ODD GENERALIZED EXPONENTIAL-INVERSE-EXPONENTIAL DISTRIBUTION: IT'S PROPERTIES AND APPLICATIONS

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DECLARATION

I declare that the work in this dissertation titled "Odd Generalized Exponential-Inverse-Exponential Distribution: Its Properties and Application" has been carried out by me in the Department of Statistics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other institution.

_____ _____

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CERTIFICATION

This dissertation titled ODD GENERALIZED EXPONENTIAL-INVERSE-EXPONENTIAL DISTRIBUTION: ITS PROPERTIES AND APPLICATION by Badamasi ABBA meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

Every challenging work needs self-efforts as well as guidance of elders especially those who were very close to one's heart. My humble effort, I dedicate to my parents, whose affection, love, encouragement and prayer of day and night made me able to get such success and honour.

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ABSTRACT

There remain many problems in real life where observed data do not follow any of the well-known probability models. To address this, there is then, a strong need to propose probability models to better capture the behaviour of some real-life phenomenon. In this dissertation, we propose a new lifetime distribution, called the Odd Generalized Exponential-Inverse-Exponential Distribution (OGE-IED) based on the odd generalized exponential generator proposed in earlier study. The statistical properties of the new distribution are studied along with its reliability functions and limiting behaviour. Also the distribution parameters were estimated using the method of maximum likelihood. We illustrate the flexibility of the proposed model by application to real data and concluded that the OGE-IED provides the best fit among the competing distributions. Finally, we recommend that this distribution can be used in modelling time-dependent events, systems, components or random variables.

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CHAPTER ONE: INTRODUCTION

1.1 Background of the Study

Statistical data modelling is an important aspect of statistics that attracts researcher's wide attention. Modelling lifetime data in areas comprising reliability analysis, engineering, economics, biological studies, environmental and medical sciences, requires an appropriate statistical model for proper actualization of the data. However, there still remains problems where the real data does not follow any of the classical or standard probability models. To address this, there is strong need then to propose new models that can better capture real-life phenomenon inherent in a given dataset. Introducing new probability models or their classes is an old practice and has ever been considered as very valuable as many other practical problems in statistics. The idea simply started with defining different mathematical functional forms, and then induction of location, scale or inequality parameters (Tahir and Cordeiro, 2016a). Addition of new shape parameter(s) expands a model into a larger family of distributions and can provides significantly skewed and heavy-tailed new distributions. It also provides greater flexibility in the form of new distributions. This induction of parameter(s) has been proved useful in exploring tail properties and also for improving the goodness-of-fit of the proposed generator family (Saboor *et al.*, 2015).

The one parameter Inverse Exponential distribution otherwise known as the Inverted Exponential distribution was introduced by Keller and Kamath (1982). It has an inverted bathtub failure rate and it is a competitive model for the Exponential distribution. It is an important probability distribution for modelling lifetime data.

1.2 Statement of the Problem

There exist several probability distributions for modelling lifetime data.However, some of these dataset do not follow any of the existing and well known standard probability distributions or at least are inappropriately described by them. This brings increased interest in proposing new univariate continuous distributions by adding one or more new shape parameter(s) to the baseline model. In this dissertation, a new probability distribution called Odd Generalized Exponential-Inverse-Exponential distribution (OGE-IED) taking inverse-exponential as the baseline distribution and using Tahir *et al.,* (2015) generator is being proposed, aimed to provide greater flexibility and create more weight to the tails of the new distribution.

1.3 Aim and objectives of the study

This work is aimed at developing a new probability density function called Odd Generalized Exponential-Inverse -Exponential distribution(OGE-IED) using Inverse-Exponential distribution as the baseline distribution.

The stated aim is to be achieved through the following objectives:

i. deriving the new distribution, named Odd Generalized Exponential-Inverse-Exponential distribution (OGE-IED) and check its validity;
ii. obtaining some statistical properties of the proposed distribution comprising moment, moment generating function, reliability function, hazard function, quantile function and distribution of its order statistics;
iii. estimating the parameters of the proposed distribution using the method of maximum likelihood estimation (MLE); and

iv. comparing the fitness of the proposed model with other competing models in the literature using real life dataset.

