. Dershowitz and Manna also defined another multiset ordering known as *reduction order*, this ordering was used to establish the transitivity property of the standard multiset ordering.

Another application of dominance ordering is found in Macdonald (1979). Here, the dominance ordering was used to study symmetric functions. DeConcini *et al*., (1980) extended dominance

ordering to an ordering on tableaux, which are combinatorial objects related to partitions, which was then used to prove a straightening law.

A more tractable multiset ordering was studied in Huet and Oppen (1980). This multiset ordering is also used in the proof of program termination and term rewriting systems. The Huet-Oppen multiset ordering is equivalent to the Dershowitz-Manna ordering, this was shown in Jouannaud and Lescanne (1982).

Baclawski (1981) defined several orders of lexicographic nature. This author studied the concept of a straightening law in commutative algebra i.e., the method of expressing elements of the algebra in terms of other elements which precede it in some ordering.

In Jouannaud and Lescanne (1982), two well-founded orderings on multisets which extend strictly Dershowitz and Manna ordering were proposed. The authors established an equivalence between the Huet-Oppen and Dershowitz-Manna multiset orderings. They also presented an efficient implementation of Dershowitz-Manna ordering using their proposed orderings. However, their orderings do not verify the monotonicity property exhibited by the Dershowitz-Manna ordering, and hence a new definition of the Dershowitz-Manna ordering based on this characterization was given. They concluded that the Dershowitz-Manna ordering is the strongest existing monotonic ordering on multisets.

Dershowitz (1982) showed how multisets can be used to construct orderings in term algebras leading to proofs of the termination of term rewriting systems.

Several ways of extending a partial order on a set $S$ to a finite multiset defined over $S$ were discussed in Krom (1985). He showed among other results that the extension is well-founded if the base set is well-founded. Two multiset orderings were defined together with their respective

transitive closures. These orderings are referred to as *bounded* and *unbounded* multiset orderings, where the bounded multiset ordering may be translated into a proof using natural numbers.

Clements (1988) extended the generalized Macaulay theorem slightly to a form convenient for dealing with $k$-families of subsets of a multiset and presented several applications.

Partially ordered multisets constitute one of the most basic models of concurrency. Process algebras and system models like Petri nets have been given ordered multiset semantics for many years and several partially ordered multiset algebras have been designed.

In Gischer (1988), partially ordered multisets were studied as a generalization of strings and their algebraic properties were also investigated. In particular the axiomatic properties of partially ordered multisets, sets of partially ordered multisets and ideals of partially ordered multisets, under operations such as concatenation, parallel composition, union and their associated closure operations were investigated. It was shown that the equational theory of sets, partially ordered multisets under concatenation, parallel composition and union is finitely axiomatizable. Similar result was obtained for ideals of partially ordered multisets which incorporated the notion of subsumption which is also known as augmentation.

Martin (1989) presented a geometric approach to multiset orderings (a generalization of Dershowitz-Manna ordering using matrix action). Martin exploited the notion of cone in $\mathbb{R}^n$ and provided a systematization of the construction and classification of various well-founded partial orderings underlying in proofs of program termination, term rewriting systems and computer algebra. The author showed that tame orderings (i.e., an ordering on a multiset that preserves both multiset union and an ordering on the ground set) can be classified geometrically. It also shown that this approach can be used to give a unified picture of different multiset orderings.

Rensink (1996) investigated a particular class of partially ordered multisets called *order-deterministic*, properly including all ordered sets which satisfied a number of interesting properties; among other things, it forms a distributive lattice under partially ordered multiset prefix, and it constitutes a *reflexive subcategory* of the categories of all ordered multisets. An algebra with a sound and $(\omega -)$ complete equational theory was developed for the order-deterministic partially ordered multisets. The operators in the algebra are *concatenation* and *join*, the latter being a variation on the more usual disjoint union of ordered multisets.

Baader and Nipkow (1998) defined a multiset extension ordering used in the proof of term rewriting system. Their ordering is expressed in terms of the multiset difference of the multisets. From the computational point of view it is envisaged that this definition is advantageous over the Dershowitz-Manna definition in the search process. The equivalence of the two definitions is true only when the underlying base set is strictly ordered.

Latapy and Phan (1999) used the dominance ordering to construct the lattice of integer partition. This proved to be very useful in getting an interesting recursive formula for the number of partitions of an integer where no closed formula was known.

Languages for manipulating partially ordered structures with duplicates were investigated in Grumbach and Milo (1999). These authors considered the partially ordered multiset data type as a general framework. Based on this, they introduced an algebra for ordered multisets which generalizes traditional algebras for (nested) sets, bags and lists. Their results showed that the use of partially ordered multisets increased the expressive power significantly. The algebra introduced constitutes a rather robust class of relational queries.

Singh and Singh (2003) outlined some insights into the concept of multisets together with a couple of combinatorial results related to multisets. These authors provided an efficient formula for determining the cardinality of the set of all multisubsets of a finite multiset $[x, y, z, \dots]_{m,p,t,\dots}$ in which $x$ occurs at most $m$ times, $y$ occurs at most $p$ times, and so on.

One of the main applications for a multiset ordering constraint is to break symmetry, especially row and column symmetry. Existing techniques for dealing with such symmetries typically eliminate only some of the symmetry, as such symmetries are very difficult to deal with since there are exponentially large number of permutations of the rows or columns to consider. Kiziltan (2004) demonstrates how multisets can be used to eliminate symmetries such that the size of the search space and consequently, time required to solve problems is considerably reduced.

In Kilibarda and Jovovic (2004), the problem of enumeration of $m$-antichains of $k$-bounded multisets on an $n$-set is considered, and a formula for calculating the cardinality of the corresponding family was obtained. These authors also considered a more general case of multiantichains and illustrates the corresponding explicit formulas for the case when $1 \leq m \leq 4$, $k \geq 1$ and $n \geq 0$.

The transitive property of the standard multiset ordering defined in Dershowitz and Manna (1979) was reaffirmed in Coupet-Grimal and Delobel (2006) using the multiset reduction order. They described the standard multiset order as its transitive closure. These results are instrumental in proving the well-foundedness of multiset path ordering (MPO).

Wehrman (2006) defined a multiset ordering (an abstraction from Dershowitz and Manna (1979)) and used it to prove a theorem of Higman (1952).

In Singh *et al.*, (2007), the authors gave a systemization of representation of multisets and basic operations under multisets. Various applications of multisets in mathematics, computer science and related areas were also described.

Conder *et al*., (2007) studied additive representability of orders on multisets which satisfy the condition of independence of equal submultisets (IES). A geometric view of these orders was considered in relation to certain combinatorial objects which they called discrete cones. Functions $f(n,k)$ and $g(n,k)$ which measure the maximal possible deviation of an arbitrary order satisfying the IES and an arbitrary almost representable order satisfying the IES, respectively from a representable order were defined. Among other results, these authors prove that $g(n,k) \leq f(n,k) \leq n$ and established that for small $n$ and $k$, the functions $g(n,k)$ and $f(n,k)$ coincide.

Ben-Amram and Codish (2008) introduced the following orders over a totally ordered base set; maximum (max), minimum (min), multiset (ms) and dual multiset (dms) orders. The defined orders are useful in termination-arguments that can be proved with size-change graphs.

Williams (2009) showed that multiset permutations can be ordered so that each permutation is a prefix shift of the previous permutation. Previous results were for permutations of sets, and the permutations of multisets whose corresponding set contains only two elements. It was also shown that multiset permutations can be generated by a loop less algorithm that uses a constant number of additional variable, where the best known loop less algorithm used a linear number of additional variables.

In Singh and Singh (2009), an alternative proof of the well-foundedness of the nested multiset ordering was introduced. It was shown that the set $M^*(S)$ of nested multisets over a given base

set S forms a cumulative type structure. Also, by exploiting the notion of sets bounded in rank, $(M^*(S), \gg^*)$ was shown to be well-founded if and only if $(S, >)$ was well-founded.

Girish and Sunil (2009) besides defining multiset relations and discussing some of their basic properties, introduced a number of concepts on multiset relations. Certain structures were also proposed and some examples were outlined.

In Singh *et al.,* (2012), an application of topological sorting to obtain linear extensions of a partially ordered class of finite multisets was described. It was shown that this ordered class of finite multisets is the intersection of its linear extensions, and any two incomparable elements of the partially ordered class of finite multisets are comparable under two of its linear extensions.

Thanh (2013) constructed algorithms for generating all iterative permutations, multi-subsets and iterative combinations of a multiset. These algorithms were parallelized using a method based on output decomposition and used to solve an optimal problem of work arrangement and an extended knapsack one.

A comprehensive study of multiset orderings, nested multiset orderings and multiset path orderings was presented in Singh and Shuaibu (2013). In particular, they illustrated how multiset path orderings admit the use of relatively simple and intuitive termination functions that lead to termination of a class of term rewriting systems.

In Ghilean *et al.,* (2014), the authors studied similarities and differences in the concepts of equivalence and order relations for sets and multisets. They also introduced the notion of an algebra on multisets and proved that a necessary but not sufficient condition for two algebras on a multiset to be isomorphic is that their root algebras are isomorphic.

Tella *et al*., (2014A) presented a comparative study of multiset orderings and the efficiency of implementations. They observed that all efforts made by various researchers particularly in the wake of resolving difficulties arising while addressing the termination issues in computer science seem to be oriented toward generalizing the notion of multiset orderings on a class of finite multisets by taking the generic set finite and partially ordered.

In recent times, the concept of multiset has become well established both in mathematics and computer science. Singh and Isah (2016) gave a modified and extended version of the extant theory of multisets by developing and applying the concept of sorts which is a unified approach.

In the last three decades, multisets have taken center place in applications, therefore it has become imperative for researchers to extend known concepts and results developed in set theoretical context to multisets.

# CHAPTER THREE

# FUNDAMENTALS OF ORDERED SETS, MULTISETS AND MULTISET ORDERINGS

In this chapter, fundamental concepts and results on classical ordered sets are presented. These structures are extensively studied in the research monograph (Trotter, 1992) and survey article (Trotter, 1995) (see also Harzheim, 2005 and Harju, 2006). Basic notions and notations of multisets and multiset orderings as presented in Blizard (1989), Singh *et al*. (2007), Singh and Isah (2016), Tella *et al*., (2014A) are explicated.

## 3.1 Notions and Notations of Partially Ordered Sets

Order theory captures the intuition of ranking that arise from instances where each element in a set can be compared to any other element and cases where some elements of a given set are incomparable. Essentially, in order to develop an ordered structure, the notion of relation plays a fundamental role. A binary relation R on a set $X$ is comprehended as $R \subseteq X \times X$.

A relation R is called:

i. A *quasi-order* (or *pre order*) if it is reflexive and transitive
ii. A *partial order* if it is reflexive, transitive and antisymmetric
iii. A *linear order* (or *total order*), if it is a partial order and for any two elements $x, y \in X$, either $(x, y) \in R \; or \; (y, x) \in R$.

**Definition 3.1.1**

For two relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, their *composition* is the relation $R \circ S \subseteq X \times Z$ defined as follows:

$$R \circ S = \{(x, z) | \exists y \in Y : (x, y) \in R \; and \; (y, z) \in S\}$$

**Definition 3.1.2**

A *partially ordered set* (we use the now standard term *poset* henceforth) is a pair $(X, \preccurlyeq)$ where $X$ is a set, and $\preccurlyeq$ is a *reflexive*, *antisymmetric*, and *transitive* binary relation on $X$. The set $X$ is called the *ground set* and $\preccurlyeq$ is called a partial order.

### 3.1.1 Notations

It is usually more convenient to use a single symbol such as $P$ to denote a poset $(X, \preccurlyeq)$. The notations $(x, y) \in P$, $xPy$, $x \preccurlyeq y$ in $P$, and $y \succcurlyeq x$ in $P$ are used interchangeably. For a set $X$ and $A \subseteq X$, $x \in P$ and $A \subseteq P$ is usually written instead of $x \in X$ and $A \subseteq X$, respectively. The order of the poset $P$ is denoted by $\preccurlyeq_p$ when several partial orders are being considered on the same ground set. The notation $x \prec y$ means $x \preccurlyeq y$ and $x \neq y$. Distinct points $x, y$ are *comparable*, denoted $x \bowtie y$, when either $x \prec y$ or $y \prec x$ in $P$, otherwise, they are *incomparable*, denoted $x||y$. The relation $\bowtie$ is reflexive and symmetric, but need not be transitive. For instance, consider the three element poset where $x \preccurlyeq y$ and $x \preccurlyeq z$, but $y||z$. A poset is *connected* if for each pair of elements $x, y \in P$ there is a sequence $x = z_0 \bowtie z_1 \bowtie z_2 \bowtie \cdots \bowtie y$ of comparable elements. An element $x \in P$ is *covered* by $y \in P$ written $x \prec: y$, if $[x, y] = \{x, y\}$, that is, $x \neq y, x \preccurlyeq y$ and there is no element $z \neq x, y$ such that $x \prec z \prec y$. Many concepts and results concerning ordered sets come in dual pairs, where one version is obtained from the other by replacing $\preccurlyeq$ by $\succcurlyeq$ throughout. The dual of a poset $P$ is usually denoted by $P^d$.

**Definition 3.1.3**

A poset $P$ is *finite* if the ground set is finite. Let $[x, y]_P = \{z | x \preccurlyeq z \text{ and } z \preccurlyeq y\}$ be the interval of the elements $x, y \in X$. If $x \not\preccurlyeq y$ then, the interval is empty. A poset is said to be *locally finite* if every interval $[x, y]_P$ is finite.

Remark 3.1.1

For this work, we will be concerned primarily with finite posets, although there are cases where related results for infinite posets are investigated.

**Definition 3.1.4**

Let $\prec$ be a binary relation on a set $X$. Then the *transitive closure* $\prec^t$ of $\prec$ is defined by $a \prec^t b$ if and only if there is a sequence $a = a_1, a_2, \ldots, a_n = b$ such that $a_1 \prec a_2 \prec \cdots \prec a_n$.

The transitive closure of a binary relation $\prec$ on a finite set $X$ is transitive.

3.1.2 Minimal (Maximal) and minimum (maximum) elements

Let $P$ be a poset and $A \subseteq P$ a nonempty subset.

An element $x \in A$ is *minimal* in $A$ if for all $y \in A$, $y \preccurlyeq x$ implies $y = x$. If $A$ has a unique minimal element then it is called the *minimum* element of $A$. If the full poset has a minimum element, it is called the bottom element of $P$ and is often denoted by $0_P$.

An element $x \in A$ is *maximal* in $A$ if for all $y \in A$, $x \preccurlyeq y$ implies $y = x$. If $A$ has a unique maximal element then it is called the maximum element of $A$. Also, if the full poset $P$ has a *maximum* element, it is called the top element of $P$, and it is often denoted by $1_P$.

**Definition 3.1.6**

A poset $P$ is said to be *well-ordered* if every nonempty subset $X \subseteq P$ has a minimum element. i.e., an element $x \in X$ such that $x \prec y$ for all $y \in X \backslash \{x\}$.

**Theorem 3.1.1**

Let $P$ be well-ordered, then each element $x \in P$ is either maximum or there exists a unique $y \in P$ such that $x \prec y$.

### 3.1.3 Order filter and order ideal

For a subset $A \subseteq P$, the sets

$\uparrow A = \{y \in P | \exists x \in A: x \preccurlyeq y\}$ and

$\downarrow A = \{y \in P | \exists x \in A: y \preccurlyeq x\}$

are the *up-set* (or the *order filter*) and the *down-set* (or *order ideal*) respectively, of $P$ generated by $A$. If $A = \{x\}$, then we write $\uparrow x \; and \; \downarrow x$. These are the principal up-set and the principal down-set of $P$ generated by the element $x$.

### 3.1.4 Bounds

An element $x \in P$ is a *lower bound* of $A$ if and only if $x \preccurlyeq y$ for all $y \in A$. Let $A^l = \{x \in P | x \preccurlyeq y \; for \; all \; y \in A\}$ be the set of all lower bounds of $A$. Then $x \in A^l$ is a *greatest lower bound* (or an *infimum*) of $A$, if $y \preccurlyeq x$ for all lower bounds $y \in A^l$. It will be denoted by $\bigwedge A$ if it exists.

An element $x \in P$ is an *upper bound* of $A$ if and only if $x \succcurlyeq y$ for all $y \in A$. Let $A^u = \{x \in P | y \preccurlyeq x \; for \; all \; y \in A\}$ be the set of all upper bounds of $A$. An element $x \in A^u$ is a *least upper bound* (or a *supremum*) of $A$, if $y \succcurlyeq x$ for all upper bounds $y \in A^u$. It will be denoted by $\bigvee A$ if it exists.

The sets $A^u$ and $A^l$ can be empty. In general, $A^u = \bigcap_{x \in A} \uparrow x$ and $A^l = \bigcap_{x \in A} \downarrow x$. If $A = \{x\}$, then $\uparrow x = \{x\}^u$ and $\downarrow x = \{x\}^l$. A poset $P$ is called a lattice when each nonempty subset $Y \subset P$ has both a least upper bound and a greatest lower bound.

3.1.5 Hasse diagrams

A poset $P$ can be represented conveniently using the covering relation. The *Hasse diagram* of an ordered set P is the ordered pair $(P, \prec)$, where $\prec$ is a lower cover relation on $P$. Each locally finite poset $P$ can be represented by a $\{0,1\}$- matrix $M$, where

$M_{xy} = 1 \Leftrightarrow x = y \; or \; x \prec y.$

The powers $M, M^2, \ldots$ signifies the number of chains from each $x$ to each $y$:

$M_{xy}^k$ is the number of ascending chains (with repetitions):

$x = x_0 \preccurlyeq_P x_1 \preccurlyeq_P \ldots \preccurlyeq_P x_k = y$ where $x_i = x_{i+1} \; or \; x_i \prec x_{i+1}$.

In particular, for a finite $P$, there is a power $M^k$, with $k \leq |P|$, such that $M_{xy}^k > 0 \Leftrightarrow x \preccurlyeq_P y$.

For instance, consider the 5-element poset on $P = \{1,2,3,4,5\}$, where the order is defined by the following matrix $M$:

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The covering relation is represented by the Hasse diagram in figure 3.1.

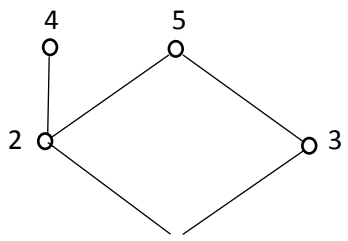





Figure 3.1 Hasse diagram of the poset *P*

**Example 3.1.1**

Frequently encountered posets include:

i. The set $\prod(X)$ of all partitions of a set $X$ forms a poset under the partition order: $\pi_1 \leq \pi_2$, if each block $Y$ of $\pi_2$ is a union of blocks of $\pi_1$. Equivalently, each block of $\pi_1$ is contained in some blocks of $\pi_2$.

ii. If $\sum$ is a set of symbols called an alphabet, then each sequence of symbols $a_1, a_2, \dots, a_n \; for \; n \geq 1$ is a word over $\sum$. The empty word which has no symbol is denoted by $\varepsilon$. The set of all words over $\sum$ is denoted by $\sum^*$. A word $u$ is a factor of $v$, denoted $u \trianglelefteq v \; if \; v = v_1 u v_2$, where $v_1 \; and \; v_2$ can be empty. Then $(\sum^*, \trianglelefteq)$ is a poset; the factor poset of $\sum^*$.

iii. A subset of $\mathbb{R}^n$ partially ordered by $(a_1, a_2, \dots, a_n) \preccurlyeq (b_1, b_2, \dots, b_n) \Leftrightarrow a_i \preccurlyeq b_i$ in $\mathbb{R}$ for $i = 1,2, \dots, n$.

iv. A set of positive integers partially ordered by division. The positive natural numbers $\mathbb{N}\backslash\{0\}$ form an infinite poset under the divisibility relation: $n \leq_{\mathbb{N}} m$ if and only if $n$ divides $m$. The divisibility posets $\mathbb{N}_n = (\{1,2, \dots, n\}, |)$ are finite subposets of this set.

Some particular instances of these examples are shown in figure 3.2.

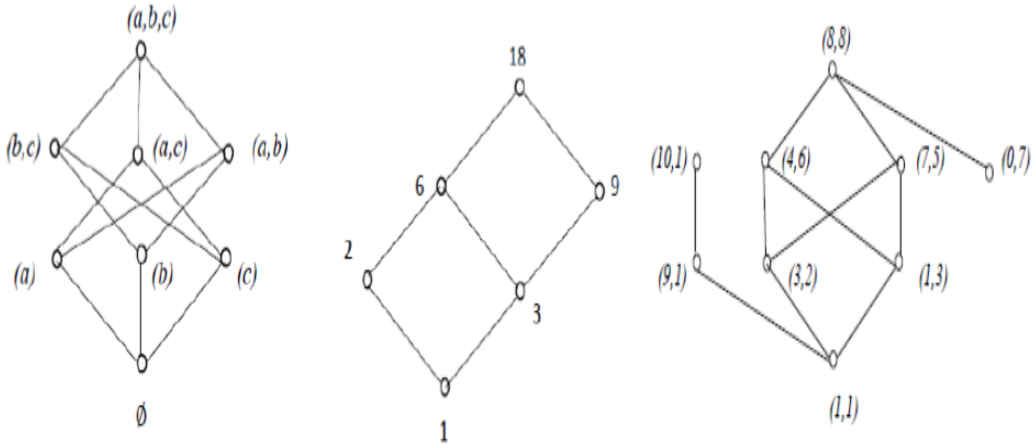



Figure 3.2 Hasse diagrams of posets (Trotter, 1995)

## 3.2 Operations on Partially Ordered Sets and their Properties

If $P_i, i \in I$ is a set of posets on a common set $X$, then $P = \bigcap_{i \in I} P_i$ is a poset. Infact $xPy$ holds only if $xP_iy$ holds for all $i \in I$ and the poset conditions for $P$ follow from the conditions for each individual poset $P_i$. On the other hand, a union $Q = \bigcup_{i \in I} P_i$ need not be a poset, since it can be that $xP_iy$ and $yP_jx$ for some $i \neq j \; and \; x \neq y$, and in $Q$ we would have both $xQy \; and \; yQx$. Also $Q$ need not be transitive and thus not even a quasi-order.

### 3.2.1 Mappings

Let $P$ and $Q$ be two posets.

a. A function $\alpha: P \to Q$ is an *order preserving mapping* (also called *monotone* or *isotone*) if for all $x, y \in P, x \preccurlyeq_P y \Longrightarrow \alpha(x) \preccurlyeq_Q \alpha(y)$.

b. An injective order preserving mapping is called an *order embedding.*

c. $\alpha$ is an order isomorphism if it is bijective and its inverse $\alpha^{-1}: Q \to P$ is order preserving. $P$ and $Q$ are isomorphic if there exists an order isomorphism between them.

**Example 3.2.1**

For a poset $P$, let $P^P$ be the set of all mappings $\alpha: P \to P$. Then $P^P$ forms a poset under the operation

$\alpha \preccurlyeq_{P^P} \beta \Leftrightarrow \alpha(x) \preccurlyeq_P \beta(x)$ for all $x \in P$.

Also, the order preserving mapping $P \to P$ forms a poset with respect to the same ordering.

**Lemma 3.2.1**

Two posets $P$ and $Q$ are isomorphic if and only if there exists a surjective mapping $\alpha: P \to Q$ such that, for all $x, y \in P$,

$$x \preccurlyeq_P y \Leftrightarrow \alpha(x) \preccurlyeq_Q \alpha(y) \qquad (3.2.1)$$

Proof

If $\alpha$ satisfies condition (3.2.1), then it is necessarily a bijection from $P$ onto $Q$. Indeed, if $\alpha(x) = \alpha(y)$ then $\alpha(x) \preccurlyeq_Q \alpha(y)$ gives $x \preccurlyeq_P y$, and symmetrically we obtain $y \preccurlyeq_P x$, thus $x = y$. Hence $\alpha$ is injective. It is surjective by hypothesis. By condition (3.2.1), $\alpha$ is isotone, and so is $\alpha^{-1}$, because (3.2.1) is equivalent to the condition

$$\alpha^{-1}(u) \preccurlyeq_P \alpha^{-1}(v) \Leftrightarrow u \preccurlyeq_Q v \qquad (3.2.2)$$

Hence $\alpha$ is an order isomorphism.

Conversely, assume that $\alpha: P \to Q$ is an order isomorphism. Then both $\alpha \; and \; \alpha^{-1}$ are isotone, and hence (3.2.1) holds.

### 3.2.2 Direct products

The direct product $P \times Q$ is the Cartesian product $\{(x, y) | x \in P, y \in Q\}$ together with the partial order $\preccurlyeq_{P \times Q}$:

$$(x_1, y_1) \preccurlyeq_{P \times Q} (x_2, y_2) \Leftrightarrow x_1 \preccurlyeq_P x_2 \text{ and } y_1 \preccurlyeq_Q y_2.$$

The Hasse diagram of the direct product of two chains of length two is depicted in figure 3.3.

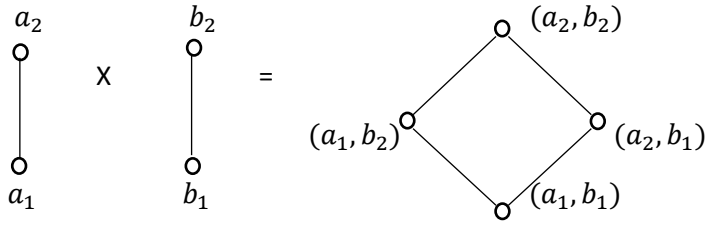



Figure 3.3 Direct product of two posets

The direct products $\prod_{i=1}^{k} P_i = P_1 \times P_2 \times ... \times P_k$ are defined inductively so that $\prod_{i=1}^{k} P_i = (\prod_{i=1}^{k-1} P_i) \times P_k$. This poset is isomorphic to $P_1 \times (\prod_{i=2}^{k} P_i)$.

