STUDY OF CHU SPACES AND CATEGORIES

BY

ALIYU,Muideen

B.Sc.Ed. (A.B.U. 2010)

M.Sc/SCI/4420/2011-2012

A THESIS SUBMITTED TO THE SCHOOL OF POSGRADUATE STUDIES, AHMADU BELLO UNIVERSITY, ZARIA, IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF MASTER DEGREE IN MATHEMATICS

DEPARTMENT OF MATHEMATICS

FACULTY OF SCIENCE

AHMADU BELLO UNIVERSITY

ZARIA, NIGERIA

JULY, 2015

DECLARATION

I hereby declare that the work in this thesis titled “Study of Chu spaces and categories” has been carried out by me in the Department of Mathematics, under the supervision of Prof. D. Singh and Dr. A.M. Ibrahim. The information derived from the literature has been duly acknowledged in the content and a list of references provided. No part of this work has been previously presented for another degree or diploma at any Institution.

_____ _____

MuideenAliyu Date

CERTIFICATION

This thesis entitled: STUDY OF CHU SPACES AND CATEGORIES by MuideenAliyu meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

_____ _____

Professor D. Singh Date

Chairman, Supervisory committee

_____ _____

Dr. A.M. Ibrahim Date

Member, Supervisory committee

_____ _____

Professor M.S Audu Date

External Examiner

_____ _____

Professor B. Sani Date

Head of Department

_____ _____

Professor Z. Hassan Date

Dean, School of Postgraduate Studies

DEDICATION

This thesis is dedicated to my late father (MallamSalihuAliyu). May the Almighty Allah grant him an eternal rest, Amin.

ACKNOWLEDGEMENT

I first thank Almighty Allah, the greatest source of knowledge who out of his infinite mercy granted me the energy, strength, wisdom and power to undertake and complete this work successfully. My special regards and appreciation go to my supervisors Prof. D. Singh and Dr. A.M. Ibrahim for their guidance and patience in giving valuable advices and constructive suggestions throughout the period of research.

I wish to express appreciations to all members of the Department of Mathematics, A.B.U. Zaria for extending their support and encouragement. In particular, I wish to extend my appreciation toProf. B. Sani, the Head of Department, Prof. B.K. Jha, Prof. J. Singh, MalAbdussamad.

Moreover, I must not forget to extend my appreciations and gratitude to Dr. A.I. Isah and Dr. A.J. Alkali for their many academic contributions in preparation of this thesis, may the Almighty Allah reward you abundantly.

My special appreciations also go to my Mother, wife, younger sister and my children for their prayers and understandings, more grease to your elbows Insha Allah.

Lastly, I would like to use this medium to extend my gratitude to Dr. Nafiu, Mal M. Shitu(Abu Sumayyah), Mal A. Samiu, relatives, friends and colleagues who in one way or the other contributed to the success of the work.

ABSTRACT

In this thesis, a comprehensive study of categories, Chu spaces and Chu categories is presented. In particular, Chu spaces on the transitive closure of a relation R on an arbitrary set A and related Chu categories are constructed, and some of their algebraic properties are studied.In addition, Chu spaces and Chu categories are constructed on $\mathbb{Z}_n$, the set of integers $modulo\ n$ .Finally, the categorical model of multisets and the Chu constructions are outlined.

TABLE OF CONTENTS

Declaration ……iii

Certification..……iv

Dedication…..……v

Acknowledgement…..……vi

Abstract……vii

Table of contents.……viii

CHAPTER ONE

General Introduction…….……..1-5

1.1 Introduction…….……1-4

1.2 Statement of the problem……..……..4

1.3 Justification……..……..……4

1.4 Aim and Objectives……..……...5

1.5 Methodology……...…….....5

1.6 Organization of the Thesis…..…...…….....5

CHAPTER TWO

Literature review……..6-15

CHAPTER THREE

Fundamentals of categories…….……16-51

3.1 The concept of category…….…….…….16-17

3.2 Examples of categories…….17-19

3.3Subcategory…...……...……...19-20

3.4 Retraction and Coretraction…….21-26

3.5 Isomorphism.…….……...26-27

3.6Functors……..……..28-33

3.7 Construction of categories ……...33-35

3.8 Universal mapping properties:……..36-37

Terminal object, Initial object and Point…….……...36-37

3.9 Product of objects in a category……..……..37

3.10 Sum of objects in a category ……..……41

3.11 Exponentiation in categories……..……42-43

3.12 Cartesian closed categories……..……..……43-45

3.13Monoidal category……..45

3.14Closed monoidal category…….48-49

3.15 Evaluation and composition……..49-50

3.16 *-Autonomous category……..……..50-51

CHAPTER FOUR

Chu spaces and Chu categories……..……..52-80

4.1 Chu spaces…….52-53

4.2 Representation of Chu spaces……...53-57

4.3 Chu Transforms…..…….…….….57-58

4.4 Isomorphism of Chu spaces……...58

4.5 Chu Category and Chu constructions…….....59-61

4.6Biextensional collapse of Chu spaces…….…….....61-65

4.7 Internal Homfunctor……..…….....65-67

4.8 Applications of Chu constructs……67-74

4.8.1 Chu constructs and Linear Logic……..67-72

4.8.2Chu constructs in Real-Life Problems ……72-73

4.8.3Chu constructs in Configuration structures……..73-74

4.9 Examples of Chu spaces and related Chu categories on $K = \{0,1\}$..……..74-79

4.10 Generality of $\boldsymbol{Chu}(\boldsymbol{Set}, 2)$…….……...79-80

CHAPTER FIVE

Categorical model of multisets……..81-88

5.1 Multisets…..…….81-82

5.1.1 Multiset Functions…….82-83

5.1.2 Empty Multiset……...83

5.1.3 Identity msetfunction.…….……..83

5.1.4 Composition of mset Functions……..84

5.1.5 Pure Multiset…….……...84

5.2 Categorical model of multisets……..……..84-88

5.2.1 Object part……..87

5.2.1 Arrow part……..87

CHAPTER SIX

Summary, conclusions and recommendation……..……89-90

6.1 Summary…….89

6.2 Conclusions…….…….89-90

6.3 Recommendation.……..90

References……..…….91-94

CHAPTER ONE

GENERAL INTRODUCTION

1.1 Introduction

The idea that mathematics involves different categories and their relationships has been implicit since long. However, it explicitly came into existence with 1945 publication of the theory of *categories, functors, and natural transformations* by Samuel Eilenberg and Saunders Mac Lane while they were specifically engaged in formalizing algebraic topology. In fact, a broader objective underlying their work was to develop an axiomatic foundation which could provide an adequate characterization of the relations between a large number of mathematical structures developed in 1930's and the processes preserving them (see {Lawvere and Schanuel, 2003}, for details).

Informally, a category $\mathbb{C}$ is an algebraic structure consisting of a class of $objects$, denoted $Ob(\mathbb{C})$, linked together by a set of *arrows*(*morphisms* or $\mathbb{C}$-*morphisms*), denoted $\mathcal{M}(X,Y)$, or $Mor_{\mathbb{C}}(X,Y)$, for every pair of objects X,Y of $\mathbb{C}$, satisfying the fundamental properties: Arrows compose *associatively*and there exists an *identity* arrow for each object. For example, $\boldsymbol{Set}$ is the category in which sets are objects and functions are morphisms. The fundamental idea of *mutability of precise mathematical structures by morphisms*{Lawvere, 2003} as well applied to categories themselves taken as structures and functors as mappings between them and further, functors taken as structures and natural transformations as mappings.

In view of the fact that $objects$ and $arrows$, the two basic ingredients for defining a category, may be abstract entities of any kind, a large number of familiar mathematical

structures can be viewed as categories. Category theory has come to occupy a central position in generalizing most of the branches of mathematics, some areas of theoretical computer science, and mathematical physics (Continuum physics, essentially).

In the sequel, the notion of Chu spaces emerged in 1979. Historically, it was Mackey {Mackey, 1945} who formalized the idea of representing *duality* as a contravariant pair of morphisms. Micheal Barr abstracted this idea to introduce the notion of Chu spaces enriched (in the sense of enriched category {Kelly, 1982}) in a symmetric monoidalcategory V, which was elaborately studied by Po-Hsiang Chu, a student of Barr, in his master's thesis. A major contribution of Chu's master's thesis has come to be known as *Chu constructions* which first appeared in print as an appendix in Barr's book titled *- *Autonomous Categories* {Barr, 1979}. In fact, Chu constructions formed the basis for constructing *- autonomous categories from autonomous categories. Note that an autonomous category is a *closed symmetric monoidal category*, while a *-autonomous category is an autonomous category with a *dualizing object*.

Informally, a Chu space over an arbitrary set K, abbreviated as *K-Chuspace* or Chu_K, is a structured mathematical object consisting of a transformable matrix whose entries are drawn from K and, rows transform forwards and columns transform backwards. Essentially, there underlies a simple form of structure and suitable morphisms, called $Chu\ transforms$. The discovery of Chu spaces, owing to their matrix orientation and hence being amenable to automation, can be considered as an outstanding development in the field of category theory.

The applications of Chu spaces abound ({Barr, 1996, 1999}, {Devarajan et al, 1999}, {Lafont and Streicher, 1991}, and {Pratt, 1999}). Chu spaces, recognized as universal mathematical objects turn out to be a unifying thread for a wide range of mathematical structures: algebraic, topological, and ordered. For example, relational structures such as sets, directed graphs (networks), posets, etc., algebraic structures such as groups, rings, fields, vector spaces, lattices and Boolean algebras, etc., and topological structures such as topological spaces, compact Hausdorff spaces, topological Boolean algebras, etc., are all represented as Chu constructs. Moreover, applying powerful tools provided by category theory, a number of categorical models of Chu spaces have been developed (see {Barr, 1979} and {Pratt, 1999} for details).

At this end, it may be emphasized that, in all these developments, the elementary theory of category of Cantor's structureless sets, developed by Lawvere {Lawvere, 2003}, and, by extension, $\boldsymbol{Chu}(\boldsymbol{Set}, K)$ (in particular, $\boldsymbol{Chu}(\boldsymbol{Set}, 2)$) {Barr, 1996}, {Pratt, 1999}, can be considered as the core of the $paradise$ of Categories.

MacLaneobserved, which appeared in his Autobiography, as follows:

The most radical aspect is Lawvere's notion of using axioms for the Category of Sets as a foundation of mathematics …

The reason being that it is *simple* and *close* since the objects are sets and morphisms are functions. Note that certain morphisms may not even be functions and hence the notions like *injectivity* or *surjectivity* would not apply. Moreover, it has the same proof-theoretic strength as that of the theory of category of categories, an appropriate framework for mathematics. The difference is that the former is defined by *discrete functor* whereas the

latter by a *structure functor*, called *adjoint* which was developed by Kan in the mid 1950's (see {Lawvere, 2003} for details).

It is remarkable of $\boldsymbol{Chu}(\boldsymbol{Set}, K)$, in particular, $\boldsymbol{Chu}(\boldsymbol{Set}, 2)$, that it realizes a rich variety of mathematical structures (see {Pratt, 1999} for details). The case $V = \boldsymbol{Set}$ was first explicitly treated by Lafont and Streicher{Lafont and Streicher, 1991}. It was pointed out that Chu spaces over $K = \{0,1\} = 2$ realizes both *topological* spaces and *coherent* spaces (introduced by J-Y Girard to model linear logic {Girard, 1987}). Also, $\boldsymbol{Chu}(\boldsymbol{Set}, 2)$, via logical relations, yields a fully complete model for multiplicative linear logic {Davarajan et al, 1999}.

In this thesis, Chu spaces and Chu categories, in particular, $\boldsymbol{Chu}(\boldsymbol{Set}, 2)$, are studied, a couple of newmodels are constructed,and the extension of the notion of Chu constructions to multisets are outlined.

1.2Statement of problem

In view of the aforesaid, it remains a significant task to construct new models of Chu spaces and Chu categories that could be applied to solve new problems.

1.3 Justification

In recent years, on one hand, many applications of Chu spaces and Chu categories have been found, and on the other, multisets have found enormous applications in mathematics and computer science. It would be significant to study the Chu concepts in detail and particularly, in multiset based structures.

1.4 Aim and Objectives

The aim of this research is to study Chu spaces and Chu categories and extend it to multisets. The proposed objectives are to:

(i) Formulate some new models of Chu spaces and Chu categories;

(ii) Describe some algebraic structures of Chu spaces;

(iii) Demonstrate application of Chu spaces to solve real-life problems;

(iv) Extend the concepts of Chu spaces and Chu categories to multisets.

1.5 Methodology

We apply the method of constructing Chu spaces as mathematical objects, and defining related Chu transforms and Chu categories.

1.6 Organization of the Thesis

The remaining chapters of this thesis are organized as follows: Chapter 2 will cover literature review, chapter 3 will contain fundamentals for developing category theory and Chu spaces, chapter 4 will be devoted to Chu spaces and related categories, chapter 5 will contain an outline of the categorical model of multisets as objects of Chu spaces, and finally, chapter 6 will present the summary of the main results obtained and some directions for future research.

CHAPTER TWO

LITERETURE REVIEW

Chu spaces first arose as an instance of a general construction, popularly called Chu's construction, described by Michael Barr and Po-Hsiang Chu [Barr, 1979]. In fact Michael Barr began a sabbatical at the ETH in Zurich in the year 1975 with the idea of studying *duality* in categories in some depth, not simply as the duality between Boolean algebras and Stone spaces or between Compact and discrete abelian groups, rather as self-dual categories such as complete semi-lattices, finite abelian groups, and locally compact abelian groups. Moreover, he was interested in the possibilities of having a category that was not only self-dual but one that had an internal homomorphism (hom) and for which the duality was implemented as the internal hom into a dualizing object.

It was also shown that the aforesaid properties hold good for complete semi-lattices but not for finite abelian groups or locally compact abelian groups. In fact, the category of finite abelian groups does have an internal hom, but lacks a dualizing object, while locally compact groups have a dualizing object, but not an internal hom that is defined everywhere. The desired properties were what he subsequently called *-autonomy.

In the sequel, the notion of Chu spaces emerged. Historically, it was Mackey [Mackey, 1945] who formalized the idea of representing *duality* as a contravariant pair of morphisms. Micheal Barr abstracted this idea to introduce the notion of Chu spaces enriched (in the sense of enriched category [Kelly, 1982]) in a symmetric monoidalcategory V, which was elaborately studied by Po-Hsiang Chu, a student of Barr, in his master's thesis. A major contribution of Chu's master's thesis has come to be known as *Chu constructions* which first appeared in print as an appendix in Barr's book

titled *- *Autonomous Categories* [Barr, 1979]. In fact, Chu constructions formed the basis for constructing *- autonomous categories from autonomous categories. Note that an autonomous category is a *closed symmetric monoidal category*, while a *-autonomous category is an autonomous category with a *dualizing object*.

Michael Barr (1979, 1996) observed that Chu categories could be exploited for addressing a number of questions in theoretical computer science. One interesting thing about Barr's paper was the modification of the definitions of the internal homomorphism and introduction of tensor product of Chu spaces. It was shown that $\boldsymbol{Chu}(\boldsymbol{Ab}, K)$, the Chu space whose objects are abelian groups with their group homomorphisms over the set K, was equivalent to a full subcategory of topological abelian groups that include all the locally compact groups, where $\boldsymbol{chu}(\boldsymbol{Ab}, K)$ denotes the full subcategory of$\boldsymbol{Chu}(\boldsymbol{Ab}, K)$.

Consequent to the introduction of Chu constructions by Barr (1996), it was found that the partial distributive lattices (PDlats) were, in a certain sense, universal to Stone duality and category of $K\text{-}ary$ relational structures could be embedded concretely in $\boldsymbol{Chu}(\boldsymbol{Set}, 2^k)$. These findings led Barr to conclude that Chu spaces were universal for concrete mathematics{Barr, 1979} and {Pratt, 1995}.

Informally, a Chu space over a set K, called the alphabet, is a structured mathematical object consisting of a transformable matrix whose entries are drawn from K and, rows transform forwards and columns transform backwards.

Lafont and Steicher (1991) made an elegant observation that the category of topological spaces embeds fully into $\boldsymbol{Chu}(\boldsymbol{Set}, 2)$ as does Girard's category of coherence spaces (see [Girard, 1987] for details), and via logical relations, $\boldsymbol{Chu}\ (\boldsymbol{Set}, 2)$ yields a fully complete

model for multiplicative linear logic (see {Devarajan et al, 1999} for details).A topological space (X, τ), where X is the set of points and τ the set of open sets, wasinterpreted as a Chu space $(X, \in, \tau)$ $over$ $\{0, 1\}$. The points of the topological space become those of the Chu space while the open sets become states and the membership relation $\in$ between points and open sets is made explicit in the Chu space. The condition that the set of open sets be closed under arbitrary union and finite intersection becomes the corresponding condition on the columns of the matrix.

Further, applications of Chu spaces were developed by Vaughan Pratt and his student Vincet Gupta (see {Gupta, 1994} for details). Chu spaces are remarkable for the wide variety of familiar structures they realize. Chu spaces have been found objects of considerable interest in theoretical computer science.

It may be significant to note that, Chu space was understood dynamically ({Gupta, 1994}), as follows:

Chu spaces transform in the manner of topological spaces, with A as the set of points, X as the set of open sets, and r as the membership relation between them, where K is the set of all possible degrees of membership of a point in an open set. The counterpart of a continuous function from (A, r, X) to (B, s, Y) is a pair (f, g) of functions $f : A \to B$, $g : Y \to X$ satisfying the *adjointness conditions* $(f(a), y) = r(a, g(y))$ for all $a \in A$ and $y \in Y$.

This shows that the function f maps points forwards at the same time as g maps states backwards. The adjointness condition makes g the inverse image function f^{-1}, while the choice of X for codomain of g corresponds to the requirement for continuous functions

that the inverse image of open sets be open. The pair (f, g) is called Chu transform or morphism of Chu spaces.

In the same place, the concept of *Dialetica spaces* was explicated. A dialetica space is a variant of Chu space in that the adjointness condition for Chu transform was replaced by $\leq$ instead of $=$, that is, $s(f(a), y) = r\,(a, g(y))$ was replaced by $s(f(a), y) \leq r\,(a, g(y))$.

Gupta (1994), while searching for a suitably general and natural model of true concurrency, introduced the notion of a normal Chu space defined as follows: A normal Chu space (A, X) over a set K consists of a set A, the underlying set or carrier, and a set X of functions $g: A \to K$, the states. A Chu morphism between normal Chu spaces $(A, X), (A', X')$ is a function $f: A \to A'$ such that for every $g: A' \to K$ in X', $gf \in X$. The essential idea in this definition was to represent the morphisms of every small category $\mathbb{C}$ as their underlying functions with their codomains expanded or astricted to a single common codomain K. Let us note that composing with an inclusion on the right is called a *restriction* and on the left an *astriction*. That is, let $B \leq B'$, composing $f: B' \to C$ with this inclusion restricts f *to* $f': B \to C$ (domain of f is restricted or shrunk to B), and composing it with $f: A \to B$ astricts f *to* $f': A \to B'$ (codomain of f is astricted or expanded to B').

In 1999, Pratt, by defining a suitable notion of morphism, showed that the class of all Chu spaces formed a category. The notion of *Chu transform* was explicated as follows:

Given two Chu spaces $\mathcal{A} = (A, r, X)$ and $\mathfrak{B} = (B, s, Y)$, a Chu transform from $\mathcal{A}$ to $\mathfrak{B}$ is a pair (f, g) consisting of functions $f: A \to B$ and $g: Y \to X$ such that $S(f(a), y) =$

$r(a, g(y))$ for all a in $A$and y in Y. This equation was termed as the *adjointness condition*.

The adjoint pairs $(f, g): \mathcal{A} \to \mathfrak{B}and(f'g'): \mathfrak{B} \to \mathcal{C}$, where let $\mathcal{C} = (C, t, Z)$, compose via $(f'g')(f, g) = (f'f, gg')$. This composite is itself an adjoint pair because for all a in $A$and $z \in Z$, $t(f'f(a), z) = s\big(f(a), g'(z)\big) = r\big(a, gg'(z)\big)$ hold. The associativity of this composition is inherited from that of composition in $\boldsymbol{Set}$, while the pair (I_A, I_X) of identity maps, respectively, on A and X is an adjoint pair and is the identity Chu transform on $\mathcal{A}$.