1.4 Significance of the Study

The new proposed distribution can be of significance in modelling lifetime data, since induction of new shape parameter(s) has been proved useful in exploring tail properties and also in improving the goodness-of-fit of the proposed generator family (Saboor *et al.*, 2015). It can be successfully applied to problems arising in several areas of research such as physical and biological sciences, reliability theory, hydrology, medicine, meteorology, engineering and survival analysis.

1.5 Limitation

The study focused only on developing new probability distribution and deriving its mathematical expressions for some selected properties of the proposed distribution such as moment generating function, quantile function, median, asymptotic behaviour, density functions for the minimum and maximum order statistics and estimating the model parameters by using the method of MLE.

1.6 Motivation

Tahir *et al.*, (2015) highlighted some special distributions that can be obtained by their Odd Generalized Exponential generator. These distributions include the Normal, Frechet, and Weibull distributions. Also, Johnson *et al.* (1994) stated that the use of four-parameter distributions should be sufficient for most practical purposes. According to them, "at least three parameters are needed but they doubted any noticeable improvement

arising from including a fifth or sixth parameter". Besides these facts, the inverse-exponential distribution is a well-known and commonly used standard continuous distribution with many areas of application. However, there has been no study on the hybrid of Odd Generalized Exponential Inverse-exponential distribution since the introduction of the Odd Generalized Exponential Generator by Tahir *et al.* (2015). Hence the need to propose an Odd Generalized Exponential- Inverse-exponential distribution (OGE-IED), study some of its properties and explore its applicability to a real life dataset.

1.7 Definition of terms

1.7.1 Quantile Function

Quantile function is use for calculating the median, skewness and kurtosis and for simulation of random numbers.

1.7.2 Skewness

Skewness is a measure of the degree of asymmetry or lack of symmetry of a distribution. It is said to be either right skewed (positively skewed) or left skewed (negatively skewed) when the measure is on either sides of zero respectively.

1.7.3 Kurtosis

A statistical technique that measures how peaked a distribution is. The kurtosis of a normal distribution is zero. If kurtosis is different from zero, then the distribution is either flatter or more peaked than normal.

1.7.4 Survival Function

The reliability function is otherwise known as the survival or survivor function. It is the probability of surviving beyond a specified time and it is obtained mathematically as the complement of the cumulative density function (CDF).

1.7.5 Hazard Function

Hazard function is also called the failure or risk function and is the probability that a system or component will fail or die in an interval of time.

1.7.6 Order Statistics

Suppose $x_1, \cdots, x_n$ is a random sample from a distribution with *pdf* $f(x)$, and let $x_1 < \cdots < x_n$ denote the corresponding order statistic obtained from this sample, then the *pdf*, $f_{i:n}(x)$ of the i^{th} order statistic can be obtained by

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \qquad (1.6.1)$$

It is used in a wide range of problems including robust statistical estimation and detection of outliers, characterization of probability distributions and goodness of fit tests, entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials.

1.7.7 Maximum Likelihood Method

Let $x_1, \cdots, x_n$ be a random sample from a population with probability density function $f(x_i, \theta)$ where θ is an unknown parameter. The likelihood function, $L(\theta)$ is defined to be the joint density of the random variables $x_1, \cdots, x_n$. That is,

$$L(\theta) = \prod_{i=1}^{n} f(x_{i,}\theta) \qquad (1.6.2)$$

The sample statistic that maximizes the likelihood function $L(\theta)$ is known as maximum likelihood estimator of θ and is denoted as $\hat{\theta}$ and the technique is called maximum likelihood method.

CHAPTER TWO: LITERATURE REVIEW

This chapter present a review of the literature on the techniques used in proposing new probability distribution, the classical distribution used (baseline distribution) and its related generalization, the generator and the probability distributions proposed so far, and finally the parameter estimation methods.

An approach to establishing flexible probability distribution is to insert appropriate probability function into a larger model by adding shape parameter (Aryal and Tsokos, 2009). Tahir *et al.*, (2015a) stated and explained two main approaches for adding new shape parameter(s) to a baseline distribution.