**Example 3.2.2**

Let $X$ be a finite set of $n$ elements and consider the subset poset $2^X$. This is isomorphic to the $n$-fold direct product $\prod^n C_2 = C_2 \times C_2 \times ... \times C_2$ of the 2-element chain $C_2 = \{0,1\}$ where $0 < 1$. Let $X = \{x_1, x_2, \dots, x_n\}$, and let $\alpha: 2^X \to \prod^n C_2$ be such that the $i^{th}$ component of $\alpha(A)$ is 1 in case $x_i \in A$. For instance, if $n = 5$, then $\alpha(\{x_2, x_3, x_5\}) = (0,1,1,0,1)$. In $\prod^n C_2$, $(a_1, a_2, \dots, a_n) \preccurlyeq (b_1, b_2, \dots, b_n) \Leftrightarrow a_i \preccurlyeq b_i$ for all $i$, this corresponds to the subset relation $2^X$.

### 3.2.3 Disjoint sum and linear sum

If $P$ and $Q$ are disjoint posets then, the disjoint sum $P + Q$ is the poset on $P \cup Q$ such that $x \prec y$ if and only if $x, y \in P$ and $x \prec y$ in $P$ or $x, y \in Q$ and $x \prec y$ in $Q$. The linear sum $P \oplus Q$ is obtained from $P + Q$ by adding the relation $x \prec y$ for all $x \in P$ and $y \in Q$.

## 3.3 Subposets

Let $P$ be a poset on a set $X$ and $A \subseteq X$ a subset of the underlying set $X$. A *suborder* $\preccurlyeq_A$ is the restriction of $\preccurlyeq_P$ to pairs of $A$:

$x \preccurlyeq_A y \Leftrightarrow x \preccurlyeq_P y$ and $x, y \in A$. The pair $(A, \preccurlyeq_A)$ is a *subposet* of $P$.

### 3.3.1 Chains and antichains

A subposet $C$ is called a *chain* in $P$, if $C$ is a linearly ordered set. The length of a chain $C$ is $|C| - 1$. A chain $C$ is *maximal* if $C$ is not strictly contained in any other chain of $P$.

Dually, a subposet $A$ of $P$ consisting of only incomparable elements is called an *antichain*. An antichain $A$ of $P$ is maximal if for all $x \in P$, $x$ is comparable to an element in $A$.

The *height* (*width*) of a poset is the maximum cardinality of a chain (antichain).

**Example 3.3.1**

The poset in figure 3.4 has height 4 and width 6.

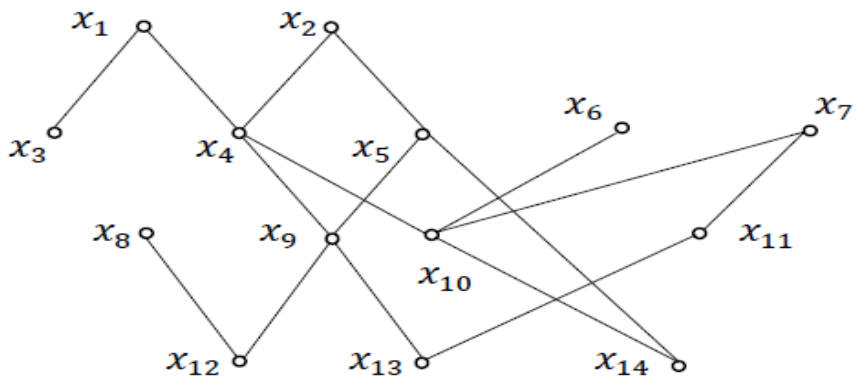



Figure 3.4 Poset with height 4 and width 6

Let $P$ be a poset on $X$, then $P$ satisfies:

i. The *finite chain condition* or FCC if every chain in $P$ is finite.

ii. The *ascending chain condition* or ACC if every strictly increasing chain $x_1 \prec_P x_2 \prec_P \ldots$ is finite.

iii. The *descending chain condition* or DCC if every strictly descending chain $x_1 \succ_P x_2 \succ_P \ldots$ is finite. A poset that satisfies DCC is well-founded.

iv. The *finite antichain condition* or FAC if every antichain of $P$ is finite.

Remark 3.3.1

A poset satisfying FCC can still have unbounded height, since there can be arbitrarily long chains in $P$. Also, the FCC implies both ACC and DCC.

**Lemma 3.3.1**

A poset $P$ satisfies the ACC if and only if every nonempty subset $X \subseteq P$ has a maximal element.

Proof

Let $A \subseteq P$ be nonempty such that $A$ has no maximal elements. Fix an element $x_0 \in A$. Since $x_0$ is not maximal, there exists an $x_1 \in A$ such that $x_0 \prec_P x_1$, and so forth, $x_0 \prec_P x_1 \prec_P x_2 \prec_P \ldots$, which gives an infinite ascending chain. Assume that every nonempty subset has a maximal element and let $x_1 \preccurlyeq_P x_2 \preccurlyeq_P \ldots$ be an ascending chain in $P$. Write $X = \{x_i | i = 1,2, \ldots\}$, and let

$x_k \in X$ be a maximal element in $X$. Hence $x_i = x_k$ for all $i \geq k$ and therefore $X$ is finite. It follows that $P$ satisfies the ACC.

A dual argument shows that a poset $P$ satisfies the DCC if and only if every nonempty subset of $P$ has a minimal element. This is equivalent to the following result:

**Example 3.3.2**

Consider the cartesian product $\mathbb{Z} \times \mathbb{Z}$ of integers with the partial order $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. This poset is locally finite. However, it does not satisfy the ACC, DCC, nor FAC. The sequence $\ldots < (-2,0) < (-1,0) < (0,0) < (1,0) \ldots$ gives a counter-example to both ACC and DCC, and $\{(-n, n) | n = 0,1, \ldots\}$ is an infinite set of incomparable elements in the poset.

**Theorem 3.3.2** (Zorn's Lemma)

Let $P$ be a poset where every chain has an upper bound, then $P$ contains a maximal element.

Remark 3.3.2

The poset $P$ is not required to be explicitly empty. However, the empty set is a chain (trivially), hence it is required to have an upper bound, thus exhibiting at least one element of $P$. Zorn's lemma plays an important role in the theory of ordered sets. It can be used to show that every

connected graph has a spanning tree. The set of all subgraphs that are trees is ordered by inclusion, and the union of a chain is an upper bound. Zorn's lemma asserts that a maximal tree must exist, which is a spanning tree since the graph is connected. This is quite useful when working with infinite posets. Zorn's lemma is equivalent to the well-ordering theorem and the axiom of choice.

### 3.3.2 Ranked posets

A poset $P$ is said to be *ranked* if every maximal chain in $P$ has the same number of points. Suppose $P$ is ranked and let $r(x)$ be the largest $i$ such that there exists a chain of $i$ points having $x$ as its least element. The value $r(x)$ is called the *rank* of $x$, and the antichains $A_i = \{x: r(x) = i\}$ are called ranks. The poset $P$ is said to be a *Sperner poset* if the width of $P$ equals the maximum cardinality of its ranks. Maximum antichains are called *maximum Sperner families*, after Sperner who first characterized them. The following classical result is due to Sperner (1982).

**Theorem 3.3.3**

For each $n \geq 1$, $2^n$ is a Sperner poset. In particular, the width of $2^n$ is $\binom{n}{\lfloor n/2 \rfloor}$.

The following decomposition theorem delineates the relationship between a maximal antichain and the size of a chain partition.

**Theorem 3.3.4** (Dilworth's theorem)

Let $P$ be a finite poset of width $n$, then there exists a partition $P = C_1 \cup C_2 \cup \ldots \cup C_n$ where each $C_i$ is a chain.

Remark 3.3.3

Several proofs and extensions are known for Dilworth's theorem (Perles, 1963; Galvin, 1994) some are algorithmic in nature (Piljs and Potharst, 2011, 2013). Mirsky (1971) proved a dual version of Dilworth's theorem in the context of antichains.

**Theorem 3.3.5** (Mirsky's theorem)

If $P$ is a poset of height $t$, then there exists a partition $P = A_1 \cup A_2 \cup \ldots \cup A_t$ where each $A_i$ is an antichain.

## 3.4 Combinatorics of Partially Ordered Sets

3.4.1 Linear extensions of a partially ordered set

Szpilrajn in his work *Sur l'extension de l'ordre partiel* showed that any order relation can be extended to a linear order. He also proved that any order relation is the intersection of its linear orders.

**Definition 3.4.1**

A linear extension of a finite poset $(X, \preccurlyeq)$ is an order-preserving bijection $\alpha: P \to \{1,2, \ldots, |P|\}$ such that $x \prec y$ in $P$ implies that $\alpha(x) \prec \alpha(y)$.

Figure 3.5 is an order diagram of a poset and some of its linear extensions.

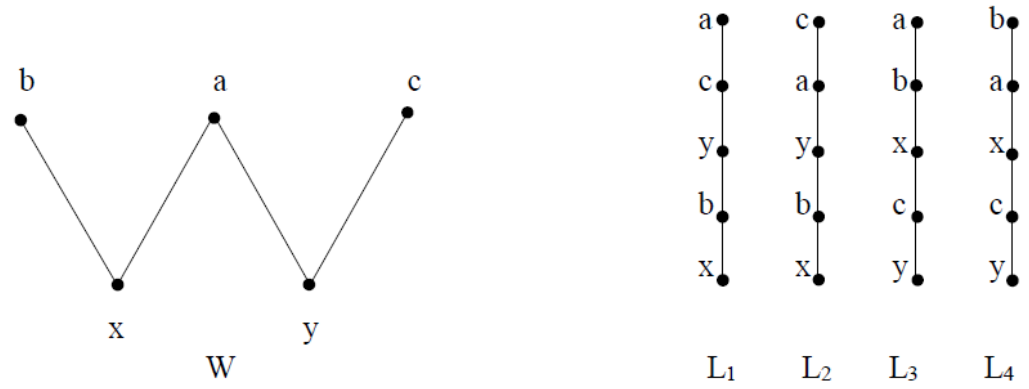



Figure 3.5 A poset $P$ and some its linear extensions (Kierstead *et al*., 1987)

Algorithmically, a linear extension $L$ of $P$ can be constructed as follows:

$SET\ R = P, \quad M = MIN(R)$

$FOR\ i = 0, \dots, t-1$

$CHOOSE\ \ x_{i+1} \in M$

$SET\ \ R = R - \{x_{i+1}\},\ \ M = MIN(R)$

END

A linear extension of $P$ is obtained from this algorithm by a suitable sequence of choices of the points $x_{i+1}$. If $L$ is a linear extension of a poset $P$, then its dual $L^d$ is a linear extension of $P^d$.

**Definition 3.4.2**

An *alternating cycle* of a poset $P$ is a subset $S = \{(x_i, y_i) | i = 1,2, \dots, k\}$ of pairs such that $x_1 || y_1 \preccurlyeq_P x_2 || y_2 \preccurlyeq_P \dots \preccurlyeq_P x_k || y_k \preccurlyeq_P x_1$, where $k \geq 1$.

Let $S$ be a subset of pairs of incomparable elements. If $S$ contains no alternating cycles, then the transitive closure $(P \cup S)^+$ is a poset. This is true also in converse.

**Theorem 3.4.1**

Let $C$ be a chain in a poset $P$, then there are linear extensions $D$ and $U$ of $P$ such that, for all $x||y$ with $x \in C \; and \; y \in P, y \preccurlyeq_D x \; and \; x \preccurlyeq_U y$.

A more restrictive class of linear extensions can be obtained by further restricting the choice of $x_{i+1}$. The notion of greedy linear extension was introduced in Cogis and Habib (1979).

3.4.2 Greedy linear extensions

A linear extension is said to be greedy if we also require that $x_{i+1}$ covers $x_j$ in P whenever possible. A simple rule that illustrates how greedy linear extensions are obtained is *climb as high as you can*. A greedy linear extension can be obtained from the following algorithm:

$SET \; R = P, \quad M = MIN(R), \quad G = M$

$FOR \; i = 0, \dots, t-1$

$CHOOSE \;\; x_{i+1} \in G$

$SET \;\; R = R - \{x_{i+1}\}, \;\; M = MIN(R)$

$IF \; M \cap U(x_{i+1}) \neq \emptyset$

$THEN \; SET \; G = M \cap U(x_{i+1})$

$ELSE\ SET\ G = M$

END

### 3.4.3 Super greedy linear extensions

A more restrictive class of linear extensions are the *super greedy linear extensions* (Kierstead *et al*., 1987). A greedy linear extension $[x_1 < x_2 < \cdots < x_l]$ of a finite ordered set P is super greedy if it is obtained from the greedy algorithm together with the additional tie breaking rules:

$T_2$: Prefer elements which cover $x_{i+1-2}$ for $x_{i+1}$.

.
.
.

$T_t$: Prefer elements which cover $x_{i+1-2}$ for $x_{i+1}$.

Such a tie breaking scheme is implemented according to its order, i.e., $T_i$ has higher priority than $T_j$ if $i < j$. A super greedy algorithm can be constructed using the following algorithm:

$SET\ R = P, \quad M = MIN(R), \quad SG = M$

$FOR\ i = 0, \dots, t-1$

$CHOOSE\ \ x_{i+1} \in SG$

$SET\ \ R = R - \{x_{i+1}\},\ M = MIN(R), \quad j = i$

$WHILE\ M \cap U(x_j) = \emptyset \quad AND \quad j \neq 0 \quad DO\ SET\ j = j - 1$

$IF\ j \neq 0$

$THEN\ SET\ SG = M \cap U(x_j)$

$ELSE\ SET\ SG = M$

END

Example 3.4.1

In figure 3.6, $L_1$ is not a greedy linear extension, $L_2$ is greedy but not super greedy, and $L_3$ is super greedy

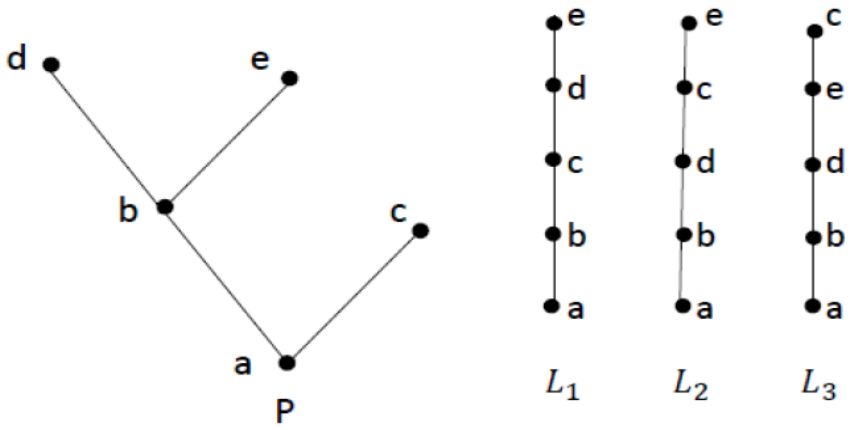
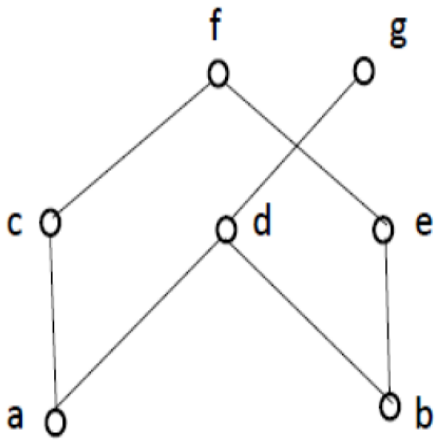



Figure 3.6 Greedy and super greedy linear extensions

3.4.4 Realizers of a partially ordered set

If $R$ is a family of linear extensions whose intersection is the order relation $\preccurlyeq_P$ on $P$, then $R$ *realizes* $P$. Figure 3.7 depicts a poset and its realizer. The set $\bigcap L_i$ for $i = \{1,2,3,4,5\}$ realizes the given poset.




$$L_1 = b < e < a < d < g < c < f$$

$$L_2 = a < c < b < d < g < e < f$$

$$L_3 = a < c < b < e < f < d < g$$

$$L_4 = b < e < a < c < f < d < g$$

$$L_5 = a < b < d < g < e < c < f$$

Figure 3.7 A poset and its realizer.

The classical theorem of Dilworth asserts that if the width of an ordered set $P$ is $n$, then $P$ can be partitioned into $n$ chains. Let $P = C_1 \cup C_2 \cup \ldots \cup C_n$ be such a partition. For each $i = 1,2, \ldots, n$ let $L_i$ be a linear extension of $P$ so that $C_i/P$ in $L_i$ (i.e., $x \succ y$ in $L_i$ whenever $x \in C_i, y \in P$ and $x||y$ in $P$) . Clearly $R = \{L_1, L_2, \ldots, L_n\}$ is a realizer for $P$.

### 3.4.5 Dimension of a partially ordered set

The *dimension* of a poset $P$ (denoted by $\dim(P)$) is the minimum cardinality of a realizer of $P$. The concept of dimension for posets was introduced by Dushnik and Miller (1941). Dilworth's decomposition theorem plays a major role in dimension theory. For example, Hiraguchi (1951) used Dilworth's decomposition theorem to prove that the dimension of a poset $P$ never exceeds its width. Another characterization for the dimension of a poset is the following theorem:

**Theorem 3.4.2** (Hiraguchi's theorem)

For any poset $P$, $\dim(P) \leq {}^{|P|}/_{2}$ for $|P| \geq 4$.

Remark 3.4.1

Every ordered set P can be represented as the intersection of some family of greedy linear extensions, this family of greedy linear extensions that realizes P is the *greedy realizer* of P. The *greedy dimension* of P, denoted $dim_G(P)$, is the least number t for which there are t greedy linear extensions of P. Also, every finite ordered set can be represented as the intersection of a family of super greedy linear extensions called a *super greedy realizer* of P. The *super greedy dimension*, denoted $dimS_G(P)$, of P is the minimum cardinality of a super greedy realizer of P. The following result was established in Kierstead and Trotter (1989).

**Theorem 3.4.2**

Let $P$ be an ordered set. Then,

$$\dim(P) = dim_G(P) = dim_{SG}(P)$$

If either of the following conditions are satisfied:

(a) $dim(P) \leq 2$

(b) $P$ is a distributive lattice

Remark 3.4.2

There is a third condition that is known to guarantee that $\dim(P) = \dim_G(P)$. The diagram of an ordered set $P$ is said to be $N$-free if $P$ does not contain points $a, b, c, d$ so that $a <: b, a <: c, d <: c$, and $d$ is not covered by $b$ in $P$. If the diagram of $P$ is $N$-free, then $\dim(P) = \dim_G(P)$. In fact, this result has been extended to $W$-free ordered sets (Kierstead *et al*., 1987).

## 3.5 Multisets (Definitions and Representation)

### 3.5.1 Definition (Multiset)

Intuitively, a *multiset* (mset, for short) is an unordered collection of objects (called elements) in which unlike the standard (Cantorian) set, repetition of elements is significant. Several names have been suggested for such a collection, this includes, bag, heap, bunch, weighted set (Hickman, 1980 and Yager, 1986). If $X = \{x_1, x_2, \ldots, x_n\}$ is a set, an mset A over $X$ is a *cardinal-valued* function i.e., $A: X \to \mathbb{N}$ such that $x \in Dom\ (A)$ implies $A(x)$ is a cardinal number and $A(x) = m_A(x) > 0$, where $m_A(x)$ denotes the frequency of the object $x$ in $A$. The number of occurences of an object in an mset is called the *multiplicity* of the object. Multiple occurrences of an element in an mset are treated without preference.

Each individual occurrence of an object in an mset is called its *element*. The *objects* of an mset are the distinguishable elements.

For an mset $M$ the *root* set (or *support*) of $M$ denoted by $M^*$ is the set $\{x \in M | M(x) > 0\}$.

The *cardinality* of an mset is the sum of the multiplicities of all its distinct elements. The class of all finite msets containing objects from a set $X$ is denoted by $M(X)$.

An mset is called *regular* or *constant* if all its objects occur with the same multiplicity.

For example, the mset $M = [a, b, c]_{2,2,2}$ is regular.

Also, an mset is called *simple* if all its objects are the same, for example, $[3]_5$ is a simple mset containing 3 as its only object.

Remark 3.5.1

In most of the known application areas, objects are allowed to repeat in an mset finitely, albeit in a theoretical development, infinite multiplicities of objects are also dealt with (Blizard, 1989 and Wildberger, 2003). Also, a considerable amount of efforts have gone into the study of msets with negative multiplicities (Blizard, 1990).

### 3.5.2 Representations of multisets

*Multiplicative form*

Following Meyer and McRobbie (1982), the use of square brackets to represent an mset has become almost standard. An mset containing one occurrence of $a$, two occurrences of $b$ and three occurrences of $c$ is represented as $[[a, b, b, c, c, c]]$ or $[a, b, b, c, c, c]$ or $[a, b, c]_{1,2,3}$ or $[a^1, b^2, c^3]$ or $[a1, b2, c3]$, depending on one's taste and suitability.

Remark 3.5.2

For our work, we shall adopt the representation

$M = [m_1x_1, m_2x_2, \dots, m_rx_r]$, where $m_i$ is the multiplicity of the element $x_i$ in $M$ for $1 \leq i \leq r$.

*Mset as a sequence*

An mset can also be represented as a sequence in which the multiplicity of an object equals the number of times the object occurs in the sequence, which is exactly the Dedekind's *frequency-number*. The idea is to construct an mset as a sequence (or function with domain $\mathbb{N}$, the set of natural numbers) and ignore the ordering of its elements, which can be done by taking all permutations of the domain of the sequence.

*Linear form*

Wildberger (2003) presented a linear notation for msets, which seems to be quite innovative especially when negative multiplicities (integral as well as rational) are to be dealt with. For example, the mset $M = [a, b, c]_{1,2,3}$ can be written as

$$M = [a] + 2[b] + 3[c]$$

Similarly, a rational mset can be represented. For example,

$$\mathbb{N} = \frac{2}{3}[5] - \frac{1}{2}[1.8]$$

In order to accommodate negative multiplicities round brackets are used, $(a)$ in an mset stands for negative of $a$, for example,

$$[2,4,(5),(5),4] = [2] + 2[4] - 2[5]$$

*Mset as a family of sets*

An mset can also be represented as a family of sets, which is altogether a generalization of the idea of a sequence. Thus, the family sets $F = F_i, i \in I$, where $F_i = F_j$, if $i = j$, which represents a

repeated element, represents an mset. Obviously, such a family $F$ is a function: $I \to \{F_i | i \in I\}$, which in turn is a sequence if $I = \mathbb{N}$.

*Mset as a Numeric-valued function*

Representation of an mset as a numeric valued function abounds, especially in application. An mset $A = [x, y, z]_{1,2,3}$ is a mapping from a ground set $S$ to $\mathbb{N}$, the set of natural numbers with zero, defined by

$$\propto (t) = \begin{cases} 1, \ if \quad t = x \\ 2, \ if \quad t = y \\ 3, \ if \quad t = z \\ 0, for\ all\ the\ remaining \ \ t \in s \end{cases}$$

In general terms, for a given ground set $S$ and a numeric set $T$, we call a mapping

$\alpha: S \to T$,

$$\begin{cases} a\ set, if\ T = \{0,1\}; \\ a\ multiset, if\ T = \mathbb{N}, the\ set\ of\ natural\ numbers: \\ a\ signed\ multiset\ (\ or, hybrid\ or\ shadow\ set)\ if\ T = Z, the\ set\ of\ integers: \\ a\ fuzzy\ (\ or\ hazy) set\ if\ T = [0,1] \subseteq R, a\ two - valued\ Boolean\ algebra. \end{cases}$$

*Mset as a generalized characteristic function*

Similar to the representation of a set by its characteristic function (function whose range is $\{0,1\}$), a multiset or hybrid set is determined by its generalized characteristic function (whose range is the set of integers, positive, negative or zero).

**Definition 3.5.1** *Dressed epsilon symbol* $\in_+$

Singh and Singh (2007) introduced the symbol $\in_+$ (called dressed epsilon, this is a binary predicate which means *belongs to at least once*). Thus, $m_A(x) = 0$ implies that $x \notin A$, and $x \in_+^k A$ implies that $x$ belongs to $A$ at least $k$ times, however $x \in^k A$ means '$x$ belongs to $A$ exactly $k$ times'.

**Definition 3.5.2** *Similar msets*

Two msets $A$ and $B$ are said to be *cognate* or *similar* if $\forall x(x \in A \Leftrightarrow x \in B)$, where $x$ is an object. Thus similar msets have equal roots but need not be equal themselves.

For example, the msets $A = [a, a, a, b, c, c]$ and $B = [a, a, b, b, c]$ are similar.

**Definition 3.5.3** *Equal msets*

Two msets $A$ and $B$ are equal, denoted by $A = B$, if and only if for

$x \in D, m_A(x) = m_B(x)$. Clearly, $A = B \Longrightarrow A^* = B^*$, but the converse need not be true.

## 3.6 Operations on Multisets and their Properties

Early references describing intuitively properties of msets, and operations under msets are Knuth (1969, 1981), Blizard (1989). More recent related works include: Petrovsky (1994, 1997), Wildberger (2003), Singh and Singh (2007).

### 3.6.1 Submultiset (also called multisubset or msubset, for short)

Let $A$ and $B$ be two msets, $A$ is called a submset (or msubset) of $B$, written as $A \subseteq B$ or $B \supseteq A$, if $m_A(x) \leq m_B(x)$ for all $x \in D$. Also, if $A \subseteq B$ and $A \neq B$, then $A$ is called a proper submset of $B$. An mset is called a *parent* in relation to its submsets. It follows from the definition of msets

that $A = B$ if and only if for all $x \in D$, $m_A(x) = m_B(x)$. Also, $A = B \Longrightarrow A^* = B^*$, but the converse need not hold. Clearly, $\subseteq$ is antisymmetric and it is a partial ordering on the class of msets defined on a given generic domain. The introduction of $\in_+$ greatly enhanced the language of msets. For example, $A \subseteq B$ implies that

$$\forall z \forall k (z \in^k A \rightarrow z \in_+^k B)$$

Clearly, $\subseteq$ is antisymmetric i.e., $A \subseteq B$ and $B \subseteq A \Longrightarrow A = B$. Also, the definition of $\subseteq$ is reflexive and transitive, hence it is a partial ordering on the class of msets defined on a given generic domain.

A *whole* submset of a given mset contains all multiplicities of common elements, while a *full* submset contains all objects of the parent mset, and accordingly, every mset contains a unique full submset, its root set. Clearly, for any two msets $A$ and $B$, if $A \subseteq B$ and $Dom(A) = Dom(B)$, then $A$ is a full submset of $B$.