It is significant to note that, the transforms and the compositions over 2 as defined above realize preordered sets, stone spaces, ordered stone spaces, topological spaces, complete semillatices, distributive lattices (but not general lattices), algebraic lattices, profinite (Stone) distributive lattices, Boolean algebras, and complete atomic Boolean algebras, to name some familiar mathematical structures.

In ({Syropoulos, 2001, 2003}), the extension of the concept of Chu constructions to multisets was introduced. It was observed that *pure multisets* could be represented as objects in the Chu category $\boldsymbol{Chu}(\boldsymbol{Rel}, 1)$, where $\boldsymbol{Rel}$ is the category of sets and binary relations between them, and 1 is a singleton set. Consiquently, pure multisets could be represented as objects in $\boldsymbol{Chu}(\boldsymbol{Set}, 2^2)$, where $\boldsymbol{Set}$ is the category of sets and functions between them.

Hughes (2004) observed a kind of similarity between the Chu construction and the Hyland-Tan double glueingconstruction $\boldsymbol{G}$. The paper establishes a more formal mathematical relationship between the two. It was shown that a double glueing on relations subsumes the Chu construction on sets. Further, a full monoidal embedding of

the category $\boldsymbol{chu}(\boldsymbol{Set}, K)$ of biextensional Chu spaces over K into$\boldsymbol{G}$ ($\boldsymbol{Rel^K}$), and a full monodial embedding of the category $\boldsymbol{Chu}$ ($\boldsymbol{Set}, K$) of Chu spaces over K into $\boldsymbol{IG}$ ($\boldsymbol{Rel^K}$), where $\boldsymbol{IG}$ was defined as the intentional double glueing construction by substituting multisets for sets in $\boldsymbol{G}$, were described. The structure of the star-autonomous category $\boldsymbol{Chu}$ ($\boldsymbol{Set}, K$) was presented as follows:

- Objects. Triples(A, r, X), where A a set of $\boldsymbol{points}$, X as set of $\boldsymbol{copoints}$ and r is a function $A \times X \to K$, the $\boldsymbol{matrix}$ or $\boldsymbol{pairing}$.
- Morphisms. A morphism$(A, r, X) \to (B, s, Y)$is a pair (f, g) of functions $f: A \to B$ and $g: Y \to X$ satisfying the $\boldsymbol{adjointness\ condition}$:

  $r(a, g(y)) = s(f(a), y)$ for all points $a \times A$ and copoints $y \in Y$.
- Duality $(A, r, X)^{\perp} = (X, r^o, A)$, where $r^0(x, a) = r(a, x)$.
- Internal hom and tensor. Given $\mathcal{A} = (A, r, X)\ and\ \mathfrak{B} = (B, s, Y)$

$$\mathcal{A} \multimap \mathfrak{B} = (hom(A, B), ev, A \times Y)$$

$$ev\big((f, g), < a, y >\big) = r\big(a, g(y)\big)\big(= s(f(a), y)\big) \in K$$

$$\mathcal{A} \otimes \mathfrak{B} = (A \times B, ev^o, \text{hom}(A, B^{\perp}))$$

$$ev^o\big(< a, b >, (h, k)\big) = r\big(a, k(b)\big)\Big(= s\big(b, h(a)\big)\Big) \in K$$

- Tensor unit. $(1, \pi_K, K)$, where $1 = \{0\}$ and $\pi_K(0, k) = k \in K$.

A biextensional collapse from $\boldsymbol{IG}$ which extends the familiar notion on Chu spaces was defined, where $\boldsymbol{IG}$ and $\boldsymbol{G}$ are the extensional double glueing and the Hyland Tan double glueing construction on $\boldsymbol{G}$, respectively. Essentially, the intentional double glueing construction was defined by substituting multisets for sets in$\boldsymbol{G}$.

Pratt(2005a)defined Chu space (X, r, A) as a structure consistingof three things: a set X of points, individuals, or subjects; a set A of states or predicates; and a look-up table or

matrix $r: X \times A \to K$ which specifies for every subject x and predicate a the value $a(x)$ of that predicate for that subject. The value of $a(x)$ is given by the matrix r as the entry $r(x, a)$. These matrix entries are drawn from a set K. The set$K$ can be as simple as $\{0,1\}$ when representing topological spaces or Booleans algebras;or it can be as complex as the set of all complex numbers when representing Hilbert spaces, or it can be something in between such as the set of 16 subsets of $\{0,1,2,3\}$ when representing topological groups.

It was shown that a topological space could be viewed as a Chu space satisfying the following three properties:

1- $K = \{0, 1\}$

2- The space is extensional. An extensional space is the one that has no repeated columns.

3- The columns are closed under arbitrary union (bitwise of any set of columns, including the empty set of columns which produces the all-zero columns) and finite intersection (bitwise of empty set of columns, along with the all ones columns as the bitwise of the empty set of columns). He interprets $r(x, a) = 1$ as meaning $x \in a$ (x *is a member of* a), and $r(x, a) = 0$ as $x \notin a$ (x *is not a member of* a).

He also discussed the interpretation of Chu space by linear logic. That is, linear logic of Girard could be seen as the language of Chu space. The basic operations in linear logic may be interpreted as acting on Chu spaces as follows: $\mathcal{A}^{\perp}$ is just the dual or the transpose of $\mathcal{A}$, where $\mathcal{A}$ is the Chu space $(A, _A, X)$, $\mathcal{A}$ is the carrier A of $\mathcal{A}$ as a Chu space, namely the normal $Chu(A, K^A)$.

$\mathcal{A} \oplus \mathfrak{B}$, for $\mathcal{A} = (A, _A, X)$, $\mathfrak{B} = (B, _B, Y)$, is $(A + B, , X \times Y)$, where is defined by $a\ (x, y) = a\ _A\ x, b\ (x, y) = b\ _B\ y$. Its unit is $(0, , 1)$.

Pratt (2005b) introduced the notion of Chu space as a binary relation from a set A to an antiset X, defined as a set which transforms via converse functions. Essentially, a Chu space is viewed as a matrix that transforms by *deletingandcopying columns* (*antimapping*) and *identifying and adjoining rows* (*ordinary mapping*). More explicitly, a Chu space $\mathcal{A}$ over a set K is defined as a triple $\mathcal{A} = (A, , X)$, consisting of set of points A and set X of states and a function $: A \times X \to K$, that is, an $A \times X\ matrix$ whose elements are drawn from K. The entry at (a, x) is written as either $(a, x)\ or\ a \quad x$.

Thus, a Chu space $\mathcal{A}$ can be viewed as being organized into rows or columns. While viewing $\mathcal{A}$ by row, A is regarded as the carrier of the structure and the row indexed by a as the complete description of the element a. The description function $: A \to K^X$ is the function satisfying $(a)(x) = (a, x)$, and assigns to each a its description. The morphisms of Chu spaces are defined in terms of *conversefunctions* or *antifunction*s specifying where the surviving columns were copied from and a *forward function* specifying where the resulting rows be sent to.

It was further shown that Chu spaces and their transforms form a remarkably well-endowed category being *concrete*, *self-dual*, *bi-complete*, and *symmetric monoidal closed*. Chu spaces can be seen to model concurrency by admitting their interpretation as *processes*. The rows and columns of a Chu space are interpreted, respectively, as the events and states of the process, while the entries of the matrix are interpreted as the interaction of events and states. Moreover, Chu space as a process may be viewed either

imperatively or *declaratively* by focusing on the columns or the rows, respectively, of the representing space.

It was also observed that in a non-extensional Chu space two states may name the same assignment. So, in this case, a Chu space can be seen as a multiset A of rows from K^X and a multiset X of columns from K^A, with "multiset" replaced by "set" when it isT_o and *extensional*. The set$K$ is usually $\{0,1\}$ and T_o is a *coextensional* Chu space.

Papadopoulos and Syropoulos (2011) described generalized topological spaces via Chu spaces. A generalized topological space consists of triplets of the form $(X, r, \mathfrak{A})$ where X is an arbitrary set of points, $\mathfrak{A}$ is an arbitrary set of open sets and $r : X x \mathfrak{A} \rightarrow I$ $(I = [0,1])$ is an arbitrary function called the *membership function*. The intuitive meaning of this function is that for all $x \in X$ and $A \in \mathfrak{A}$, $r\,(x, A)$ is the degree to which x is a member of the open set A. Any such triplet will be called a generalized topological space (or GTS for short).

As pointed out by Pratt (2005a), Chu spaces unify a wide range of mathematical structures, including the following:

- Relational structures such as sets, directed graphs, posets and small categories.
- Algebraic structures such as groups, rings, fields, modules, vector spaces, lattices and Boolean algebras.
- Topologized versions of the above, as topological spaces, compact Hansdorff spaces, logically compact abelian groups and topological Boolean algebras.

In this Thesis, we study the fundamentals of Chu spaces, Chu transforms, and Chu categories on sets and multisets. As category theory provides a powerful tool for

understanding the internal structure of Chu spaces, we present an overview of Category Theory as a background material for developing Chu categories.

CHAPTER THREE

FUNDAMENTALS OF CATEGORIES

The content of this chapter is largely abstracted from ([Barr and Wells, 1999], [Ivancevic and Ivancevic, 2009], [Kuku, 1992], [Laan, 2003], [Lawvere, 2003], and [Turi, 2001]).

3.1 The concept of category

Informally, a category $\mathbb{C}$ is an algebraic structure consisting of a class of $objects$, denoted $Ob(\mathbb{C})$, linked together by a set of $arrows(morphisms$ or $\mathbb{C}-morphisms)$, denoted $\mathcal{M}(X,Y)$, or $Mor_{\mathbb{C}}(X,Y)$, for every pair of objects X,Y of $\mathbb{C}$, satisfying the fundamental properties: For any pair (X,Y) of $\mathbb{C}$, arrows to be composed $associatively$ and the existence of an $identity$ arrow for each object. This fundamental idea of $mutability\ of\ precise\ mathematical\ structures\ by\ morphisms$ as well applied to categories themselves taken as structures and functors as mappings between them and further, functors taken as structures and natural transformations as mappings.

In view of the fact that $objects$ and $arrows$, the two basic ingredients for defining a category, may be abstract entities of any kind, a large number of familiar mathematical structures can be viewed as categories. Category theory has come to occupy a central position in generalizing most of the branches of mathematics some areas of theoretical computer science, and mathematical physics (Continuum physics, essentially).

Formally, a category $\mathbb{C}$ consists of the following:

(i) A class $ob(\mathbb{C})$, of *objects* of $\mathbb{C}$.

(ii) A class $Hom(\mathbb{C})$ of *morphisms* or *maps* or *arrows*of $\mathbb{C}$.

(iii) A binary operation called *composition* of *morphisms*, such that for any three objects $A, B, C \in \mathbb{C}$, there exists a mapping $Hom(A,B) \times Hom(B,C) \longrightarrow Hom(A,C)$ such that if $f \in Hom(A,B)$ and $g \in Hom(B,C)$, then $gof \in Hom(A,C)$, (i.e., the composite of $f: A \longrightarrow B$ and $g: B \longrightarrow C$ is written $gof: A \longrightarrow C$ or $gf: A \longrightarrow C$), governed by two axioms

(a) Associativity: if $f: A \longrightarrow B$, $g: B \longrightarrow C$ and $h: C \longrightarrow D$ are morphisms, then $h(gf) = (hg)f$, and

(b) Identity: For every object X, there exists a morphism $Id_X: X \longrightarrow X$ called the identity morphism for X, such that for every morphism $f: A \longrightarrow B$, we have $Id_B of = f = foId_A$.

A category is called $small$ if the collection of its objects and collection of its arrows constitute $sets$; otherwise it is $large$. A category with the property that the collection of arrows from $A$to B, denoted$Hom(A,B)$, is a set for all objects A and B, is called $locally\ small$. For example, the category of finite sets denoted $\boldsymbol{Finset}$ is $locally\ small$.

If A and B are two objects of a category $\mathbb{C}$, then the set of all arrows of $\mathbb{C}$ that have source A and target B is donated $Hom_{\mathbb{C}}(A,B)$ or just $Hom(A,B)$ if the category is clear from context. Thus for each triple A, B, C of objects, composition induces a function $Hom\ (A,B) \times Hom\ (B,C) \Rightarrow\ Hom\ (A,C)$.

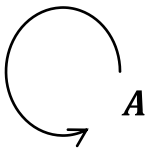
3.2 Examples of Categories

(i) The simplest category is the empty category. This category has neither an object nor an arrow.

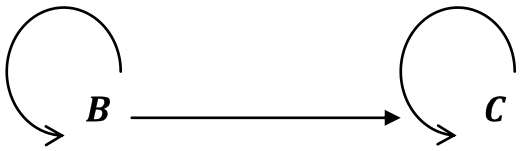
(ii) The simplest nonempty category is the category with only one object and a single arrow, denoted $\mathbf{1}$. This category is useful in the description of category of monoid. The single arrow is forced to be the identity arrow on object.

(iii) The category $\mathbf{2}$is the category with two objects and a single arrow.

The categories in (i) and (ii) can be diagrammatically represented as follows:



Category 1



Category 2

(iv) The category of sets, denoted $\boldsymbol{Set}$, is the category whose objects are $sets$ and whose arrows are functions between them. The composition of arrows in this category is the usual function composition and the identity function from A to $A$is denoted by Id_A.

(v) The category of finite sets, denoted $\boldsymbol{Fins}$,has finite sets as objects and functions between finite sets as arrows.

(vi) The category $\boldsymbol{Pfn}$ of sets and partial functions has all sets as objects and all partial functions as arrows.

(vii) The category $\boldsymbol{Rel}$ of sets and relations has sets as objects and relations as arrows.

(viii) The category $\boldsymbol{Pos}$, has partially ordered sets as objects and morphisms(arrows) as monotonic functions.

(ix) The category $\boldsymbol{Top}$, has topological spaces as objects and morphisms (arrows) as continuous functions. Another category could be obtained when morphisms (arrows) are restricted to open continuous functions.

(x) The category $\boldsymbol{Vec}$, has vector spaces as objects and morphisms (arrows) aslinear maps.

(xi) The category $\boldsymbol{Lat}$, has lattices as objects and morphisms(arrows) aslatticeshomomorphisms.

(xii) The category $\boldsymbol{Grp}$, has groups as objects and morphisms(arrows) as group homomorphisms.

(xiii) The category $\boldsymbol{Mon}$, has monoids as objects and morphisms(arrows) asmonoidhomomorphisms.

(xiv) The category $\boldsymbol{BoolAlg}$, has Boolean algebras as objects and morphisms(arrows) as Boolean algebra homomorphism.

(xv) The category $\boldsymbol{Sem}$, has semigroups as objects and morphisms (arrows) assemigroup homomorphism.

3.3Subcategory

A subcategory $\mathbb{D}$ of a category $\mathbb{C}$ is a category for which:

S-1 All the objects of $\mathbb{D}$ are objects of $\mathbb{C}$ and all the arrows of $\mathbb{D}$ are arrows of $\mathbb{C}$. In other words, $\mathbb{D}_o \subseteq \mathbb{C}_o$ and $\mathbb{D}_1 \subseteq \mathbb{C}_1$, where $\mathbb{D}_o, \mathbb{C}_o$, $\mathbb{D}_1$ and $\mathbb{C}_1$ are the collections of objects and arrows of $\mathbb{D}$ and $\mathbb{C}$, respectively.

S-2 The source and target of an arrow of $\mathbb{D}$ are the same as its source and target of $\mathbb{C}$. It follows that for objects A and B of $\mathbb{D}$, $Hom_{\mathbb{D}}(A,B) \subseteq Hom_{\mathbb{C}}(A,B)$.

S-3 If A is an object of $\mathbb{D}$ then its identity Id_A in $\mathbb{C}$ is in $\mathbb{D}$.

S-4 If $f: A \to B$ and $g: B \to C$, then the composite gof (in $\mathbb{C}$) is in $\mathbb{D}$, called the composite in $\mathbb{D}$. In other words, a category $\mathbb{D}$ is a subcategory of a category $\mathbb{C}$ if

(i) $\mathbb{D} \subset \mathbb{C}$;

(ii) For any two objects A, B of $\mathbb{D}$, $Hom_{\mathbb{D}}(A,B) \subset Hom_{\mathbb{C}}(A,B)$;

(iii) the composition of any two morphisms in $\mathbb{D}$ is the same as their composition in $\mathbb{C}$; and

(iv) the identity morphism Id_A of any object A *in* $\mathbb{D}$ is the same as that in $\mathbb{C}$.

For example, the category $\boldsymbol{Fin}$ of finite sets and all functions between them is a subcategory of $\boldsymbol{Set}$, and in turn $\boldsymbol{Set}$ is a subcategory of category of sets and partial functions between sets (Pfn).

If $\mathbb{D}$ is a subcategory of $\mathbb{C}$ and for every pair of objects A, B of $\mathbb{D}$, $Hom_{\mathbb{D}}(A,B) = Hom_{\mathbb{C}}(A,B)$, then $\mathbb{D}$ is a full subcategory of $\mathbb{C}$. In other words, in addition to the conditions (i) through (iv) above,

(v) $Hom_{\mathbb{D}}(A,B) = Hom_{\mathbb{C}}(A,B) for\ all\ A, B\ in \mathbb{D}$ is satisfied.

Thus, $\boldsymbol{Fin}$ and $\boldsymbol{Mon}$ are both *full* subcategories of $\boldsymbol{Set}$and $\boldsymbol{Grp}$, respectively, while $\boldsymbol{Set}$and$\boldsymbol{Lat}$ are not *full* subcategories of $\boldsymbol{Pfn}$ and $\boldsymbol{Pos}$, respectively.

If $\mathbb{D}$ is a subcategory of $\mathbb{C}$ with the same objects, then $\mathbb{D}$ is a *wide* subcategory of $\mathbb{C}$.

For example, $\boldsymbol{Set}$ is a wide subcategory of the category $\boldsymbol{Pfn}$.

It may be noted that the category of monoids is a subcategory of the category of semigroups that is neither wide nor full.

Product of categories

If $\mathbb{D}$ and $\mathbb{C}$ are categories, the *product* $\mathbb{C} \times \mathbb{D}$ is the category whose objects are all ordered pairs of the form (C, D), with C an object of $\mathbb{C}$ and D an object of $\mathbb{D}$, and in which an arrow $(f, g): (C, D) \rightarrow (C', D')$ is a pair of arrows $f: C \rightarrow C'$ *in* $\mathbb{C}$ and $g: D \rightarrow D'$ *in* $\mathbb{D}$. The identity of (C, D) is (Id_C, Id_D). If $(f', g'): (C', D') \rightarrow (C'', D'')$ is another arrow, then the composite is defined as

$(f', g')(f, g) = (f'of, g'og) : (C, D) \rightarrow (C'', D'')$.

The Dual of a category

Given a category $\mathbb{C}$, another category called *dual* or *opposite* of $\mathbb{C}$, denoted $\mathbb{C}^{opp}$, is constructed by reversing all the arrows. More explicitly, the $dual$ or $opposite$ $\mathbb{C}^{opp}$ of a category $\mathbb{C}$ is constructed as follows:

D-1 the objects and arrows of $\mathbb{C}^{opp}$ are the objects and arrows of $\mathbb{C}$

D-2 If $f: A \rightarrow B$ *in* $\mathbb{C}$, then $f: B \rightarrow A$ *in* $\mathbb{C}^{opp}$ i.e., the source and the target get reversed

D-3 If $h = gof$ *in* $\mathbb{C}$, then $h = fog$ *in* $\mathbb{C}^{opp}$

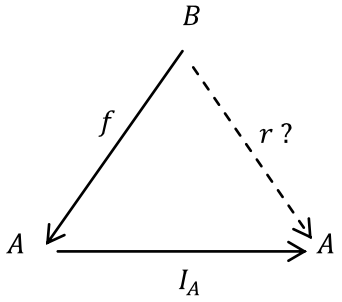
It is immediate to see that the identity arrows have to be the same in both $\mathbb{C}$ and $\mathbb{C}^{opp}$

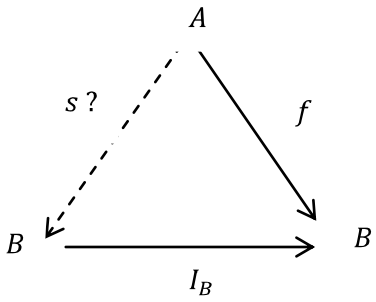
3.4 Retraction and Coretraction

Let $f: A \rightarrow B$, then

- a*retraction* for f is a map $r: B \rightarrow A$ for which $rof = I_A$.
- a*coretractiontion* or *section* for f is a map $s: B \rightarrow A$ for which $fos = I_B$.