In the first approach, Marshall and Olkin (1997) pioneered the generalization of probability distributions by inducing a parameter to the survival function, $S(x) = 1 - G(x)$, where $G(x)$ is the cumulative distribution function (CDF) of any baseline distribution. Gupta *et al.*, (1998) induced one parameter to the CDF, $G(x)$ of a baseline distribution to define the Exponentiated- G ("Exp-G" for short) class of distributions based on Lehmann-type alternatives (see Lehmann 1953). Following Gupta *et al.*'s proposed family of distributions, Gupta and Kundu (1999) studied two-parameter generalized-exponential (GE) distribution as an extension of the exponential distribution based on Lehmann type I alternative. The GE distribution is also known as the Exponentiated exponential (EE) distribution. As one of the early generalizations of the exponential distribution, the GE distribution has received increased attention and many researchers studied its various properties and also offered some comparisons with other distributions. In fact, the GE distribution has been proven to be a good alternative to the gamma, Weibull and log-normal distributions, all of them being two-parameter

distributions. The GE distribution can be used effectively for analyzing lifetime data with monotonic hazard rate function but unfortunately it cannot be used if the hazard rate function is upside-down, J or reversed-J shapes.

The second approach is achieved by adding two or more shape parameters. This approach of generalization was pioneered by Eugene *et al.*, (2002) and Jones (2004) who defined the beta-generated (beta-G) class from the logit of the beta distribution. Further work on generalized distributions are the Kumaraswamy-G (Kw-G) by Cordeiro and de Castro (2011), McDonald-G (Mc-G) by Alexander *et al.*, (2012), Odd gamma-G type 3 by Torabi and Montazari (2012), Odd Exponentiated Generalized (odd exp-G) by Cordeiro *et al.*, (2013), transformed-transformer (T-X) (Weibull-X and gamma-X) by Alzaatreh *et al.*, (2013), Exponentiated T-X by Alzaghal *et al.*, (2013), gamma-G type 1 by Zografos and Balakrishanan (2009) and Amini *et al.*, (2014), gamma-G type 2 by Ristic and Balakrishanan (2012) and Amini *et al.*, (2014), logistic-G by Torabi and Montazari (2014), Odd Weibull-G by Bourguignon *et al.*, (2014), Exponentiated half-logistic by Cordeiro *et al.*, (2014), T-X{Y}-quantile based approach by Aljarrah *et al.*, (2014) and TR{Y} by Alzaatreh *et al.*, (2014), a new Weibull-G by Tahir *et al*., (2015b) and Kumaraswamy odd log-logistic-G by Alizadeh *et al.*, (2015), Lomax-G by Cordeiro *et al.*, (2014), logistic-X by Tahir *et al.*, (2016b).

A one-parameter Inverse Exponential distribution introduced by Keller and Kamath (1982) has an inverted bathtub failure rate and was found to be competitive in comparison to Exponential distribution. It is one of the distributions that are used in modelling lifetime data. Recently, several generalization of inverse-exponential distribution were obtained. Abouammoh and Alshangiti (2009) introduced a generalized

version of the inverted exponential distribution (GIED). The study investigated some statistical properties and reliability functions of the GIED. Furthermore, it can be considered as the inverse of the well-known generalized exponential distribution (GED) introduced by Gupta and Kundu (1999). Oguntunde *et al.*, (2014) studied a three-parameter distribution called the Kumaraswamy Inverse Exponential distribution (K-IED). K-IED parameters were estimated and an explicit expression for the i^{th} moment was given. The distribution is believed to be leptokurtic. A one-parameter Inverse Exponential distribution is being identified as a special case of K-IE distribution when the two additional parameters$a$ and b have taken a unit value of one. It is expected that, the K-IE distribution would have an advantage over the Beta Inverse Exponential distribution in terms of tractability because the former does not involve any special function like the incomplete beta function ratio. Singh and Goel (2015) developed a three-parameter beta inverted exponential distribution which contains generalized inverted exponential and inverted exponential distributions as special sub models. The distribution can be applied effectively in modelling lifetime data, due to its ability to accommodate non-monotonic, uni-modal and inverse bathtub-shaped hazard functions. Oguntunde and Adejumo (2015) defined a two-parameter distribution called Transmuted Inverse Exponential distribution as a generalization of the one-parameter Inverse Exponential distribution. The probability model is positively skewed and uni-modal; the proposed distribution has an inverted bathtub failure rate. Explicit expressions for the r^{th} moment and the moment generating function was derived. It was further observed that, the moment for the distribution only exists for $0 < r < 1$. By the behaviour of the hazard function, it indicates that the model is a competitive probability model for other