### 3.6.2 Direct product of msets

The direct product of two msets $A$ and $B$, denoted by $A \times B$, is the mset consisting of all ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$. Clearly,

$m_{A \times B}\big((a, b)\big) = m_A(a) m_B(b)$ or $|A \times B| = |A| \times |B|$

For example, if $A = [a, a, c]$ and $B = [b, b]$ are two msets, then $A \times B$ is given by

$[a, a, c] \times [b, b] = [(a, b), (a, b), (a, b), (a, b), (c, b), (c, b)] = [(a, b), (c, b)]_{4,2}$

### 3.6.3 Power mset

The power mset of a given mset $A$, denoted by $\widetilde{\wp}(A)$, is the mset of all submsets of $A$. For example, suppose $A = [x, y]_{2,1} = [x, x, y]$, then,

$\widetilde{\wp}(A) = [\emptyset, \{x\}, \{x\}, [x]_2, \{y\}, \{x, y\}, \{x, y\}, [x, y]_{2,1}]$.

In this sense, $|\widetilde{\wp}(A)| = 2^{|A|}$ for any mset $A$. Howbeit, a good number of researchers (see Blizard, 1989; Hickman, 1980), have pointed out that it is not necessary to admit repeated elements into a power mset. Hence a power mset needs to be called a power set only and denoted by $\wp(A)$. Accordingly, for $A = [x, y]_{2,1}$; $\wp(A) = [\emptyset, \{x\}, [x]_2, \{y\}, \{x, y\}, [x, y]_{2,1}]$ and hence $|\wp(A)| < 2^{|A|}$, which implies that Cantor's power set theorem: $|A| < |\wp(A)|$ fails.

### 3.6.4 Union of msets

Let $A$ and B be two msets over a given domain set $D$. Then $A$ union $B$, denoted by $A \cup B$, is the mset defined by

$m_{A \cup B}(x) = m_A(x) \cup m_B(x) = maximum(m_A(x), m_B(x))$

if such a maximum exists, otherwise, the minimum is taken which always exists. It follows that for any given mset $M$ there exists an mset $N$ which contains elements of elements of $M$, where the multiplicity of an element $z$ in $N$ is the maximum multiplicity of $z$ as an element of elements of $M$ along with the above stipulation on the existence of such a maximum. We denote this fact by $N = \cup M$. For example, if $A = [2,3,4,4], B = [1,4,3,3]$ then, $A \cup B = [1,2,3,3,4,4]$

Also, it follows that for a finite mset $M$, the maximum multiplicity of elements of elements of $M$ always exists. However, for certain infinite sets like $M = \{\{y\}, [y]_2, [y]_3, \dots\}$, the maximum

multiplicity of elements of elements of $M$ does not exist and hence, $\cup M = \{y\}$. It is obvious by the definition of the union that multiplicity of any $y \in M \neq \emptyset$ is irrelevant to $\cup M$, and hence

$\cup M = \cup M^*$ (Blizard, 1989).

Note that $A \cup B = B \cup C$ does not imply that $A = B$. Similarly, the same condition holds for intersection.

3.6.5 Intersection of msets

Given two msets $A$ and $B$ defined over a domain set $D$, their intersection $A \cap B$ is the mset defined by $m_{A \cap B}(x) = m_A(x) \cap m_B(x) = minimum(m_A(x), m_B(x))$, for all $x \in D$.

That is, $A \cap B$ is the smallest mset which is contained in both $A$ and $B$. An object occurring $a$ times in $A$ and $b$ times in $B$ occurs minimum $(a, b)$ times in $A \cap B$. This minimum always exists.

For example, if $A = [3,3,3,4,4], B = [1,4,3,3]$, then $A \cap B = [3,3,4]$

In general, for a given mset $X$, $Dom(\cap X) = \cap \{Dom(A): A \in X\}$ and $z \in \cap X$ implies that the multiplicity of $z$ is the minimum of its multiplicities as element of elements of $X$. Note that $\cap X \subseteq \cup X$ for any mset $X$.

3.6.6 Arithmetic difference of msets

Let $A$ and $B$ be two msets defined over a common domain $D$, and $B \subseteq A$, then the difference of $B$ from $A$, otherwise known as the relative complement of $B$ in $A$, denoted by $A - B$, is the mset such that

$m_{A-B}(x) = m_A(x) - m_B(x)$ for all $x \in D$.

In general, $m_{(A-B)}(x) = m_A(x) - m_{(A\cap B)}(x)$, for all $x \in D$

$$= max\,[m_A(x) - m_B(x), 0]$$

It is also sometimes called the relative complement of $B$ in $A$. If $B \nsubseteq A$, this definition still holds good. It should be noted that in mset context, an element may occur both in an mset and its complement. The removal of an element $x$ from an mset $A$ gives rise to a new mset $A' = A - x$ such that $m_{A'}(x) = max[\, m_A(x) - 1,0]$. Similarly, the symmetric difference, $A \,\Delta\, B$, is defined by,

$$m_{A\Delta B}(x) = m_{(A-B)}(x) \cup m_{(B-A)}(x) = |m_A(x) - m_B(x)|$$

### 3.6.7 Complementation

Let $A \in X^n$. The complement of $A$, denoted by $\bar{A}$ *or* $A^c$, in $X^n$ is a member of $X^n$ such that $m_{\bar{A}}(x) = n - m_{\ A}(x)$ for all objects $x$ in $X$. Alternatively, let $\mathcal{J} = \{A_1, A_2, \ldots\}$ be a family of msets. The maximum mset $Z$ is defined such that $m_z(x) = max m_A \in J$, for all objects $x$ and all A$\in J$. Then, $\bar{A} = Z - A$ such that $m_{\bar{A}}(x) = m_Z(x) - m_A(x)$, for all objects $x$.

### 3.6.8 Addition or sum or merge

Given two msets $A$ and $B$ defined over a common domain $D$, the direct sum or arithmetic addition of $A$ and $B$ denoted by $A + B$ or $A \uplus B$ is the mset defined by,

$m_{A \uplus B}(x) = m_A(x) + m_B(x)$, for all $x \in D$. That is an object occurring $a$ times in $A$ and $b$ times in $B$, occurs $a + b$ times in $A \uplus B$.

For example, if $A = [a, a, b, b, d, d, d]$ and $B = [a, b, c, c, ]$, then

$A \uplus B = [a, a, a, b, b, b, c, c, d, d, d]$

It follows that $|A \uplus B| = |A \cup B| + |(A \cap B)|$. The insertion of an element $x$ into an mset $A$ gives rise to a new mset $A' = A + x$ such that $m_{A'}(x) = m_A(x) + 1$ and $m_{A'}(y) = m_A(y)$ for all $y \neq x$.

Some of the consequences of the definition above may not conform to the laws of classical set theory. Consider for instance two msets $A$ and $B$, given by $A = [x, y]_{3,5}$ and $B = [x, y]_{2,3}$,with $A - B = [x, y]_{1,2}$. Clearly $A - B$ is contained in $B$, contradicting the classical law: $(A - B) \cap B = \emptyset$.

3.6.9 Properties of mset operations

i. $A \uplus B = B \uplus A,\ A \cup B = B \cup A \quad A \cap B = B \cap A$ (Commutativity)

ii. $A \uplus (B \uplus C) = (A \uplus B) \uplus C,\ A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C$ (Associativity)

iii. $A \cup A = A,\ A \cap A = A,\ but\ \ A \uplus A \neq A$ (Idempotency)

iv. $A \cup \phi = A,\ A \cap \phi = \phi,\ but\ \ A \uplus \phi \neq A$ (Identity laws)

v. $A \uplus (B \cup C) = (A \uplus B) \cup (A \uplus C)$, (Distributivity)

$$A \uplus (B \cap C) = (A \uplus B) \cap (A \uplus C),$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Observe that the operation ⊎ is stronger than ∪ and ∩ in the sense that, neither ∪ nor ∩ distributes over ⊎, whereas ⊎ distributes over ∪ and ∩.

### 3.6.10 Functions between multisets

Functions between msets are actually functions between their root sets since identical elements cannot be mapped to non-identical elements.

The function $f: A \rightarrow B$ is injective if and only if

i. $f: A^* \rightarrow B^*$ is injective, and

ii. $\forall z(z \in A^* \Rightarrow m_A(z) \leq m_B(f(z)))$

The function $f: A \rightarrow B$ is surjective if and only if

i. $f: A^* \rightarrow B^*$ is surjective, and

ii. $\forall z(z \in A^* \Rightarrow m_A(z) \geq m_B(f(z)))$

The function $f: A \rightarrow B$ is bijective if and only if

i. $f: A^* \rightarrow B^*$ is bijective, and

ii. $\forall z(z \in A^* \Rightarrow m_A(z) = m_B(f(z)))$

Remark 3.6.1

Msets of equal cardinality need not have a bijection between them. For example $[a,b]_{1,2}\ and\ [a,b,c]$ both contain three elements, but there can be no bijection between them because the objects and their multiplicities are different in the domain and the range of any such function. In other words, there is no bijection between their roots, viz; $f: \{a,b\} \rightarrow \{a,b,c\}$ can not be a bijection. Also, Schroder – Bernstein theorem fails.

## 3.7 Multiset Orderings

In this section, various orderings developed for comparing msets in $M(S)$ are discussed. Mset orderings have been studied for years. Dershowitz and Manna (1979) presented an mset ordering used in proving termination of programs and in term rewriting systems. This ordering is usually referred to as the standard mset ordering.

### 3.7.1 Dershowitz-Manna multiset ordering (Dershowitz and Manna, 1979)

For two msets $M\ and\ N$ defined over a set $S$, the Dershowitz-Manna mset ordering is defined thus:

$M \ll N$ if there exists two msets $X\ and\ Y$ in $M(S)$ such that,

i. $\emptyset \neq X \subseteq N$

ii. $M = (N \backslash X) + Y$, and

iii. $X$ dominates $Y$, that is, for all $y \in Y$, there is some $x \in X$ such that $y < x$.

In other words, for any two finite msets $M\ and\ N$ defined over a set $S$, $M \ll N$ if $M$ can be obtained from $N$ by replacing one or more elements in $N$ by any finite number of elements taken from $S$, each of which is smaller than one of the replaced elements. Dershowitz and Manna showed that the ordering on $S$ induces an ordering on the finite msets over $S$. A well-founded set $(S, >)$ consists of a set $S$ and an ordering $>$ defined on $S$ such that no infinite descending sequences of elements of $S$ can occur. It is shown here that the ordering $>$ on any given well-founded set $S$ can be extended to form a well-founded ordering $\gg$ on the finite msets over $S$. The Dershowitz-Manna mset ordering enables the use of relatively simple and intuitive termination functions in otherwise difficult termination proofs. This mset ordering was used in particular to prove the termination of production systems, programs defined in terms of rewriting rules. An important property of the Dershowitz-Manna ordering is its monotonicity and as observed by Jouannaud and Lescanne (1982), it is the strongest monotonic ordering on msets. Characterization of the Dershowitz-Manna ordering based on monotonicity has been proposed. Dershowitz-Manna mset ordering is difficult to use when proving that two multisets are not related by inclusion.

### 3.7.2 Huet-Oppen multiset ordering (Huet and Oppen, 1980)

A more tractable definition of mset ordering was proposed in Huet and Oppen (1980) as follows:

Let $M\ and\ N$ be two msets over a set $S$, then

$M \ll N$ if and only if $M \neq N\ and\ [M(y) > N(y) \Longrightarrow (\exists x \in S)x \succ y\ and\ M(x) < N(x)]$

The Huet-Oppen ordering is total and becomes a lexicographic ordering if the ordering $<$ on the ground set is total (Martin, 1989), it is also well-founded if and only if $(S, <)$ is well-founded. This mset ordering is also used in the proof of program termination and term rewriting systems.

**Lemma 3.7.1**

The Dershowitz-Manna mset ordering is equivalent to the Huet-Oppen definition.

### 3.7.3 Jouannaud-Lescanne multiset ordering **(**Jouannaud and Lescanne, 1982)

Jouannaud and Lescanne (1982) presented two mset orderings $\ll_1$ and $\ll_2$, which extend strictly the Dershowitz-Manna ordering as follows:

1. Assume that the partition $\overline{M} = \{M_i | i = 1 \dots p\}$ of the mset $M$ satisfies the following conditions:
   i. $x \in M_i \Longrightarrow M_i(x) = M(x)$.
   ii. $x \in M_i$ and $y \in M_i \Rightarrow x \; and \; y$ are incomparable.
   iii. $\forall i \in [2, \dots, p] x \in M_i \Rightarrow (\exists y \in M_{i-1}) y \succ x$.

The partition is built by first computing the mset $M_1$ of all the maximal elements and then recursively computing the partition of $M - M_1$. Let $M \; and \; N$ be two msets with set based partition in lexicographical order, $\overline{M} \; and \; \overline{N}$ respectively, then $M \ll_1 N$ if and only if $\overline{M} \ll_1^{lex} \overline{N}$.

**Example 3.7.1**

If $a \prec b$ and $M = \{1, a, b\}$ then $M_1 = \{1, b\}, M_2 = \{a\}$. If $N = \{1,1, b\}$ then $N_1 = \{1,1, b\}$ and $M \ll_1 N$.

Clearly, $\ll_1$ is an ordering because lexicographic extension preserves the ordering. Jouannaud and Lescanne proved that this ordering contains the Dershowitz-Manna ordering as follows:

**Lemma 3.7.2**

$M \ll N$ implies $M \ll_1 N$.

The contrary is not true. From the last example $M(a) = 1 > N(a) = 0$. However, the only element in $N$ greater than $a$ is $b$ with one occurrence in both $M$ and $N$. Thus $M$ and $N$ cannot be compared using Dershowitz-Manna ordering.

On the other hand, the multiset ordering $\ll_1$ has a serious drawback, which limits its use in some usual cases, where incremental ordering is required: it is not monotonic. For instance in example 3.7.1, assume that $1 \prec a \prec b$. That increased the basic ordering by adding the new pair $1 \prec a$. This gives,

$M_1 = \{b\}, \quad M_2 = \{a\}, \quad M_3 = \{1\}$

$N_1 = \{b\}, \quad N_2 = \{1,1\}$

thus, $M$ and $N$ cannot be compared using $\ll_1$.

A different method for building a partition from a set is to require that different occurrences of the same element belong to different multisets of the partition, for example to assume that all the multisets of the partitions are sets.

2. In this case assume that the partition $\bar{M} = \{S_i | i = 1 \dots p\}$ satisfies the following conditions:

   i. $S_i$ is a set, that is $S_i(x) \leq 1$.

   ii. $x \in S_i$ and $y \in S_i \Rightarrow x \; and \; y$ are incomparable.

   iii. $\forall i \in [2, \dots, p] x \in S_i \Rightarrow (\exists y \in S_{i-1}) y \succcurlyeq x$

The difference between 1 and 2 is in the first condition. In the same way, the partition is built by first computing the set $S_1$ of maximal elements and then recursively computing the partition of $M - S_1$.

**Example 3.7.2**

$M = \{a, a, 2, 2, b, 1, 1\}$ with $a \prec b$ and $1 \prec 2$ and $S_1 = \{2, b\}$, $S_2 = \{2, a\}$, $S_3 = \{1, a\}$,

$S_4 = \{1\}$.

Let $\ll_2$ be the following ordering:

$S \ll_2 T$ if and only if $S \neq T$ and $(\forall x \in S)(\exists y \in T) y \succcurlyeq x$.

$\bar{N} = \{T_i | i = 1 \dots q\}$ will be the partition of $N$. If sets are considered as a particular case of msets, $\ll_2$ is $\ll$ on the sets of incomparable elements.

**Definition 3.7.2**

Let $M$ and $N$ be multisets. Then $M \ll_2 N$ if and only if $\bar{M} \ll_2^{lex} \bar{N}$.

**Example 3.7.3**

Suppose $a$ and $b$ are incomparable. If $N = \{a, b\}$ and $T_1 = \{a, b\}$, if $M = \{b, b\}$ then $S_1 = \{b\}$, $S_2 = \{b\}$ and $M \ll_2 N$.

The following result also holds.

**Lemma 3.7.3**

$M \ll N$ implies $M \ll_2 N$

The converse is false. Consider the last example: $M = \{b, b\}$ and $N = \{a, b\}$.

Suppose $a \prec b$ with the same $M$ and $N$ as in example 3.8.3. We have, $S_1 = \{b\},\ S_2 = \{b\}$ and $T_1 = \{b\},\ T_2 = \{a\}$. Thus $N \ll_2 M$. Similarly, $\ll_2$ is not monotonic.

The Jouannaud-Lescanne ordering is well-founded if the ordering on the generic set is well-founded.

3.7.4 Melven Krom multiset ordering (Krom, 1985)

Melven Krom defined two mset orderings together with their respective transitive closures.

For two msets $M\ and\ N$ defined over a set $S$, Krom defined $M \ll^n N$ if and only if

i. there exists exactly one element $y \in S$ such that $M(y) < N(y)$

ii. for this one element, $M(y) + 1 = N(y)$

iii. if $M(x) > N(x)\ then\ x < y\ in\ S$

iv. $\sum(M(x) - N(x)) \leq n$ where the sum is over all $x \in S\ such\ that\ M(x) > N(x)$

For the binary relation $\ll^\infty$, if condition iv is omitted then, $M \ll^\infty N$.

An extension of the partial order $\prec$ on $S$ to an ordering on $M(S)$ is the ordering $\prec_\epsilon$ defined thus:

$M \prec_{\in} N$ if and only if there exists $x \in S$ such that $N(x) > 0$ and for any $y \in S$, if $M(y) > 0$ then there exists $z \in S$ such that $y \prec z$ and $N(z) > 0$ i.e.,

$N \neq \emptyset \wedge \forall y \in M \rightarrow \exists z \in N \wedge y \prec z.$

Krom showed that the ordering on $M(S)$ is well-founded if $(S, <)$ is well-founded.

### 3.7.5 Martin multiset ordering (Martin, 1989)

The following mset ordering was presented by Ursula Martin:

For two msets $M\ and\ N$ defined on $S$, if $<$ is a strict order on $S$ and $A$ an $n\ by\ n$ matrix over $\mathbb{N}$ indexed by the elements of $S$, then $M \ll N$ if and only if,

$f_x(M) \leq f_x(N)\ \forall x \in S$ where $f_x(N) = \sum_{y \geq x} N(y)$

Martin's classification is based on the notion of a tame ordering i.e. an ordering on $M(S)$ with those properties that arise naturally in termination proofs. This geometric approach is used to give a unified picture of different mset orderings. Martins showed that the dominance ordering is the unique weakest possible tame ordering on $M(S)$ (see Martin, 1989 (theorem 5.1)). The author also showed that the strongest possible tame ordering on $M(S)$ is either the standard mset ordering, or an ordering got from it by the action of an invertible matrix (Martin, 1989 (theorem 6.2)). This ordering is well-founded and monotonic (Tella *et al*., 2014 ).

### 3.7.6 Wehrman multiset ordering (Wehrman, 2006)

Let $(S, \preccurlyeq)$ be a reflexive transitive ordering. Wehrman defined its mset extension $(M(S), \prec\prec)$ as follows:

$M \prec\prec N$ if and only if $M = (N - X) + Y$ for some $X, Y \in M(S)\ with\ \emptyset \neq X \subseteq N$ and for all $y \in Y$ there exists $x \in X$ with $y \preccurlyeq x$, and $x \preccurlyeq y$ for at most $|X|$ of the elements $y \in Y$.

This mset ordering is well-founded if and only if $(S, \preccurlyeq)$ is well-founded. In particular $(M(\mathbb{N}), \prec \prec)$ over $(\mathbb{N}, \leq)$ is well-founded, also $(M(S), \prec\prec)$ is a monotonic mset ordering.

3.7.7 Savaglio-Vannucci multiset ordering (Savaglio and Vannucci, 2007)

Let $M, N \in M(S)$, then $M \gg N$ if and only if $M(x) > N(x)$ for all $x \in S$ i.e. the mset $M$ dominates the mset $N$ if and only if $N$ is a proper submset of $M$. In this case, the ordering $\gg$ is independent of the underlying order on the base set $S$. Also $\gg$ cannot be total since the inclusion relation on $M(S)$ is not total.

## 3.8 The Concept of Partially Ordered Multisets

A number of researchers have proposed that partially ordered multisets (for short, pomsets) can be used as mathematical representations of runs of a concurrent system. In recent times, researches have been centered on partial orders in order to aid in understanding the semantics of parallel programs (Pratt, 1986; Gischer, 1988; Rensink, 1996; Morin, 2010).

As noted in Gischer (1988), the notion of a pomset is to have a trace of a process i.e., some sort of sequencing of activities called tasks. A task can be thought of as some activity which can be performed several times and have a duration. One consequence of tasks having duration in time is that they may overlap in time, thus we cannot assign sequence to them; neither occurs before

the other. In consequence of the repeatability of tasks, any trace will have to allow repeated symbols. This brings up a tricky technical point of stating in a formal model that one occurrence of task $A$ precedes another occurrence of task $A$ while still another occurrence of $A$ overlaps in time with both the previous occurrences of $A$. Pomsets are a natural model for parallel and distributed computation. An approach to this is to define an underlying partial order of distinct vertices, and then to label them with the tasks we have in mind for each of them, while the identity of the underlying partial order is abstracted away leaving only its structure.

The following definition of a pomset is based on labeled partial orders, pomsets are considered abstractly without taking into account the nature of the elements that are being ordered. The elements are assumed to be collected in a set $\sum$.

### 3.8.1 Labeled partial orders and pomsets

A *labeled partially ordered set* (or *lposet*) over $\sum$ is a structure $p = \langle V, <, l \rangle$, where

i. $V$ is an arbitrary set of vertices;

ii. $< \subseteq V \times V$ is an irreflexive and transitive ordering relation;

iii. $l: V \to \sum$ is a labeling function.

Two lposets $p = \langle V_p, <_p, l_p \rangle$ and $q = \langle V_q, <_q, l_q \rangle$ over the same set of labels $\sum$ are isomorphic if there exists a bijection $f: V_p \to V_q$ such that for all $v, w \in V_p$, $v <_p w$ if and only if $f(v) <_q f(w)$ and $l_p(v) = l_q(f(w))$.

A pomset is an isomorphism class of lposets. The class of pomsets over $\sum$ is denoted $POM|\sum|$ $(= LPO|\sum|/\cong)$. A pomset $\langle V, <, l \rangle$ is finite if $V$ is finite.

Remark 3.8.1

Standard types such as sets, bags (msets), posets, and lists are sub-types of the pomset type and are obtained by restricting the map $l$ and the kind of order in the pomset. For instance, an mset is a pomset with the empty partial order on its vertices. A set is an mset with an injective labelling function. With the preceding definitions, many varied set-theoretic structures can be dealt with.

Intuitively, a pomset is a string in which the total order has been relaxed to a partial order. Since strings are clearly a special case of pomsets, one way to study the latter is by generalizing and extending existing theories about the former. This is indeed the approach that one generally finds in the literature.

3.8.2 Strings

The partial ordering in the case of strings is actually total. Strings over a given set of elements are free monoids, meaning that they are freely generated by the signature $\sum_{str} = \langle \varepsilon, \cdot \rangle$ with the following properties:

$$\varepsilon \cdot x = x \tag{1}$$

$$x \cdot \varepsilon = x \tag{2}$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{3}$$

$\varepsilon$ denotes the *empty string* and $\cdot$ *concatenation of strings*. The pomsets that can be generated in this way are precisely those $p$ whose ordering is total i.e., such that either $v \leq_p w$ or $w \leq_p v$ for all $v, w \in V_p$. The empty string is modeled by the empty pomset $[\emptyset, \emptyset, \emptyset]$, a single element $e \in \sum$

by $[\{0\}, \emptyset, \{(0, \mathrm{e})\}]$ (where 0 is a simple placeholder without intrinsic meaning), and concatenation of $p$ and $q$ is defined by

$$p \cdot q = [V_p \cup V_q, <_p \cup (V_p \times V_q) \cup <_q, l_p \cup l_q],$$

where $p$ and $q$ are disjoint i.e., they are chosen such that $V_p \cap V_q = \emptyset$.

### 3.8.3 Multisets

Msets (or bags) over a given set of elements are another special case of partially ordered multisets. Here the elements are actually completely unordered. Msets are known to constitute free commutative monoids; that is, they are freely generated by the signature $\sum_{mul} = \langle \varepsilon, \uplus \rangle$ with the following properties:

$$\varepsilon \uplus x = x \tag{4}$$

$$x \uplus y = y \uplus x \tag{5}$$

$$(x \uplus y) \uplus z = x \uplus (y \uplus z) \tag{6}$$

$\varepsilon$ now denotes the empty mset and $\uplus$ multiset addition; the latter is associative and commutative, and has the empty mset as its neutral element. The pomsets that can be generated in this way are precisely those without any ordering whatsoever, i.e, those $p$ with $<_p = \emptyset$. The empty mset and single-element msets are modeled in the same way as the empty string and single-element strings. Mset addition is modeled by disjoint pomset union:

$$p \uplus q = [V_p \cup V_q, <_p \cup <_q, l_p \cup l_q]$$

Where again the representatives $p$ and $q$ should be disjoint.

### 3.8.4 Mazurkiewicz traces

An interesting mixture of strings and msets can be found in the *Mazurkiewicz traces* (Mazurkiewicz, 1989), sometimes also called *partially commutative monoids*. Here a set with structure $\langle \sum, I \rangle$, where $I \subseteq \sum \times \sum$ is an irreflexive and symmetric independency relation, is considered. This relation controls the degree to which the concatenation operator, denoted $\odot$, (to distinguish it from string concatenation) is commutative:$d \odot e = e \odot d$ precisely when $d$ and $e$ are independent. Mazurkiewicz traces are freely generated by $\sum_{Maz} = \langle \varepsilon, \odot, I \rangle$ with properties:

$$\varepsilon \odot x = x \tag{7}$$

$$(x \odot y) \odot z = x \odot (y \odot z) \tag{8}$$

The ordering in the pomsets generated by the above signature satisfies the following condition: for all $v, w \in V_p$, if $v \not<_p w \not<_p v$ then $l_p(v) I l_p(w)$, whereas if $v <_p w$ then $\exists u \in V_p . v <_p u \leq_p w$ and $\sim(l_p(v) I l_p(u))$. This implies that in principle only the dependent elements are ordered; some additional orderings must be due to transitive closure.