Represented diagrammatically as follows:





Given f and $r$with $rof = I_A$, then r is a section of f and f is a section for r.

Properties of Retraction and Section

Theorem 3.1 [Lawvere and Schanuel, 2003]

If a map $f: A \rightarrow B$ has a section, then for any T and for any map $y: T \rightarrow B$, there exists a map $x: T \rightarrow A$ such that $fox = y$.

Proof

Let s be a section map for the map $f: A \rightarrow B$ such that $fos = I_B$, where I_B is the identity map on B.

Define a map $x = soy$.

It is immediate to see that x satisfy the required equation i.e,

$$fox = fo(soy) = (fos)\ oy = I_B\ oy = y.$$

It follows that if a map $f: A \rightarrow B$ satisfies the *conclusion* above then for any y there exists a map x such that $fox = y$.

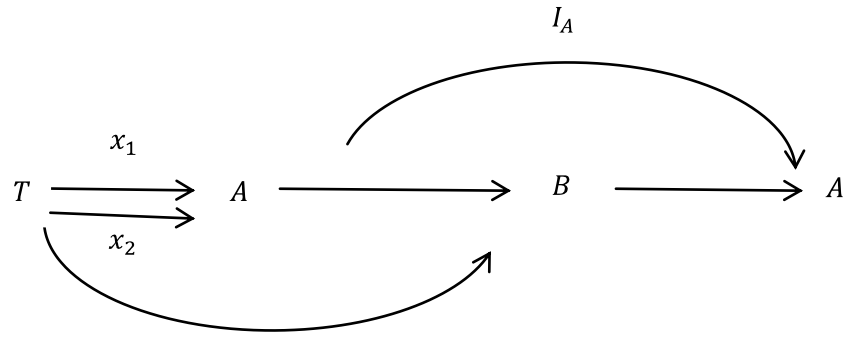
Theorem 3.2 [Lawvere and Schanuel, 2009]

If a map $f: A \rightarrow B$ has a retraction. Then for any set T and for any pair of maps $x_1: T \rightarrow A$, $x_2: T \rightarrow A$ from any set T to A. If $fox_1 = fox_2$,then$x_1 = x_2$

Proof

Let r be a retraction for $f: A \rightarrow B$, then by definition $rof = I_A$, where I_A is identity map on A.

Now, using the assumption that x_1 and x_2 are such that f composes with them to get the same map $T \rightarrow B$, we can compose further with r as follows:



$x_1 = I_A o x_1 = (rof) o x_1 = ro(fox_1) = ro(fox_2) = (rof) o x_2 = I_A o x_2 = x_2.$

Monomorphism

A morphism $f: A \longrightarrow B$ in a category $\mathbb{C}$ is called a *monomorphism* if, for every pair of maps $x_1: T \rightarrow A$ and $x_2: T \rightarrow A$, $fox_1 = fox_2$, implies $x_1 = x_2$.

Epimorphism

A morphism $f: A \longrightarrow B$ in a category $\mathbb{C}$ is called an *epimorphism* if for every pair of maps $x_1: T \rightarrow A$ and $x_2: T \rightarrow A$, $x_1 of = x_2 of$ implies $x_1 = x_2$.

It is significant to note that, in category theory, both *monomorphism* and *epimophism* are *cancellation properties*. The cancellation laws for monomorphisms and epimorphisms are customarily called *left* and *right* cancellation.

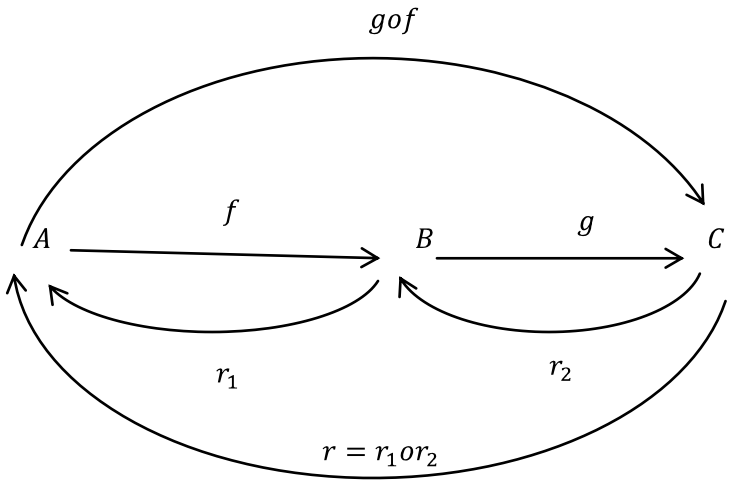
Theorem 3.3

If $f: A \rightarrow B$ has a retraction and $g: B \rightarrow C$ has a retraction, then the map $gof: A \rightarrow C$ has a retraction. That is, the composition of two maps each having a retraction has a retraction.

Proof

Let r_1 and r_2 be retractions for the maps f and g, respectively. Then by definition, $r_1 of = I_A$ and $r_2 og = I_B$.

The diagram below depicts the behaviour of the maps.



In order to show that gof also has a retraction we need to show that the composition of r and gof is the identity map on A i.e.,

$$ro(gof) = (r_1 \, o \, r_2)o(gof) = r_1o(r_2og)of = r_1oI_Bof = r_1of = I_A.$$

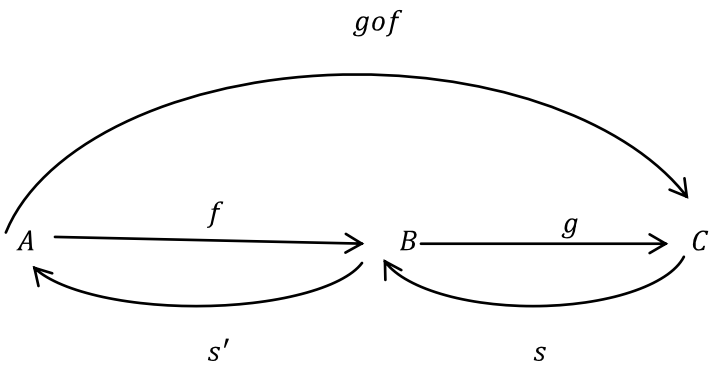
Hence the proof.

Theorem 3.4

The composite of two maps that have sections (coretraction) has a section (coretarction)

Proof

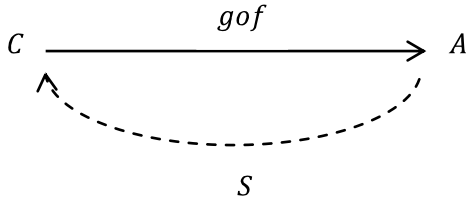
Let s and s' be sections for the maps $f: A \to B$and $g: B \to C$, respectively.We want to show thatthe composite gof also has a section and it is ss'.



$s$is a section for g implies $gos = I_C$, where I_C is the identity map on C.

Similarly, s' is a section for f implies$fos' = I_B$, where I_B is the identity map on B.

Now it remains to show that s is a section for gof. i.e., $s: C \to A$ i.e., $(gof)os = I_C$.



We need to show that $s'os$ is a section for gof.

Since $(gof)o(s'os) = go(fos')os = goI_Bos = go(I_Bos) = gos = I_C$, then the map $s'os$ is a section for gof. Hence the result.

Theorem 3.5(Uniqueness of inverses) [Lawvere and Schanuel, 2009]

If f has both retraction r and a section s, then $r = s$.

Proof

Since $f: A \to B$ has both a retraction and a section, the following equations hold:

$rof = I_A$ and $fos = I_B$.

Identity laws and the associativity give:

$r = roI_B = ro(fos) = (rof)os = I_Aos = s.$

$\Rightarrow r = s$.

3.5 Isomorphism

A map $f: A \to B$ is called an *isomorphism* if there exists another map $f^{-1}: B \to A$ which is both a retraction and a section for f i.e.,

$fof^{-1} = I_B$, and $f^{-1}of = I_A$.

The map f^{-1} is called the inverse map for f.

Theorem 3.6

If $f: A \to B$ and $g: B \to C$ are isomorphisms in a category with inverses $f^{-1}: B \to A$ and $g^{-1}: C \to B$, then gof is an isomorphism with inverse $f^{-1}og^{-1}$.

Proof

By section- retraction $f^{-1}of = I_A$, $fof^{-1} = I_B$ and

$g^{-1}og = I_B$, $gog^{-1} = I_C$.

Moreover, $(gof)\,(gof)^{-1} = (gof)o\,(f^{-1}og^{-1}) = g\ o\ (fof^{-1})o\ g^{-1} =$

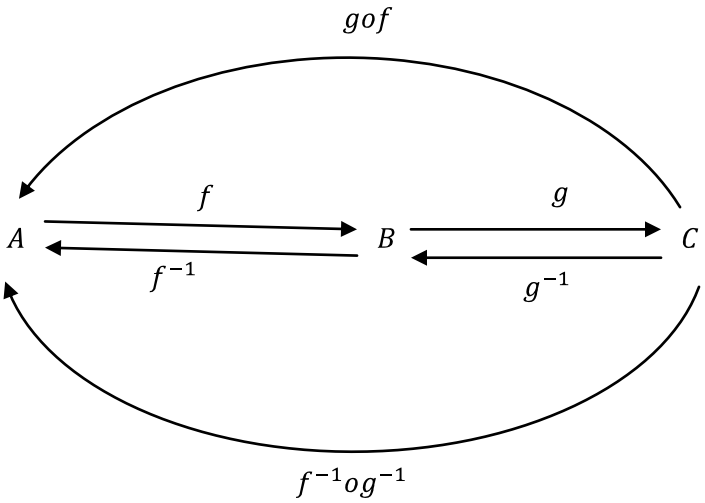
$goI_Bog^{-1} = gog^{-1} = I_C$ (1)

Similarly, $(gof) = (f^{-1}og^{-1})o\,(g\ of) =$

$f^{-1}o\,(g^{-1}og)o\,f\ =\ f^{-1}o\,I_B\ o\,f\ =\ f^{-1}\ o\,I_A$ (2)

From (1) and (2) we see that gof is also an isomorphism.

Diagrammatically,



This theorem is sometimes called the $shoe - sock\ theorem$ (see [Barr and Wells, 1999] and[Lawvere, 2009] for details)

Isomorphism of any two objects in a category

Suppose A and B are any two objects of a category$\mathbb{C}$. Then an arrow $f: A \to B$ is said to be an isomorphism in $\mathbb{C}$, if it has an inverse which is both a retraction and a section for f. In that case, A is said to be isomorphic to B, written as $A \cong B$in $\mathbb{C}$.

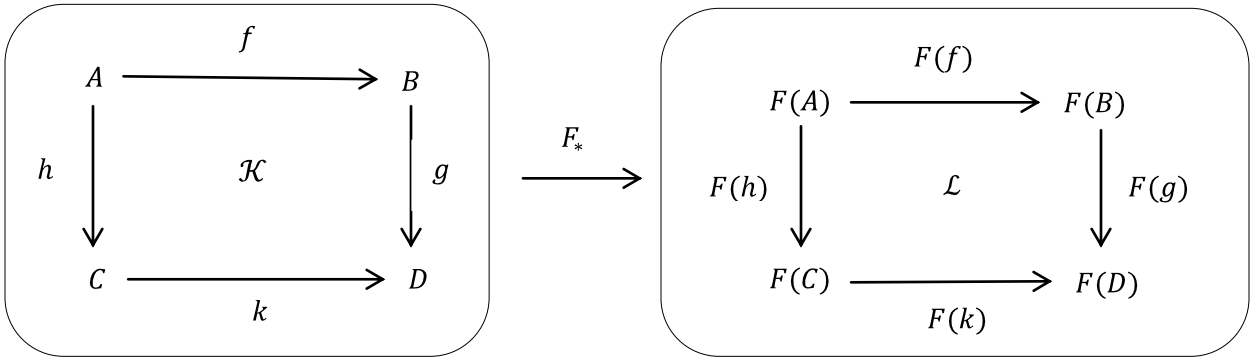
Automorphism

An arrow $f: A \to A$ in a category (with the source and target the same) is called an endomorphism. If it is invertible, it is called an automorphism.

3.6*Functors*

A functor$F$ from a category $\mathbb{C}$ to a category $\mathbb{D}$ is a homomorphism which preserves identities and composition. It plays the same role as monoidhomomorphisms for monoids and monotone maps for posets: it preserves the structures that a category has.

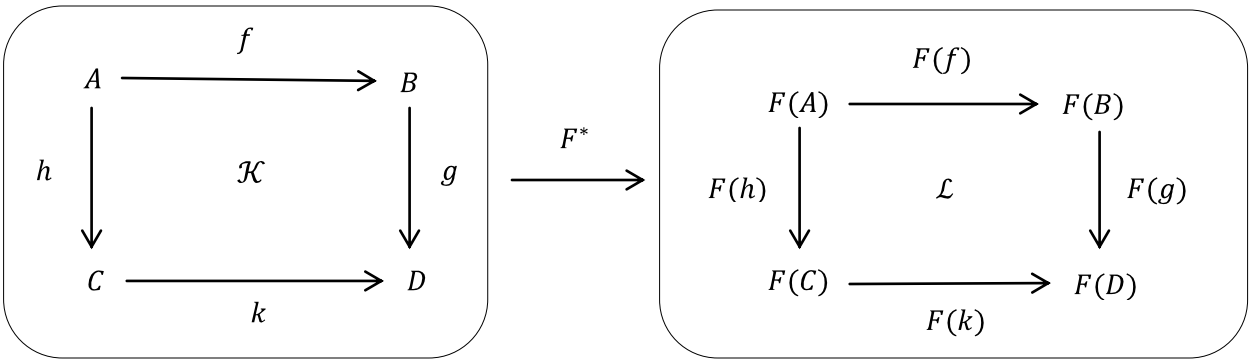
Moreover, a functor is a generic *picture* projecting all objects and morphisms of a source category into a target category. Let $\mathcal{K} = (ob(\mathcal{K}), Mor(\mathcal{K}))$ be a source(or domain) category and $\mathcal{L} = (ob(\mathcal{L}), Mor(\mathcal{L}))$ be a target (or codomain) category, where $ob(\mathcal{K}), ob(\mathcal{L}), Mor(\mathcal{K}), Mor(\mathcal{L})$ are classes of objects and arrows in the categories $\mathcal{K}$ and $\mathcal{L}$ respectively. Then a functor $F = (F_o, F_M)$ is defined as a pair of functions $F_o: ob(\mathcal{K}) \longrightarrow ob(\mathcal{L})$and $F_M: Mor(\mathcal{K}) \longrightarrow Mor(\mathcal{L})$, preserving the structure of $\mathcal{K}$ in $\mathcal{L}$. That is, a *covariant functor*$F_*: \mathcal{K} \longrightarrow \mathcal{L}$ is a picture in the target category $\mathcal{L}$ of the source category $\mathcal{K}$ such that the following categorical symmetry is achieved:



and satisfies the following properties:

1. For any objects $A, B \in \mathcal{K}$: if $f \in Mor_{\mathcal{K}}(A, B)$, then $F_M(f) \in Mor_{\mathcal{L}}(F_o(A), F_o(B))$. That is, if there is an arrow between objects A, B of $\mathcal{K}$; then there is an arrow between $f(A), f(B)$ of $\mathcal{L}$
2. For all $A \in ob(\mathcal{K})$, $F_M(Id_A) = Id_{F_o(A)}$. That is the identity arrow of A in $\mathcal{K}$ is the identity of $f(A)$ in $\mathcal{L}$;
3. For all $f, g \in Mor(\mathcal{K})$, if $cod(f) = dom(f)$, then $F_M(gof) = F_M(g)oF_M(f)$.

Similarly, a contravariantfunctor$F^*: \mathcal{K} \longrightarrow \mathcal{L}$ is a picture in the target category $\mathcal{L}$ of the source category $\mathcal{K}$ such that the following categorical symmetry is achieved:



and satisfies the following properties:

1. For any objects $A, B \in \mathcal{K}$: if $f \in Mor_{\mathcal{K}}(A, B)$, then $F_M(f) \in Mor_{\mathcal{L}}(F_o(B), F_o(A))$. That is, if there is an arrow between objects A, B of $\mathcal{K}$; then there is an arrow between $f(B), f(A)$ of $\mathcal{L}$
2. For all $A \in ob(\mathcal{K})$, $F_M(Id_A) = Id_{F_o(A)}$. That is the identity arrow of A in $\mathcal{K}$ is the identity of $f(A)$ in $\mathcal{L}$;
3. For all $f, g \in Mor(\mathcal{K})$, if $cod(f) = dom(f)$, then $F_M(gof) = F_M(f)oF_M(g)$.

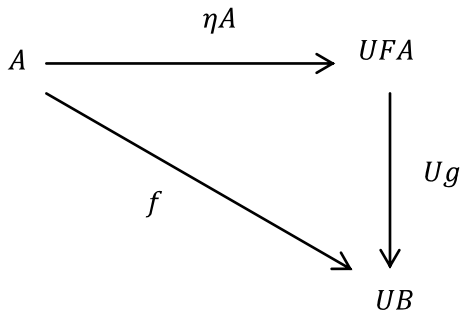
Examples 3.6a

1. For any category $\mathbb{C}$, the *identity*functor$Id_{\mathbb{C}}: \mathbb{C} \longrightarrow \mathbb{C}$, defined by $Id_{\mathbb{C}}(A) = A$ on every object A of $\mathbb{C}$ and $Id_{\mathbb{C}}(f) = f$ on every morphism f of $\mathbb{C}$ is a covariant functor;
2. If $\mathcal{D}$ is a subcategory of $\mathbb{C}$, then the *inclusion*functor$F: \mathcal{D} \longrightarrow \mathbb{C}$ defined by $F(\mathcal{D}) = \mathcal{D}$ is a covariant functor;
3. The functor$Q: \boldsymbol{Set} \longrightarrow \boldsymbol{Set}$ that takes every set A to its power set $\wp(A)$ and a mapping $f: A \longrightarrow B$ to the inverse image mapping $f^{-1}: \wp(B) \longrightarrow \wp(A)$, which maps a subset C of B to $f^{-1}(C) \subseteq A$ is a contravariant functor.

3.6.1 Types of Functors

1. **Forgetful Functor**: is a functor that forgets some properties or structures of a category in another category. For example, the functor$U: \boldsymbol{Grp} \longrightarrow \boldsymbol{Set}$, which maps a group homomorphism $f: G \longrightarrow H$ to its underlyning function of sets $U(f): U(G) \longrightarrow U(H)$, is a forgetful functor. It forgets about the group structure and forgets the fact that the morphisms are group homomorphism. These functors are sometimes called the *underlying*functors. Another example is the functor$V$: $\boldsymbol{Sem} \rightarrow \boldsymbol{Set}$ which forgets all the structure of a semigroup.
2. **Endo-Functor**: The $F: \mathbb{C} \longrightarrow \mathbb{C}$ is called an *endo-functor***.** Thatis, the functor that maps every object of itself to itself and every morphism of itself to itself. A typical example is the *identity*functor.
3. **AdjointFunctors**:Let $\mathbb{A}$ and $\mathbb{B}$ be categories. If $F: \mathbb{A} \longrightarrow \mathbb{B}$ and $G: \mathbb{B} \longrightarrow \mathbb{A}$are functors, thenthe functor$F$ is called the *left adjoint* of the functor$U$ denoted $F \vdash$

U, and U is called the *right adjoint* of F, denoted $F \dashv U$, provided there exists a natural transformation $\eta: Id \longrightarrow UF$ such that for any objects $A \in \mathbb{A}$, $B \in \mathbb{B}$ and any arrow $f: A \longrightarrow UB$, there is a unique arrow $g: FA \longrightarrow B$ such that the diagram



commutes.