probability models having inverted bathtub failure rates like the one parameter Inverse Exponential distribution and can serve as an alternative. It was further mentioned that, the new probability model can be used for modelling cases with low risk at the initial stage, with risk increasing with time and then later decreasing; for instance, cases of breast and bladder cancers (among others). Abouammoh and Alshangiti (2016) proposed a new distribution called Modified Generalized Inverse Exponential distribution (MGIED) which is a modification of Generalized Inverse Exponential distribution (GIED) proposed by Abouammoh and Alshangiti (2009). The MGIED can serve better than the existing GIED.

Recently, Tahir *et al.*, (2015a) proposed a new generator for the class of univariate distributions called the odd generalized exponential-G (OGE-G) and studied each of the OGE-Weibull (OGE-W), the OGE-Fréchet (OGE-Fr) and the OGE-Normal (OGE-N) models. This method is more flexible because the hazard rate shapes could be increasing, decreasing, bathtub and sometimes upside down bathtub. The main aim for employing the technique advanced by the OGE generator is to propose more flexible and highly skewed distributions that are not fat-tailed for modelling real data (Tahir *et al.*, 2015). Thereafter, many OGE-G distributions have appeared in the literature. El-Damcese *et al.*, (2015) proposed a new lifetime probability distribution, called the Odd Generalized Exponential-Gompertz distribution (OGE-GD) and studied its different properties. Some statistical properties of this distribution have been derived and discussed. A real dataset was used to compare the OGE-G with other known distributions such as Exponential (E), Generalized Exponential (GE), Gompertz (G), Generalized Gompertz (GG) and Beta-Gompertz (BG). Applications on some of real data showed that the OGE-G was the best

fit compared with other distributions under study. Rosaiah *et al.*, (2016) studied a new probability distribution called odd generalized exponential log logistic distribution (OGE-LLD). The statistical properties of the distribution have been studied and the model parameters have also been estimated using the method of maximum likelihood. Appropriateness of fitting the OGE-LLD has been established by analyzing a real life data set.

El-Damcese *et al.*, (2016) studied the odd generalized exponential linear failure rate distribution (OGE-LFRD). The pdf, the CDF, the hazard rate function and the reversed hazard function for the new probability distribution as well as the explicit expression for the moments were provided. They estimated the parameters of the distribution by maximum likelihood method. A lifetime dataset was used to compare the OGE-LFR with other known distributions such as Exponential (E), Generalized Exponential (GE), Linear Failure Rate (LFR), New Generalized Linear Exponential (NGLE) and Transmuted Linear Exponential (TLE). Applications on set of real data showed that the OGE-LFR is the best distribution for fitting these lifetime data compared with ED, GED, LFRD, NGLED and TLED distributions. Abdelall (2016) came up with a proposed probability distribution called odd generalized exponential modified-Weibull distribution (OGE-MWD) using the generator by Tahir *et al.*, (2015), he studied some mathematical properties of the new distribution such as moment, order statistics and method of Maximum Likelihood Estimation was employed to estimate the parameters of the model.The study also compared the new model with the existing ones to ascertain the effectiveness of the new distribution and based on the result obtained it was observed that, the distribution (OGE-MWD) provides better fit to the data than the other

distributions. These include exponential, generalized exponential, linear failure rate and Weibull distributions.