The independence relation is extended to pomsets by putting $pIq$ if and only if $l_p(v) I l_q(w)$ for all $v \in V_p$ and $w \in V_q$. The empty trace and single-element traces are modeled in the same manner; the partially commutative concatenation operation is defined by

$$p \odot q = [V_p \cup V_q, <_p \cup <_{pq} \cup <_q, l_p \cup l_q]$$

Where $p$ and $q$ are disjoint representatives and $v <_{pq} w$ if and only if there exist $v' \in V_p$ and $w' \in V_q$ such that $v \leq_p v', \sim(l_p(v') I l_q(w'))$ and $w' \leq_q w$. The only difference with respect to ordinary string concatenation lies in the fact that essentially only the dependent vertices of $p$ and $q$ are ordered.

### 3.8.5 Series-parallel pomsets

Probably the most intensively studied approach to obtain a more extensive theory of pomsets is the direct combination of the algebras of strings and msets, where the neutral elements of both are made to coincide. This leads to the theory of series-parallel or N-free pomsets (Grabowski, 1981; Pratt, 1986; Gischer, 1988; Aceto, 1991). Series parallel pomsets are freely generated by the signature $\sum_{sp} = \langle \varepsilon, \cdot, \uplus \rangle$ with the following properties:

$$\varepsilon \cdot x = x \tag{1}$$

$$x \cdot \varepsilon = x \tag{2}$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{3}$$

$$\varepsilon \uplus x = x \tag{4}$$

$$(x \uplus y) \uplus z = x \uplus (y \uplus z) \tag{5}$$

$$x \uplus y = y \uplus x \tag{6}$$

Concatenation (serial composition) and disjoint union (parallel composition, hence series-parallel) do not interact at all. The models that can be generated using this signature are $N$-free: if $v <_p v' >_p w <_p w'$ for distinct $v, v', w, w' \in V_p$ then $v' < w'$ or $w < v$ or $v < w'$.

The empty and singleton pomsets are $N$-free, and $N$-freedom is preserved by concatenation and disjoint union.

### 3.8.6 Forests

Forest (i.e., msets of trees) are pomsets with the special property that all predecessors of a given element are totally ordered. Algebraically this can be seen as an extension of msets with an associative concatenation operator with respect to which the empty forest is left cancellative (rather than left and right neutral as in strings), and which distributes over addition from the right. This operator is denoted by $\diamond$. Forests are freely generated by the signature $\sum_{tr} = \langle \varepsilon, \diamond, \uplus \rangle$ with the following properties:

$$\varepsilon \uplus x = x \tag{1}$$

$$(x \uplus y) \uplus z = x \uplus (y \uplus z) \tag{2}$$

$$x \uplus y = y \uplus x \tag{3}$$

$$\varepsilon \diamond x = \varepsilon \tag{4}$$

$$(x \diamond y) \diamond z = x \diamond (y \diamond z) \tag{5}$$

$$(x \uplus y) \diamond z = x \diamond z \uplus y \diamond z \tag{6}$$

The above algebra was presented by Baeten and Weijland (1990) with the additional axiom

$x \uplus x = x$ . Intuitively, concatenation of two forests $p$ and $q$ appends $q$ to all the termination points of $p$, which are essentially its maximal elements. The pomsets generated by this system are hierarchial orders with termination, i.e., of the form $p = [V, <, l, \intercal]$ such that $u < w > v$ implies $u \leq v$ or $v \leq u$ for all $u, v, w \in V$, and where the termination points are modeled by the extra component $\intercal \subseteq max_< V$. The empty forest is modeled by $[\emptyset, \emptyset, \emptyset, \emptyset]$ and single-element forests are modeled by $[\{0\}, \emptyset, \{(0, e)\}, \{0\}]$. Forest addition coincides with mset addition.

### 3.8.7 Order-deterministic pomsets

For order-deterministic pomsets, a new operator called *join* replaces pomset union. The resulting signature is given by $\sum_{det} = \langle \varepsilon, \cdot, \sqcup \rangle$ with the following properties:

$$\varepsilon \cdot x = x \tag{1}$$

$$x \cdot \varepsilon = x \tag{2}$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{3}$$

$$\varepsilon \sqcup x = x \tag{4}$$

$$(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z) \tag{5}$$

$$x \sqcup y = y \sqcup x \tag{6}$$

$$x \cdot (y \sqcup x) = (x \cdot y) \sqcup (x \cdot z) \tag{7}$$

Pomsets generated by this algebra are order-deterministic as in the case of forests interpreted as labeled transition systems; such a transition system is called deterministic if every transition is

completely determined by its source node in combination with its label. Also, in order-deterministic pomsets, every vertex is completely determined by its set of predecessors and its label. The order-deterministic pomsets proper include all Mazurkiewicz traces (and therefore all posets) but not all forests or series-parallel pomsets.

### 3.9 Mset Order Relations

The mset order relations and quasi-mset order relations are denoted by the symbol $\leq:\leq$, and the strict orders by $<:<$. The mset with elements drawn from $X$ is given by $\{k_1/x_1, k_2/x_2, \ldots., k_n/x_n\}$ (Girish and Sunil, 2009; Ghilezan *et al*., 2014).

**Definition 3.9.1**

$Let\ M_1\ and\ M_2$ be two msets drawn from a set $X$; then the Cartesian product of $M_1$ and $M_2$ is defined as

$$M_1 \times M_2 = \{(m/x, n/y)/mn: x \in^m M_1, y \in^n M_2\}.$$

The Cartesian product of three or more nonempty msets is obtained by generalizing the definition of the Cartesian product of two msets. That is, the Cartesian products $M_1 \times M_2 \times \ldots \times M_n$ of nonempty mset $M_1, M_2 \ldots M_n$ is the mset of all ordered n-tuples $(m_1, m_2 \ldots., m_n)$ where $m_i \in^{ri} M_i, i = 1,2 \ldots n\ \ and\ \ (m_1, m_2 \ldots., m_n) \in^p \ \ M_1 \times M_2 \times \ldots \times M_n\ with$

$$p = \prod r_i, \quad r_i = \ C_M\ (m_i), i = 1,2, \ldots . n.$$

**Definition 3.9.2**

A submset $R\ of\ M \times M$ is said to be an mset relation on $M$ if every member $m/x,\ n/y\ of\ R$ has count $C_1(x,y) \cdot\ C_2(x,y)$ . The notation $m/x\ \ R\ \ n/y$ denotes $m/x$ related to $n/y$ .

**Definition 3.9.3**

The domain and range of the mset relation $R$ on $M$ is defined as follows:

$Dom\ R = \{x \in^r\ M\colon \exists y\ \in^s\ M\ \ such\ that\ \ r/x\ \ R\ s/y\}$ , where

$C_{Dom\ R}(x) = \sup\ \{\ C_1(x,y)\colon x \in^r\ M\}.$

$Ran\ R = \{y \in^s M\colon \exists x\ \in^r\ M\ \ such\ that\ \ r/x\ R\ s/y\},$ where

$C_{Ran\ R}(x) = \sup\ \{\ C_2(x,y)\colon y \in^s\ M\}.$

**Definition 3.9.4**

The identity mset relation in any mset $M$ is the set of all pairs in $M \times M$ with equal co-ordinates and it is denoted by $I_M$.

**Definition 3.9.5**

A partition of a nonempty mset $M$ is a collection $P$ of nonempty submset of $M$ such that

i. Each element of $M$ belongs to one of the msets in $P$

ii. If $M_1$ and $M_2$ are distinct elements of $P$ then $M_1 \cap M_2 = \emptyset$.

# CHAPTER FOUR

# PARTIALLY ORDERED MULTISETS

In this chapter, a new ordered multiset structure $\mathcal{M} = (M, \preccurlyeq\leq)$ is developed. Substructures of the ordered multiset are also constructed. We consider generalization of known results that hold for ordered sets and obtain some properties that are peculiar to ordered multisets. In particular, by employing set-based partitioning, the width and height of a partially ordered multiset $\mathcal{M}$ are characterized.

## 4.1 The Ordering $\preccurlyeq\leq$

In this section, we introduce the ordering $\preccurlyeq\leq$ on a mset $M \in M(S)$ and present properties satisfied by the ordered structure $(M, \preccurlyeq\leq)$, which we call a partially ordered multiset. Here, the ordering induced by the underlying base set plays a vital role in defining hierarchies between the points of $M$.

4.1.1 Preliminaries

We begin by fixing some notations.

A *point* $m_i x_i$ in an mset $M$ will denote an object in $M$ together with its multiplicity. We write $m_i x_i \bowtie m_j x_j$ whenever $m_i x_i$ and $m_j x_j$ are *comparable* under the defined order in $\mathcal{M}$ and $m_i x_i || m_j x_j$ whenever $m_i x_i$ and $m_j x_j$ are *incomparable* in $\mathcal{M}$. As usual the class of all finite msets defined over a set $S$ will be denoted by $M(S)$.

**Definition 4.1.1**

Let $S$ be a universe set and $P = (S, \preccurlyeq)$ a poset. Suppose $M \in M(S)$, then for any pair of points $m_i x_i \; and \; m_j x_j$ in $M$, we say $m_i x_i \preccurlyeq\!\leq m_j x_j$ if and only if $x_i \preccurlyeq x_j$ in $P$. The points $m_i x_i$ and $m_j x_j$ coincide if and only if $x_i = x_j$. Moreover, $m_i x_i \bowtie m_j x_j$ if and only if $m_i x_i \preccurlyeq\!\leq m_j x_j \vee m_j x_j \preccurlyeq\!\leq m_i x_i$ otherwise $m_i x_i || m_j x_j$.

Remark 4.1.1

The partial ordering on $S$ induces a partial ordering on $M$. The condition $m_i x_i == m_j x_j$ if and only if $x_i = x_j$ implies that $m_i = m_j$, this follows from the principle of uniqueness of the multiplicity of an object in an mset.

**Definition 4.1.2**

The *strict* order associated with $\preccurlyeq\!\leq$ is given by $\prec\!<$, where $m_i x_i \prec\!< m_j x_j$ implies that $m_i x_i \preccurlyeq\!\leq m_j x_j$ and $m_i x_i \neq\!= m_j x_j$.

**Definition 4.1.3**

The ordering $\preccurlyeq$ on $M$ is said to be *reflexive* if and only if $m_i x_i \preccurlyeq m_i x_i$ for all $m_i x_i \in M$, *symmetric* if and only if $m_i x_i \preccurlyeq m_j x_j$ implies $m_j x_j \preccurlyeq m_i x_i$, *antisymmetric* if and only if $m_i x_i \preccurlyeq m_j x_j \wedge m_j x_j \preccurlyeq m_i x_i$ implies that $m_i x_i == m_j x_j$, and *transitive* if and only if $m_i x_i \preccurlyeq m_j x_j \wedge m_j x_j \preccurlyeq m_k x_k$ implies $m_i x_i \preccurlyeq m_k x_k$.

**Definition 4.1.4**

The ordering $\preccurlyeq$ is called a *quasi-mset order* (or a *pre-mset order*) if it is reflexive and transitive, and a *strict mset order* if it is irreflexive and transitive. The ordering $\preccurlyeq$ is called a *partial mset order* (or simply *mset order*) if it is reflexive, antisymmetric and transitive. Also, $\preccurlyeq$ is a *linear* (or *total*) mset order if it is a partial mset order and for all pairs of point $m_i x_i, m_j x_j$ in $M$, we have $m_i x_i R m_j x_j \vee m_j x_j R m_i x_i$.

Remark 4.1.2

We will be concerned primarily with finite pomsets, howbeit, some of the results obtained extend to infinite cases.

**Definition 4.1.5**

A pomset $\mathcal{M}$ is a pair $(M, \preccurlyeq)$, where $M \in M(S)$, and $\preccurlyeq$ is a partial mset order defined on $M$.

**Theorem 4.1.1**

Let $(S, \preccurlyeq)$ be a poset and $M \in M(S)$. Then $\mathcal{M} = (M, \preccurlyeq)$ is a pomset.

Proof

For any $m_i x_i$ in $M$, since $x_i \preccurlyeq x_i$ we have $m_i x_i \preccurlyeq m_i x_i$, implying that $(M, \preccurlyeq)$ is reflexive.

Let $m_i x_i \preccurlyeq\leq m_j x_j$ and $m_j x_j \preccurlyeq\leq m_i x_i$ in $\mathcal{M}$. Then, $x_i \preccurlyeq x_j$ and $x_j \preccurlyeq x_i$, and hence $x_i = x_j$.

In particular, $m_i x_i == m_j x_j$, hence $\preccurlyeq\leq$ is antisymmetric.

Let $m_i x_i, m_j x_j, m_k x_k$ be points in $M$ such that $m_i x_i \preccurlyeq\leq m_j x_j$ and $m_j x_j \preccurlyeq\leq m_k x_k$.

We have $x_i \preccurlyeq x_j \preccurlyeq x_k$.

Hence, $m_i x_i \preccurlyeq\leq m_k x_k$

Thus transitivity holds.

Therefore, $(M, \preccurlyeq\leq)$ is a pomset. □

Remark 4.1.3

With the ordering $\preccurlyeq\leq$, the mset order diagram presented in figure 4.1a can be simplified to the mset order diagram in figure 4.1b.

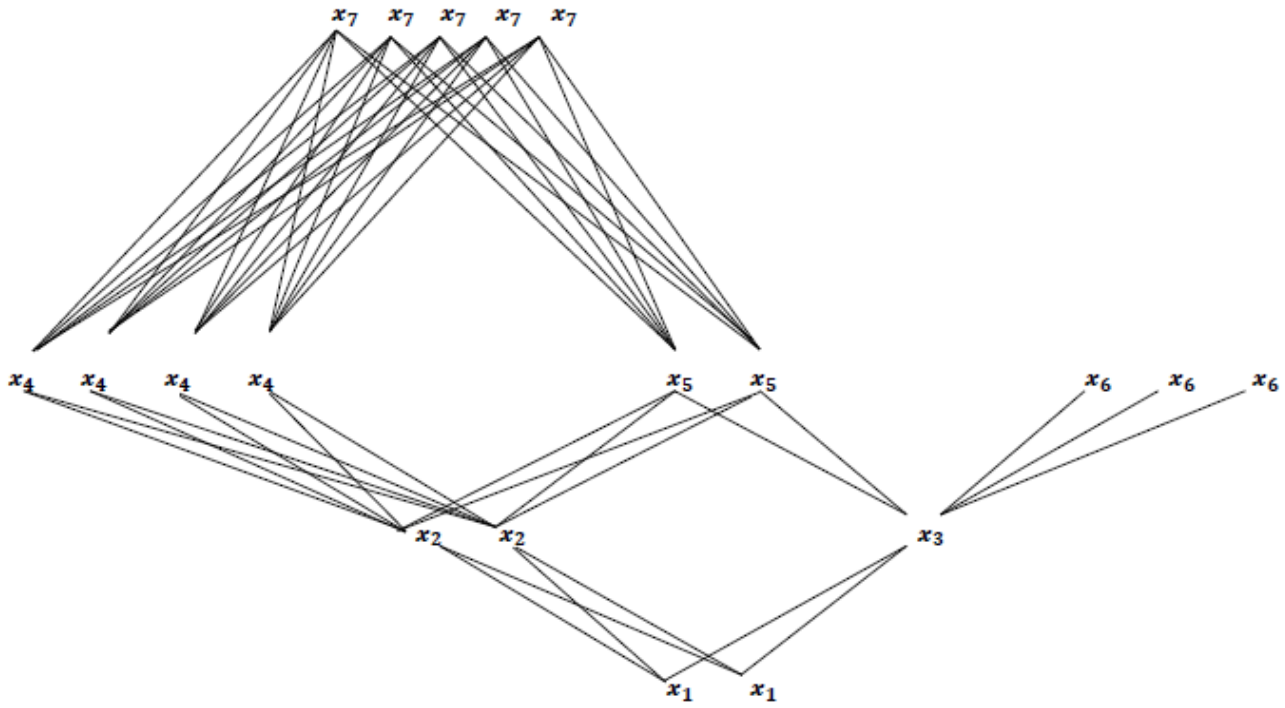



Figure 4.1a Mset order diagram

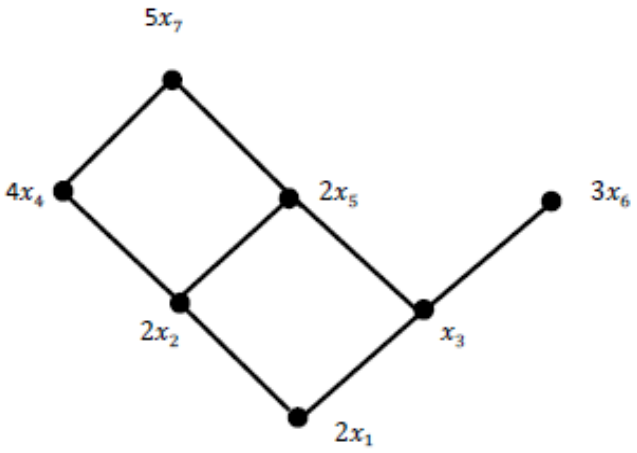



Figure 4.1b Simplified mset order diagram

**Definition 4.1.6**

For two mset orders $\preccurlyeq_1\leq_1$ and $\preccurlyeq_2\leq_2$ on an mset $M$, the mset order $\preccurlyeq\leq$ is said to be an intersection of $\preccurlyeq_1\leq_1$ and $\preccurlyeq_2\leq_2$ if and only if $m_ix_i \preccurlyeq\leq m_jx_j \Longrightarrow m_ix_i \preccurlyeq_1\leq_1 m_jx_j \wedge m_ix_i \preccurlyeq_2\leq_2 m_jx_j$, for all $m_ix_i, m_jx_j \in M$.

**Theorem 4.1.2**

If $\mathcal{M} = (M, \preccurlyeq_1\leq_1)$ and $\mathcal{N} = (M, \preccurlyeq_2\leq_2)$ are pomsets with base sets $(S, \preccurlyeq_1)$ and $(S, \preccurlyeq_2)$, then $\mathcal{M} \cap \mathcal{N} = (M, \preccurlyeq\leq)$ is also a pomset, where $\preccurlyeq\leq = \preccurlyeq_1\leq_1 \cap \preccurlyeq_2\leq_2$.

Proof

For any point $m_ix_i$ in $M$, clearly $m_ix_i \preccurlyeq_1\leq_1 m_ix_i$ and $m_ix_i \preccurlyeq_2\leq_2 m_ix_i$ since $\preccurlyeq_1\leq_1$ and $\preccurlyeq_2\leq_2$ are partial mset orders.

Thus, $m_ix_i \preccurlyeq\leq m_ix_i$ (reflexive property).

Let $m_ix_i$ and $m_jx_j$ be points in $M$ such that

$m_ix_i \preccurlyeq\leq m_jx_j$ and $m_jx_j \preccurlyeq\leq m_ix_i$.

(1)

From (1) we have,

$m_ix_i \preccurlyeq_1\leq_1 m_jx_j$ and $m_jx_j \preccurlyeq_1\leq_1 m_ix_i$.

(2)

Since $\preccurlyeq_1\leq_1$ is antisymmetric, we have

$m_i x_i == m_j x_j$.

(3)

Similarly,

$m_i x_i \preccurlyeq_2 \leq_2 m_j x_j$ and $m_j x_j \preccurlyeq_2 \leq_2 m_i x_i$ imply $m_i x_i == m_j x_j$. (4)

From (2) - (4) we can conclude that,

$m_i x_i \preccurlyeq \leq m_j x_j$ and $m_j x_j \preccurlyeq \leq m_i x_i$ imply $m_i x_i == m_j x_j$.

Therefore, $\preccurlyeq \leq$ is antisymmetric.

For transitivity,

let $m_i x_i, m_j x_j$ and $m_k x_k$ be points in $M$ such that,

$m_i x_i \preccurlyeq \leq m_j x_j$ and $m_j x_j \preccurlyeq \leq m_k x_k$.

We need to show that $m_i x_i \preccurlyeq \leq m_k x_k$.

Now,

$m_i x_i \preccurlyeq \leq m_j x_j$ and $m_j x_j \preccurlyeq \leq m_k x_k$ imply

$m_i x_i \preccurlyeq_1 \leq_1 m_j x_j$ and $m_j x_j \preccurlyeq_1 \leq_1 m_k x_k$.

Since $\preccurlyeq_1 \leq_1$ is transitive, we have $m_i x_i \preccurlyeq_1 \leq_1 m_k x_k$.

(5)

Similarly,

$m_i x_i \preccurlyeq_2 \leq_2 m_j x_j$ and $m_j x_j \preccurlyeq_2 \leq_2 m_k x_k$ imply $m_i x_i \preccurlyeq_2 \leq_2 m_k x_k$. (6)

From (5) and (6), we obtain $m_i x_i \preccurlyeq\leq m_k x_k$ , hence $\preccurlyeq\leq$ is transitive.

Therefore, $\mathcal{M} \cap \mathcal{N} = (M, \preccurlyeq\leq)$ is a pomset. □

**Theorem 4.1.3**

Let $(S, \preccurlyeq)$ be a poset. An mset $M \in M(S)$ is partially ordered if and only if its root set is a subposet of $(S, \preccurlyeq)$.

Proof

Suppose $M \in M(S)$ is partially ordered. Thus, for $m_i x_i \in M$, $m_i x_i \preccurlyeq\leq m_i x_i$ holds. By definition this implies

$x_i \preccurlyeq x_i$ for all $x_i \in M^*$, with $i \in [1, n]$. (1)

Also, for all $m_i x_i, m_j x_j \in M$,

we have $m_i x_i \preccurlyeq\leq m_j x_j \wedge m_j x_j \preccurlyeq\leq m_i x_i \Longrightarrow m_i x_i == m_j x_j$.

Again by $\preccurlyeq\leq$, it must be the case that

$x_i \preccurlyeq x_j \wedge x_j \preccurlyeq x_i \Longrightarrow x_i = x_j$ for all $x_i, x_j \in M^*$. (2)

Now, let $m_i x_i, m_j x_j, m_k x_k$ be any three points in $M$. Since $M$ is partially ordered we have

$m_i x_i \preccurlyeq\leq m_j x_j \wedge m_j x_j \preccurlyeq\leq m_k x_k \Longrightarrow m_i x_i \preccurlyeq\leq m_k x_k$, and

$x_i \preccurlyeq x_j \wedge x_j \preccurlyeq x_k \Longrightarrow x_i \preccurlyeq x_k$ for all $x_i \in M^*$. (3)

From (1) through (3), it follows that $(M^*, \preccurlyeq)$ is a subposet of $(S, \preccurlyeq)$.

The converse is straightforward. Suppose that $(M^* \preccurlyeq)$ is a subposet of $(S, \preccurlyeq)$. Clearly, $x_i \preccurlyeq x_i$ for all $x_i \in M^*$. Let $m_i$ be the multiplicity of $x_i$ in $M \in M(S)$. From the definition of $\preccurlyeq\leq$, we have $m_i x_i \preccurlyeq\leq m_i x_i$ (reflexivity of $\preccurlyeq\leq$). Also, $x_i \preccurlyeq x_j \wedge x_j \preccurlyeq x_i \Longrightarrow x_i = x_j$ for all $x_i, x_j \in M^*$, this in turn gives, $m_i x_i \preccurlyeq\leq m_j x_j \wedge m_j x_j \preccurlyeq\leq m_i x_i \Longrightarrow m_i x_i == m_j x_j$ (antisymmetry of $\preccurlyeq\leq$). Finally, for all $x_i, x_j, x_k \in M^*$, we will have $x_i \preccurlyeq x_j \wedge x_j \preccurlyeq x_k \Longrightarrow x_i \preccurlyeq x_k$. Again, it follows that $m_i x_i \preccurlyeq\leq m_j x_j \wedge m_j x_j \preccurlyeq\leq m_k x_k \Longrightarrow m_i x_i \preccurlyeq\leq m_k x_k$ (transitivity of $\preccurlyeq\leq$). □

**Example 4.1.1**

Let $M = [2x_1, 4x_2, 3x_3, 5x_4, 2x_5]$ be an mset. Suppose the ordering on $M$ is given by

$2x_1 \preccurlyeq\leq 4x_2 \preccurlyeq\leq 2x_5$

$2x_1 \preccurlyeq\leq 3x_3$

$2x_1 \preccurlyeq\leq 5x_4$

Then, $(M, \preccurlyeq\leq)$ is a pomset.

## 4.2 Substructures of $(M, \preccurlyeq\leq)$

In this section, mset chains and mset antichains are defined on the ordered multiset structure. Some new definitions and results are presented.

**Definition 4.2.1**

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset. A point $m_i x_i$ in $M$ is *maximal* in $\mathcal{M}$ if for any other point $m_j x_j \in M$ with $m_i x_i \preccurlyeq\leq m_j x_j$ we have $m_i x_i == m_j x_j$. Similarly, a point $m_i x_i$ in $M$ is *minimal* if for any other point $m_j x_j \in M$ with $m_j x_j \preccurlyeq\leq m_i x_i$ we have $m_i x_i == m_j x_j$. If such points are unique, they are called *maximum* and *minimum* respectively. The set of maximal and minimal points of $\mathcal{M}$ are denoted by MAX($\mathcal{M}$) and MIN($\mathcal{M}$), respectively.

**Lemma 4.2.1**

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset. If $\mathcal{M}$ is totally ordered then maximal and maximum points coincide.

Proof

Let $m_i x_i \; and \; m_j x_j$ be points in $M$ such that $m_i x_i$ is a maximal point in $\mathcal{M}$ and $m_j x_j$ is a maximum point in $\mathcal{M}$.

Since $\mathcal{M}$ is totally ordered, we will have either $m_i x_i \preccurlyeq\leq m_j x_j$ or $m_j x_j \preccurlyeq\leq m_i x_i$.

Now, suppose that $m_i x_i \preccurlyeq\leq m_j x_j$, then, by definition of a maximal point

$m_i x_i == m_j x_j$.

Similarly, the other case follows.

□

A similar argument holds for minimal and minimum points if $\mathcal{M}$ is totally ordered.