This definition asserts that there is a functional way to convert any arrow $f: A \longrightarrow UB$ to an arrow $g: FA \longrightarrow B$ in such a way that g solves the equation $f = U(?)o\ \eta A$, and that the solution is unique.

3.6.2 Full, Faithful and Embedding Functors:

Let $f: A \longrightarrow A'$ be a mapping between any two objects of a Category $\mathbb{A}$ and let FA and FA' be objects in the category $\mathbb{B}$. Then a functor $F: \mathbb{A} \longrightarrow \mathbb{B}$is said to be *full* (respectively, *faithful*) if the mapping $\Phi: Mor_{\mathbb{A}}(A, A') \longrightarrow Mor_{\mathbb{B}}(FA, FA')$ given by $\Phi(f) = F(f)$ is surjective (respectively, injective).

Moreover, the functor as defined above is an *embedding* if $F: Mor(\mathbb{A}) \longrightarrow Mor(\mathbb{B})$is injective.

Examples 3.6b

1. The functor$F: \boldsymbol{Rmod} \longrightarrow \boldsymbol{Set}$is a faithful functor;
2. The inclusion functor is an embedding;
3. The functor$P : \boldsymbol{Mon} \rightarrow \boldsymbol{Sem}$which embeds the category of monoids into the category of semi groups by forgetting that a monoid has an identity is an example of embedding and underlying functor.

3.6.3 Natural Transformations

Natural transformation provides a compact way of transforming one functor into another while respecting the internal structures (i.e., composition of morphisms) of the category involved. Essentially, natural transformations can be regarded as morphisms between functors.

Formally, let $F, G: \mathbb{A} \longrightarrow \mathbb{B}$ be two functors from a category $\mathbb{A}$ to a category $\mathbb{B}$. Then a natural transformation $\eta: F \Rightarrow G$ from F to G is a family $(\eta_A: F(A) \longrightarrow G(A))_{A \in A_o}$ of morphisms of $\mathbb{B}$, indexed by the objects of $\mathbb{A}$, such that $\eta_{A'} F(f) = G(f)\eta_A$ for every morphism $f: A \longrightarrow A'$ in $\mathbb{A}$, i.e., the following diagram commutes

$$\begin{array}{ccc} A & F(A) & \xrightarrow{\eta_A} & G(A) \\ \downarrow f & \downarrow F(f) & & \downarrow G(f) \\ A' & F(A') & \xrightarrow[\eta_{A'}]{} & G(A') \end{array}$$

Note that the morphismη_A for an object $A \in \mathbb{A}_o$ is called the component of the natural transformation ηat A.

It is reasonable to expect any underlying functor$U$ to be faithful and that if f is an isomorphism and $U(f)$ is an identity arrow then f is an identity arrow.

Examples of natural transformations3.6c

(1) For every functor$F: \mathbb{A} \longrightarrow \mathbb{B}$, the identity natural transformations$Id_F: F \Rightarrow F$, defined as $(Id_F)_A = Id_{F(A)}$, exists.

(2) The map which takes an arrow of a graph to its source is a natural transformation from A to N where A and N are the collection of arrows and nodes of the graph, respectively (see {Barr and Wells, 1999} for details).

3.7 Construction of Categories

3.7.1Category of relations

The category$\boldsymbol{Rel}$ has the class of sets as objects and binary relations as morphisms. An *arrow (or morphism)*$R : A \rightarrow B$ in this category is a relation between the sets A and B i.e., $R \subseteq A \times B$.

The composition of any two relations $R: A \rightarrow B$ and $S: B \rightarrow C$ is given by: $(a, c) \in S \; o \; R$ if and only if for some $b \in B, (a, b) \in R$ and $(b, c) \in S$.

The identity morphism is given by Id_A and it is the relation $Id_A: A \rightarrow A$ such that for each $a \in A$, $(a, a) \in Id_A$.

Properties

(1) Category $\boldsymbol{Rel}$ has the category $\boldsymbol{Set}$ as a (wide) subcategory.

(2) The categorical product in $\boldsymbol{Rel}$ is given by the disjoint union of sets rather than the cartesian product as it is in the category $\boldsymbol{Set}$, and so is the coproduct.

(3) The category $\boldsymbol{Rel}$ is a *monoidal closed category*. The *monoidal product* $A \otimes B$ and the *internal hom* $A \Rightarrow B$ are given by the Cartesian productof sets.

3.7.2 Category of Monoids and Semigroups

A monoid is a semigroup with an identity element. It is commutative if its binary operation is commutative.

An identity element e for a semigroup S is an element of S that satisfies the equation $se = es = s$ for all $s \in S$.

Remarks 3.7a

(1) The identity element of a semigroup is unique if it exists.

(2) It follows that a $monoid$ is not allowed to be empty: it must contain an element.

(3) The identity element of the monoidis x^0.

(4) The laws $S^k S^n = S^{k+n}$ and $(S^k)^n = S^{kn}$ then hold for all non-negative k and n.

3.7.3 Homomorphisms of Semigroups and Monoids

Let S and T be any two semigroups, a function $h: S \to T$ is a homomorphism of semigroups if $h(s, s') = h(s)(s')$, for all $s, s' \in S$

A homomorphism of monoids is a semigroup homomorphism between monoids that preserves the identity elements: If e is the identity elements of the domain, $h(e)$ must be identity element of the codomain.

Examples 3.7

(1) The identity function on any monoid is a monoid homomorphism.

(2) If M is a monoid and S is a submonoid, the inclusion function from S to M is a monoid homomorphism.

(3) A function that takes an even integer to 0 and an odd integer to 1 is also a monoid homomorphism.

i.e., $f(2n) = 0$ when n is an even integer

$f(2n - 1) = 1$ when n is an odd integer

This is a monoid homomorphism from the monoid of integers on multiplication to the set $\{0,1\}$ on multiplication.

Remarks 3.7b

(1) It is easy to see that identity functions are homomorphism and

(2) The composition of homomorphisms yields homomorphisms.

3.7.4 The Category $\boldsymbol{Sem}$ and $\boldsymbol{Mon}$

(1) The category of semigroups and semigrouphomomorphisms is denoted by $\boldsymbol{Sem}$.

(2) The category of monoids and monoidhomomorphisms is denoted by $\boldsymbol{Mon}$.

3.7.5 Isomorphism of semigroups

If a semigroup homomorphism has an inverse which is also a homomorphism, then we say that the semigroup homomorphism is an isomorphism. In this case the two semigroups in question have the same abstract structure and are said to be isomorphic.

Remarks 3.7c

(1) As we shall see later, the property of possessing an inverse is taken to define the categorical notion of isomorphism.

(2) There may in general be many different isomorphisms between isomorphic semigroups.

3.8 Universal mapping properties

3.8.1 Terminal object

An object T in a category $\mathbb{C}$ is called a *terminal object* of $\mathbb{C}$ if for each object X of $\mathbb{C}$ there is exactly one $\mathbb{C} - mapX \to T$.

3.8.2 Initial object

An object T in a category $\mathbb{C}$ is called an *initial object* of $\mathbb{C}$ if for each object X of $\mathbb{C}$ there is exactly one $\mathbb{C} - mapT \to X$. This appears to be the dual notion of the terminal object of the category $\mathbb{C}$.

The following hold:

(i) Both the initial and the terminal objects are unique up to isomorphism.

(ii) An object A in a category $\mathbb{C}$ is terminal in $\mathbb{C}$ if and only if A is initial in $\mathbb{C}^{opp}$.

(iii) The initial and the final objects in any category if they exist are denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively.

Example 3.8

The initial object in a preordered set is the bottom element, if it exists; the terminal object is the top element. In $\boldsymbol{Set}$, the initial object is the empty set, while the terminal object is the singleton set.

Similarly, the initial object in $\boldsymbol{Cat}$ is the category $\mathbf{0}$, while the final object is the category $\mathbf{1}$.

3.8.3 Point

If $\mathbf{1}$is a terminal object of a category $\mathbb{C}$ and if X is any object of the category $\mathbb{C}$, then a $\mathbb{C} - map\mathbf{1} \to X$ is called a point of X.

3.8.4 Binary operations and Action

A *binary operation* on an object A is a map $A \times A \to A$

An action of an object A on an object X is a map $A \times X \to X$

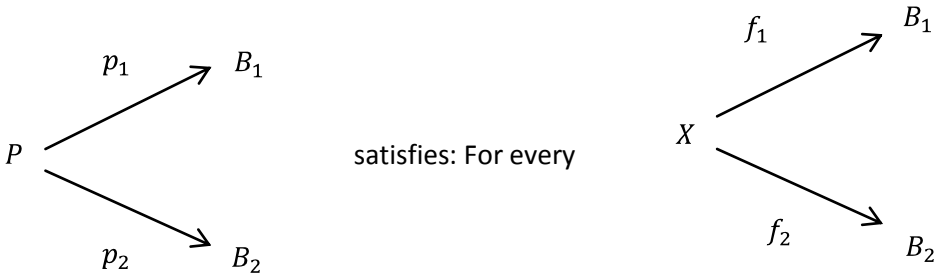
For example, if $\aleph = \{0,1,2, \ldots\}$ is the set of natural numbers, then addition and multiplication are binary operations on $\aleph$ respectively. That is, $\delta : \aleph \times \aleph \to \aleph$, where $\delta(x, y) = x + y$ or $\delta(x, y) = xy$ for each $(x, y) \in \aleph \times \aleph$ respectively are the binary operations on $\aleph$.

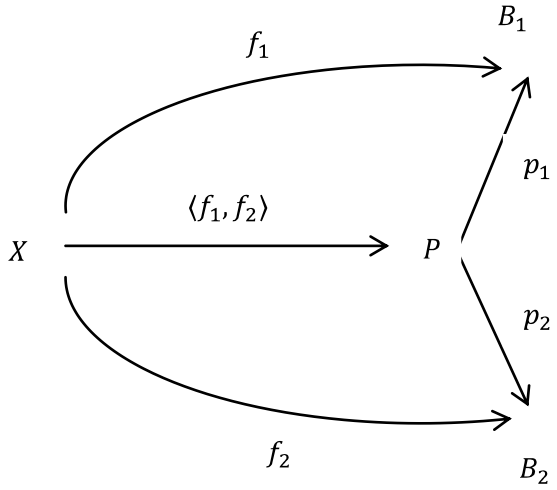
On the other hand, an action of $\mathbf{1}$ on X is just a given endomap of X, since $1 \times X = X$, where $\mathbf{1}$ is the category with one object and exactly one arrow.

3.9 Product of objects in any category $\mathbb{C}$

Let B_1 and B_2 be any two objects of a category $\mathbb{C}$. The *product* of B_1 and B_2 in $\mathbb{C}$ is an object Pin $\mathbb{C}$, along with a pair of maps, $p_1 : P \to B_1$ and $p_2 : P \to B_2$, in $\mathbb{C}$ satisfying;

For every object X and maps$f_1 : X \to B_1,\ f_2 : X \to B_2$, there is exactly one map $f : X \to P$ for which $f_1 = p_1 f$and $f_2 = p_2 f$. Pictorially,



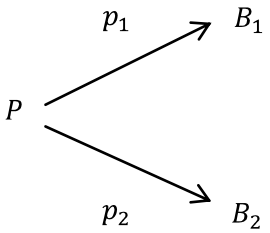


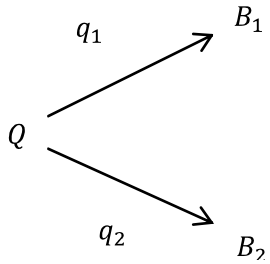
The maps p_1 and p_2 are called the *projection maps* for the product P. Also, the map f is uniquely determined by the maps f_1 and f_2 and sometimes denoted by $\langle f_1, f_2 \rangle$.

In general, an object P together with a pair of maps $p_1: P \to B_1,\ p_2: P \to B_2$ is said to be a product of B_1 and B_2 if for each object X and each pair of maps $f_1: X \to B_1,\ f_2: X \to B_2$, there is exactly one map $f: X \to P$ for which both $f_1 = p_1 f$ and $f_2 = p_2 f$.

The uniqueness theorem for product of objects in a category [Barr and Wells, 1999]

If both

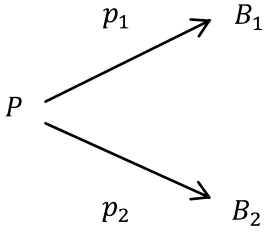


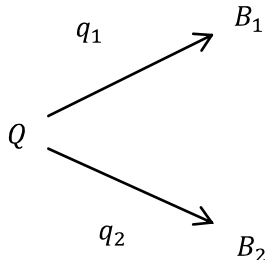


are product projection pairs (i.e., the $p's$ as well as the $q's$ satisfy the universal mapping property), then there is exactly one map $f: P \to Q$ for which $q_1 f = p_1$ and $q_2 f = p_2$. In particular, the map f is an *isomorphism*.

Proof

Suppose





are two products of B_1 and B_2, respectively. We need to show that there is exactly one map $f: P \to Q$ up to isomorphism for which $q_1 f = p_1$ and $q_2 f = p_2$.

Since P is a product, by the universal property there is exactly one map $h: Q \to P$ for which $p_1 h = q_1$ and $p_2 h = q_2$. We have to show that this map h is an *isomorphism*. For this purpose, it suffices to find an inverse for it. But Q is a product implies that there is exactly one map $k: P \to Q$ for which $q_1 k = p_1$ and $q_2 k = p_2$.

Now, we have to show that k is really the inverse of h . i.e., $hk = I_P$, where I_P is the identity map of P.

Since

$$p_1(hk) = (p_1 h)k = q_1 k = p_1$$

and

$$p_2(hk) = (p_2 h)k = q_2 k = p_2$$

This means that hk is the unique map which composed with p_1 gives p_1 and composed with p_2 gives p_2. But there is another map with this properties which is the identity of P.

However, the exactly one map in the definition of product implies that hk is the identity of P. This proves half of the theorem. The other half is that kh is the identity of Q, which follows in the same way from

$$q_1(kh) = (q_1 k)h = p_1 h = q_1$$

and

$$q_2(kh) = (q_2k)h = p_2h = q_2$$

3.10 Sum of any two objects in a category

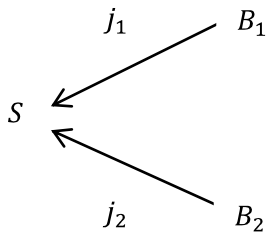
The dual notion of the definition of product of any two objects in a category is called the *sum* or *coproduct* of the two objects in the resulting category.

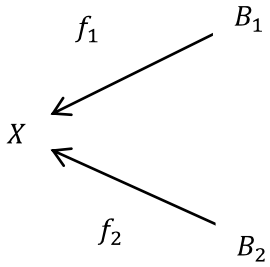
Let B_1 and B_2 be any two objects of a category $\mathbb{C}$. Then the *sum or coproduct* of B_1 and B_2 in $\mathbb{C}$ is

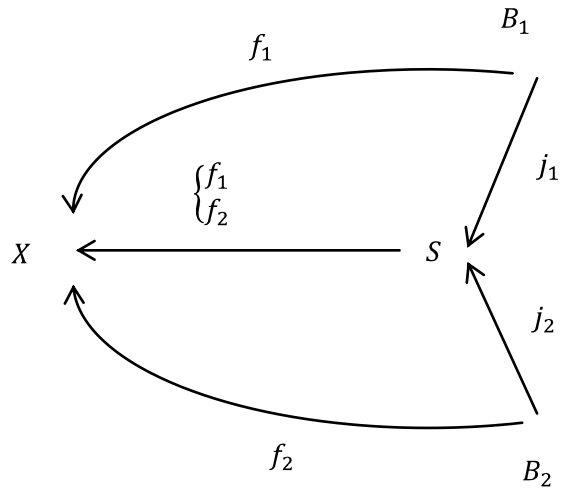
P-1an object S in $\mathbb{C}$, and

P-2a pair of maps, $j_1: B_1 \to S$ and $j_2: B_2 \to S$, in $\mathbb{C}$ satisfying:

For every object X and every pair of maps $f_1: B_1 \to X$, $f_2: B_2 \to X$, there is exactly one map $f: S \to X$ for which $f_1 = fj_1$ and $f_2 = fj_2$:







Remarks3.10

(1) The maps j_1 and j_2 are called the *projection maps* for the sum S. Also, the map f is uniquely determined by the maps f_1 and f_2 and denoted $\begin{cases} f_1 \\ f_2 \end{cases}$.

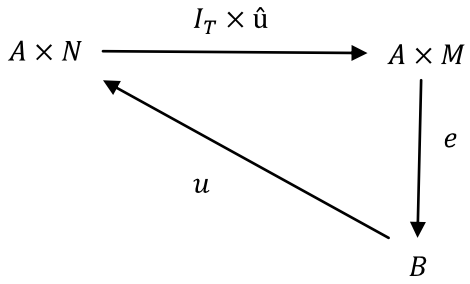
(2) In $\boldsymbol{Set}$ (category of sets and functions between sets), the disjoint union of sets is the coproduct. It therefore satisfies the *universal mapping property*. This also means that the disjoint union is the *categorical dual* of the Cartesian product construction.

3.11 Exponentiation in categories

3.11.1 Map objects

Let A and B be any two objects in a category $\mathbb{C}$ equipped with *product*. Then an object M together with an evaluation map $e: A \times M \to B$, is called an *object* of *maps* from A to

B, if M and e satisfy: for each object N and each map $u: A \times N \to B$, there is exactly one map, denoted $\hat{u}: X \to M$, such that $u = e(I_T \times \hat{u})$, in other words, the following diagram



commutes.

Remarks 3.11

(1) The map $\hat{u}$ uniquely determines u and is sometimes called the*name of* u.

(2) In some literature, the object map M is referred to as *function spaces or exponential objects* and given a special symbolB^A with $e: A \times B^A \to B$, so that there is *exactly one condition* on the evaluation map e, abbreviated $e: A \times B^A \to B$ and induces $\frac{N \to B^A}{A \times N \to B}$

3.12 Cartesian Closed Categories

Intuitively, a category with *product* or *sum*, *terminal object* and *object maps* is called a *Cartesian closed category* with a stronger universal mapping property i.e., *distributivity*.

A category $\mathbb{C}$is called a *cartesian closed category*if $\mathbb{C}$ satisfies the following:

1. There is a terminal object **1**in $\mathbb{C}$.

2. For each pair of objects A and B of $\mathbb{C}$, there is a product denoted $A \times B$ with projections $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$.
3. For every pair of objects A and B, there is a *map object* denoted B^A, and an arrow $e : B^A \times A \to B$ called the *evaluation map*, with the property that for any arrow $f : C \times A \to B^A$, there is a unique arrow $\lambda f : C \to B^A$ such that the composite

$$C \times A \xrightarrow{\lambda f \times A} B^A \times A \xrightarrow{e} B$$

is f.

In some literatures, B^A has been denoted by $[A \to B]$

Examples3.12

1. $\boldsymbol{Set}$,is a cartesian closed category. Because, for any sets $A$and B, B^A is the set of all functions from A to B, and the evaluation map e is the map from $A$to B.In this category,B^A is denoted$Hom_{\boldsymbol{Set}}(A, B)$.
2. The category $\mathcal{G}$ of graphs(the category whose objects are graphs and whose morphisms are homomorphisms of graphs)isacartesian closed category.
3. The category $F(\mathbb{C}, \boldsymbol{Set})$ is a cartesian closed category, where $\mathbb{C}$ is any small category. The objects of this category are functors while the morphisms are natural transformations.

4. The category $\boldsymbol{Cat}$, is a cartesianclosed category. It is similar to the category $\boldsymbol{Set}$, only that the object map B^A is taken to be the category whose objects are functors from A to B and whose arrows are natural transformations between A and B.