**Theorem 4.2.2**

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset defined over a poset $(S, \preccurlyeq)$. A point $m_i x_i$ in $M$ is maximal in $\mathcal{M}$ if and only if the generating object is maximal in a subposet $(X, \preccurlyeq)$ of $(S, \preccurlyeq)$, where $X = M^*$.

Proof

Let $X = M^*$ and suppose $m_i x_i$ in $M$ is maximal in $\mathcal{M}$ then, for any point $m_j x_j$ in $M$ such that

$m_i x_i \preccurlyeq\leq m_j x_j$ we have, $m_i x_i == m_j x_j$ (1)

From (1), we have $x_i \preccurlyeq x_j$ implies $x_i = x_j$

Therefore, $x_i$ is maximal in $(M^*, \preccurlyeq)$

Next, assume that $x_i$ is maximal in $(M^*, \preccurlyeq)$

Then,

$x_i \preccurlyeq x_j \implies x_i = x_j$ for any $x_j$ in $(M^*, \preccurlyeq)$ (2)

Let $m_i$ be the multiplicity of $x_i$ in $M$ and let $m_j$ be the multiplicity of $x_j$ in $M$.

From (2), we have $m_i x_i \preccurlyeq\leq m_j x_j$ in $\mathcal{M}$

Also, by the uniqueness of multiplicity of an object in $M$, $x_i = x_j$ implies that $m_i x_i == m_j x_j$

Hence, we have $m_i x_i \preccurlyeq\leq m_j x_j \implies m_i x_i == m_j x_j$

Therefore, $m_i x_i$ is maximal in $\mathcal{M}$ □

**Definition 4.2.2**

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and $N$, a submset of $M$. A suborder $\preccurlyeq\leq_{\mathcal{K}}$ is the restriction of $\preccurlyeq\leq$ to pairs of points in the submset $N$ of $M$ such that,

$n_i x_i \preccurlyeq\leq_{\mathcal{K}} n_j x_j \Leftrightarrow m_i x_i \preccurlyeq\leq m_j x_j$, where $n_i x_i, n_j x_j \in N$ and $n_i \leq m_i$. The pair $(N, \preccurlyeq\leq_{\mathcal{K}})$ is called a subpomset of $\mathcal{M}$.

**Definition 4.2.3**

A subpomset $\mathsf{C}$ of a pomset $\mathcal{M} = (M, \preccurlyeq\leq)$ is called an *mset chain* if $\mathsf{C}$ is linearly (or totally) ordered.

A subpomset $\mathcal{A}$ of $\mathcal{M}$ is called an *mset antichain* if no two points in $\mathcal{A}$ are comparable.

A pomset $\mathcal{M}$ is *connected* (or is an mset chain) if $m_i x_i \bowtie m_j x_j$ for all distinct pairs of points $m_i x_i, m_j x_j \in M$. $\mathcal{M}$ is an mset antichain if $m_i x_i || m_j x_j$ for all distinct pairs of points $m_i x_i, m_j x_j$ in $M$.

Remark 4.2.1

The set of all mset chains (mset antichain) of a pomset is partially ordered by the mset inclusion relation.

**Definition 4.2.4**

An mset chain $C$ in a pomset $\mathcal{M}$ is *maximal* if it is not strictly contained in any other mset chain of $\mathcal{M}$. An mset chain $\mathsf{C}_i$ in a pomset $\mathcal{M}$ is a *maximum mset chain* if $|\mathsf{C}_i| > |\mathsf{C}_j|$ for all other mset

chains $C_j$ in the pomset $\mathcal{M}$. A maximal mset antichain is defined analogously. An mset antichain in $\mathcal{M}$ is a *maximum mset antichain* if it contains the maximum number of points.

Remark 4.2.2

The number of incomparable points rather than the cardinality of an mset antichain is critical in defining a maximum mset antichain. The incomparable points play a major role in characterizing ordered structures.

A pomset may contain more than one maximal mset chain. Also, in a pomset, maximal and maximum mset chains may coincide. The following example illustrates this.

**Example 4.2.1**

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ and let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be the root set for the mset $M = [2x_1, 3x_2, 4x_3, 6x_4, 8x_5, 16x_6]$ where $X$ is partially ordered as follows: $x_1 \preccurlyeq x_3 \preccurlyeq x_5 \preccurlyeq x_6$, $x_1 \preccurlyeq x_4$, and $x_2 \preccurlyeq x_4$.

The following are mset chains in $\mathcal{M}$:

$C_1 = [2x_1, 4x_3, 8x_5, 16x_6]$

$C_2 = [2x_1, 6x_4]$

$C_3 = [3x_2, 6x_4]$

$C_4 = [2x_3, 4x_5]$

$C_1$, $C_2$ and $C_3$ are maximal mset chains. Where $C_1$ is the maximum.

**Definition 4.2.5**

A pomset $\mathcal{M} = (M, \preccurlyeq \leq)$ is said to be well-ordered if for any submset $N$ of $M$, there exists a point $n_i x_i$ in $N$, such that $n_i x_i$ is the minimum point with respect to the defined order.

**Lemma 4.2.3**

Every well-ordered pomset is an mset chain.

Proof

Let $\mathcal{M} = (M, \preccurlyeq \leq)$ be a pomset and $m_i x_i, m_j x_j$ be any arbitrary pair of distinct points in $M$. Since $\mathcal{M}$ is well-ordered, the submset $[n_i x_i, n_j x_j]$ has a minimum point.

Thus, either $n_i x_i \prec\prec n_j x_j$ or $n_j x_j \prec\prec n_i x_i$.

Since this condition holds for every pair of distinct points in $M$, it follows that $\mathcal{M}$ is totally ordered. □

## 4.3 Bounds of Pomsets

In this section, the notion of bounds is extended to subpomsets of $\mathcal{M}$, some definitions and new results are presented.

**Definition 4.3.1**

Let $\mathcal{K} = (N, \preccurlyeq\leq_{\mathcal{K}})$ be a subpomset of a pomset $\mathcal{M} = (M, \preccurlyeq\leq)$. A point $m_i x_i \in M$ is an upper bound for $\mathcal{K}$ if $m_i x_i \succcurlyeq\geq n_j x_j$ for all points $n_j x_j$ in $N$. Dually, $m_i x_i \in M$ is a lower bound of $\mathcal{K}$ if $m_i x_i \preccurlyeq\leq n_j x_j$ for all points $n_j x_j$ in $N$.

**Lemma 4.3.1**

If an mset chain $C$ is maximal in a pomset $\mathcal{M}$, then $C$ necessarily contains its upper bound.

Proof

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and let $C = (N, \preccurlyeq\leq_C)$ be a maximal mset chain in $\mathcal{M}$. Since $C$ is linearly ordered, for some $i$ we will have a point $n_i x_i \in N$ such that $n_i x_i >\!\succ n_j x_j$ for all other points $n_j x_j \in N$. This implies that $n_i x_i$ is a maximum point. Suppose a point $m_k x_k \notin N$ is an upper bound for $C$. Now $C$ is maximal implies that for any point $m_k x_k \notin N$, we would have either

$m_k x_k || n_i x_i$ or $m_k x_k \preccurlyeq\leq n_i x_i$ since $n_i x_i$ is the maximum point.

If $m_k x_k || n_i x_i$, then $m_k x_k$ cannot be an upper bound for $C$.

Now, suppose that $m_k x_k \preccurlyeq\leq n_i x_i$, by the definition of an upper bound we have a contradiction, hence the result. □

**Theorem 4.3.2**

Let $\mathcal{M}$ be a pomset and let $\mathcal{C}$ be a collection of all maximal mset chains in $\mathcal{M}$. If $K$ is an mset containing all upper bounds of the elements of $\mathcal{C}$. Then any two distinct points in $K$ are incomparable.

Proof

Let $C_1, \dots, C_n$ be the maximal mset chains in $\mathcal{M}$. Suppose that $m_1x_1, m_2x_2, \dots\ m_nx_n$ are upper bounds for the mset chains $C_1, C_2, \dots, C_n$, then $K = [m_1x_1, \dots, m_nx_n]$.

Let $m_ix_i\ and\ m_jx_j$ be distinct points in $K$, then there exists maximal mset chains $C_i$ and $C_j$ in $\mathcal{C}$ such that $m_ix_i$ is an upper bound for $C_i$ and $m_jx_j$ is an upper bound for $C_j$ say.

Now, $C_i \cup [m_jx_j]$ is not an mset chain since $C_i$ is maximal in $\mathcal{M}$. Similarly, $C_j \cup [m_ix_i]$ is not an mset chain.

Assume that $m_ix_i \bowtie m_jx_j$, then either $m_ix_i \prec\prec m_jx_j$ or $m_jx_j \prec\prec m_ix_i$ holds.

Suppose $m_ix_i \prec\prec m_jx_j$. Now, $m_ix_i$ is an upper bound for $C_i$ implies that $m_ix_i \succcurlyeq m_kx_k$ for all other points $m_kx_k \in C_i$. By transitivity, it follows that, $m_jx_j \succ\succ m_kx_k$ for all $m_kx_k \in C_i$, which is a contradiction since $C_i$ is maximal in $\mathcal{M}$.

A similar argument holds for the case $m_jx_j \prec\prec m_ix_i$ in $C_j$.

Hence it must be the case that $m_ix_i || m_jx_j$.

Since $m_ix_i, m_jx_j$ are arbitrary points in $K$, therefore, no two points in $K$ are comparable. □

**Definition 4.3.2**

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and $\mathcal{S} = (N, \preccurlyeq\leq)$ a subpomset of $\mathcal{M}$. Suppose that $K$ is a set containing all upper bounds of the subpomset $\mathcal{S}$, then a point $m_i x_i$ in $K$ is a *least upper bound* for $\mathcal{S}$ if $m_i x_i$ is a minimum point in $K$.

**Definition 4.3.3**

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and $\mathcal{F} = (L, \preccurlyeq\leq)$ a subpomset of $\mathcal{M}$. Suppose that $T$ is a set containing all lower bounds of the subpomset $\mathcal{F}$, then a point $m_i x_i \in T$ is a *greatest lower bound* for $\mathcal{F}$ if $m_i x_i$ is a maximum point in $T$.

## 4.4 Height and Width of a Pomset

The height and width of a pomset are studied in this section. By exploiting set-based partitioning a set of necessary and sufficient condition is constructed and employed in extending the classical theorem of Dilworth and its dual to ordered msets.

**Definition 4.4.1**

The *height* of a pomset $\mathcal{M}$ denoted by $\hbar$ is the cardinality of maximum mset chain in $\mathcal{M}$.

The *width* of a pomset $\mathcal{M}$ denoted by $\varpi$ is the number of points in a maximum mset antichain in $\mathcal{M}$.

Remark 4.4.1

Ordered structures are usually characterized based on the values of $\hbar$ and $\varpi$. These characterizations are very useful when it is necessary to provide an embedding of the partial order into a linear order in solving application problems. The number of mset chains in a chain

partitioning of $\mathcal{M}$ can be described in relation to the width of $\mathcal{M}$. Likewise, the number of mset antichains in an antichain partitioning of a pomset $\mathcal{M}$ can be described with respect to the height of $\mathcal{M}$. Dilworth's theorem (Dilworth, 1950), and its dual (Mirsky, 1971) describe these relationships in the classical setting.

To characterize the width and height of a pomset, the following result (theorem 4.6.1), which is, a necessary and sufficient condition for these characterizations is first established. The idea of a set-based partition studied in Jouannaud and Lescanne (1982) is exploited.

**Theorem 4.4.1**

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and let $\mathrm{C}_i, \mathcal{A}_j$ be mset chains and mset antichains in $\mathcal{M}$, respectively with $i, j \in \{1,2, \dots, n\}$. Then $\left|\mathrm{C}_i \cap \mathcal{A}_j\right| \leq 1$ for any $i, j$, if and only if the partitions of the mset antichains are such that each occurrence of the generating object of a point $m_i x_i$ belongs to a different partition i.e. $x_i, x_j \in \mathcal{A}_j \Longrightarrow x_i \neq x_j$.

Proof

Assume that $|C_i \cap \mathcal{A}_j| \leq 1$. Now, $C_i \cap \mathcal{A}_j$ is either empty or has only one point for any $i, j$. Let the points $l_1 x_1, \dots, l_n x_n$ be in $\mathcal{A}_j$, with $l_i \leq m_i$. The case where $\left|C_i \cap \mathcal{A}_j\right| < 1$ is trivial. Suppose $C_i \cap \mathcal{A}_j \neq \emptyset$ and let $l_i x_i$ in $\mathcal{A}_j$ be a point in $C_i \cap \mathcal{A}_j$. Now $|C_i \cap \mathcal{A}_j| \leq 1$ implies that $l_i \not> 1$. Hence it must be the case that $l_i = 1$. We can apply this process inductively on all points $l_1 x_1, \dots, l_n x_n \in \mathcal{A}_j$ since each point $l_i x_i \in \mathcal{A}_j$ must belong to a different mset chain $C_i$. Hence all points in $\mathcal{A}_j$ will be of the form $l_i x_i$ with $l_i = 1$ . Therefore, $x_i, x_j \in \mathcal{A}_j \Longrightarrow x_i \neq x_j$.

Next, assume the converse. Clearly, for each point $l_i x_i \in \mathcal{A}_j$, $l_i \not> 1$, otherwise we will have a contradiction. If $C_i \cap \mathcal{A}_j = \emptyset$, the result follows. Now assume that $C_i \cap \mathcal{A}_j$ is not empty and suppose that $|C_i \cap \mathcal{A}_j| > 1$. Then there will be points say $x_1, \dots, x_n$ of $\mathcal{A}_j$, with $n \leq |\mathcal{A}_j|$ in $C_i \cap \mathcal{A}_j$. This implies that $x_1, \dots, x_n$ are comparable since they are also points in $C_i$ which is a contradiction. Hence $C_i \cap \mathcal{A}_j$ is empty or $|C_i \cap \mathcal{A}_j| = 1$. Therefore, $|C_i \cap \mathcal{A}_j| \leq 1$.

□

The next result is an extension of Dilworth's theorem to ordered msets

**Theorem 4.4.2**

Let $\mathcal{M} = (M, \preccurlyeq \leq)$ be a pomset defined over a partially ordered base set. Then $\mathcal{M}$ can be partitioned into exactly $\varpi$ mset chains, where $\varpi$ is the width of the pomset $\mathcal{M}$.

Proof

The case where $\mathcal{M}$ contains only one point $m_i x_i$ is trivial. Suppose the assertion is true for all pomsets $\mathcal{N}_i, i = 1,2, \dots, k$ with $|\mathcal{N}_i| < |\mathcal{M}|$ for each $i$ and let $\mathcal{M} = \mathcal{N}_k \cup [m_i x_i]$, this implies that $|\mathcal{M}| = |\mathcal{N}_k| + |m_i x_i|$. If $\mathcal{A}$ is an mset antichain in $\mathcal{M}$ containing only one point $m_i x_i$, then the assertion is true. Now assume that $\mathcal{A}$ contains more than one point and let $\mathcal{C}$ be a maximal mset chain in $\mathcal{M}$, then $\varpi - |\mathcal{A}| \leq width(\mathcal{M} \setminus \mathcal{C}) \leq \varpi$. Let F be the subpomset $\mathcal{M} \setminus \mathcal{C}$, if F has width $\varpi - |\mathcal{A}|$, by the induction hypothesis F can be partitioned into $\varpi - |\mathcal{A}|$ mset chains, together with $\mathcal{C}$ gives a partition into at most $\varpi$ mset chains.

Furthermore, if the pomset $\mathcal{M}$ is partitioned into $n$ mset chains then, $n = \varpi$. Observe that since $\varpi$ is the cardinality of a maximum mset antichain, every point in that mset antichain must belong

to a different mset chain. Taking $n < \varpi$ will imply that there exist $m_i x_i, m_j x_j \in \mathsf{C}_i$ for some $i, j$ with $m_i x_i || m_j x_j$, which is a contradiction. □

Dually, we present an extension of Mirsky's theorem to pomsets as follows:

**Theorem 4.4.3**

Let $\mathcal{M} = (M, \preccurlyeq \leq)$ be a pomset. Then $\mathcal{M}$ can be partitioned into exactly $\hbar$ mset antichains where $\hbar$ is the height of the pomset $\mathcal{M}$.

Proof

If $\mathcal{M}$ is an mset antichain, we have a trivial case. Next, assume that the theorem holds for pomsets of height $t$ where $t < \hbar$. Define $\mathcal{H}$ to be the mset of all maximal points of $\mathcal{M}$. Clearly $\mathcal{H}$ is an mset antichain in $\mathcal{M}$ and every maximal mset chain in $\mathcal{M}$ contains exactly one point $m_i x_i$ from $\mathcal{H}$ which is also the maximum point in that mset chain. Let $\mathcal{B}$ be the pomset $\mathcal{M} \backslash \mathcal{H}$, height of $\mathcal{B}$, denoted height $(\mathcal{B})$, will be $\hbar - (height\ of\ \mathcal{H})$. By the induction hypothesis, height $(\mathcal{B}) < \hbar$ implies that $\mathcal{B}$ is partitioned into $\hbar - (height\ of\ \mathcal{H})$ mset antichains. Therefore the pomset $\mathcal{B}$ together with $\mathcal{H}$ is partitioned into at most $\hbar$ mset antichains.

Theorem 4.4.1 guarantees that for subpomsets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ with $\mathcal{M} = \bigcup_{i=1}^{n} \mathcal{A}_i$ (where each $\mathcal{A}_i$ is a mset antichain in $\mathcal{M}$) the integer $n$ must be equal to $\hbar$. Using a fewer number of partitions will imply that more than one point in a maximal mset chain belong to some $\mathcal{A}_i$ which is a contradiction. □

The following example illustrates theorems 4.4.2 and 4.4.3.

**Example 4.4.1**

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and $M = [2x_1, 6x_2, 2x_3, 5x_4, 3x_5, x_6]$. Suppose that the ordering $\preccurlyeq\leq$ on $M$ is defined as follows:

$2x_1 \preccurlyeq\leq 6x_2,\ 2x_3 \preccurlyeq\leq 5x_4,\ 2x_1 \preccurlyeq\leq 3x_5.$

The pomset $\mathcal{M}$ has $\varpi = 4\ and\ \hbar = 8$.

Observe that, in an mset chain partitioning of $\mathcal{M}$, there are exactly 4 mset chains as follows:

$\mathrm{C}_1 = [2x_1, 6x_2], \mathrm{C}_2 = [2x_3, 5x_4], \mathrm{C}_3 = [3x_5], \mathrm{C}_4 = [x_6].$

In view of theorem 4.4.1, an mset antichain partitioning of $\mathcal{M}$ has exactly 8 antichains as follows:

$\mathcal{A}_1 = \{x_2, x_4, x_5, x_6\},\ \mathcal{A}_2 = \{x_2, x_4, x_5\},\ \mathcal{A}_3 = \{x_2, x_4, x_5\},\ \mathcal{A}_4 = \{x_2, x_4\},\ \mathcal{A}_5 = \{x_2, x_4\},$

$\mathcal{A}_6 = \{x_2\},\ \mathcal{A}_7 = \{x_1, x_3\},\ \mathcal{A}_8 = \{x_1, x_3\}$

Figure 4.2 is an order diagram of $\mathcal{M}$.

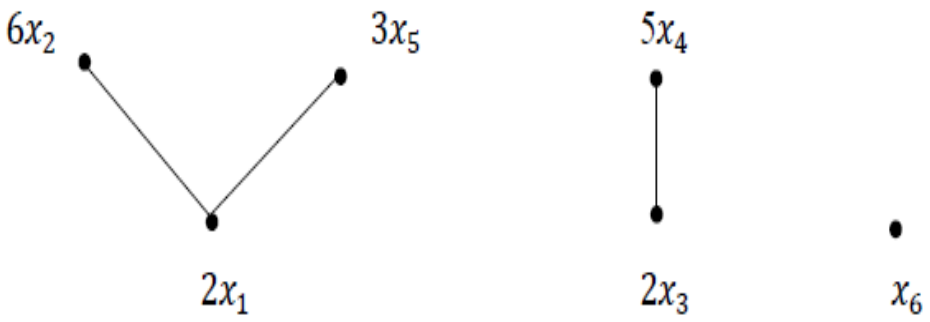



Figure 4.2 Order diagram of the pomset $\mathcal{M}$

## 4.5 The Ordering $\preccurlyeq\leq'$

In this section, we investigate the condition of two points in an mset being comparable if and only if the generating objects as well as their multiplicities are comparable in the same direction. In the sequel, we introduce the notion of semimset chains and present some results.

**Definition 4.5.1**

Let $(S, \preccurlyeq)$ and $(\mathbb{N}, \leq)$ be ordered sets. For any pair of points $m_i x_i \; and \; m_j x_j$ in $M \in M(S)$, $m_i x_i \preccurlyeq\leq' m_j x_j$ if and only if $x_i \preccurlyeq x_j \wedge m_i \leq m_j$. The points $m_i x_i \; and \; m_j x_j$ coincide i.e., $m_i x_i == m_j x_j$ if and only if $x_i = x_j$. Distinct points $m_i x_i \; and \; m_j x_j$ in $M$ are said to be comparable if and only if $m_i x_i \prec\prec' m_j x_j \vee m_j x_j \prec\prec' m_i x_i$, otherwise they are incomparable.

Remark 4.5.1

Observe that $m_i = m_j$ follows from $x_i = x_j$.

The following properties are defined based on definition 4.5.1:

The relation $\preccurlyeq\leq'$ on $M \in M(S)$ is said to be *reflexive* if and only if $m_i x_i \preccurlyeq\leq' m_i x_i$ for any point $m_i x_i \in M$. The relation $\preccurlyeq\leq'$ is *symmetric* if and only if $m_i x_i \preccurlyeq\leq' m_j x_j$ implies that $m_j x_j \preccurlyeq\leq' m_i x_i$ for all $m_i x_i \; and \; m_j x_j$ in $M$. The relation $\preccurlyeq\leq'$ is *antisymmetric* if and only if $m_i x_i \preccurlyeq\leq' m_j x_j \wedge m_j x_j \preccurlyeq\leq' m_i x_i$ implies that $m_i x_i == m_j x_j$ for all $m_i x_i \; and \; m_j x_j$ in $M$, *transitive* if and only if $m_i x_i \preccurlyeq\leq' m_j x_j \wedge m_j x_j \preccurlyeq\leq' m_k x_k$ implies that $m_i x_i \preccurlyeq\leq' m_k x_k$.

**Definition 4.5.2**

A pomset $\mathcal{M}$ is a pair $(M, \preccurlyeq\leq')$, where $M$ is an mset and $\preccurlyeq\leq'$ is a partial order on $M$.

**Theorem 4.5.1**

For any $M \in M(S)$, $\mathcal{M} = (M, \preccurlyeq \leq')$ is a pomset.

Proof

For any $m_i x_i$ in $M$, $x_i \preccurlyeq x_i$ and $m_i \leq m_i$. Thus, $m_i x_i \preccurlyeq \leq' m_i x_i$ and $(M, \preccurlyeq \leq')$ is reflexive.

For antisymmetry, suppose that $m_i x_i \preccurlyeq \leq' m_j x_j$ and $m_j x_j \preccurlyeq \leq' m_i x_i$ in $\mathcal{M}$. Then, $x_i \preccurlyeq x_j$ and $x_j \preccurlyeq x_i$

Thus, $x_i = x_j$ (1)

Similarly, $m_i \leq m_j$ and $m_j \leq m_i$

Thus, $m_i = m_j$ (2)

(1) and (2) imply $m_i x_i == m_j x_j$, therefore $\preccurlyeq \leq'$ is antisymmetric.

Let $m_i x_i, m_j x_j, m_k x_k$ be points in $M$ such that $m_i x_i \preccurlyeq \leq' m_j x_j \preccurlyeq \leq' m_k x_k$

Thus, $x_i \preccurlyeq x_j \preccurlyeq x_k$

In particular, $x_i \preccurlyeq x_k$ (3)

Similarly,

$m_i \leq m_j \leq m_k$

this implies,

$m_i \leq m_k$ (4)

(3) and (4) imply

$m_i x_i \preccurlyeq\leq' m_k x_k$, thus $\preccurlyeq\leq'$ is transitive.

Therefore $(M, \preccurlyeq\leq')$ is a pomset.

□

**Theorem 4.5.2**

Let $(S, \preccurlyeq_1)$ and $(S, \preccurlyeq_2)$ be two posets. If $\mathcal{M} = (M, \preccurlyeq_1 \leq_1{}')$ and $\mathcal{N} = (M, \preccurlyeq_2 \leq_2{}')$ are pomsets such that $M \in M(S)$ then, $\mathcal{M} \cap \mathcal{N} = (M, \preccurlyeq\leq')$ is a pomset. Where $\preccurlyeq\leq' = \preccurlyeq_1 \leq_1{}' \cap \preccurlyeq_2 \leq_2{}'$.

Proof

Now for any point $m_i x_i$ in $M$, clearly $m_i x_i \preccurlyeq_1 \leq_1{}' m_i x_i$ and $m_i x_i \preccurlyeq_2 \leq_2{}' m_i x_i$ since $\preccurlyeq_1 \leq_1{}'$ and $\preccurlyeq_2 \leq_2{}'$ are partial orders.

Thus, $m_i x_i \preccurlyeq\leq' m_i x_i$ (reflexive property)

Let $m_i x_i$ and $m_j x_j$ be points in $M$ such that

$m_i x_i \preccurlyeq\leq' m_j x_j$ and $m_j x_j \preccurlyeq\leq' m_i x_i$ (1)

From (1) we have,

$m_i x_i \preccurlyeq_1 \leq_1{}' m_j x_j$ and $m_j x_j \preccurlyeq_1 \leq_1{}' m_i x_i$ (2)

Since $\preccurlyeq_1 \leq_1{}'$ is antisymmetric, we have

$m_i x_i == m_j x_j$ (3)

Similarly, from (1) we have,

$m_i x_i \preccurlyeq_2 \leq_2{}' m_j x_j$ and $m_j x_j \preccurlyeq_2 \leq_2{}' m_i x_i$ (4)

Since $\preccurlyeq_2 \leq_2{}'$ is antisymmetric, we have

$m_i x_i == m_j x_j$ (5)

From (2) - (5) we can conclude that,

$m_i x_i \preccurlyeq \leq' m_j x_j$ and $m_j x_j \preccurlyeq \leq' m_i x_i$ implies that $m_i x_i == m_j x_j$

Therefore, $\preccurlyeq \leq'$ is antisymmetric.

Let $m_i x_i, m_j x_j$ and $m_k x_k$ be points in $M$ such that,

$m_i x_i \preccurlyeq \leq' m_j x_j$ and $m_j x_j \preccurlyeq \leq' m_k x_k$

We need to show that $m_i x_i \preccurlyeq \leq' m_k x_k$

Now,

$m_i x_i \preccurlyeq \leq' m_j x_j$ and $m_j x_j \preccurlyeq \leq' m_k x_k$ implies that

$m_i x_i \preccurlyeq_1 \leq_1{}' m_j x_j$ and $m_j x_j \preccurlyeq_1 \leq_1{}' m_k x_k$

Since $\preccurlyeq_1 \leq_1{}'$ is transitive, we have

$m_i x_i \preccurlyeq_1 \leq_1{}' m_k x_k$ (6)

Similarly,

$m_i x_i \preccurlyeq_2 \leq_2{}' m_j x_j$ and $m_j x_j \preccurlyeq_2 \leq_2{}' m_k x_k$

Since $\preccurlyeq_2 \leq_2{}'$ is transitive, we have

$m_i x_i \preccurlyeq_2 \leq_2{}' m_k x_k$ (7)

From (6) and (7) above we have,

$m_i x_i \preccurlyeq\leq' m_k x_k$

Now, $m_i x_i \preccurlyeq\leq' m_j x_j$ and $m_j x_j \preccurlyeq\leq' m_k x_k$ implies that $m_i x_i \preccurlyeq\leq' m_k x_k$, hence $\preccurlyeq\leq'$ is transitive.

Therefore, $\mathcal{M} \cap \mathcal{N} = (M, \preccurlyeq\leq')$ is a pomset. □

**Example 4.5.1**

Let $S = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be the root set for the mset $M = [2x_1, 3x_2, 4x_3, 6x_4, 8x_5, 16x_6]$ where $\mathbb{N}$ is ordered by divisibility, and the ordering on $S$ is given by $x_1 \preccurlyeq x_3 \preccurlyeq x_5 \preccurlyeq x_6$, $x_1 \preccurlyeq x_4$, and $x_2 \preccurlyeq x_4$.

By the definition of $\preccurlyeq\leq'$, the following are mset chains in $\mathcal{M}$:

$\mathcal{C}_1 = [2x_1, 4x_3, 8x_5, 16x_6]$

$\mathcal{C}_2 = [2x_1, 6x_4]$

$\mathcal{C}_3 = [3x_2, 6x_4]$

$\mathcal{C}_4 = [4x_3, 8x_5]$

The following are mset antichains in $\mathcal{M}$:

$\mathcal{A} = [2x_1, 3x_2]$

$\mathcal{A} = [6x_4, 16x_6]$

$\mathcal{A} = [3x_2, 8x_5]$

Remark 4.7.2

Observe that with the ordering $\preccurlyeq\leq'$, an element $x_i$ being maximal in $(S, \preccurlyeq)$ does not necessarily imply that $m_i x_i$ is maximal in $(M, \preccurlyeq\leq')$ and vice versa. The following example illustrates this:

**Example 4.5.2**

Let $\mathcal{M}$ be the pomset with only two points say $[10x_1, 5x_2]$ where $x_1 \leq x_2$ in $(S, \preccurlyeq)$. Now $x_1 \leq x_2$ and $10 > 5$ (natural ordering on $\mathbb{N}$), thus $10x_1$ and $5x_2$ are incomparable in $\mathcal{M}$. The point $10x_1$ is maximal in $(M, \preccurlyeq\leq')$ but $x_1$ is not maximal in $(S, \preccurlyeq)$.

## 4.6 Semimset Chains

With the ordering $\preccurlyeq\leq'$, cases where the incomparable pairs of points in a given event satisfy condition i or ii below abound.

i. $((x_i \preccurlyeq x_j) \vee (x_j \preccurlyeq x_i)) \wedge ((m_i \not\leq m_j) \vee (m_j \not\leq m_i))$

ii. $((x_i \not\preccurlyeq x_j) \vee (x_j \not\preccurlyeq x_i)) \wedge ((m_i \leq m_j) \vee (m_j \leq m_i))$

We define subpomsets whose incomparable points are of the form i or/and ii only as follows:

**Definition 4.6.1**

Let $\mathcal{M} = (M, \preccurlyeq\leq')$ be a pomset. A subpomset $\zeta$ of the pomset $\mathcal{M}$ is called a semimset chain if $m_i x_i \preccurlyeq\not\leq' m_j x_j \vee m_i x_i \not\preccurlyeq\leq' m_j x_j$ holds for all incomparable points $m_i x_i \; and \; m_j x_j$ in $M$.

**Example 4.6.1**

Let $\mathcal{M} = (M, \preccurlyeq \leq')$ be a pomset and $M = [6x_1, 4x_2, 6x_3, 5x_4, 4x_5, ]$ with the following order:

$x_1 \preccurlyeq x_2 \preccurlyeq x_5$, $x_3 \preccurlyeq x_4$ and $1 < 3 < 5$, $2 < 4 < 6$.

The following are subpomsets of $\mathcal{M}$:

$H_1 = [2x_1, 4x_2, 3x_5]$

$H_2 = [x_3, 3x_2, 5x_4]$

$H_3 = [3x_3, 2x_4, 4x_5]$

$H_1$ and $H_2$ are semimset chains in $\mathcal{M}$, while $H_3$ is an mset antichain.

Observe that,

For the subpomset $H_1$, we have,

$2x_1 \preccurlyeq \leq' 4x_2$

$2x_1 \preccurlyeq \not\leq' 3x_5$

$4x_2 \preccurlyeq \not\leq' 3x_5$

For the subpomset $H_2$ we have,

$x_3 \not\preccurlyeq \leq' 3x_2$

$x_3 \preccurlyeq \leq' 5x_4$

$3x_2 \preccurlyeq \leq' 5x_4$

For the subpomset $H_3$ we have,

$3x_3 \preccurlyeq \not\trianglelefteq' 2x_4$

$3x_3 \not\preccurlyeq \not\trianglelefteq' 4x_5$

$2x_4 \not\preccurlyeq \trianglelefteq' 4x_5$

**Definition 4.6.2**

A semimset chain $\zeta$ in a pomset $\mathcal{M}$ is maximal if it is not strictly contained in any other semimset chain of $\mathcal{M}$, and a maximum semimset chain if it is of maximum cardinality in $\mathcal{M}$.

**Theorem 4.6.1**

An mset antichain $\mathcal{A}$ in a pomset $\mathcal{M}$ is a semimset chain if its root set is linearly ordered.

Proof

Let $\mathcal{A}^*$ denote the root set of $\mathcal{A}$. Now, $\mathcal{A}$ is an mset antichain implies that $m_i x_i || m_j x_j$ for all $m_i x_i, m_j x_j$, hence neither $m_i x_i \preccurlyeq \trianglelefteq' m_j x_j$ nor $m_j x_j \preccurlyeq \trianglelefteq' m_i x_i$ holds in $\mathcal{M}$ for all $i, j$. But for all $x_i, x_j \in \mathcal{A}^*$, we have $(x_i \preccurlyeq x_j) \vee (x_j \preccurlyeq x_i)$, suppose $\mathcal{A}^* = x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n$.

Since $\mathcal{A}$ is an mset antichain, and $x_i \preccurlyeq x_j$ for all $x_i, x_j$ with $i \leq j$, then it must be the case that $m_i \not\trianglelefteq m_j$ holds for all points $m_i x_i, m_j x_j$. This implies that for all $m_i x_i, m_j x_j \in \mathcal{A}$, we have $m_i x_i \preccurlyeq \trianglelefteq' m_j x_j$. Therefore $\mathcal{A}$ is a semimset chain. □

**Corollary 4.6.2**

A semimset chain $\zeta$ in a pomset $\mathcal{M}$ is an mset antichain if for all points $m_i x_i, m_j x_j \in \zeta$, we have $m_i x_i \preccurlyeq\leq' m_j x_j \vee m_i x_i \preccurlyeq\not\leq' m_j x_j$.

Proof

Suppose $m_i x_i \preccurlyeq\leq' m_j x_j \vee m_i x_i \preccurlyeq\not\leq' m_j x_j$ for all $m_i x_i, m_j x_j \in \zeta$. The result is straightforward from the fact that $m_i || m_j x_j$ for all $m_i x_i, m_j x_j \in \zeta$. □

Remark 4.6.1

Observe that in this case a linear ordering on $(S, \preccurlyeq)$ cannot induce a linear mset ordering on $(M, \preccurlyeq\leq')$. Embedding the partial order $\preccurlyeq\leq'$ into a linear order when the generating objects and their multiplicities are comparable but with reversed orders needs further vindication. We investigate combinatorics of pomsets in the next chapter using the ordering $\preccurlyeq\leq$.

# CHAPTER FIVE

# COMBINATORICS OF PARTIALLY ORDERED MULTISETS

Notions of combinatorics of ordered multisets are investigated in this chapter. The classical extension theorem for ordered sets is generalized. Also, an algorithm for generating mset linear extensions of ordered multisets is constructed and implemented on a partially ordered multiset. Finally, some characterizations for the dimension of an ordered multiset are presented.

## 5.1 Multiset Linear Extensions

In this section, the notion of mset linear extension is introduced and a classical extension theorem is extended to ordered mset.

**Definition 5.1.1**

Let $M \in M(S)$ and $\mathcal{M} = (M, \preccurlyeq \leq_{\mathcal{M}})$, $\mathcal{N} = (M, \preccurlyeq \leq_{\mathcal{N}})$ be two pomsets defined over partially ordered base sets $P$ and $Q$, respectively, where $P$ and $Q$ have the same ground set. Then $\mathcal{N}$ is an *mset extension* of $\mathcal{M}$ if $m_i x_i \preccurlyeq \leq_{\mathcal{M}} m_j x_j$ implies that $m_i x_i \preccurlyeq \leq_{\mathcal{N}} m_j x_j$ i.e. the relation $\preccurlyeq \leq_{\mathcal{M}}$ is contained in $\preccurlyeq \leq_{\mathcal{N}}$. An mset extension of $\mathcal{M}$ that is a linear mset order is called an *mset linear extension* of $\mathcal{M}$.

The following theorem is an extension of Szpilrajn's result to ordered msets.

**Theorem 5.1.1**

Let $\mathcal{M} = (M, \preccurlyeq \leq_{\mathcal{M}})$ be a pomset with a partially ordered base set and let $m_i x_i, m_j x_j$ be points in $M$ with $m_i x_i || m_j x_j$ in $\mathcal{M}$. Then there exists a pomset $\mathcal{N}$ extending $\mathcal{M}$ such that $m_i x_i \preccurlyeq \leq_{\mathcal{N}} m_j x_j$.

In order to prove this theorem, proofs of the following two complementary results are first outlined:

**Lemma 5.1.2**

Let $P$ be a poset and let $x, y$ be incomparable elements in $P$. Then there exists a poset $Q$ extending $P$ such that $x \preccurlyeq_Q y$.

Proof

The proof is a consequence of Szpilrajn's extension theorem: *Every partial order is contained into a total order.*

Let $Q = P \cup (\downarrow x \times \uparrow y)$, i.e., $Q$ is obtained from $P$ by adjoining P with all pairs of points $(u, v)$ for which $u \preccurlyeq_P x$ and $y \preccurlyeq_P v$. Since $x \preccurlyeq_Q y$, we need to show that the ordering on $Q$ is a partial order. Now $\downarrow x \cap \uparrow y = \emptyset$, since if $z$ is in this intersection, then $y \preccurlyeq_P z$ and $z \preccurlyeq_P x$ would yield $y \preccurlyeq_P x$, a contradiction. $Q$ is reflexive since $P \subseteq Q$. To show that the order on $Q$ is antisymmetric, suppose that $u \preccurlyeq_Q v$ and $v \preccurlyeq_Q u$ with $u \neq v$. By the definition of $Q$, either $u \preccurlyeq_P v$ or, $u \preccurlyeq_P x$ and $y \preccurlyeq_P v$. Similarly, either $v \preccurlyeq_P u$ or, $v \preccurlyeq_P x$ and $y \preccurlyeq_P u$. Suppose $u \preccurlyeq_P v$, then $v \preccurlyeq_P u$ cannot hold, otherwise we will have $u = v$, which is a contradiction. Hence $v \preccurlyeq_P x$ and $y \preccurlyeq_P u$. However, $y \preccurlyeq_P u \preccurlyeq_P v \preccurlyeq_P x$ gives a contradiction i.e., $y \preccurlyeq_P x$. Similarly, the case $v \preccurlyeq_P u$ gives a contradiction. Suppose, $u \preccurlyeq_P x$ and $y \preccurlyeq_P v$ also, $v \preccurlyeq_P x$ and $y \preccurlyeq_P u$, which contradicts the fact that $\downarrow x \cap \uparrow y = \emptyset$. Hence $u \preccurlyeq_P v$ and $v \preccurlyeq_P u$ must imply that $u = v$.

To show transitivity, assume that $u \preccurlyeq_Q v$ and $v \preccurlyeq_Q w$. Now, either $u \preccurlyeq_P v$ or, $u \preccurlyeq_P x$ and $y \preccurlyeq_P v$. Also, either $v \preccurlyeq_P w$ or, $v \preccurlyeq_P x$ and $y \preccurlyeq_P w$. Again we have to consider the following cases. First assume that $u \preccurlyeq_P v$, if also $v \preccurlyeq_P w$, then the claim follows from transitivity of $P$. Now suppose that $v \preccurlyeq_P x$ and $y \preccurlyeq_P w$. Also $u \preccurlyeq_P x$, and this implies that $u \in \downarrow x$ and $w \in \uparrow y$ hence $u \preccurlyeq_Q w$ as required. Next, suppose that $u \preccurlyeq_P x$ and $y \preccurlyeq_P v$. Now, if $v \preccurlyeq_P w$, then also $y \preccurlyeq_P w$ and this implies that $u \in \downarrow x$ and $w \in \uparrow y$, hence $u \preccurlyeq_Q w$. The case $v \preccurlyeq_P x$ and $y \preccurlyeq_P w$ gives a contradiction since $\downarrow x \cap \uparrow y = \emptyset$.

The next result gives necessary and sufficient condition for a pomset $\mathcal{N}$ to be an mset extension of $\mathcal{M}$.

**Lemma 5.1.3**

Let $\mathcal{M}$ and $\mathcal{N}$ be two pomsets defined over posets $P$ and $Q$, respectively, such that $P$ and Q have a common ground set. Then $\mathcal{N}$ is an mset extension of $\mathcal{M}$ if and only if $Q$ is an extension of $P$.

Proof

Let $\mathcal{M} = (M, \preccurlyeq_{\mathcal{M}})$, and $\mathcal{N} = (M, \preccurlyeq_{\mathcal{N}})$ be pomsets over $P$ and $Q$, respectively. Suppose that $\mathcal{N}$ is an mset extension of $\mathcal{M}$. Then,

$m_i x_i \preccurlyeq_{\mathcal{M}} m_j x_j$ implies $m_i x_i \preccurlyeq_{\mathcal{N}} m_j x_j$. (1)

Now,

$m_i x_i \preccurlyeq_{\mathcal{M}} m_j x_j \Leftrightarrow x_i \preccurlyeq_P x_j$, since $\mathcal{M}$ is defined over $P$. (2)

Similarly, $m_i x_i \preccurlyeq_{\mathcal{N}} m_j x_j \Leftrightarrow x_i \preccurlyeq_Q x_j$. (3)

From (1) through (3) we have $x_i \preccurlyeq_P x_j \Leftrightarrow m_i x_i \preccurlyeq_{\mathcal{M}} m_j x_j \Longrightarrow m_i x_i \preccurlyeq_{\mathcal{N}} m_j x_j \Leftrightarrow x_i \preccurlyeq_Q x_j$.

Hence $x_i \preccurlyeq_Q x_j$ whenever $x_i \preccurlyeq_P x_j$.

Therefore, $Q$ is an extension of $P$.

Next, assume that $Q$ is an extension of $P$, then we must have

$x_i \preccurlyeq_Q x_j$ whenever $x_i \preccurlyeq_P x_j$.

(4)

Now, $x_i, x_j$ are also elements of $M$, hence we must have $m_i, m_j \in \mathbb{N}$ with $M(x_i) = m_i$ and $M(x_j) = m_j$.

We have,

$x_i \preccurlyeq_P x_j \Longrightarrow m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j$ for any pair of points $m_i x_i, m_j x_j \in M$.

(5)

Also,

$x_i \preccurlyeq_Q x_j \Longrightarrow m_i x_i \preccurlyeq\leq_{\mathcal{N}} m_j x_j$, for some pomset $\mathcal{N}$.

(6)

Now, from (4) through (6), we have $m_i x_i \preccurlyeq\leq_{\mathcal{N}} m_j x_j$ whenever $m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j$.

Therefore, $\mathcal{N}$ is an mset extension of $\mathcal{M}$.

□

We now give the proof of theorem 5.1.1 using lemmas 5.1.2 and 5.1.3.

Proof

Let $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$ be a pomset defined over a partially ordered base set $P$. Suppose there exist points $m_i x_i$ and $m_j x_j$ in $M$ with $m_i x_i || m_j x_j$ in $\mathcal{M}$. Then we have $\neg((m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j) \wedge (m_j x_j \preccurlyeq\leq_{\mathcal{M}} m_i x_i))$. This implies that neither $x_i \preccurlyeq_P x_j$ nor $x_j \preccurlyeq_P x_i$ holds in $P$. Therefore, $x_i || x_j$ in $P$. By lemma 5.1.2, there exists a poset, say $Q$, extending $P$ such that $x_i \preccurlyeq_Q x_j$. Now, $x_i$ and $x_j$ are objects in $M$, with multiplicities $m_i$ and $m_j$, respectively. We

have $x_i \preccurlyeq_Q x_j$ implies $m_i x_i \preccurlyeq\leq_{\mathcal{N}} m_j x_j$ for some $\mathcal{N}$ defined over the poset $Q$. Suppose that $\mathcal{N}$ is given by the structure $(M, \preccurlyeq\leq_{\mathcal{N}})$, it can be verified that $\mathcal{N}$ is a pomset. It follows from lemma 5.1.3 that $\mathcal{N}$ is an mset extension of $\mathcal{M}$. □

**Lemma 5.1.4:** A generalization of theorem 5.1.1

Every finite pomset has a linear mset extension.

Proof

For a pomset $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$, let $\mathcal{U}$ denote the mset containing all pairs $m_i x_i, m_j x_j$ in $M$ with $m_i x_i || m_j x_j$ in $\mathcal{M}$. The claim follows by applying the construction in theorem 5.1.1 inductively on all pairs $m_i x_i, m_j x_j \in \mathcal{U}$ to obtain a pomset $\mathcal{N}$ having an mset linear order $\preccurlyeq\leq_{\mathcal{N}}$ on it. The order $\preccurlyeq\leq_{\mathcal{N}}$ contains all mset extensions of the original mset order $\preccurlyeq\leq_{\mathcal{M}}$. □

**Example 5.1.1**

Let $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$ be a pomset where $M = [3x_1, 5x_2, 2x_3, x_4, x_5, 4x_6, 3x_7]$. Suppose that the mset order on $M$ is defined as follows:

$3x_1 \preccurlyeq\leq_{\mathcal{M}} 5x_2 \preccurlyeq\leq_{\mathcal{M}} x_4 \preccurlyeq\leq_{\mathcal{M}} 3x_7$

$3x_1 \preccurlyeq\leq_{\mathcal{M}} 2x_3 \preccurlyeq\leq_{\mathcal{M}} 4x_6$

$3x_1 \preccurlyeq\leq_{\mathcal{M}} 2x_3 \preccurlyeq\leq_{\mathcal{M}} x_5 \preccurlyeq\leq_{\mathcal{M}} 3x_7$

$3x_1 \preccurlyeq\leq_{\mathcal{M}} 5x_2 \preccurlyeq\leq_{\mathcal{M}} x_5 \preccurlyeq\leq_{\mathcal{M}} 3x_7$.

Lemma 5.1.4 establishes that the set $\mathcal{L}$ of mset linear extensions of the pomset $\mathcal{M}$ is not empty. Figure 5.1 is an order diagram of $\mathcal{M}$ and some of its possible mset linear extensions. Observe

that $\ell_1$, $\ell_2$, $\ell_3$, $\ell_4$, $\ell_5$, $\ell_6$, $\ell_7$ and $\ell_8$ are all linear extensions of $\mathcal{M}$, since for each $i$, $m_i x_i \preccurlyeq \leq_{\ell_i} m_j x_j$ whenever $m_i x_i \preccurlyeq \leq_{\mathcal{M}} m_j x_j$.

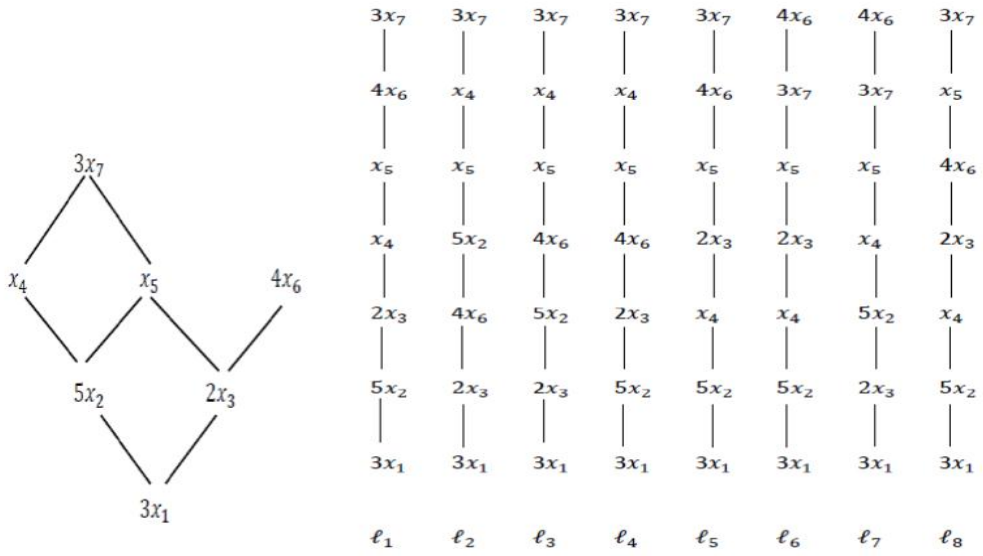



Figure 5.1. The pomset $\mathcal{M}$ and some of its mset linear extensions

## 5.2 The M-LIN Algorithm

Various algorithms for generating linear extensions of ordered sets and minning posets from linear orders abound in the literature (Varol and Rotem, 1981; Bianco *et al.*, 1997; Korsh and Lafollette, 2002; Fernandez *et al.*, 2013). In this section, a heuristic algorithm for generating mset linear extensions is constructed and implemented on a randomly generated pomset.

Let $\mathcal{M} = (M, \preccurlyeq \leq)$ be a pomset defined over a partially ordered base set $P$. The subposet $(M^*, \preccurlyeq)$ of $P$ will be a root poset for $(M, \preccurlyeq \leq)$. An mset linear extension of the pomset $\mathcal{M}$ is an mset

linear order $\mathcal{L} = (M, \preccurlyeq_{\mathcal{L}})$ induced by the linear order $L = (S, \preccurlyeq_L)$ on $P = (S, \preccurlyeq)$, such that $m_i x_i \preccurlyeq_{\mathcal{L}} m_j x_j$ whenever $m_i x_i \preccurlyeq m_j x_j$ in $\mathcal{M}$, and $m_i x_i \preccurlyeq_{\mathcal{L}} m_j x_j \vee m_j x_j \preccurlyeq_{\mathcal{L}} m_i x_i$, for all $m_i x_i, m_j x_j$ in $M$. The mset linear order $\mathcal{L}$ can be obtained through the following procedure.

Given an mset $M$

1. Choose $x \in M^*$ if $freq(x)$ in $M$ is greater than 0
2. Choose $M(x) = freq(x)$ for each $x \in M^*$
3. Choose a minimal element $x_1$ in $M^*$
4. Choose $m_1$ in $M(x)$ such that $M(x_1) = m_1$
5. Choose $m_1 x_1$ to be minimal in $\mathcal{M}$
6. Given that $m_1 x_1, m_2 x_2, \dots, m_i x_i$ have been chosen, choose a minimal element $x_{i+1}$ in $M^* \backslash \{x_1, x_2, \dots, x_i\}$
7. Choose $m_{i+1}$ in $M(x)$ such that $M(x_{i+1}) = m_{i+1}$

The following is a heuristic algorithm for generating mset linear orders $[m_1 x_1, m_2 x_2, \dots, m_n x_n]$ of the pomset $\mathcal{M}$. We call this algorithm *M-LIN (1)*. The algorithm is split into two parts, viz., algorithm *M-LIN 1.1* and algorithm *M-LIN 1.2*.

### 5.2.1 ALGORITHM M-LIN (1)

Algorithm M-LIN 1.1(M$[x_1, \dots, x_n]$)

Create Array $M^*, M(x)$

For $i \leftarrow 0$ to $len(M) - 1$

temp← $M[i]$

if temp is not in $M^*$

$M^*[] \leftarrow temp$

$freq \leftarrow 0$

for $j \leftarrow 0$ to $len(M) - 1$

if temp= $M[j]$

$freq = freq + 1$

end if

end for

$multiplicity[] \leftarrow freq$

end if

end for

Algorithm M-LIN 1.2

Create Array $CM$

$T = M^*$

$q = multiplicity$

While $(len(T) > 0)$

$X = Min(T)$

$i = T.Index\ Of(X)$

$Y = q[i]$

If $(Y > 1)$

$CM[] \leftarrow Y.X$

Else

$CM[] \leftarrow X$

$T.Remove\ (X)$

end while

Return $CM$

Remark 5.2.1

Algorithm M-LIN 1.1 creates two arrays from the input set, viz., the root set and the multiplicity arrays.

Algorithm M-LIN 1.2 generates the mset linear order from the two arrays based on the defined order while maintaining all preexisting hierarchies.

**Example 5.2.1**

Suppose the pomset in example 5.1.1 is ordered by *divisibility*, where the mset $M =$ [2,5,2,3,6,6,12,1,8,8,1,3,2,1,8,12,3,12,3].