3.13 Monoidal category

A monoidal category $\mathbb{C}$ is a sextuple $(\mathbb{C},\otimes, a, \mathbf{1}, r, l)$satisfying the following axioms:

(i) $\otimes: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a *bifunctor* called the *tensor product*;

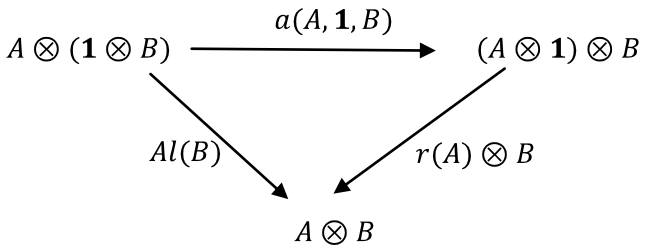
(ii) an object $\mathbf{1} \in \mathbb{C}$ called the *unit object* or the *tensor object*;

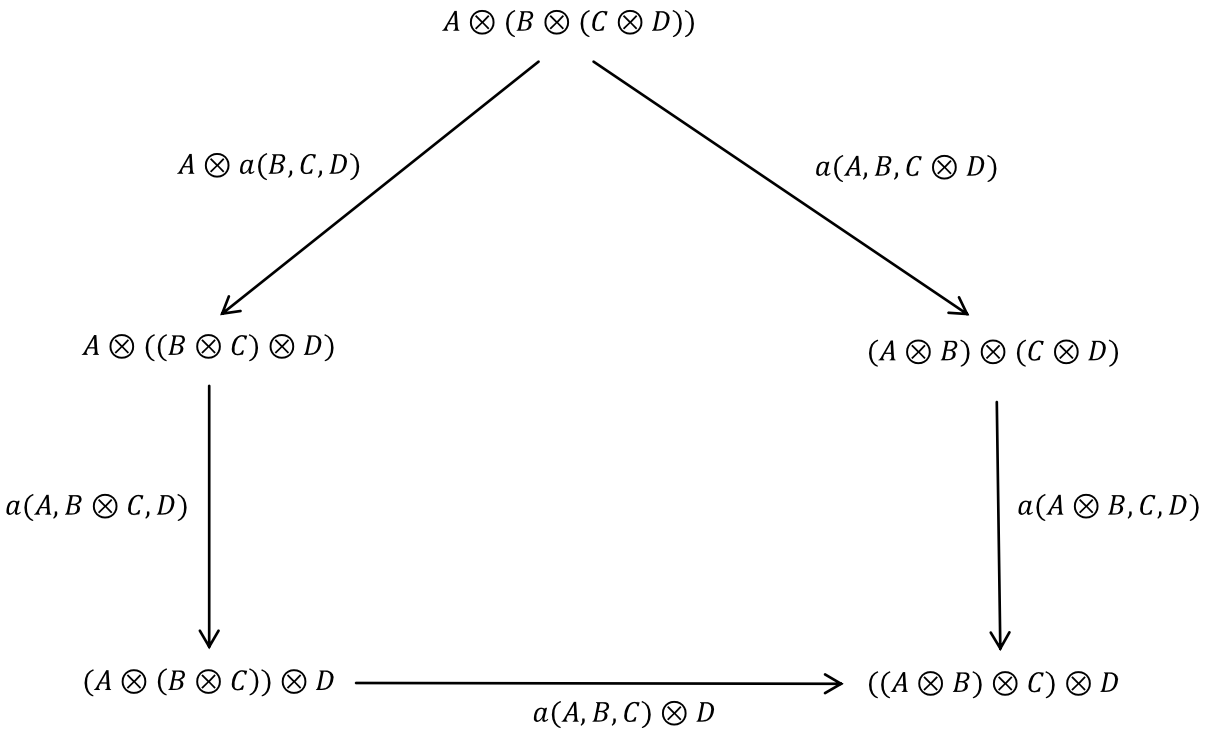
(iii) a natural isomorphism $a_{A,B,C}: A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ called the *associator*, $for\ all\ A, B, C \in \mathbb{C}$;

(iv) a natural isomorphism$r_A: A \otimes \mathbf{1} \to A$ called the *right unitor*, $for\ all\ A, \in \mathbb{C}$; and

(v) a natural isomorphism $l_A: \mathbf{1} \otimes A \to A$ called the *left unitor*

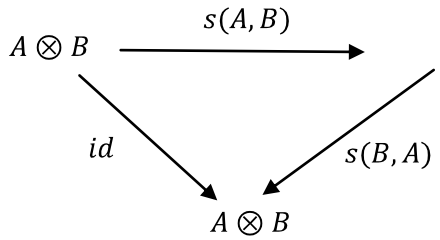
such that the following diagrams commute.

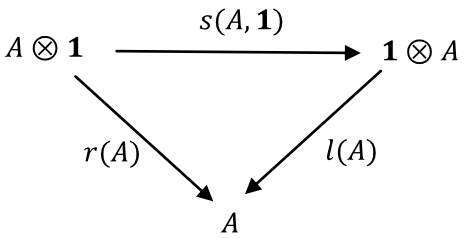


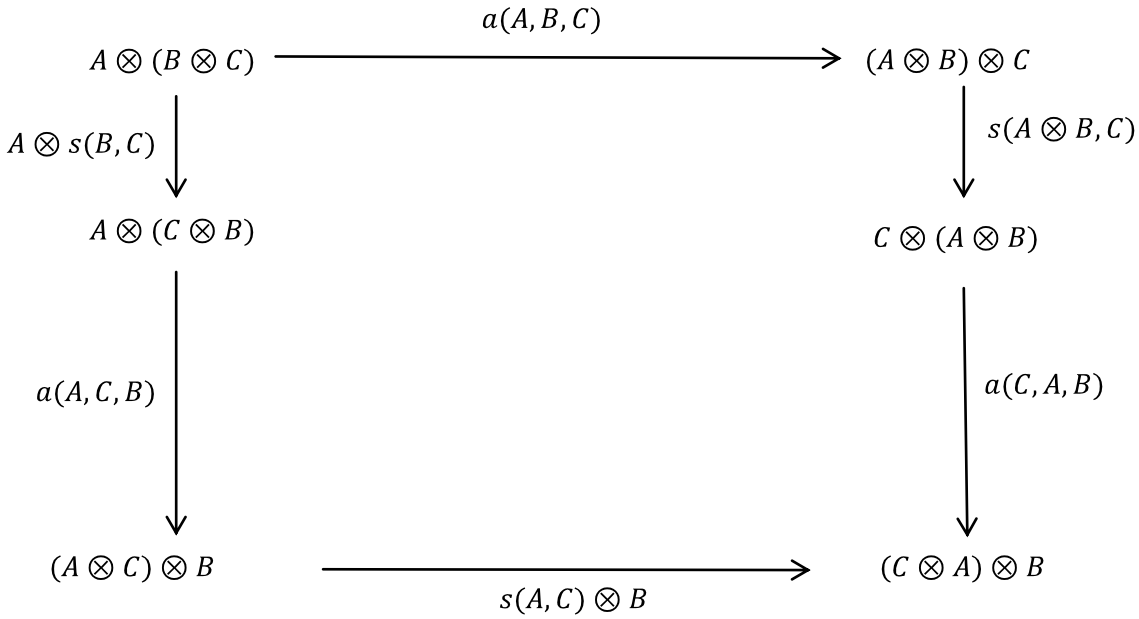


A monoidal category is called *symmetric* if, in addition, it has the natural isomorphism:

(vi) $s(A, B): A \otimes B \to B \otimes A$ and the following diagrams commute:







Examples3.13

The notion of Monoidal category is ubiquitous, it can be seen almost everywhere in the category theory. Here are some examples.

1. The category $\boldsymbol{Sets}$of sets is a monoidal category with the tensor product as the usual Cartesian product of sets and the unit object as a singleton set. The structure morphismsa, l, r are the usual functions between sets.

2. Any category with *finite products* is a *symmetric monoidal category* with the product as the monoidal product and the *terminal object* as the unit. Such a category is sometimes called a cartesianmonoidal category.
3. Any category with *finite coproducts* is monoidal with the coproduct as the monoidal product and the *initial object* as the unit. Such a monoidal category is usually called *cocartesianmonoidal category*.
4. The category$\boldsymbol{Cat}$ of small categories is also monoidal with the product category, where the category with one object and only its identity map is the unit.
5. In general, any Cartesian closed category is a symmetric monoidal category.

More examples can be found in ({Barr, 1979, 1996}, {Barr and Wells, 1999}, and {Pratt, 2004}).

3.14 Closed Monoidal category

A monoidal category $\mathbb{C}$ is said to be closed if for every object $A$of $\mathbb{C}$, the functor $A \otimes -$ has *a right adjoint*.

Formally, a closed monoidalcategory is a category $\mathbb{C}$ equipped with

(i) an object map $[A \to B]$, i.e., B^Ain the context of $\boldsymbol{Set}$; and

(ii) a morphism,$eval_B^A$: $[A \to B] \otimes A \to B$, for every objects $A, B \in \mathbb{C}$

satisfying the following universal mapping property: for every morphism

$$f: X \otimes A \to B$$

there exists a unique morphism

$$g: X \to [A \to B]$$

such that $f = eval_B^A \, o(g \otimes I_A)$.

It can easily be shown that this construction defines a functor $[-,-]: \mathbb{C}^{op} \otimes \mathbb{C} \to \mathbb{C}$. This functor is called the *internal Homfunctor*, and the object $[A \to B]$ is called the *internal Hom* of A and B.

This means that there exists a bijection, called *currying*, between the Hom-sets i.e.,

$$Hom_C(A \otimes B, C) \cong Hom_C(A, C^B)$$

that is natural in both A and C.

It is significant to note that, several other explications for the *internal Hom* are in common use. If the tensor product on $\mathbb{C}$ is the cartesian product, the usual notation is B^A(the set of all functions from $A$to B) and this object is called the *object map* or the *exponential object*. In $\boldsymbol{Set}$, the internal hom is just B^A while in $\boldsymbol{Rel}$, the internal hom is not B^Abut the disjoint union of both A and B.

3.15 Evaluation and composition

The following morphisms could be derived in any closed monoidal category using the adjunction. For each pair of objects $A, B \in \mathbb{C}$, there is an identity map $A \multimap B \to A \multimap B$ which corresponds to an evaluation map $eval_B^A: A \otimes (A \multimap B) \to A$ and which internalizes the evaluation map. This internalization is natural in both $A$and B. We can

then derive an arrow $k_{A,B,C}$ defined as the map$(A \multimap B) \otimes (B \multimap C) \to A \multimap C$ which corresponds to the map

$$A \otimes (A \multimap B) \otimes (B \multimap C) \xrightarrow{eval_B^A \otimes id} B \otimes (B \multimap C) \xrightarrow{eval_C^B} C$$

under the adjointness.

It follows from above that the arrow $k_{A,B,C}$ which is also natural in the variables A, B and C internalizes the composition arrow.

3.16 $*-$*Autonomous Category*

A *-*autonomous category* is a symmetric monoidal closed category with a *dualizing object*, i.e., a symmetric closed monoidal category equipped with a given *duality funtor*.

Let $\mathbb{C}$ be a symmetric monoidal closed category. Then a functor$(-)^*: \mathbb{C}^{opp} \to \mathbb{C}$ is a *duality functor* if there is an isomorphism $d(A,B): A \multimap B \to B^* \multimap A^*$, natural in A and B, such that for all objects A, B and C, the following diagram

$$\begin{array}{ccc} (A \multimap B) \otimes (B \multimap C) & \xrightarrow{k_{A,B,C}} & A \multimap C \\ \downarrow{\scriptstyle d(A,B) \otimes d(B,C)} & & \downarrow{\scriptstyle d(A,C)} \\ (B^* \multimap A^*) \otimes (C^* \multimap B^*) & \xrightarrow[k_{C^*,B^*,A^*} os]{} & C^* \multimap A^* \end{array}$$

commutes.

The map $s = s(B^* \multimap A^*, C^* \multimap B^*)$.

3.16.1 Properties of *-autonomous category

Given the duality funtor,

(i) $\otimes$ and $\multimap$ are related by the following isomorphisms:

$$A \otimes B \cong (A \multimap B^*)^* \qquad A \multimap B \cong (A \otimes B^*)^*$$

(ii) There is another monoidal structure in a *-autonomous category denoted $\oplus$,defined by $A \oplus B \cong (A^* \multimap B^*)^*$. The unit for this monoidal structure is denoted $\mathbf{1}^*$. It follows by the definition of $\oplus$ and right unitor, the following isomorphism $A \oplus \mathbf{1} = (A^* \otimes B^*)^* \cong (A^* \oplus \mathbf{1}^*)^* \cong A^{**} \cong A$ holds.

(iii) The object $\mathbf{1}^*$ which would therefore be denoted $\mathbf{0}$ is the dualizing object and can also be defined in such a way that it correspond to the adjunction of the evaluation arrow defined above.

Examples 3.16

(1) The category $\boldsymbol{Rel}$ of sets and relations discussed above is a *-autonomous category;

(2) The Chu spaces obtained from arbitrary meet and join of a semilatticeis a *-autonomous category.

Details of *-autonomous category may be found in ([Barr, 1979] and [Barr and Wells, 1999]).

CHAPTER FOUR

CHU SPACES AND CHU CATEGORIES

In this chapter, largely following ([Barr, 1996], [Henry, 2003], [Devarajan, 1999], [Pratt, 1994, 1995, 1999, 2004, 2005a, 2005b, 2005c, 2005d] and [Zhang, 2004]), the concept of Chu spaces and Chu categoriesare explicated and exemplified.

4.1 Chu spaces

A $K-Chu\ space$ is a triple $\mathcal{A} = (A, r, X)$ consisting of sets $A\ and\ X$ and a function $r: A \times X \rightarrow K$. Essentially, this amounts to defining a Chu space as an $A \times X$ matrix whose entries are drawn from K, or a $K-valued$ binary relation between A and X. The function r is intended to provide a structure to the object Chu it defines by relating the elements of A with the elements of X. The entries at (a, x) is written $r(a, x)$ or arxor $a(x)$. The elements of A are called $points, subjects\ or\ individuals$ depending on the context, while the elements of X are considered as states or predicates.

The Chu space $\mathcal{A}$ may be organized either into rows or columns. If $\mathcal{A}$ is organized into rows, the set A is regarded as the *domain* or the *carrier* of the structure and the row, indexed by $a,$ as the complete description of the element a. In this case, the function $: A \rightarrow K^X$ is used for sorting the row entries satisfying the condition $(a)(x) = r(a, x)$, and it assigns to each point a its description. Notationally, is written for the description of the point a, namely (a), and for the set $\{: a \in A\}$ of all rows of the Chu space where $\{: a \in A\}$ is the set of all descriptions of the point $a\ in\ A$. Whenever is injective, that is, when there is no repeated rows, the Chu space $\mathcal{A}$ is called $separable$or $coextensional$.

On the other hand, if $\mathcal{A}$ is organized into columns, A is viewed as consisting of variables with values ranging over the set K and the column indexed by x as one of the permitted assignments to these variables. The function ř: $X \rightarrow K^A$ satisfies $\check{r}(x)(a) = r(x,a)$ and it is used for sorting the columns' entries. These entries appear as duals of and . However, if ř is injective, that is, when there is no repeated columns, the Chu space $\mathcal{A}$ is called $extensional$.

A Chu space is said to be *biextensional*if it is both extensional and coextensional.

It may be noted that the ground set K could be as meagre as the empty set or a singleton, or as big as the set of all complex numbers (while representing Hilbert spaces), or something in between such as the set of 2^n elements of an $n-set$ while representing topological groups (see [Pratt, 1999] for details).

4.2 Representation of Chu spaces

Let : $A \rightarrow K^X$ and ř: $X \rightarrow K^A$ be the left and right transposes of the Chu spaces $r: A \times X \rightarrow K$. We call and ř*representations* of A and X respectively, If $(a)(x) = \check{r}(x)(a) = r(a,x)$, for all $a \in A, x \in X$. That is, for each point $a \in A$, (a) represents a as a function from X to K i.e. of type K^X. Dually ř(x) represents state x as a function from $A$to K.

Two points$a_1, a_2 \in A$ are said to be *equivalent*, written as $a_1 \equiv a_2$, If $(a_1) = (a_2)$. Dually, two states $x_1, x_2 \in X$ are *equivalent*, denoted $x_1 \equiv x_2$, if $\check{r}(a_1) = \check{r}(a_2)$. In other words the identity of points and states is determined by their representations i.e., by and ř respectively.

is call a *faithfull representation* of A whenever the function: $A \rightarrow K^x$ is injective i.e., when no two distinct points are equivalent and the Chu space $\mathcal{A}$ separable. Dually, if ř: $Y \rightarrow K^A$ is injective, ř is called a *faithfull representation* of X and the Chu space $\mathcal{A}$ extensional.

Definition 4.1

Any Chu space $\mathcal{A} = (A, r, X)$ gives rise to an order on A, the specialization order $\leq_r$ is defined as follows: $x \leq_r y$ iff for every $x \in X(arx \Longrightarrow ary)$. The specialization order then induces the following equivalence relation $\equiv_r$ on A. i.e., $a \equiv_r b$ iff $a \equiv_r b$ and $b \equiv_r a$ i.e., iff for every $x \in X$ $(arx \Leftrightarrow bry)$.

The reader should not confuse arx as another form of expression other than $r(a, x)$ above.

A typical example of this can be exhibited by considering a normal Chu space

Definition 4.2

A Chu space $\mathcal{A} = (A, r, X)$ is said to be normal if $X \subseteq 2^A$ and r is the membership relation restricted to A, that is, arx iff $a \epsilon x$, where 2^A is the power set of A.

If $A = \{a, b, c\}$, then A can be represented as a normal Chu space over the set $K = \{0, 1\}$ by taking $X = 2^A$. This Chu space satisfies only the trivial property 2^A , signifiying that sets have no internal structure. The matrix representation is as follows:

	$\{\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a, b, c\}$
a	0	1	0	0	1	1	0	1
b	0	0	1	0	1	0	1	1

c	0	0	0	1	0	1	1	1

Figure 2

If we impose an order on A by defining $a \leq b$ to hold just when it holds in every column (taking $0 \leq 1$ as usual), then we get a non trivial order relation $\leq_r$ which is reflexive, transitive and antisymmetric with respect to the above specialization order. i.e., a *poset* satisfying $a \leq a; a \leq b$ and $b \leq c$ implies $a \leq c$.and $a \leq b \leq a$ implies $a = b$ for all a, $b, c \in A$.

According to Pratt (1999), this last antisymmetry is the poset counterpart of separability for Chu spaces. Moreover, by deleting more columns, we can further increase the structure of the space. For instance, if we require that the set of rows be closed under bitwise disjunction (join $\vee$), then we speak of a join semi lattice and if it is closed under bitwise conjunction (meet $\wedge$), then we get a meet semilattice.

Also, if it is closed under both operations, then it will define a lattice. Finally, if the rows are closed under complementation then we have a Boolean algebra.

In view of the above discussion, we construct semilattices by taking $\mathbb{Z}_n$ and the transitive closure of any relation R as objects in $\boldsymbol{Chu}(\boldsymbol{Set}\, 2)$. These constructions of which form the basis of our paper would be discussed later.

The followings below show the matrix representations of some Chu spaces. We use a, b, c, d to denote the elements of A, while v, w, x, y, z denote the elements of X. The Chu spaces $(a), (b)$, and (d) are all *biextensional*, since there are neither repeated rows

nor repeated columns, while the Chu spaces (c) and (e) are only extensional and saparable, respectively.

	w	x	y	z
a	1	0	1	1
b	0	1	1	1
c	1	1	1	0

(a)

	w	x	y	z
a	1	0	1	1
b	0	1	1	1
c	1	1	1	0
d	1	0	1	0

(b)

	x	y
a	1	1
b	1	0
c	0	1
d	1	1

(c)

	x	y

a	0	0
b	0	1
c	1	0
d	1	1

(d)

	w	x	y	z
a	1	0	1	1
b	0	1	0	1
c	1	1	1	0
d	1	0	1	0

(e)

Figure 2

4.3 Chu Transforms

Let $\mathcal{A} = (A, r, X)$ and $\mathfrak{B} = (B, s, Y)$ be any two Chu spaces. The *Chu transforms* or morphism of Chu spaces $\mathcal{A}$ and $\mathfrak{B}$ is a pair (f, g) of functions $f: A \rightarrow B$ and $g: Y \rightarrow X$, satisfying the adjointness condition: $s(f(a), y) = r\big(a, g(y)\big)$, for all $a \in A$ and $y \in Y$.