If we denote a point $m_i x_i$ in $M$ by $m_i/x_i$ then the pomset $\mathcal{M}$ can be written as,

$3/1 \preccurlyeq_{\mathcal{M}} 3/2 \preccurlyeq_{\mathcal{M}} 2/6 \preccurlyeq_{\mathcal{M}} 3/12$

$3/1 \preccurlyeq_{\mathcal{M}} 3/2 \preccurlyeq_{\mathcal{M}} 3/8$

$3/1 \preccurlyeq_{\mathcal{M}} 4/3 \preccurlyeq_{\mathcal{M}} 2/6 \preccurlyeq_{\mathcal{M}} 3/12$

$3/1 \preccurlyeq_{\mathcal{M}} 1/5$

The set $\mathcal{J} = \{(3/2,4/3), (3/2,1/5), (4/3, 1/5), (1/5, 2/6), (1/5,3/8\ ), (1/5, 3/12), (2/6,3/8), (3/8, 3/12)\}$

represents the incomparable pairs in $\mathcal{M}$.

Figure 5.2 is an order diagram of the pomset in example 5.2.1.

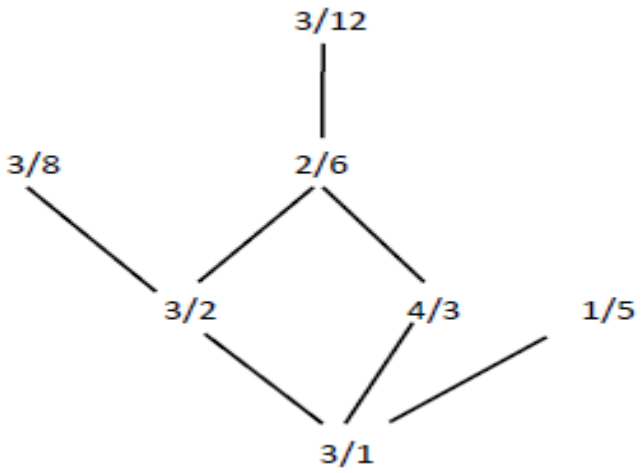



Figure 5.2 Order diagram of pomset in example 5.2.1

Based on the defined ordering, the following exact algorithm is constructed for generating mset linear extensions of $\mathcal{M} = (M, \preccurlyeq \leq_{\mathcal{M}})$.

### 5.2.2 ALGORITHM M-LIN (2)

Algorithm M-LIN 2.1 (M=[2,5,2,3,6,6,12,1,8,8,1,3,2,1,8,12,3,12,3])

Create Array $M^*, Multiplicity$

For $i \leftarrow 0$ to $len(M) - 1$

temp$\leftarrow M[i]$

if temp is not in $M^*$

$M^*[] \leftarrow temp$

$freq \leftarrow 0$

for $j \leftarrow 0$ to $len(M) - 1$

if temp$= M[j]$

$freq = freq + 1$

end if

end for

$multiplicity[] \leftarrow freq$

end if

end for

Algorithm M-LIN 2.2

Create Array $mset\ Linear\ extension$

Sort ($M^*[0 \dots n-1], multiplicities[0, \dots, n-1]$)

for $i = 0$ to $n-2$ {

min $\leftarrow i$

for $j \leftarrow i+1$ to $n-1$ {

if ($M^*[min] modulo\ M^*[j] = 0$) min$\leftarrow j$

}

if $min \neq i$ then swap ($M^*[min], M^*[i]$); $swap\ (multiplicity[min], multiplicity[i])$

}

$q = multiplicity$

While ($len(M^*) > 0$)

$X = M^*[min]$

$i = M^*.Index\ Of(X)$

$Y = q[i]$

$mset_linear_extension[] \leftarrow Y.X$

$T.Remove\ (X)$

$T.Remove(Y)$

end while

Return $mset_linear_extension$

Algorithm M-LIN 2.1 generates the following arrays:

$M^*$: [2,5,3,6,12,1,8]

Multiplicity: [3,1,4,2,3,3,3]

Algorithm M-LIN 2.1 runs in quadratic time $O(n^2)$.

Algorithm M-LIN 2.2 sorts the root set based on the *divisibility* relation, it obtains the multiplicity of the minimal element at each stage and returns $m_i/x_i$. The output is an mset linear extension of the pomset $\mathcal{M}$.

Mset Linear Extension1 of $\mathcal{M}$ => $[3/1,\ 1/5,\ 4/3,\ 3/2,\ 2/6, 3/12, 3/8]$

It is decidable and runs in polynomial time.

The mset linear extension of $\mathcal{M}$ is generated with a complexity $O(n^2)$.

Figure 5.3 is a screen shot of the generated mset linear extension based on the defined input.

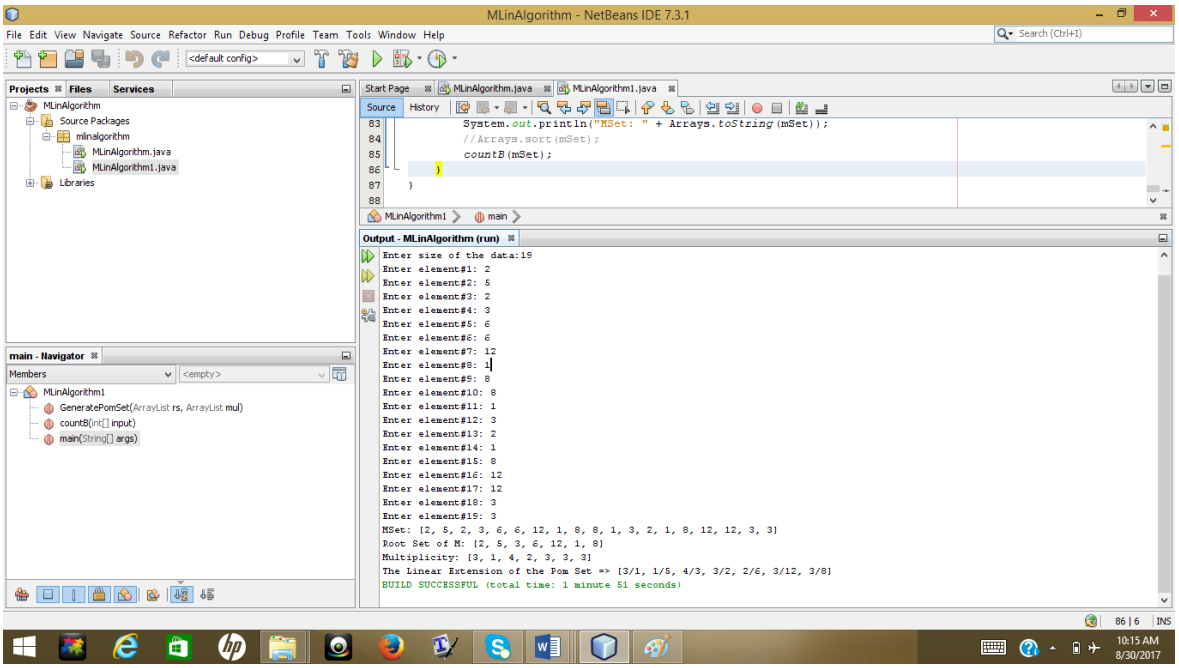



Figure 5.3 Screen shot of mset linear extension1

A new set of mset linear extension can be generated by interchanging the position of the elements:

Input: [1, 3, 3, 1, 6, 1, 3, 8, 2, 6, 3, 8, 12, 5, 8, 2, 12, 12, 2]

$M^*$: [1, 3, 6, 8, 2, 12, 5]

Multiplicity: [3, 4, 2, 3, 3, 3, 1]

Mset Linear Extension 2 of $\mathcal{M}$ => [3/1, 4/3, 3/2, 3/8, 2/6, 3/12, 1/5]

Figure 5.4 is a screen shot of the generated mset linear extension based on the new input.

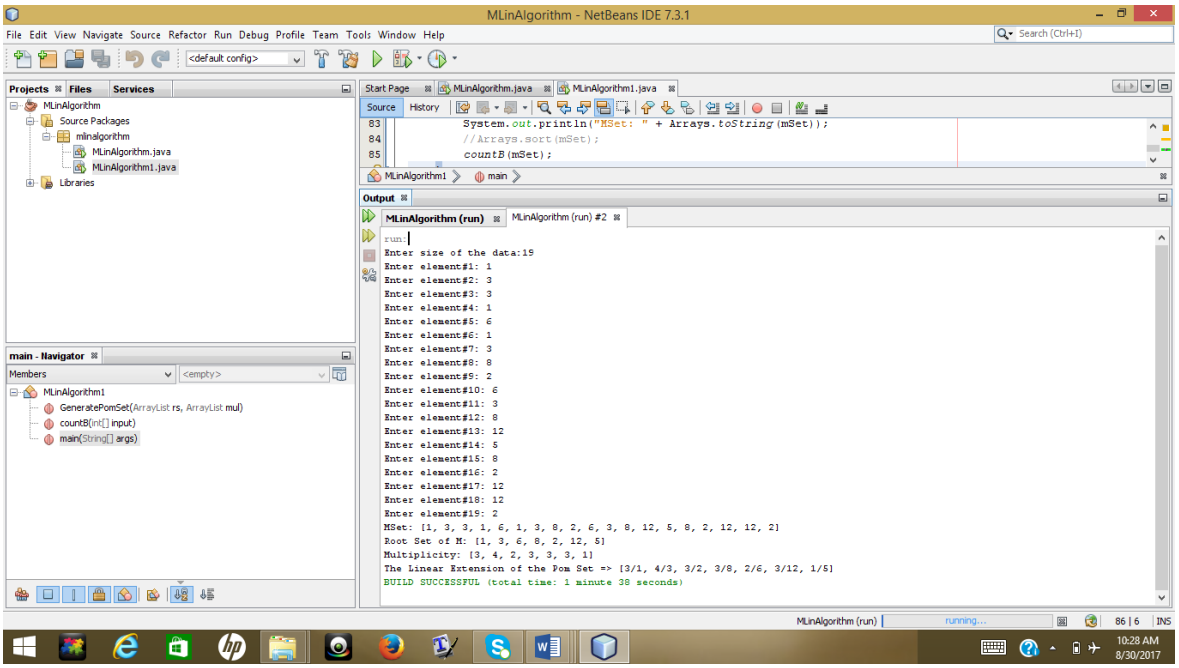



Figure 5.4 Screen shot of mset linear extension 2

For optimization, a more restrictive class of mset linear extensions can be obtained by restricting the choice of $m_{i+1}x_{i+1}$via certain constraints.

## 5.3 Mset Realizers

Let $\mathcal{J} = \{(m_i x_i, m_j x_j) \in M: m_i x_i || m_j x_j \; in \; \mathcal{M} \}$. Recall that $m_i x_i \preccurlyeq\leq m_j x_j$ in $\mathcal{M}$ if and only if $x_i \preccurlyeq x_j$ in the base set $P$. For any pair $m_i x_i, m_j x_j$ in $\mathcal{J}$, we must have $x_i || x_j$ in $P$. By Szpilrajn's extension theorem for ordered sets, there exist two linear extensions $l_1, l_2$ with $x_i \preccurlyeq x_j$ in $l_1$ and $x_j \preccurlyeq x_i$ in $l_2$. Consequently, $m_i x_i \preccurlyeq\leq_{\ell_1} m_j x_j$ and $m_j x_j \preccurlyeq\leq_{\ell_2} m_i x_i$, for some linearly ordered mset extensions $\ell_1$ and $\ell_2$ of $\mathcal{M}$ induced by $l_1, l_2$. An *mset realizer* of a pomset is defined as follows:

**Definition 5.3.1**

A family $\{\ell_1, \ell_2, \dots, \ell_t\}$ of mset linear extensions of $\mathcal{M}$ is called an *mset realizer* of $\mathcal{M}$ if

$\mathcal{M} = \ell_1 \cap \ell_2 \cap \dots \cap \ell_t$, i.e., for each pair $m_i x_i || m_j x_j$ in $\mathcal{M}$ there exist $\ell_i, \ell_j$ in $\bigcap_{i=1}^{t} \ell_i$ such that $m_i x_i \preccurlyeq_{\ell_i} m_j x_j$ and $m_j x_j \preccurlyeq_{\ell_j} m_i x_i$.

**Theorem 5.3.1**

Every finite pomset $\mathcal{M}$ is the intersection of mset linear orders on $\mathcal{M}$.

Proof

Let $\mathcal{L}$ be the set containing all mset linear extensions of the pomset $\mathcal{M}$ and let $\omega = \bigcap_{\mathcal{M} \subseteq \mathcal{L}} \mathcal{L}$. By Lemma 5.1.4, the family of such mset linear extensions is not empty. Also, $\omega$ is partially ordered, and $\mathcal{M} \subseteq \omega$. For any pair $m_i x_i, m_j x_j$ such that $m_i x_i || m_j x_j$ in $\mathcal{M}$, there exists an mset linear order $\ell_i$ with $\mathcal{M} \subseteq \ell_i$ such that $m_i x_i \preccurlyeq_{\ell_i} m_j x_j$ (theorem 5.1.1). Also, there exists another mset linear order $\ell_j$ for which $\mathcal{M} \subseteq \ell_j$ and $m_j x_j \preccurlyeq_{\ell_j} m_i x_i$. This implies that $m_i x_i || m_j x_j$ in $\omega$. Hence $\omega = \mathcal{M}$. □

Figure 5.5 is an example of a pomset $\mathcal{M}$ and its mset realizer. i.e., $\mathcal{M} = \ell_1 \cap \ell_2 \cap \ell_3 \cap \ell_4 \cap \ell_5 \cap \ell_6$.

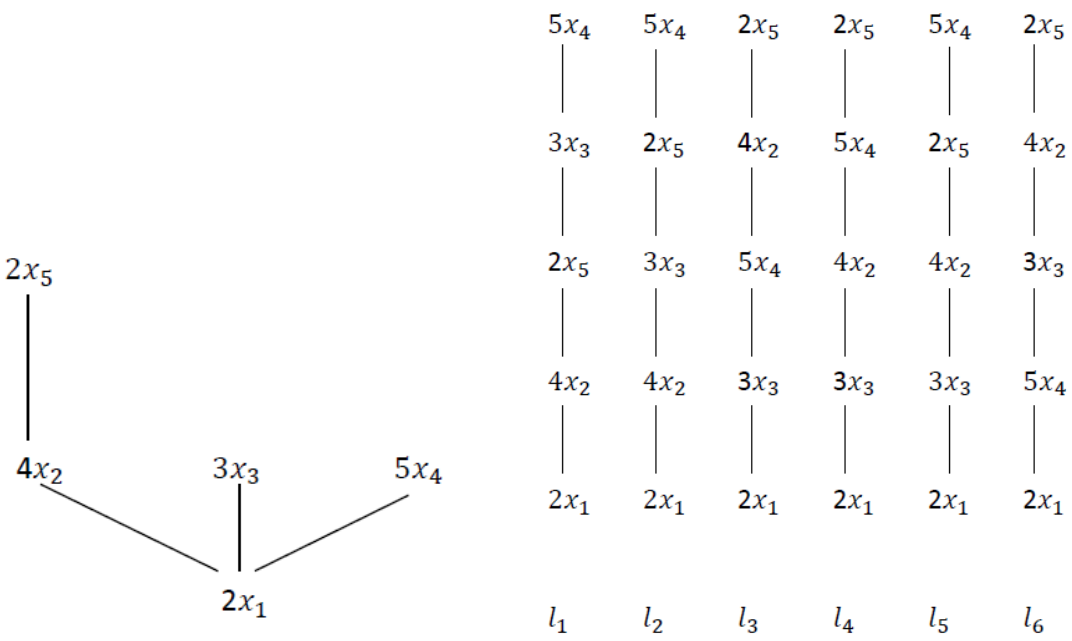



Figure 5.5. A pomset $\mathcal{M}$ and its realizer

The following result shows that an mset realizer is bounded in terms of the realizer of the underlying generic set.

**Theorem 5.3.2**

Let $\mathcal{M}$ be a pomset defined over a poset $P$. If $\mathcal{R} = \{\ell_1, \ell_2, \dots \ell_t\}$ and $R = \{l_1, l_2, \dots, l_s\}$ are realizers for $\mathcal{M}$ and $P$, respectively, then $t \leq s$.

Proof

Given a pomset $\mathcal{M} = (M, \preccurlyeq \leq)$, where $M$ has points $m_1x_1, m_2x_2, \dots, m_nx_n$. Let $\mathcal{J} = \{(m_ix_i, m_jx_j) \in M | m_ix_i || m_jx_j \; in \; \mathcal{M}\}$. By theorem 5.1.1, for any pair $m_ix_i, m_jx_j$ in $\mathcal{J}$, there exist mset linear extensions $\ell_i, \ell_j$ with $m_ix_i \prec\prec_{\ell_i} m_jx_j$ and $m_jx_j \prec\prec_{\ell_j} m_ix_i$. This implies

$t \leq 2n$ for $n$ incomparable pairs in $\mathcal{M}$. Now, for any pair $m_i x_i, m_j x_j$ in $M$ such that $m_i x_i || m_j x_j \; in \; \mathcal{M}$, there is a corresponding pair $x_i || x_j$ in $(S, \preccurlyeq)$. By lemma 5.1.2, there exists linear extensions $l_i, l_j$ say, such that $x_i \preccurlyeq_{l_i} x_j$ and $x_j \preccurlyeq_{l_j} x_i$. Similarly, $s \leq 2m$ for m incomparable pairs in $P$. Let $\mathcal{U} = \{(x_i, x_j) \in S | x_i || x_j \; in \; P\}$. If for each pair $(x_i, x_j) \in \mathcal{U}$, there exists a corresponding pair $(m_i x_i, m_j x_j) \in \mathcal{J}$, then $m = n$. By the definition of $\preccurlyeq\leq$, each linear order $\ell_i$ is induced by the linear order $l_i$, this implies $t = s$, otherwise $t < s$, hence the result. □

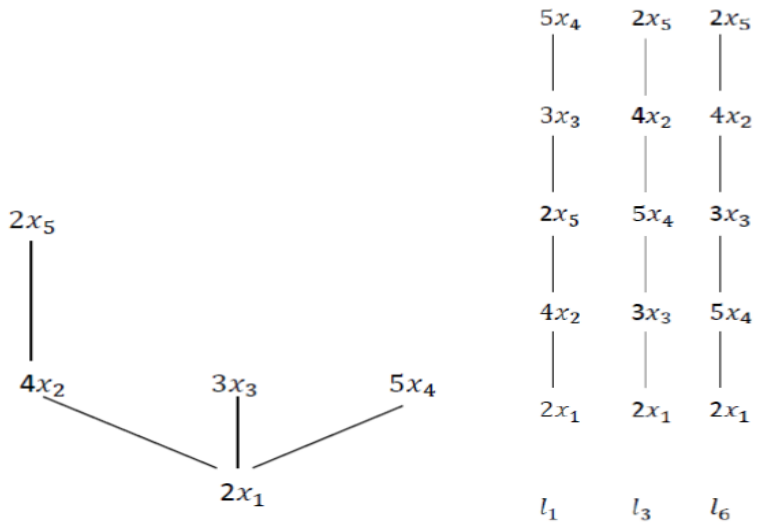



Figure 5.6. $\mathcal{M}$ and its realizer (using three linear orders)

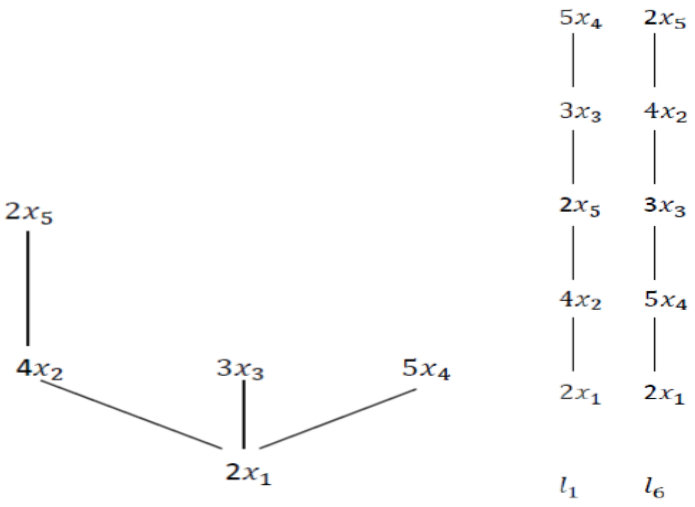


Figure 5.7. $\mathcal{M}$ and its realizer (using two linear orders)

The sets $\{\ell_1, \ell_3, \ell_6\}$ and $\{\ell_1, \ell_6\}$ in figures 5.6 and 5.7 are also mset realizers for $\mathcal{M}$. The fact that different sets of linear orders can realize a given ordered structure leads to another interesting combinatorial parameter known as *dimension*.

## 5.4 Dimension of a Partially Ordered Multiset

In this section, the concept of dimension of a pomset is introduced. The relationship between the dimension of a pomset and the dimension of the underlying base set is investigated and some characterizations are established.

**Definition 5.4.1**

Let $\mathcal{M}$ be a pomset, the dimension of $\mathcal{M}$, denoted by $\dim(\mathcal{M})$, is the least positive integer $t$ for which $\mathcal{M} = \bigcap_{i=1}^{t} \ell_i$, with $\mathcal{M} \subset \ell_i$ for each $i$, where $\ell_i$ is an mset linear order on $\mathcal{M}$.

**Proposition 5.4.1**

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset with dimension, $\dim(\mathcal{M})$, defined over a partially ordered base set $P = (S, \preccurlyeq)$ with dimension $\dim(P)$. Then for any $M \in M(S)$, $\dim(\mathcal{M}) \leq \dim(P)$.

Proof

The result follows from the proof of the preceding theorem. If for each pair $(x_i, x_j) \in \mathcal{U}$, there exists a corresponding pair $(m_i x_i, m_j x_j) \in \mathcal{J}$, then $m = n$ (for $m$ and $n$ incomparable pairs in $\mathcal{J}$

and $\mathcal{U}$, respectively). Consequently $t = s$. This implies that the minimum cardinality for which $\bigcap \ell_i = \mathcal{M}$ is the same for $\bigcap l_i = P$. Therefore $\dim(\mathcal{M}) = \dim(P)$.

Next, assume that $\dim(\mathcal{M}) > \dim(P)$, this implies that $m > n$ i.e., there exist $(m_i x_i, m_j x_j) \in \mathcal{J}$ with no corresponding generating pairs $(x_i, x_j) \in \mathcal{U}$, this is a contradiction. Hence, $\dim(\mathcal{M}) < \dim(P)$. Therefore $\dim(\mathcal{M}) \leq \dim(P)$ as required. □

**Proposition 5.4.2**

Let $\mathcal{M} = (M, \preccurlyeq \leq)$ be a pomset defined over a poset $P = (S, \preccurlyeq_P)$, and $Q$ a poset induced by a subset $X$ of $S$. If the subset $X = M^*$, then dimension of $\mathcal{M}$ is equal to the dimension of the subposet $Q$.

Proof

Suppose $X = M^*$ and let $\preccurlyeq_Q$ be the restriction of $\preccurlyeq_P$ to elements of $X$. For any $x_i, x_j$ in $X$ with $x_i \preccurlyeq_Q x_j$ there exist points $m_i x_i, m_j x_j$ in $M$, with $m_i x_i \preccurlyeq \leq m_j x_j$ in $\mathcal{M}$. Hence for all $x_i || x_j$ in $Q$, we have $m_i x_i || m_j x_j$ in $\mathcal{M}$. If $m$ and $n$ are the number of incomparable pairs in $\mathcal{M}$ and $Q$, respectively, then $m = n$. It follows from theorem 5.3.2 and proposition 5.4.1 that $\dim(\mathcal{M}) = \dim(Q)$. □

**Example 5.4.1**

From example 5.3.1 above, the sets $\mathcal{R} = \{\ell_1, \ell_3, \ell_5, \ell_6\}$, $\mathcal{R} = \{\ell_1, \ell_3, \ell_6\}$ and $\mathcal{R} = \{\ell_1, \ell_6\}$ are all realizers for the pomset $\mathcal{M}$ and by definition 5.4.1, $\dim(\mathcal{M}) = 2$.

Remark 5.4.1

If the pomset $\mathcal{M}$ is linearly ordered, then $\dim(\mathcal{M}) = 1$, since $\mathcal{M}$ itself is a linear order. Also, if we let $\mathcal{M}^d$ denote the dual of the pomset $\mathcal{M}$, then $\dim(\mathcal{M}) = \dim(\mathcal{M}^d)$. If $\mathcal{M}$ is a nontrivial mset antichain, then $\dim(\mathcal{M}) = 2$, since the set $\{\ell, \ell^d\}$ realizes the pomset $\mathcal{M}$, with $m_i x_i \prec <_\ell m_j x_j$ and $m_j x_j \prec <_{\ell^d} m_i x_i$ for the points $m_i x_i || m_j x_j$ in $\mathcal{M}$.

**Theorem 5.4.3**

Let $\mathcal{M} = (M, \preccurlyeq \leq)$ be a pomset with dimension $\dim(\mathcal{M})$. Then, $\dim(\mathcal{M})$ is at most the width of $\mathcal{M}$.

Proof

Let $\mathcal{M}$ be a pomset defined over a poset $P = (S, \preccurlyeq_P)$. Let $m$ and $\varpi$ be the dimension and width of $\mathcal{M}$, respectively. Let $Q = (M^*, \preccurlyeq_Q)$ be a poset such that there exists an embedding of $Q$ in $P$, this implies $x_i \preccurlyeq_Q x_j$ whenever $x_i \preccurlyeq_P x_j$. Let $n$ and $w$ be the dimension and width of $Q$, respectively. It follows from Dilworth's decomposition theorem that,

$$\dim(Q) \leq width(Q) \quad (1)$$

by proposition 5.4.2, we have,

$$\mathrm{m} = n \quad (2)$$

Since for each $x_i || x_j \; in$ Q there exists $m_i x_i || m_j x_j$ in $\mathcal{M}$, then

$$\varpi = w \quad (3)$$

Consequently, (1) through (3) imply $m \leq \varpi$. □

**Lemma 5.4.4**

Let $\mathcal{M}$ and $\mathcal{N}$ be pomsets defined over partially ordered base sets $P$ and $Q$, respectively. If $\mathcal{M}$ is contained in $\mathcal{N}$, then $\dim(\mathcal{M}) \leq \dim(\mathcal{N})$.