Let $\mathcal{C} = (C, t, Z)$ be another Chu space. Then the Chu transform of $\mathfrak{B} and\, \mathcal{C}$ is a pair (f', g') of functions $f': B \rightarrow C$ and $g': Z \rightarrow Y$ satisfying the *adjointness* condition: $t(f'(b), z) = s(b, g'(z))$, for all b in B and z in Z.

Since $f: A \rightarrow B$, $g: Y \rightarrow X, f': B \rightarrow C$, and $g': Z \rightarrow Y$, we have $f'f: A \rightarrow C$ and $gg': Z \rightarrow X$, and hence their composition $(f'f, gg'): A \rightarrow C$.

It is straightforward to see that the composition satisfies the adjointness condition; $r\big(a, gg'(z)\big) = s\big(f(a), g'(z)\big) = t(f'f(a), z)$, for $all\ a \in A\ and\ z \in Z$, and hence it is a Chu transform.

The associativity of composition defined above is inherited from the one in $\boldsymbol{Set}$. The pair (I_A, I_X) of the identity maps on A and X respectively, is an adjoint pair: $I_A: A \rightarrow A$ and $I_X: X \rightarrow X$ satisfying $(I_A(a), x) = r(a, I_X(x))$, $for\ all\ a\ in\ A\ and\ x\ in\ X$. Thus the pair (I_A, I_X) is a Chu transform and serves as the identity on $\mathcal{A}$.

4.4 Isomorphism of Chu Spaces

It may be interesting to note that, the isomorphism of Chu spaces follows from the adjointness condition of the Chu transforms. Informally, two Chu spaces $\mathcal{A} = (A, r, X)$ and $\mathcal{B} = (B, s, Y)$ are said to be isomorphic if A is isomorphic to Y and r and s have the same value on corresponding pairs. Thus, an isomorphism between the Chu spaces $\mathcal{A} = (A, r, X)$ and $\mathcal{B} = (B, s, Y)$ is just a pair of functions $f: A \rightarrow B$ and $g: Y \rightarrow X$ such that both f and g are bijective functions and $r\big(a, g(y)\big) = s(f(a), y)$ for all $a \in A, y \in Y$.

4.5 Chu Category and Chu constructions

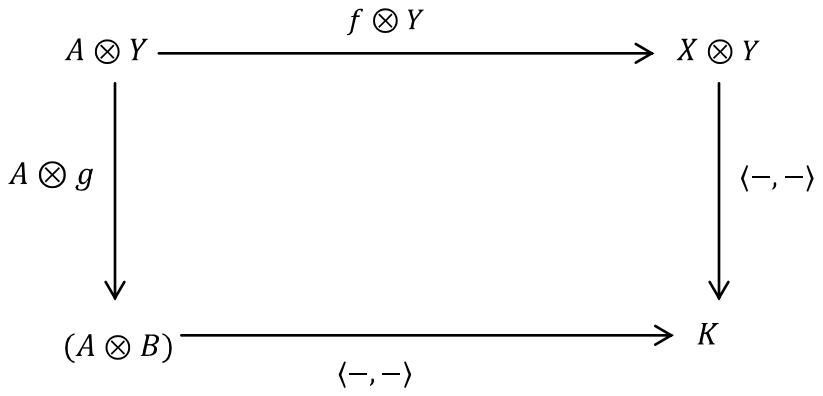
The Chu construction is the general idea for constructing a **-autonomous category* from an *autonomous category*. Note that an *autonomous category* is a symmetric monoidal

closed category, while a *-*autonomous category* is an autonomous category with a *dualizing* object.

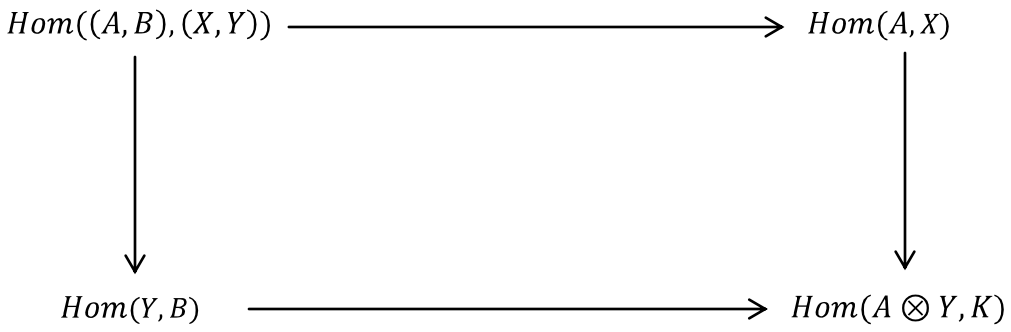
In the sequel, the Chu construction was named after Po-Hsiang Chu, a student of Micheal Barr, who gave the construction in his master's thesis at MCGill University.

Let K be a fixed object of a closed symmetric monoidal category $\mathbb{C}$. We define a category denoted $Chu(\mathbb{C}, K)$ as follows: An object of this category is a triple $(A, B, \langle -, - \rangle)$ consisting of objects $A, B \in \mathbb{C}$, where $\langle -, - \rangle$ is a pairing function such that $\langle -, - \rangle : A \otimes B \to K$ is an arrow of $Chu(\mathbb{C}, K)$. The pairing function is usually omitted and the pair (A, B) is written as an object of $Chu(\mathbb{C}, K)$ if it is understood from the context. This notation means a pairing from $A \otimes B$to K.

A morphism$(f, g) : (A, B) \to (X, Y)$ of $Chu(\mathbb{C}, K)$ consists of a pair of functions $f : A \to X$ and $g : Y \to B$ such that the diagram



commutes or equivalently, the following diagram



is a pullback. This signifies that the morphisms from $(A,B) \rightarrow (X,Y)$ consist of all pairs $(f,g) \in Hom(A,B) \times Hom(X,Y)$ which give the same element of $Hom(A \otimes Y,K)$.

It follows that Chu spaces over K together with Chu transforms form a *category*, denoted $\boldsymbol{Chu}(\boldsymbol{Set},K)$ or simply $\boldsymbol{Chu_K}$, where K is a fixed object of the category $\boldsymbol{Set}$.In other words, thecategory $\boldsymbol{Chu}(\boldsymbol{Set},K)$ has *Chu spaces* over the set K as *objects* and *Chu transforms* as *morphisms*.

The *initial* objects of the category $\boldsymbol{Chu}(\boldsymbol{Set},K)$ are all Chu spaces with the empty set of points and the empty set of states. Its *monics* are those Chu transforms (f,g) for which f is an injection and g a surjection, and dually for its *epis*. Moreover, its isomorphisms are those (f,g) for which f and g are both bijections i.e., isomorphisms in the $\boldsymbol{Set}$(see {Pratt, 1999}, for details).

Recall that when K is a two-element set $\{0,1\}$, we have$\boldsymbol{Chu}(\boldsymbol{Set},2)$. The Category $\boldsymbol{Chu}(\boldsymbol{Set},2)$ realizes a large number of mathematical structures such as preordered sets, Stone spaces, ordered Stone spaces, topological spaces, locales, complete semillatices, distributive lattices (but not general lattices), algebraic lattices, frames, profinite (Stone)

distributive lattices, Boolean algebras, and complete atomic Boolean algebras, and so on (see [Pratt, 1999], for details). For instance, a topological space can be represented as a Chu space over the set $K = 2 = \{0,1\}$: if X stands for the set of open subsets of A, then for any $a \in A$ and $x \in X$, the expression $r(a,x)$ is equal to one if a belongs to x, and zero otherwise, where $r: A \times X \rightarrow 2$. Also, Chu spaces over the set $K = 2 = \{0,1\}$realizes a $poset$ by imposing an order relation on the set K i.e., by defining $a \leq b$ to hold just when it holds in every column by taking $0 \leq 1$, as usual, in each row.

Any Chu space could be made extensional and saparable accordingly by identifying repeated columns or rows of the space– this process is called standardization. In some literatures, biextensional collapse has been referred to as the standardization of Chu spaces. However, the former is frequently used while constructing subcategories of the category $\boldsymbol{Chu_K}$.

4.6Biextensional collapse of Chu spaces

Biextensional collapse of a Chu space is a way of identifying equal rows from equal columns.

Now let us define a relation on A as follows:

$a_1 \sim a_2$if and only if, for all $a \in A$, $r(a_1,x) = r(a_2,x)$.

Then this relation is an equivalence relation. However, a Chu space $\mathcal{A} = (A,r,X)$ is said to be saparable just when this relation is identity on A.

Remarks4.6

(i) If $(A/\sim, s, X)$ is the Chu space obtained from the equivalence relation defined above, the map $(k, Id_A): (A, r, X) \longrightarrow (A/\sim, s, X)$ is a Chu transform satisfying the adjointness condition $s([a], x) = r(a, x)$, where $k: A \longrightarrow A/\sim$ is the quotient map.

(ii) The Chu space obtained from $(A/\sim, s, X)$ is saparable;

(iii) If (A, r, X) is extensional, then $(A/\sim, s, X)$ is biextensional;

(iv) Whenever an equivalence relation $\sim$ is defined on the set A, then the Chu space $\mathcal{A}$ obtained is saparable exactly when the relation is the identity.

Let us describe how to find the biextensional collapse of a Chu space.

Now, let $\mathcal{A} = (A, r, X)$ be a $K - valued$ Chu space and$\tilde{q}$be the equivalence class of a point or state q.Denote $\tilde{A}$ and $\tilde{X}$as the quotients of A and X by $\sim$ respectively. That is,

$$\tilde{A} = A/\sim \equiv \{\tilde{a}: a \in A\}$$

$$\tilde{X} = X/\sim \equiv \{\tilde{x}: x \in X\}$$

So, the biextensional collapse of a Chu space $\mathcal{A} = (A, r, X)$ is the Chu space $\tilde{\mathcal{A}} = (\tilde{A}, \tilde{r}, \tilde{X})$ identified by $\tilde{r}: \tilde{A} \times \tilde{X} \longrightarrow K$ and satisfying $\tilde{r}(\tilde{a}, \tilde{x}) = r(a, x)$.

For example, the bottom 2-by-2 Chu space of figure 2 is the biextensional collapse of the 5-by-3 Chu space above it. Note that we have written the elements of X on the top while the elements of A by the column-left. This convension is taken to ease our transformations.

	x	y	z

a	0	0	0
b	0	0	0
c	0	0	0
d	0	1	1
e	0	1	1

	$\{x\}$	$\{y,z\}$
$\{a,b,c\}$	0	0
$\{d,e\}$	0	1

Since the quotient $\tilde{r}$ satisfies the adjointness condition via $\tilde{r}(\tilde{a},\tilde{x}) = r(a,x)$, we remark as follows that the biextensional collapse is functorial from the category $\boldsymbol{Chu}(\boldsymbol{Set},K)$ to the category $\boldsymbol{bChu_K}$(the category determined by the biextensional Chu spaces) and the morphism $(f,g)\colon \mathcal{A} \longrightarrow \mathcal{B}$ transforms to $(\tilde{f},\tilde{g})\colon \tilde{\mathcal{A}} \longrightarrow \tilde{\mathcal{B}}$, were $\tilde{f}(\tilde{a}) = \widetilde{f(a)}$ and $\tilde{g}(\tilde{y}) = \widetilde{g(y)}$ for all $a \in A, y \in Y$.

Theorem 4.1

If the pair of functions $(f,g)\colon (A,r,X) \longrightarrow (A/\sim, s, Y)$ is a Chu transform, then the map $f$preserves $\sim$. That is, for all $a_1, a_2 \in A$, $a_1 \sim a_2 \Rightarrow f(a_1) \sim f(a_2)$.

Proof

$(f,g):(A,r,X)\longrightarrow(B,s,Y)$is a Chu transform implies $f:A\to B$ and $g:Y\to X$ such that adjointness condition $s(f(a),y)=r\big(a,g(y)\big)$ is satisfied.

Now, for any $y\in Y$:

$s(f(a_1),y)=r\big(a_1,g(y)\big)=r\big(a_1,g(y)\big)=s(f(a_2),y)$.

We remark as follows that, the categories determined by the separated, extensional, andbiextensional Chu spaces denoted $\boldsymbol{sChu_K}$, $\boldsymbol{eChu_K}$, and $\boldsymbol{bChu_K}$, respectively, are *full subcategories* of $\boldsymbol{Chu_K}$.

Theorem4.2

The category $\boldsymbol{eChu_K}$ determined by the extensional Chu spaces is a *full embedding*of $\boldsymbol{bChu_K}$.

Proof

Recall that a category $\mathcal{D}$*fully embeds* in to the category $\mathbb{C}$, if $\mathcal{D}$ is a subcategory of $\mathbb{C}$ and there is a functor $F:Hom_{\mathcal{D}}\longrightarrow Hom_{\mathbb{C}}$ which is injective. As a particular case, the inclusion functor suffices. That is, we only need to prove that the functor $F:\boldsymbol{bChu_K}\hookrightarrow\boldsymbol{eChu_K}$has a *left adjoint*.

The unit of the adjunction is the Chu transform $(k,Id_A):(A,r,X)\longrightarrow(A/\sim,s,X)$ discoursed above, while the theorem above guarantees that given a Chu transform $(f,g):(A,r,X)\longrightarrow(B,s,Y)$ to a biextensional Chu space, we can factor it through the quotient space $(A/\sim,s,X)$.

The functor$P:\boldsymbol{bChu_K}\longrightarrow\boldsymbol{eChu_K}$given by this adjunction sends morphisms

$$(f, g): (A, r_1, X) \longrightarrow (B, r_2, Y)$$

to

$$(f/\sim, g): (A/\sim, s_1, X) \longrightarrow (B/\sim, s_1, Y)$$

where $f/\sim([x]) = [f(x)]$.

It may be noted that, the functor$P$ is called the biextensional collapse. On the other hand, an equivalence relation could be defined on the Chu transforms in every $hom\text{-}set$ in $\boldsymbol{eChu_K}$ by:

$(f, g) \sim (f', g')$if and only if $\forall x.\, f(a) \sim g(a)$.

$P(f, g) = P(f', g')$if and only if $(f, g) \sim (f', g')$.

It may be noted that, the map f transforms covariantly while g transforms contravariantly.

4.7 Internal Homfunctor

The set of all maps between any two Chu spaces $\mathcal{A} = (A, r, X)$ and $\mathcal{B} = (B, s, Y)$ can itself be turned into a Chu space. This Chu space is what we called the internal hom $functor$. If $\mathcal{C} = (C, t, Z)$ represent the Chu space, then C would be the set of all maps from $\mathcal{A}$ to $\mathcal{B}$ and Z is the Cartesian product of $A$and Y. That is, $Z = (A \times Y)$. Now, if $(f, g) \in C$, and $(a, y) \in Z$, then $t\big((f, g), (a, y)\big) = s(f(a), y) = r(a, g(y))$. We will denote the Chu space $\mathcal{C} = (C, t, Z)$by $\mathcal{A} \multimap \mathcal{B}$.

Properties of $\mathcal{A} \multimap \mathcal{B}$

(i) The internal $hom\ functor \mathcal{A} \multimap \mathcal{B}$ between any Chu spaces $\mathcal{A}$ and $\mathcal{B}$ may not be extensional even if both $\mathcal{A}$ and $\mathcal{B}$ are extensional;

(ii) However, if extensionality is required, we can just identify equal rows by the method of biextensional collapse;

Theorem 4.3

Let $\mathcal{A} \multimap \mathcal{B}$ be the internal hom functor between any two Chu spaces $\mathcal{A}$and $\mathcal{B}$. If $\mathcal{B}$ is saparable, then $\mathcal{A} \multimap \mathcal{B}$ is also saparable.

Proof

Let $\mathcal{B} = (B, s, Y)$**.** Then $\mathcal{B}$is saparable if for any distinct points $b, b' \in B$, for all $y \in Y$, $[s(b, y) = s(b', y)] \Rightarrow b = b'$. So, for any distinct elements $(f, g), (f', g') \in C$, there is some $a \in A$ such that $f(a) \neq f'(a)$. Now, the saparability property of $\mathcal{B}$, there is $y \in Y$ which saparates $f(a)$ and $f'(a)$. Then (a, y)saparates $(f, g), (f', g')$.

Theorem4.4 [Gupta, 1994]

If $\mathcal{A} = (A, r, X)$ is a Chu space, then the dual of $\mathcal{A}$ denoted $\mathcal{A}^{\perp}$ is isomorphic to the internal hom $functor\ \mathcal{A} \multimap \perp$. i.e., $\mathcal{A}^{\perp} \cong \mathcal{A} \multimap \perp$, where $\perp = (\perp_A, \perp_r, \perp_X)$ with $\perp_A = \{a, b\}$, $\perp_X = \{*\}$ and $\perp_r (a,*) = 0$, $\perp_r (b,*) = 1$.

Proof

Let $\mathcal{A} \multimap \perp$ be represented by the Chu space $\mathcal{B} = (B, s, Y)$. For each element $x \in X$, define a map $f_x: \mathrm{A} \longrightarrow \perp_A$ by $f_x(\mathrm{a}) = \mathrm{r}(a, x)$, and a map $g_x: \perp_X \longrightarrow \mathrm{X}$ by $g_x(*) = x$. It follows that (f_x, g_x) is a Chu transform, so for each $x \in X$ we have given a Chu transform. It is clear from the definition of a Chu transform that these are all the maps from $\mathcal{A}$ to $\perp$. Thus $B \cong X$.

By definition of $\mathcal{A} \multimap \perp$, $Y = A \times \{*\} \cong A$. Now, This means that the dual of $r(a, x)$denoted $\check{r}(x, a) = r(a, x) = r\big(a, g_x(x)\big) = s((f_x, g_x), (a, *))$. Thus the Chu spaces are isomorphic.

This isomorphism is natural in $\mathcal{A}$. Thus, if we have a map $h = (h, k): \mathcal{B} \longrightarrow \mathcal{A}$, and the isomorphism above is denoted by $\eta\mathcal{A}: \mathcal{A}^{\perp} \longrightarrow \mathcal{A} \multimap \perp$, then $\eta\mathcal{B}\ o\ h^{\perp} = (h \multimap \perp)\ o\ \eta\mathcal{A}$. Given any $\in X, \eta\mathcal{B}\ o\ h^{\perp}(x) = \eta\mathcal{B}\big(k(x)\big) = f_{k(x)}$. Now for $b \in B$, $f_{k(x)}(b) = s\big(b, k(x)\big) = r(h(b), x) = f_x(h(b))$. This is exactly $(h \multimap \perp)\mathrm{o}\, \eta\mathcal{A}(x)(b)$. If this argument is also shown on the right adjoint of the maps, then we have a natural isomorphism.

4.8 Applications of Chu constructs

4.8.1 Chu constructs and Linear Logic

Girard's linear logic provides a powerful tool for modeling Chu spaces. This logic and its operations on Chu spaces make $\boldsymbol{Chu_K}$ a $* - autonomous$ (a closed symmetric monoidal self-dual) category (see [Barr, 1979] for details). This generality is easily achieved when $K = 2 = \{0,1\}$. The connectives of linear logic are seen as functors rather than mere operations of logical connectives. From its discovery, linear logic has found many

applications in the computer science, e.g., it is employed in the description of concurrent systems, in the design of new programming languages that don't need garbage collection, proof theory (see [Pratt, 2005b] for details).

The linear logic for Chu space interpretation consists of four operations and two constants via $\mathcal{A} \oplus \mathcal{B}$called the additive disjunction and$\mathcal{A} \otimes \mathcal{B}$called the tensor product or the multiplicative conjunction, associated with this binary operation is their respective units or zeroary operations0 and 1.