Proof

Suppose $\mathcal{M} \subseteq \mathcal{N}$ and let $m$ and $n$ be the dimensions of $\mathcal{M}$ and $\mathcal{N}$, respectively. If $\mathcal{N}$ is an mset chain, then $n = 1$. Now, $\mathcal{M} \subseteq \mathcal{N}$ implies that $\mathcal{M}$ is also an mset chain, hence $m = 1$. Suppose $\mathcal{N}$ is not an mset chain, and let $\varpi_{\mathcal{M}}$ and $\varpi_{\mathcal{N}}$ denote the widths of $\mathcal{M}$ and $\mathcal{N}$, respectively, $\mathcal{M} \subseteq \mathcal{N}$ implies $\varpi_{\mathcal{M}} \leq \varpi_{\mathcal{N}}$. Suppose $\bigcap_{i=1}^{s} \ell_i = \mathcal{M}$ and $\bigcap_{i=1}^{t} \ell_i = \mathcal{N}$, since $\varpi_{\mathcal{M}} \leq \varpi_{\mathcal{N}}$, clearly, $s \leq t$. Therefore, $\dim(\mathcal{M}) \leq \dim(\mathcal{N})$. □

Remark 5.4.2

Lemma 5.4.4 shows that the dimension of a pomset is monotonic. Another important property of the dimension of a pomset is that, the removal of a point $m_i x_i$ from $M$ can lower $\dim(\mathcal{M})$ by at most 1. The following result also holds:

**Corollary 5.4.5**

Let $\mathcal{M}$ be a pomset defined over a poset $P$, and $Q = (X, \preccurlyeq)$ a subposet of $P$. Then $\dim(\mathcal{M})$ is at most $k/2$ for $k \geq 4$, where $k = |X|$ and $X$ is isomorphic to $M^*$.

Proof

Let $m$ and $n$ be the dimensions of $\mathcal{M}$ and $Q$, respectively, and suppose $X \cong M^*$. By proposition 5.4.2 we have,

$m = n$ . (1)

It follows from Hiraguchi's theorem that

$\dim(Q) \leq {}^{|X|}/_{2}$ $if$ $|X| \geq 4$ (2)

Equations (1) and (2) imply

$m \leq {}^{|X|}/_{2}$ for $|X| \geq 4$

Therefore $\dim(\mathcal{M}) \leq {}^{k}/_{2}$ for $k \geq 4$. □

## CHAPTER SIX

## SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

### 6.1 Summary

The theory of partially ordered structures was studied in this work. A new partially ordered multiset structure was constructed exploiting the ordering induced by the underlying generic set. Substructures of the ordered multiset were also constructed with some new results. New concepts

were defined particularly in the area of combinatorics of ordered structures. Consequently, some new results were obtained on the defined ordered multiset structure.

Fundamentals of ordered sets, which included, operations on partially ordered sets and their properties, were presented. Existing orderings on multisets were also explicated. Some classical theorems of ordered sets were investigated and these were extended to ordered multisets. Various formulations of partially ordered multisets were also described.

The proposed ordered multiset structure $(M, \preccurlyeq\leq)$ was used to extend results on ordered sets to ordered multisets. Substructures of the partially ordered multiset were formulated and investigated via the defined suborder. In particular, Dilworth's decomposition theorem for partially ordered sets and its dual were extended to partially ordered multisets under restricted conditions.

By using a partially ordered base set, the concepts of linear extensions, realizers and dimension were extended to ordered multisets. Significantly, the classical extension theorem for ordered sets was extended to ordered multisets, this formed the basis for establishing some new results on the ordered multiset structure. Among other results, it was shown that any partial mset order can be sorted into an mset linear order while maintaining all preexisting relations. Consequently, an algorithm for generating mset linear extensions of partially ordered multisets was constructed and implemented. It was also shown that the concept of realizers is well-defined on the new ordered multiset structure. Finally, some characterizations for the dimension of an ordered multiset were obtained on the ordered multiset structure.

## 6.2 Conclusions

Not very surprisingly, multisets are found quite apt to represent partial orders. The new ordered multiset structure constructed via the ordering induced by the underlying generic set was suitable for extending concepts and properties, particularly in the area of combinatorics, that hold on the classical ordered sets to multisets. New results were obtained through generalization of known results that hold for ordered sets to multisets, some properties that are peculiar to ordered multisets were also established.

### 6.2.1 Contributions to knowledge

The following results are the main contributions of this thesis:

1. Constructing an ordered multiset structure $\mathcal{M} = (M, \preccurlyeq\leq)$, and proving its following properties:
   i. $\mathcal{M}$ is a pomset.
   ii. The intersection of two pomsets is also a pomset.
   iii. An mset $M$ in $M(S)$ is partially ordered if and only if its root set is a subposet of $P$. where $M(S)$ is the class of finite msets defined over a set $S$, and $P$ is a poset with ground set $S$.
   iv. A point $m_i x_i$ is maximal in $\mathcal{M}$ if and only if $x_i$ is maximal in the root set.
   v. If $\mathcal{M}$ is linearly ordered then maximal and maximum (resp. minimal and minimum) points coincide.

2. Formulating substructures of $\mathcal{M}$ and explicating the following properties;
   i. Every well-ordered pomset is an mset chain.

ii. A maximal mset chain necessarily contains its upper bound.

iii. If $\mathcal{C}$ is a collection of all maximal mset chains in a pomset $\mathcal{M}$, and $K$ is an mset containing all upper bounds of the elements of $\mathcal{C}$. Then any two distinct points in $K$ are incomparable.

iv. A necessary and sufficient condition for $\mathsf{C}_i \cap \mathcal{A}_j$ to have a maximum of one element for any $i, j$. Where $\mathsf{C}_i, \mathcal{A}_j$ are mset chains and mset antichains in the pomset $\mathcal{M}$, respectively.

v. Dilworth's decomposition theorem and its dual were extended to ordered msets by characterizing the width and height of the pomset $\mathcal{M}$ exploiting set-based partitioning, into minimum number of mset chains and antichains, respectively.

3. The extension theorem was generalized through the following results:

   i. A pomset $\mathcal{N}$ is an mset extension of $\mathcal{M}$ if and only if the poset $Q$ is an extension of $P$, where $\mathcal{M}$ and $\mathcal{N}$ are defined over $P$ and $Q$, respectively, such that $P$ and Q have a common ground set.

   ii. For any pair of points $m_i x_i, m_j x_j \in M$ with $m_i x_i, m_j x_j$ incomparable in $\mathcal{M}$, there exists a pomset $\mathcal{N}$ extending $\mathcal{M}$ such that $m_i x_i \preccurlyeq_{\mathcal{N}} m_j x_j$.

   iii. Every finite pomset has an mset linear extension.

4. A heuristic algorithm (Algorithm M-LIN (1)) and an exact algorithm (Algorithm M-LIN (2)) for generating mset linear extensions were constructed and evaluated on a randomly generated ordered multiset structure.

5. The following results on mset realizers and dimension were established on the defined ordered multiset structure:
    i. A partial mset order is the intersection of its set of linearizations.
    ii. The size of an mset realizer is bounded by the size of the realizer of its base set $P$ i.e., $|\mathcal{R}| \leq |R|$, where $\mathcal{R}$ and $R$ are realizers for the pomset $\mathcal{M}$ and the poset $P$, respectively.
    iii. For any $M \in M(S)$, $\dim(\mathcal{M}) \leq \dim(P)$, where $\dim(\mathcal{M})$ and $\dim(P)$ denote the dimensions of $\mathcal{M}$ and $P$, respectively.
    iv. $\dim(\mathcal{M}) = \dim(Q)$, if $Q = (X, \preccurlyeq)$ and $X = M^*$.
    v. The dimension of the pomset $\mathcal{M}$ is monotonic.
    vi. Dimension of $\mathcal{M}$ is at most the width of $\mathcal{M}$.

### 6.3 Recommendations

Partially ordered multisets help in understanding the semantics of parallel programs and are found suitable to model runs of concurrent systems. In view of numerous and diverse practical applications of ordered multisets, there is a growing need to develop a wider class of ordered multiset structures, for researchers to investigate and adopt the structure that is most suitable to model the application problem at hand. With further investigations, our defined ordered structures can be extended to compare multisets in the class of finite multisets defined over a set $S$. Also, the structure proposed in section 4.5 needs further vindication as it promises to be useful in modelling complex application problems.

## References


Aceto, L. (1991). Full abstraction for series-parallel pomsets. Edited by S. Abramsky and T. S. E. Maibaum. *Lecture Notes in Computer Science. Springer-Verlag Berlin*, 493: 1-25.

Bachmair, L., Dershowitz, N., & Hsiang, J. (1986). Orderings for equational proofs, in: *Proc. IEEE Symp. on Logic in Computer Science, Cambridge, MA*, 346-357.

Baclawski, K. (1981), Rings with lexicographic straightening law. *Adv. In Math*., 39: 185-213.

Baeten, J. C. M., & Weijland, W. P. (1990). *Process Algebra*. Cambridge University Press, Cambridge.

Behrendt, G. (1988). Maximal antichains in partially ordered sets. *ARS Combinatoria,* 25C: 149-157.

Ben-Amram, A., & Codish, M. (2008). A SAT-based approach to size change termination with global ranking functions. *14th Intl. Conference on tools and algorithms for construction and analysis of systems (TACAS), LNCS, Springer Verlag*.

Bertet, K., Gustedt, J., & Morvan, M. (2003). Weak-order extensions of an order. *Theoretical Computer Science*, 304: 249-268.

Bianco, L., Dell'Olmo, P., & Giordani, S. (1997). An optimal algorithm to find the jump number of partially ordered sets. *Computational Optimization and Applications*, 8(2): 197-210.

Blizard, W. (1989). Multiset theory. *Notre Dame Journal of Formal Logic*, 30: 36-66.

Blizard, W. (1990). Negative membership. *Notre Dame Journal of Formal Logic*, 31: 346-368.

Bouchitte, V., & Habib, M. (1987). NP-Completeness properties about linear extensions. *Order,* 4: 143-154.

Bouchitte, V., Habib, M., & Jegou, R. (1985). On the greedy dimension of a partial order. *Order*, 1: 219-224.

Brandt, J. (1982). Cycles of partitions. *Proc. American Mathematical Society*, 85: 483-486.

Brightwell, G., & Winkler, P. (1991). Counting linear extensions. *Order*, 8(3): 225-242.

Carmo, R., Donadelli, J., Kohayakawa, Y., & Laber, E. (2004). Searching in random partially ordered sets. *Theoretical Computer Science*, 321: 41-57.

Caro, Y., & Tuza, Z. (1988). Decomposition of partially ordered sets into chains and antichains of given size. *Order*, 5: 245-255.

Chen, Y. (2006). An efficient search algorithm for partially ordered sets. *Proceedings of the IASTED International Conference: Advances in Computer Science and Technology*. January 23-25, Puerto Vallarta, Mexico.

Clements, G. F. (1988). On multisets *k*-families. *Discrete Mathematics*, 69:153-164.

Cogis, O., & Habib, M. (1979). Nombre de sauts et graphes serie-paralleles, *RAIRO Inform. Theorique,* 13:3-18.

Conder, M., Marshall, S., & Slinko, A. (2007). Orders on multisets and discrete cones. *Order,* 24:277-296.

Coupet-Grimal, S., & Delobel, W. (2006). An effective proof of the well-foundedness of the multiset path ordering. *Applicable Algebra in Engineering, Communication and Computing, 17(6): 453-469.*

DeConcini, C., Eisenbud, D., & Procesi, C. (1980). Young diagrams and determinantal varieties. *Inventiones mathematicae*, 56:129-165.

Dershowitz, N. (1982). Orderings for term rewriting systems. *Theoretical Computer Science*, 17: 279-301.

Dershowitz, N., & Manna, Z. (1979). Proving termination with multiset orderings. *Communications of the Journal of Association for Computing Machinery*, 22: 465-476.

Dilworth, R.P. (1950). A decomposition theorem for partially ordered sets. *Annals of Mathematics*, 51(1): 161-166.

Donaldson, D., & Weymark, A. J. (1998). A quasiordering is the intersection of orderings. *Journal of Economic Theory*, 78: 382-387.

Dushnik, B., & Miller, E.W. (1941). Partially ordered sets. *American Journal of Mathematics,* 63: 600-610.

Ehrenborg, R., Kitaev, S., & Perry, P. (2011). A spectral approach to consecutive pattern-avoiding permutations. *Journal of Combinatorics,* 2: 305-354.

Faigle, U., & Schrader, R. (1984). Minimizing completion time for a class of scheduling problems. *Information Processing Letters*, 19(1): 27-29.

Fanchon, J., & Morin, R. (2002). Regular sets of pomsets with autoconcurrency *Proceedings of the 13$^{th}$ International Conference on Concurrency Theory, 402-417.*

Felsner, S., & Trotter, W. T. (2005). Posets and planar graphs. *J. Graph Theory*, 49:273-284.

Felsner, S., Raghavan, V., & Spinrad, J. (2003). Recognition algorithms for orders of small width and graphs of small Dilworth number. *Order*, 20(4): 351-364.

Felsner, S., Trotter, W. T., & Wiechert, V. (2015). The dimension of posets with planar cover graphs. *Graphs and Combinatorics*, 31(4): 927-939.

Fernandez, P., Heath, L., Ramakrishnan, N., Tan, M., & Vergira, J. P. (2013). Mining posets from linear orders. *Discrete Mathematics, Algorithms and Applications,* 5(4):1-15.

Fishburn, P. C. (1973). *The theory of social choice*. Princeton, N. J.: Princeton University Press.

Fishburn, P. C., & Trotter, W. T. (1992). Linear extensions of semiorders: A maximization problem. *Discrete Math*., 103:25-40.

Floyd, R. W. (1967). Assigning meanings to programs. *Proc. Symp. in Applied Mathematics (J. T. Schwartz, ed.), American Mathematical Society*, 19:19-32.

Frank, A. (1980). On chain and antichain families of a partially ordered set. *Journal of Combinatorial Theory*, Series B (29): 176-184.

Freese, R. (1974). An application of Dilworth's lattice of maximal antichains. *Discrete Mathematics*, 7: 107-109.

Galvin, F. (1994). A proof of Dilworth's chain decomposition theorem. *The American Mathematical Monthly*, 101(4): 352-353.

Ghilezan, S., Pantovic, J., & Vojvodic, G. (2014). Binary relations and algebras on multisets. *Publications De L 'Intitut Mathematique,* 95(109): 111-117.

Girish, K. P, & Sunil, J. J. (2009). General relationship between partially ordered multisets and their chains and antichains. *Mathematical communications*, 14(2): 193-205.

Gischer, J. L. (1988). The equational theory of pomsets. *Theoretical Computer Science,* 61:199-224.

Golumbic, M. C. (1980). Coloring and other problems on comparability graphs. *Algorithmic Graph Theory and Perfect Graphs*, New York: Academic Press: 132-135.

Grabowski, J. (1981). On partial languages. *Fundamenta Informaticae*, 6: 427-498.

Grumbach, S., & Milo, T. (1999). An algebra for pomsets. *Information and Computation*, 150: 268-306.

Guenver, G., & Rampon, J. (2004). Split orders. *Discrete Mathematics*, 276(1-3):249-267.

Harju, T. (2006). Lecture notes in ordered sets. Department of Mathematics, University of Turku, Finland.

Harzheim, E. (2005). Ordered sets. *Advances in Mathematics (Springer) 7, New York: Springer*.

Hasson, B. (1968). Choice structures and preference relations. *Syntheses*, 18: 443-458.

Hickman, J. L. (1980). A note on the concept of multiset. *Bulletin of Australian Mathematical Society*, 22:211-217.

Higman, G. (1952). Ordering by divisibility in abstract algebras. *Proc. London Math. Soc.* 12(3): 326-336.

Hiraguchi, T. (1951). On the dimension of partially ordered sets. *Science Report of Kanazawa University*, 1: 77-94.

Huet, G., & Oppen, D. C. (1980). Equations and rewrite rules: a survey in formal Languages perspectives and open problems, *Ed. Book R., Academic Press*.

Joret, G., Micek, P., Milans, K. G., Trotter, W. T., Walczak, B., & Wang, R. (2016). Tree-Width and Dimension. *Combinatorica*, 36(4):431-450.

Joret, G., Micek, P., Trotter, W. T., Wang, R., & Wiechert, V. (2016). On the dimension of posets with cover graphs of tree-width 2. *Order*. DOI:10.1007/s11083-016-9395-y.

Jouannaud, J-P., & Lescanne, P. (1982). On multiset orderings. *Information Processing Letters*, 15: 57-63.

Kelly, D. (1981). On the dimension of partially ordered sets. *Discrete Math.*, 35:135-156.

Kelly, D., & Trotter, W. (1982). Dimension theory for ordered sets. *Proceedings of the NATO Advanced study Institute*, 83: 171-211.

Kierstead, H. A., & Trotter, W. T. (1989). Super-greedy linear extensions of ordered sets in: *Combinatorial Mathematics, eds. G. Bloom et al., Ann. New York Acad. Sci.* 555:262-271.

Kierstead, H. A., & Trotter, W. T. (1985). Inequalities for the greedy dimensions of ordered sets. *Order*, 2: 145-164.

Kierstead, H. A., Trotter, W. T., & Zhou, B. (1987). Representing an ordered set as the intersection of super greedy linear extensions. *Order*, 4: 293-311.

Kilibarda, G. (2015). Enumeration of certain classes of antichains. *Publications De L'Institut Mathematique*, 97(111): 69-87.

Kilibarda, G., & Jovovic, V. (2004). Antichains of multisets. *Journal of integer sequences*, 7(1), Article 04.1.5

Kiziltan, Z. (2004). Symmetry breaking ordering constraints. *Dissertation, Uppsala University Publication, Sweden.*

Kloch, K. (2007). Online dimension of partially ordered sets. *Reports on Mathematical Logic*, 42: 101-116.

Knuth, D. E. (1969). *The Art of Computer Programming (Seminumerical Algorithms)*, volume 2. Addison-Wesley, Reading, Massachusetts, first edition.

Knuth, D. E. (1973). *The Art of Computer Programming*: *sorting and searching*, volume 3. Addison- Wesley, Reading, Massachusetts, second edition.

Knuth, D. E. (1981). *The Art of Computer Programming (Seminumerical Algorithms)*, volume 2. Addison-Wesley, Reading, Massachusetts, second edition.

Korsh, J. F., & Lafollete, P. (2002). Loopless generation of linear extensions of a poset. *Order*, 19(2): 115-126.

Krom, M. (1985). Partial ordering for sets of multisets. *Algebra Universalis*, 21:156-162.

Latapy, M., & Phan, H. D. (1999). The lattice of integer partition and its infinite extension. *LNCS, special issue, proceedings of ORDAL '99.*

Linial, N., & Saks, M. (1985). Searching ordered structures. *Journal of Algorithms*, 6: 86-103.

Lovasz, L. (1972). Normal hypergraphs and the perfect graph conjecture. *Discrete Mathematics*, 2(3): 253-267.

Macdonald, I. G. (1979). Symmetric Functions and Hall Polynomials. *Oxford University Press*.

Martin, U. (1989). A geometrical approach to multiset orderings. *Theoretical Computer Science*, 67: 37-54.

Mazurkiewicz, A. (1989). Basic notion of trace theory, in linear time, branching time and partial orders in logic and models for concurrency. Edited by J. W. de Baker, W. P. de Roever, and G. Rozenberg. *Lecture Notes in Computer Science, Springer-Verlag, Berlin*, 354: 285-363.

Meyer, R. K., & McRobbie, M. A. (1982). Multisets and relevant implication I and II. *Australasian Journal of Philosophy*, 60:107-139 and 265-281.

Mirsky, L. (1971). A dual of Dilworth's decomposition theorem. *American Mathematical Monthly*, 78(8): 876-877.

Morin, R. (2010). Unambiguous shared memory systems. *International Journal of Foundations of Computer Science*, 21(4): 317-330.

Muirhead, R. F. (1903). Some methods applicable to identities and inequalities of symmetric algebraic functions of $n$ letters. *Proc. Edinburgh Math. Soc*. 21: 144-157.

Paskac, R., & Kuncak, V. (2008). Linear arithmetic with stars. *Computer-Aided Verification (CAV) vol. 5123 LNCS.*

Perles, M. A. (1963). A proof of Dilworth's decomposition theorem for partially ordered sets. *Isreal J. Math*., 1:105-107.

Petrovsky, A. B. (1994). *An Axiomatic Approach to Metrization of Multiset Space. Multiple Criteria Decision Making.* New York, Springer-Verlag.

Petrovsky, A. B. (1997). *Structuring Techniques in Multiset Spaces.* Springer.

Pijls, W., & Potharst, R. (2011). Dilworth's theorem revisited; An algorithmic proof. *Econometric Institute Research papers.* No EI 2011-13.

Pijls, W., & Potharst, R. (2013). Another Note on Dilworth's decomposition theorem. *Journal of Discrete Mathematics.*

Pratt, V. R. (1984). The pomset model of parallel processes: Unifying the temporal and the spatial. *Proceedings of CMU/SERC Workshop on Analysis of Concurrency, Pittsburgh. Springer-Verlag Lecture Notes in Computer*. LNCS 196: 180-196.

Pratt, V. R. (1986). Modelling concurrency with partial orders. *International Journal of Parallel Programming*, 15: 33-71.

Pretzel O. (1985). Problem presented at Oberwolfach conference on Combinatorics and Ordered sets.

Pruesse, G., & Ruskey, F. (1994). Generating linear extensions fast. *SIAM Journal on Computing*, 23: 373-386.

Rensink, A. (1996). Algebra and theory of order-deterministic pomsets. *Notre Dame Journal of Formal Logic*, 37(2): 283-320.

Rival, I. (1983). Optimal linear extensions by interchanging chains. *Proceedings of the American Mathematical Society*, 89(3): 387-394.

Rival, I., & Zaguia, N. (1987). Greedy linear extensions with constraints. *Discrete Mathematics*, 63(2): 249-260.

Savaglio, E., & Vannucci, S. (2007). On multidimensional inequality in partitions of multisets. *Quaderno de Dipartimento di Economica Politica N504. University of Siena*.

Schroder, B. S. W. (2003). *Ordered sets; An introduction*. Birkhauser, Boston, USA.

Singh, D., & Isah, A. I. (2016). Mathematics of multisets: a unified approach. *Afri. Mat*., 27(1): 1139-1146.

Singh, D., & Shuaibu, A. M. (2013). Multiset path orderings and their application to termination of term rewriting systems. *Mathematical Theory and Modelling*, 3(5): 96-110.

Singh, D., & Singh, J.N. (2003). Some combinatorics of multisets. *Int. J. Math. Educ. Sci. Technol*., 34(4): 489-499.

Singh, D., & Singh, J.N. (2009). An alternative proof of the well-foundedness of the nested multiset ordering. *International Mathematical Forum*,4(8): 359-362.

Singh, D., Ibrahim, A.M., Yohanna, T., & Singh, J.N. (2007). An overview of the applications of multisets. *Novi Sad Journal of Mathematics*, 37(2): 73-92.

Singh, D., Yohanna, T., & Singh, J. N. (2012). Topological sorts of a multiset ordering. *International Journal of Computer Science and Software Technology*, 5(2): 101-105.

Streib, N., & Trotter, W. T. (2014). Dimension and height for posets with planar cover graphs. *European Journal of Combinatorics*, 35: 474-489.

Suzumura, K. (1976). Remarks on the theory of collective choice. *Economica*, 43: 381-390.

Syslo, M. M. (1987). Minimizing the jump number for partially ordered sets: A graph-theoretic approach. *Discrete Applied Mathematics*, 63:279-295.

Syslo, M. M. (1995). On some new types of greedy chains and greedy linear extensions of partially ordered sets. *Discrete Applied Mathematics*, 60: 349-358.

Szpilrajn E., (1930). Sur'l extension de l'order partiel. *Fund. Math*, 16: 386-389.

Tella, Y., Singh, D., & Singh, J. N. (2014A). A comparative study of multiset orderings. *International Journal of Mathematics and Statistics Invention*, 2(5): 59-71.

Tella, Y., Singh, D., & Singh, J. N. (2014B). Some aspects of topological sorting. *Convenant Journal of Informatics and Communication Technology*, 2(1): 28-45.

Thanh, H. C. (2013). Parallel combinatorial algorithms for multi-sets and their applications. *International Journal of Software Engineering and Knowledge Engineering*, 23(1): 81-99.

Trotter, W. T. (1992). *Combinatorics and partially ordered sets: Dimension Theory*, The Johns Hopkins University Press.

Trotter, W.T. (1995). Partially ordered sets, in : R.L. Graham, M. Grotschel, L. Lovasz (Eds.), *Handbook of combinatorics, Elsevier*, 433-480.

Trotter, W. T. (1996). Graphs and partially ordered sets. *Congressus Numerantium*, 116: 253-278.

Trotter, W. T. (2017). Trees and circle orders. *Abhandlungen aus dem Mathematischen Seminar der Universitat Hamburg*. DOI:10.1007/s12188-016-0176-8

Trotter, W. T., & Moore, J. I. (1977). The dimension of planar posets. *J. Combin. Theory Ser. B*, 21:51-67.

Trotter, W. T., & Wang, R. (2015). Planar posets, dimension, breadth and the number of minimal elements. *Order*, 31: 927-939.

Varol, Y. L., & Rotem, D. (1981). An algorithm to generate all topological sorting arrangements. *Computer Journal*, 24(1): 83-84.

Wehrman, I. (2006). Higman's theorem and the multiset order (unpublished note). *Available at: http://www.cs.utexas.edu/~iwehrman/pub./ms-wqo.pdf*.

Wildberger, N.J. (2003). A new look at multiset. *School of Mathematics, UNSW Sydney, 2052 Australia*.

Williams, A. (2009). Loopless generation of multiset permutations using a constant number of variables by prefix shift. *Proceedings of the Twentieth Annual ACM-SIAM symposium on Discrete Algorithms, SODA 2009, New York, USA*, January, 4-6, pp 987-996.

Yager, R. R. (1986). On the theory of bags. *International Journal of General Systems*, 13:23- 37.