The dual of $\mathcal{A} \oplus \mathcal{B}$ and $\mathcal{A} \otimes \mathcal{B}$ are $\mathcal{A} \& \mathcal{B}$ and $\mathcal{A} \parr \mathcal{B}$ respectively. The dual of $!\mathcal{A}$ (off course) is $?\mathcal{A}$ (why-not). The linear implication $\mathcal{A} \multimap \mathcal{B}$ is defined as $(\mathcal{A} \otimes \mathcal{B})^{\perp\perp}$ or $\mathcal{A}^{\perp} \parr \mathcal{B}^{\perp}$ while the intuitionistic implication $\mathcal{A} \to \mathcal{B}$ is defined as $!\mathcal{A} \multimap \mathcal{B}$. Pratt uncovers that none of these operations has a number-theoritic interpretation. The dual or transpose of$\mathcal{A} = (A, r, X)$is$\mathcal{A}^{\perp} = (X, \check{r}, A)$satisfying$\check{r}(x, a) = R(a, x)$.

The transpose of the pair of functions (f, g) is $(f, g)^{\perp}$and it is defined as(g, f):. Linear negation is involutive i.e. $A^{\perp\perp} = A$ this shows that $(f, g)^{\perp\perp} = (f, g)$.

By definition, for any two Chu spaces $\mathcal{A} = (A, r_A, X)$ and $\mathcal{B} = (B, r_B, Y), \mathcal{A} \oplus \mathcal{B}$ is defined as $(A + B, r, X \times Y)$ where r is defined by $a\, r(x, y) = a\, r_A x$ and $b\, r(x, y) = b R_B y$. It *unit*is $(0, !, 1)$.

The tensor product $\mathcal{A} \otimes \mathcal{B}$ of $\mathcal{A} = (A, r, X)$ and $\mathcal{B} = (B, s, Y)$ is $(A \times B, t, F)$ where $F \subset Y^A \times X^B$ is the set of all pairs (h, k) of functions $h: A \to Y, k: B \to X$ such that $s(b, h(a)) = r(a, k(b))$ for all $a \in A, b \in B,$ and $t: (A \times B) \times F \to K$ and it is given by

$t((a,b),(h,k)) = s(b,h(a))(= r(a,k(b)))$. Its unit is the $1\times K$ normal Chu space whose one row is the identity function on K.

Given $(f,g):\mathcal{A}\to\mathcal{B}$ and $(f',g'):\mathcal{A}'\to\mathcal{B}'$define(f,g)$\otimes(f',g'):\mathcal{A}\otimes\mathcal{A}'\to\mathcal{B}\otimes\mathcal{B}'$ on points by $(a,a')\to(f(a,)\big(f'(a')\big)$,and on states by $(h':B\to Y',k:B'\to Y)\to(g'h'f:A\to X',gk'f':A'\to X)$.

Recall that in the sense of topological representation of a Chu space discoursed above, a Chu space could be completely specified by a set A and a subset of its power set $X\subseteq 2^A$. However, power set is a complete Boolean algebra, so if the cardinality of the set A is small, then we can depict 2^A as a *Hasse* diagram.So that the vertices of $2^A$are represented as soft masses or holes according to whether they are present or absent. This representation is what is referred to as the *partial Boolean Algebra*(pBA) representation.

If in the Chu, arbitrary meets and joins are present and that every maximal chain has a second least element, then we speak of a *complete atomicBoolean Algebra*$(CABA)$. Note that the $CABA$ is useful for determining homomorphisms between partial distributive lattices representation of Chu spaces$(pdLat)$.

Note also that, if A is infinite, then the power set of A is a *complete atomicBoolean Algebra*$(CABA)$. As all finite Boolean algebras are *complete* and *atomic*, not all infinite Boolean algebras are *complete* and *atomic* an example is just the Boolean algebra consisting of all the finite and cofinite subsets of the natural number since ℵ is not complete.

An alternative way to specify a Chu space is via formula, of infinitary propositional Boolean logic. This representation makes it possible to see clearly the properties of Chu spaces.

Let us assign a Boolean propositional formula to each extensional Chu space as follows. We denote the variables of the formula by the elements of the set A. For each $x \in X$. We form a clause $\bigwedge\{a: a \in A, r(a,x) = 1\} \bigwedge \quad \bigwedge\{ \quad a: a \in A,\ r(a,x) = 0 \}$. So the disjunction of all such clauses then gives the proposional Boolean formula. We call this the *complete disjunctive normal form*$(CDNF)$ of a propositional formula.

The Boolean propositional formulae of the biextensional Chu spaces above are:

(i) $a\bar{b}c \vee \bar{a}bc \vee abc \vee ab\bar{c}$;

(ii) $\bar{a}\bar{b}cd \vee \bar{a}bc\bar{d} \vee abcd \vee ab\bar{c}\bar{d}$; and

(iii) $\bar{a}\bar{b}cd \vee \bar{a}b\bar{c}d$respectively.

Note that the $CDNF$ form is not an efficient way to write out the propositional formula, however, any formula logically equivalent to the $CDNF$ formula is an equally valid representation. This may be obtained by the rules of inference.

It should also be noted that the rows of the matrix are the satisfying assignments of the propositional formula. That is, each row makes the clause corresponding to its state true, so the whole formula becomes true. Moreover, any other assignment differs from each column on at least one variable, makes the each clause false. So in general, given a propositional formula, we can construct its Chu space as the set of all satisfying assignments.

The set of all satisfying assignments of a Chu space denoted Γ would now be regarded as the logical $CDNF$ representation of the Chu space.

If Γ and Δ are the logical $CDNF$ representations of two Chu spaces, then there is a *logical transformation* between them.

The logical transformation between them is a map $f: Var(\Gamma) \longrightarrow Var(\Delta)$ such that $\Delta \longrightarrow f(\Gamma)$ is a logical tautology, where $f(\Gamma)$ is the formula obtained by substituting each variable a of Γ by its image $f(a)$ and $Var(\Gamma)$ and $Var(\Delta)$ are the set of variables in the formulas Γ and Δ respectively.

The meaning of the representations is that every formula in the theory of the which are consequences of Γ is mapped to a formula in the theory of the target which are logical consequences of Δ.

Theorem4.5

Chu maps are in $one\ to\ one$ correspondence with logical transformations.

Proof

Let $\mathcal{A} = (A, r, X)$ and $\mathcal{B} = (B, s, Y)$ be two biextensional Chu spaces and let Γ and Δ be their respective representations. If $(f, g): (A, r, X) \longrightarrow (B, s, Y)$ is a Chu map, then we define a logical transformation f via its first component. Now consider any satisfying assignment $y \in$ Yfor Δ. then $g(y) \in X$, so it forms a satisfying assingnment for Γ, because of the adjointness condition. So $\Delta \longrightarrow f(\Gamma)$ is a logical tautology.

Conversely, given a logical transformation f, it determines a unique map $g: Y \longrightarrow X$, since every satisfying assignment of Δ is a satisfying assignment of $f(\Gamma)$, so every $y \in Y$ is mapped to $x \in X$ such that $f(x) = y \cap f(A)$.

4.8.2 Chu constructs in Real - Life Problems

Example 1

A soccer team A in a soccer field X may be interpreted as the Chu space $\mathcal{A} = (A, X, r)$, where

(i) A is a finite set consisting of eleven elements;

(ii) X is an infinite set such that every point in the field is an element of X;

(iii) (a, x)means the player a is having the ball at the position x in the soccer field and the matrix $r(a, x)$ is the probability that the player a kicks a goal from the position x in the soccer field.

Example 2

The structure $\mathcal{U} = (A, X, r)$may be interpreted as Chu space, where

(i) A is the set of students of the University $\mathcal{U}$;

(ii) X is the set of courses being taken at the University $\mathcal{U}$;

(iii) (a, x)means the student a is taking the course a, and the matrix $r(a, x)$ is interpreted as the grade of the student a in the course x. For example $r(a, x) = 1$ if the student a gets an "A" in the course x. In this example,

the function $f: A \times X \longrightarrow [0,1]$ takes only five values: $A = 1$, $B = \frac{3}{4}$, $C = \frac{1}{2} D = \frac{1}{4}$ and $F = 0$.

4.8.3 Chu constructs in Configuration Structures

A configuration structure is an asynchronous set transition system consisting of the pair (E, r) with

(i) E a set of events

(ii) $r \subseteq \wp(E) \times \wp(E)$ a set of transitions, satisfying the following axioms:

1. $x\ r\ y \rightarrow x \subseteq y$ and
2. $x\ r\ y \wedge x \subseteq x' \subseteq y' \subseteq y \rightarrow x'\ r\ y'$.

A set $x \in \wp(E)$ with $x\ r\ x$ is called configuration.

The set of configurations of (E, r) is denoted C.

If $x\ r\ y$, then by the requirement of axiom 2, $x, y \in C$.

When an asynchronous system is used to represent a concurrent system it is called a configuration structure.

Configuration structures have been variously referred to as set systems, asynchronous set transitions, and ST-configuration structures (see {Gupta, 1994} and {Zhang, 2004}).

It has also been referred to as asynchronous multiset transition systems in which transition relation (r) is considered as the relation between multisets of events, rather than sets.

Configuration structures have been used for several models of concurrency (see {Zhang, 2004} for details).

It may be significant to observe that extensional Chu spaces can be interpreted as an asynchronous set systems with the Chu spaces determined by $x\, r\, y \leftrightarrow x \subseteq y$. This is called the *subset interpretation* with the property that the resulting set transition systems be asynchronous.

4.9 Examples of Chu spaces and related Chu Categories on $K = \{0,1\}$

At the outset, let us recall a motivating example of Chu space provided in {Henry, 2003}. $(A, r, \wp(A))$, where $\wp(A)$ is the power set of A, is a Chu space over $\{0,1\}$ for any set A. the function r is defined as $r(a, x) = 1$ if and only if $a \in X$ and zero otherwise, where $x \in \wp(A)$ and $a \in A$.

In the following, adapting from ([Henry, 2003] and [Pratt, 1999]), two new Chu spaces and related Chu categories are constructed.

Theorem 4.6

The structure $(A, r, tc(R_A))$ is a Chu space over $\{0,1\}$, where $tc(R_A)$ is the transitive closure of an arbitrary relation R_A on a set A.

Proof

The structure$(A, r, tc(R_A))$ defines a Chu space over the set $K = \{0,1\}$ with $r(a, x) = 1$ if and only if $a \in x$ and zero otherwise, for all $a \in A$, $x \in tc(R_A)$**.**

The Chu transform of the aforesaid Chu spaces is defined as follows: Let $\mathcal{A} = (A, r, tc(R_A))$, $\mathcal{B} = (B, s, tc(R_B))$ and $\mathcal{C} = (C, t, tc(R_C))$ be Chu spaces. Then the Chu transforms between $\mathcal{A} = (A, r, tc(R_A))$ and $\mathcal{B} = (B, s, tc(R_B))$ is the pair (f, g) of functions $f: A \rightarrow B$ and $g: tc(R_B) \rightarrow tc(R_A)$ satisfying the adjointness condition:

$s(f(a), y) = r\big(a, g(y)\big), for\ all\ a \in A, y \in tc(R_B).$

The identity maps on A and $tc(R_A)$ respectively, are given by the pair $\big(I_A, I_{tc(R_A)}\big)$. This pair also satisfies the adjointness condition and it is the Chu transform on $\mathcal{A}$.

Moreover, for a pair of functions (f', g') between the Chu spaces $\mathcal{B}$ and $\mathcal{C}$, It can be likewise shown that this pair meets the adjointness condition and hence is a Chu transform between $\mathcal{B}$ and $\mathcal{C}$.

The composition $(f, g)o(f', g') = (f'f, gg')$ follows from$r\big(a, gg'(z)\big) = s\big(f(a), g'(z)\big) = t(f'f(a), z)$for all $a \in A, z \in tc(R_c)$.

Thus the Chu spaces obtained in this way over the set $K = \{0,1\}$ with their adjoint pairs form a category.

For example, let $A = \{a, b, c, d\}$ and let $R_A = \{(a, a), (b, a), (c, b), (d, c)\}$. Hence, $tc(R_A) = \{(a, a), (b, a), (c, a), (c, b), (d, b), (d, c), (d, a)\}$. The function r is defined as $r(a, x) = 1$ if and only if $a \in A, x \in tc(R_A)$. The resulting Chu space over $K = \{0,1\}$ is as follows:

Table 1: Chu Matrix

	(a,a)	(b,a)	(c,a)	(c,b)	(d,b)	(d,c)	(d,a)
a	1	1	1	0	0	0	1
b	0	1	0	1	1	0	0
c	0	0	1	1	0	1	0
d	0	0	0	0	1	1	1

Note that the Chu space in Table(1) is biextensional.

It is straightforward to see that if the columns corresponding to $(a,a),(c,b),(d,b),$ (d,c) are deleted (see {Henry, 2003}, {Pratt, 1999} for details), then another Chu space is obtained whose first row is the *join* (bitwise disjunction, denoted $\vee$) of the other rows viz.,

	(b,a)	(c,a)	(d,a)
a	1	1	1
b	1	0	0
c	0	1	0
d	0	0	1

Let $R_i, i = \overline{1,4}$, denote the rows of the matrix in Table(2). An order $\leq$, as usual, can be defined on Table(2) as follows: $R_j \leq R_k$ (component-wise), $j \geq k, j, k = \overline{1,4}$.

It is easy to see that the Chu space in Table(2) is a $Poset$under $\leq$. Also this Chu space under $\vee$(bitwise disjunction) satisfies commutativity, idempotency and associativity. Hence it is a *join-semilattice* i.e., a semigroup that is commutative and idempotent (see [Pratt, 1999] for a detailed description of operations on Chu spaces).

Note, however, that this Chu space is not a *meet-semilattice*.

Theorem4.7

The structure $(\mathbb{Z}_n\backslash\mathbf{0}, r, \mathbb{Z}_n)$over$\{0,1\}$is a Chu space, where $\mathbb{Z}_n$ is the set of integers congruence modulo n

Proof

The matrix $r: A \times X \rightarrow 2$ under the operation $a \mid (a + x)$, read as $adivides(a + x)$ where $a \in A, x \in \mathbb{Z}_n$, defines a Chu space.

The Chu transform of the aforesaid Chu spaces is defined as follows: Let $\mathcal{A} = (\mathbb{Z}_n\backslash\mathbf{0}, r, \mathbb{Z}_n)$, $\mathcal{B} = (\mathbb{Z}_m\backslash\mathbf{0}, s, \mathbb{Z}_m)$, and $\mathcal{C} = (\mathbb{Z}_l\backslash\mathbf{0}, t, \mathbb{Z}_l)$ be Chu spaces. Then the Chu transforms between $\mathcal{A}$ and $\mathcal{B}$ is the pair (f, g) of functions $f: \mathbb{Z}_n\backslash\mathbf{0} \rightarrow \mathbb{Z}_m\backslash\mathbf{0}$ and $g: \mathbb{Z}_m \rightarrow \mathbb{Z}_n$ satisfying the adjointness condition: $s(f([a]), [y]) = r\big([a], g([y])\big)$, for all$[a] \in \mathbb{Z}_n, [y] \in \mathbb{Z}_m$.

The identity maps on $A = \mathbb{Z}_n\backslash\mathbf{0}$ and $X = \mathbb{Z}_n$ respectively, are given by the pair $\big(I_{A=\mathbb{Z}_n\backslash\mathbf{0}}, I_{X=\mathbb{Z}_n}\big)$. This pair also satisfies the adjointness condition and it is the Chu transform on $\mathcal{A}$.

Moreover, for a pair of functions (f', g') between the Chu spaces $\mathcal{B}$ and $\mathcal{C}$, It can be likewise shown that this pair meets the adjointness condition and hence is a Chu transform between $\mathcal{B}$ and $\mathcal{C}$.

The composition $(f, g)o(f', g') = (f'f, gg')$ follows from$r\big([a], gg'([z])\big) = s\big(f([a]), g'([z])\big) = t(f'f([a]), [z])$for all$[a] \in \mathbb{Z}_n, [z] \in \mathbb{Z}_l$.

Thus the Chu spaces obtained in this way over the set $K = \{0,1\}$ with their adjoint pairs form a category.

For example, let $\mathbb{Z}_4 = \{[\mathbf{0}], [\mathbf{1}], [\mathbf{2}], [\mathbf{3}]\}$, in standard notation.

Let $X = \mathbb{Z}_4$, $A = \mathbb{Z}_4 \backslash 0$,and $K = 2 = \{0,1\}$. The matrix $r: A \times X \longrightarrow 2$ under the operation $a \mid (a + x)$, read as $adivides(a + x)$ where $a \in A, x \in X$, defines a Chu space:

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix} \qquad (1)$$

If the structure of the Chu space (1) is pruned by deleting the column corresponding to [**1**] of $\mathbb{Z}_4$, another Chu space is obtained whose first row is the join (bitwise disjunction, denoted $\vee$) of the other two rows viz.,

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \qquad (2)$$

Let $R_i, i = \overline{1,3}$, denote the rows of the matrix in (2). An order $\leq$, as usual, can be defined on (2) as follows: $R_j \leq R_k$ (component-wise), $j \geq k, j, k = \overline{1,3}$.

It is easy to see that the Chu space in (2) is a $Poset$under $\leq$. The Chu space under $\vee$ (bitwise disjunction) satisfies commutativity, idempotency and associativity. That is, if

$p = 1 \quad 1 \quad 1,$
$q = 1 \quad 1 \quad 0,$and
$t = 1 \quad 0 \quad 1,$

Then $p \vee p = p$, etc., $p \vee q = q \vee p$, etc., $p \vee (q \vee r) = (p \vee q) \vee r$, etc., hold.

Hence, Chu space(2) represents a *semilattice* i.e., a semigroup that is commutative and idempotent.

Note that the Chu space(2) under the *meet operation* is not a *semilattice*.

It is observed that $\mathbb{Z}_n, n = 2,3,4,$ and $K = \{0,1\}$,give rise to *biextensional* Chu spaces. For $\mathbb{Z}_n, n > 4$, under the aforesaid operation and $K = \{0,1\}$, the resulting Chu spaces are not biextensional.

4.10 Generality of $Chu(Set, 2)$

(1) A topological space can be defined by just considering the underlying set K of the Chu space as the set consisting of the elements 0,1., that is, by taking K={0,1}. Lafont and Streicher point out that Chu spaces over 2 realize both topological spaces and coherent spaces (see [Lafont and Streicher, 1991] for details).

In this way, a topological space can be represented as a Chu space over the set K={0, 1} without imposing any structure on the set. For instance, if $\mathcal{X}$ stands for the set of open subsets of A, then for any $a \in A \; and \; X \in \mathcal{X}$ the expression $r(a, X)$ is equal to one if a belongs to the open subset X, and zero otherwise, where $r: A \times \mathcal{X} \to 2$.

(2) A generalized topological space GTS is a space consisting of the triplets (X, R, A), where X an arbitrary set is called the set of points,A is an arbitrary binary function called the membership function. The intuitive meaning of the function is that for all $x \in X$ and $A \in \mathcal{A}$, $r(x, A)$ is the degree to which x is a member of the open set A.

(3) Also, the functorϕ: $X \rightarrow (X, \wp(X)$; $eval$: $X \times \wp(X) \rightarrow 2)$ is a fully embedding of $\boldsymbol{Set}$ in to $\boldsymbol{Chu}(\boldsymbol{Set}, \boldsymbol{2})$

(4) Chu spaces over the set 2 realize a $poset$ by imposing an order relation on the set $K = \{0, 1\}$ such that $a \leq b$ holds for $0 \leq 1$, where $\leq$ is the usual ordering. In this way, the category of posets denoted $\boldsymbol{POS}$ takes 2 to be the partially ordered set of $K = \{0,1\}$ with 1 indicating the members and 0 the non-members of the space (see [Pratt, 1999, 1995, 2005] for details).

CHAPTER FIVE

CATEGORICAL MODEL OF MULTISETS

This chapter includes a brief introduction to multisets, related Chu categories, and some applications.

5.1Multisets

The materials presented in this section are mostly abstracted from ({Singh et al, 2012}, {Monro, 1987}, and {Syropoulos, 2001,2003}).

Intuitively, a set is any collection M of definite and distinct objects m of our intuition or thought (which would be called 'elements' of M) in to a whole.

The Cantorian set ignores repetition of elements in a set, for example, the collection $\{2,1,3,1,3\}$ is reduced to $\{2,1,3\}$.

If repetition of elements are allowed and not collapsed, then we speak of a multiset (mset for short), for example, $\{2,1,3,1,3\}$. In other words, an unordered collection of objects with repetition allowed is called a multiset. It is customary to be noted that the distinct elements of any mset are referred to as the *objects* of the mset. Thus it follows that, if $A = [2,2,2,3,3,4,5,5]$, then objects of A are simply $2,3,4$ and 5. In mathematics, the notion of **multiset** (or **bag**) is a generalization of the notion of set in which members are allowed to appear more than once. For example, there is a unique set that contains the elements a and b and no others, but there are many multisets with this property, such as the multiset that contains two copies of a and one of b or the multiset that contains three

copies of both a and b. In fact, multisets are more common than sets. The multiplicity of an element of a multiset is the number of times the element occurs in the multiset. The total number of elements in a multiset$M$, including repeated memberships, is the *cardinality* or *size* of the multiset, denoted $C(M)$ or $|M|$. For example, in the multiset $\{a, a, b, b, b, c\}$, the multiplicities of the elements $a, b,$ and c are respectively 2, 3, and 1, while the cardinality of the multiset itself is 6. To distinguish between sets and multisets, a notation that incorporates capital brackets is sometimes used i.e, the multiset$\{2,2,3\}$ is represented as $[2,2,3]$. The use of square brackets to represent an mset has become almost standard. Thus an mset containing one occurrence of a, two occurences of b, and three occurrences of c is notationally written as $[[a, b, b, c, c, c]]$ or $[a, b, b, c, c. c]$ or $[a, b, c]_{1,2,3}$ or $[a^1, b^2, c^3]$ or $[a1, b2, c3]$, depending on one's taste and convenience. For simplicity the third notation would be used except in the mset functions where other notation like $[1/a, 2/b, 3/c]$ is used.

5.1.1 Multiset Functions

In order to describe Chu categories in multiset environment, we need to apply the equivalence relation approach to multiset introduced in [Monro, 1987] and further used in [Syropoulos, 2003].

In this approach, an mset can be viewed as a pair (X_o, ρ), where X_o is a set and ρ an equivalence relation on X_o . The set X_o is called the field of the multiset. Elements of X_o in the same equivalence class will be said to be of the same sort; elements in different equivalence classes will be said to be of different sorts.

The pair $(X_o, \emptyset)$, where $\emptyset$ denotes the empty relation and X_o is a set, is an ordinary set.

Let $X = (X_o, \rho)$ and $Y = (Y_o, \sigma)$ be two multisets. Then a *multisetfunction(or morphism)* from X to Y, written as $f: X \to Y$, is a function $f: X_o \to Y_o$ which respects sorts. i.e., if $x, x' \in X_o$and $x \rho x'$, then $f(x) \sigma f(x')$.

For example, let $X = [x, x', x'', r, s]$and $Y = [y, h, m, y']$. Then the function $f: X \to Y$ defined by $f(x) = y, f(x') = f(x'') = y', f(r) = h,$ and $f(s) = m$ is an mset function while the function $g: [x, x', x'', r, s] \to [y, h, m, y']$ defined by $f(x) = y, f(x') = h, f(x'') = y', f(r) = h,$ and$f(s) = m$ is not an mset function since $x, x' \in X_o$ and $x \rho x'$but $g(x) \not\sigma g(x')$.

5.1.2 Empty multiset

The pair (ϕ_o, ϕ), where ϕ_o is an *empty* set and ϕ an *empty relation* on ϕ_o is called an *empty multiset*. In other words, an empty set together with an empty equivalence relation defined on it determines an empty multiset.

It has been shown in [Singh et al, 2012] that the following hold for multiset functions:

Let $X = (X_o, \rho)$ and $Y = (Y_o, \sigma)$ be multisets. Then,

(i) $f: X \to Y$ is *injective(one to one)*iff$f: X_o \to Y_o$ is injective;

(ii) $f: X \to Y$ is *surjective(onto)*iff$f: X_o \to Y_o$ is surjective;

(iii) $f: X \to Y$is*bijective(injective and surjetive)*iff$f: X_o \to Y_o$ is bijective.

5.1.3 Identity mset function

Let $X = (X_o, \rho)$ be a multiset. An mset function $f: X \to X$ is called an *identity mset* function on X, denoted I_X, whenever $f: X_o \to Y_o$ and $I_X(x) = x$ for all $x \in X_o$.

5.1.4 Composition of mset functions

Let $X = (X_o, \rho), Y = (Y_o, \sigma)$ and $Z = (Z_o, \gamma)$ be multisets and $f: X \to Y$ and $f: Y \to Z$ be multiset fuctions. Then the composition of f and $g$denoted gof, is the mset function $gof: X \to Z$ which respects sorts. i.e., if $x, x' \in X_o$and $x\rho x'$, then $gof(x)\ \gamma\ go(x')$, satisfying $gof(x) = g(f(x))$, for all $x \in X_o$.

Obviously, multisets and multiset functions form a category denoted $\boldsymbol{Mul}$.

5.1.5 Pure Multiset

Let D be a set. A pure multiset over D is just a pair $\langle D, f\rangle$, where D is a set and $f: D \to \mathbb{N}$ is a function. This functional approach to the notion of multiset was intuitively introduced in the beginning of the section.

Note that the pair $\langle A, \mathcal{X}_A\rangle$ is a pure multiset where A is an ordinary set and $\mathcal{X}_A$ is the characteristic function from A.

5.2Categorical model of multisets

The content of this section is largely abstracted from [Syropoulos, 2003]. Moreover, the general structure of Chu construction will be applied to construct categorical model of multisets.

Let $\mathbb{C}$ be a category. A functor $E: \mathbb{C}^{op} \to \boldsymbol{Set}$is called a *presheaf*on $\mathbb{C}$. Thus, a presheaf on $\mathbb{C}$ is a contravariant functor. However, a presheaf on $\mathbb{C}$ with natural transformations as morphisms form a category denoted $\boldsymbol{Psh}(\mathbb{C})$. If C is a set(i.e., a discrete category); then the presheaf on $F: C \to \boldsymbol{Set}$ denotes a pure multiset, since $F(c)$ is a set whose cardinality

is equal to the number of times c occurs in the pure multiset. So, for any set C, the category $\boldsymbol{Psh}(C)$ denotes the category of all pure multisets of C.

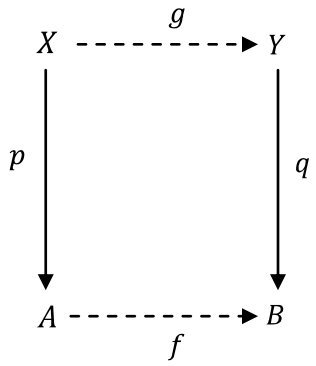
Suppose that $F: A \to \boldsymbol{Set}$ is a presheaf and that A is a set. Then if we form the set $X = \bigcup_{i \in A} X_i$, where $X_i = F(i)$, we can define $p: X \to A$. This function is equivalent to the presheaf F. Moreover, $p^{-1}(a)$ (i.e., the preimage of p) is the set of copies of a in the pure multiset. Now, we can define a category of all possible pure multisets:

Definition 5.1

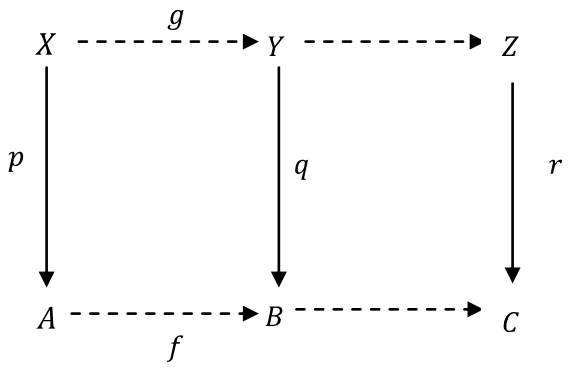
Category $\boldsymbol{Bags}$is a category of all possible pure multisets.

(i) The objects of the category are pairs (A, p), where $p: \bigcup_{i \in A} X_i \to A$.

(ii) An arrow between two objects (A, p) and (B, q) is a pair (f, g), where $f: A \to B$ and $g: X \to Y$, such that the following diagram commutes:



(iii) Suppose that $(f, g): (A, p) \to (B, q)$ and $(f', g'): (B, q) \to (C, r)$ are two arrows; then $(f, g) o (f', g') = (f' o f, g' o g)$ such that in the following diagram



the outer rectangle commutes if and only if the inner squares commute.

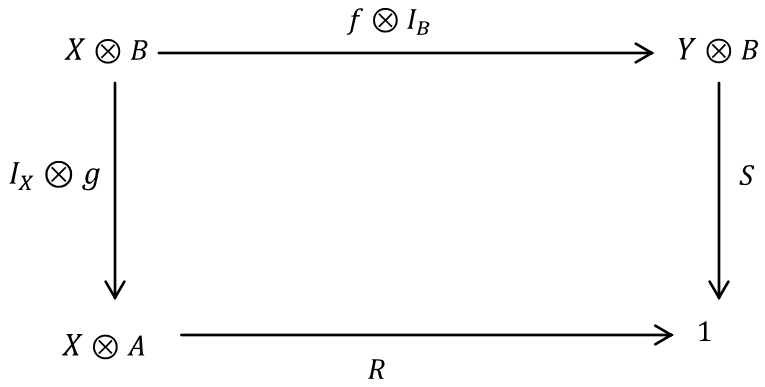
(iv) The arrow $(A,p) \xrightarrow{(I_X, I_A)} (A,p)$ is the identity arrow.

This definition has indeed provided the categorical representation of both forms of multisets. We will now focus on how to represent multisets as objects in some *Chu category*.

Let us recall that $\boldsymbol{Rel}$ is a category of sets and binary relations between the sets. Multisets have been represented as an object in the category $\boldsymbol{Chu}(\boldsymbol{Rel}, 1)$ where 1 is a singleton set.

The objects of this category are triplets (X, R, A), where X and A are arbitrary sets and R is a binary relation in $X \times A$, i.e., $R \subseteq X \times A$. so, if $x \in X$ is related to some $a \in A$, then we write xRa, otherwise, we simply write $x \backslash R\ a$.

The morphism between any two objects $\mathcal{A} = (X, R, A)$ and $\mathcal{B} = (Y, S, B)$ of $\boldsymbol{Chu}(\boldsymbol{Rel}, 1)$ is a pair of relations (f, g), where $f \subseteq X \times Y$ and $g \subseteq B \times A$ satisfying the diagram below:



However, in order to represent multisets as objects in $\boldsymbol{Chu}(\boldsymbol{Rel}, 1)$, it suffices to define the object and the arrow parts of the functor $\mathcal{M}$.

5.2.1 Object part

The functor$\mathcal{M}$ maps each object (A, p) in $\boldsymbol{Bags}$in to the Chu space $(A, \tilde{p}, X)$, where $X = \bigcup_{i \in A} X_i$, i.e., $X = dom\ p$, and $\tilde{p}$ is the graph of the function p, and apX_i if and only if the multiplicity of a is equal to the cardinality of X_i.

5.2.2 Arrow part

Let (A, p) and (B, q) be two objects in $\boldsymbol{Bags}$.

Now, suppose that $(f, g): (A, p) \longrightarrow (B, q)$ is an arrow between these objects. Then $\mathcal{M}(f, g) = (\tilde{f}, \tilde{g}^{-1})$, where $\tilde{f}$ is the graph of the function f and $\tilde{g}^{-1}$ the inverse of the graph of the function g.

Theorem5.1 [Syropoulos, 2003]

The functor$\mathcal{M}$ is injective on objects and faithful.

Proof

$\mathcal{M}$ is injective is obvious from its definition since $\mathcal{M}$ maps each object (A, p) in $\boldsymbol{Bags}$in to the Chu space $(A, \tilde{p}, X)$, where $X = \bigcup_{i \in A} X_i$, i.e., $X = dom\ p$, and $\tilde{p}$ is the graph of the function p, and apX_i if and only if the multiplicity of a is equal to the cardinality of X_i.

Now, it remains to show that $\mathcal{M}$ is faithful on the objects. Let (f, g) and (f', g') be two parallel morphisms in $\boldsymbol{Bags}$ suchthat $(f, g) \neq (f', g')$; then $\mathcal{M}(f, g) \neq \mathcal{M}(f', g')$, since the graph of two different functions are different. This proves that $\mathcal{M}$ is faithful.

Pratt(1999), showed that the category $\boldsymbol{Str}_n$ of $n\text{-}ary$ relational structures fully embeds into $\boldsymbol{Chu}(2^n)$. Using the same recipe, since $\boldsymbol{Mul}$ is a sub-category of $\boldsymbol{Str}_2$, the category $\boldsymbol{Mul}$ has been represented as an object of $\boldsymbol{Chu}(2^2)$ (see{Pratt, 1999} for details).

CHAPTER SIX

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

6.1 SUMMARY

In this thesis, consisting of six chapters, a systematic study of the theory of Chu categories, some of its algebraic structures and their applications were presented. Since category theory happens to be the stepping stone to the theory of Chu spaces and Chu categories, some of its basic concepts were explicated, exemplified, and extended to multisets.

6.2 Conclusions

The followings are the main findings:

1. The structure$(A, r, tc(R_A))$ defines a Chu space over the set $K = \{0,1\}$ with $r(a, x) = 1$ if and only if $a \in x$ and zero otherwise, for all $a \in A$, $x \in tc(R_A)$,and the Chu spaces obtained with their adjoint pairs form a category.
2. The structure $(\mathbb{Z}_n \backslash \mathbf{0}, r, \mathbb{Z}_n)$ under the operation $a \mid (a + x)$, read as $a divides (a + x)$ where $a \in A, x \in \mathbb{Z}_n$, defines a Chu space, and the Chu spaces obtained with their adjoint pairs form a category.

It is shown that, both the structures $(A, r, tc(R_A))$ and $(\mathbb{Z}_n \backslash \mathbf{0}, r, \mathbb{Z}_n)$ over the set $K = \{0,1\}$ could be made a *Poset* and a *join-semilatice* by defining a suitable Chu morphism and an order $\leq$ as usual.

It was observed that $\mathbb{Z}_n, n = 2,3,4$, and $K = \{0,1\}$,give rise to *biextensional* Chu spaces. For $\mathbb{Z}_n, n > 4$, under the aforesaid operation and $K = \{0,1\}$, the resulting Chu spaces are not biextensional [Proofs and other details are on pages 74-79].

3. Two examples were constructed to demonstrate the application of Chu spaces.

6.3 Recommendation

Chu spaces and Chu categories are relatively new areas in mathematics and, until recently, almost all the developments that have taken place in the theory are largely in the set theoretic environment. It seems a promising area of research to study Chu categories inmultiset environment.

REFERENCES

Abramsky, S. (2009). Big Toy Models: Representing Physical Systems as Chu spaces. Technical Report,*Oxford University computing laboratory*, pp. 1-18.

Andres, A., Aristizabal, P., Rueda, C., Valencia, F.D. (2009). Technical Report, NTCC Semantic based on Chu spaces. pp. 1-23.

Barr, M. (1979).*-autonomous categories.Lecture Notes in Mathematics, vol. 752. *Springer-Verlag*.

Barr, M. (1996).The Chu Construction.*Theory and Applications of Categories*, vol.2 no.2, pp. 17-35.

Barr, M. (2006). The Chu construction: History of an idea.*Theory and Applications of Categories*, vol.17, no.1, pp. 10-16.

Barr, M. and Wells, C.(1999). Category Theory.*Lecture Notes for ESSLLI (European summer School of Logic, Language and information*), pp. 1-128.

Devarajan, H., Hughes, D., Plotkin, G., and Pratt, V. (1999).Full Completeness of the Multiplicative Linear Logic of Chu Spaces. Proc. IEEE, *Logic in Computer Science*, pp. 1-10.

Girard, J.-Y.(1987). Linear Logic.*Theoretical Computer Science*, vol.50, no.1, pp. 1-102.

Gupta, V. (1994). Chu spaces: A Model of Concurrency. Ph.D. Dissertation, *Stanford University*, pp. 1-90.

Henry, (2003).Examples of Chu space. [online] *Planet-math.org/exampleofchuspace.htm*, accessed 22/12/2014.

Ivancevic, V.G. and Ivancevic, T.T. (2009).Applied categories and Functors for undergraduates. Lecture notes, pp. 1-28.

Kelly, G.M. (1982). Basic Concepts of Enriched Category Theory.*London Math.Soc. Lecture Notes, Cambridge University Press*, 64.

Kuku, A.O. (1992). *Abstract Algebra*.Ibadan University press, Chapter 8, pp. 353-387.

Laan, V. (2003).Introduction to Category theory. Lecture notes, pp. 1-52.

Lafont, Y., and Streicher, T. (1991).Games Semantics for Linear Logic.In Proc. *6th Annual IEEE Symp.on Logic in Computer Science, Amsterdam*, pp. 43-49.

Lawvere, F. W. and Schanuel, S. H. (2003).*Conceptual Mathematics: A first introduction to Categories, (second edition)*.Cambridge University Press, pp.1-205.

Lawvere, F.W. (2003). An Interview with Lawvere, F.W., http://www.mat.uc.pt/-Picado/lawvere/interview.Pdf, accessed 12/12/2014.

Mackey, G. (1945). On Infinite Dimensional Vector Spaces*Trans. Amer. Math. Soc.*, vol.57, pp. 155-207.

Monro, G.P. (1987) The concept of multiset.*ZeitchritfurmathematischeLogik und Grundlagen der mathematic [mathematical logic quarterly]*, vol.33, no.2, pp. 171-178.

Pratt, V.R. (1994). Chu spaces: Automata with quantum aspects.*Department of Computer Science, Stanford University, CA94305*, pp. 1-17.

Pratt, V.R. (1995). The Stone Gamut: A coordinatization of mathematics. *Dept. of Computer Science, Stanford University, CA 94305*, pp. 1-13.

Pratt, V.R. (1999). Chu spaces. Lecture notes for the School of Category Theory and Applications, *University of Coimbra*, pp. 1-70.

Pratt, V.R. (2004). Chu spaces as a semantic bridge between linear logic and mathematics.*Dept. of Computer Science, Stanford University, CA94305*, pp1-40.

Pratt, V.R. (2005a). Chu spaces from the representational viewpoint.*Dept. of Computer Science, Stanford University, CA 94305-9045*, pp. 1-18.

Pratt, V.R. (2005b) Chu spaces and their interpretation as concurrent objects.*Dept. of Computer Science, Stanford University, CA94305*, pp. 1-16.

Pratt, V.R. (2005c) Chu realizes all small concrete categories.*Dept. of Computer Science, Stanford University, CA94305*, pp. 1-4.

Pratt, V.R. (2005d). Rational Mechanics and Natural Mathematics.*Dept. of Computer Science, Stanford University, CA 94305-2140*, pp. 1-16.

Rosen, K.H.(1999). *Discrete Mathematics and Its Applications*.MCGraw-Hill, Singapore, Fourth edition.

Singh, D., Isah, A.I., and Ibrahim, A.M. (2012).A survey of the Development of the Theory of Categories.*International Journal of Innovation, Management and Technology*, vol. 3, no. 2, pp. 156-159.

Syropoulos, A. (2001). Mathematics of multiset.*C.S. Calude et al. (Eds.): Multiset processing, LNCS 2235*, PP. 347-358.

Syropoulos, A. (2003). Categorical models of multisets.*Romanian Journal of Information Science and Technology*, Vol. 6, no. 3-4, pp. 393-400.

Turi, D. (2001). Category theory. Lecture Notes, *Laboratory for foundation of Computer Science, University of Edinburgh*, pp.1-58.

Wildberger, N. J. (2003). A new look at Multisets.School of Mathematics, UNSW, *Sydney* 2052, Australia, pp.1-21.

Zhang, G.Q. (2004). Chu spaces, Concept Lattices, and Domains.*Electronic Notes in Theoretical Computer Science*, vol. 83, pp. 1-17.