In addition, Maiti and Pramanik (2015) proposed new probability distribution called Odds Generalized Exponential – Exponential Distribution (OGE-ED) using a different link function called T-X family of distribution proposed by Alzaatreh *et al.*, (2013). The structural and reliability properties of this distribution have been studied and inference on parameters have also been conducted. The distribution has been compared with some standard distributions having two parameters through simulation study and the superiority of the proposed distribution was established. The appropriateness of fitting the odds generalized exponential - exponential distribution has also been established by analyzing a real life dataset. Maiti and Pramanik (2016) have also studied Odd Generalized Exponential Pareto distribution (OGE-PD).

Kantar *et al.*, (2011) discussed different parameter estimation methods which include: maximum likelihood method, least square method, weighted least square method, moments method, the method based on quantiles and maximum spacing method for the Weibull distribution which are widely accepted in wind power study and concluded that in most cases method of moment outperformed the other methods, ML method as second. Abbas and Yincai (2012) estimated scale parameter for Frechet distribution with known shape using the maximum likelihood estimation and Probability weighted moment estimation. Comparisons were made between the different estimators based on simulation study and observed that ML estimator performs better than other estimators in terms of biases for all cases considered. Shawky *et al.*, (2009) studied the Exponentiated Pareto distribution with the aim of comparing different methods of estimations. The different

methods of estimations used are maximum likelihood estimation (MLE's), method of moment estimation (MME's), percentile-based estimation (PCE's), least square estimation (LSE's) and weighted least square estimation (WLSE's). They presented results of some numerical experiments to compare the performance of different estimators proposed mainly with respect to their biases and root mean squared errors (RMSE's) for different sample sizes and for different parameter values. When comparing the performance of all the estimators, it was observed that, as far as biases or RMSE's are concerned, the MLE performed better in almost all the cases.

Hence based on the reviewed literature, this research intends to develop new probability distribution: a hybrid of generalized exponential and inverse-exponential distributions using the Odd generalized exponential family of distributions by Tahir *et al.*, (2015a) called Odd Generalized Exponential- Inverse-Exponential Distribution (OGE-IED).

CHAPTER THREE: METHODOLOGY.

3.1 The Definition of the Generator

For any continuous distribution with cumulative distribution function (CDF), $G(x,\omega)$ and probability density function (pdf), $g(x,\omega)$, Tahir *et al.*, (2015a) proposed a new generator for the class of univariate distributions called the Odd Generalized Exponential-G (OGE-G) family of distributions that provides greater flexibility in modelling of real data sets. The cumulative distribution function (CDF) of the OGE-G family of distributions according to Tahir *et al.*, (2015a) is defined as follows:

Given a cumulative distribution function (CDF) of Generalized Exponential distribution

$$F(x) = (1 - e^{-\alpha x})^{\beta} \qquad (3.1.1)$$

The CDF of the OGE-G is obtained by replacing the variable in equation (3.1.1) with $G(x,\omega)/\bar{G}(x,\omega)$, as given in equation (3.1.2) below:

$$F(x) = \left(1 - e^{-\alpha\frac{G(x,\omega)}{\bar{G}(x,\omega)}}\right)^{\beta} \qquad (3.1.2)$$

where α and β are two additional parameters. $G(x,\omega)$ is the CDF of any univariate continuous distribution defined on parameter vector ω and $\bar{G}(x,\omega)$ is the survival function of $G(x,\omega)$.

The above OGE generator can be interpreted as follows: Let Y be a lifetime random variable having a certain continuous distribution $G(x,\omega)$. The odds ratio that a system (or component) following the lifetime Y will die (failure) at time x is $G(x,\omega)/\bar{G}(x,\omega)$. If the variability of these odds of death is represented by the random variable X and is assumed to follow Generalized Exponential distribution with cumulative distribution function

(CDF), $F(x) = (1 - e^{-\alpha x})^{\beta}$, where α is the scale parameter and β is the shape parameter. Hence, this can be represented as:

$$P(Y \leq x) = P\left(Y \leq \frac{G(x,\omega)}{\bar{G}(x,\omega)}\right) = F(x,\alpha,\beta,\omega) = \left(1 - e^{-\alpha\frac{G(x,\omega)}{\bar{G}(x,\omega)}}\right)^{\beta} \quad (3.1.3)$$

The main motivation for employing the techniques advanced by the *OGE*-generator is to propose more flexible and highly skewed distribution that are not long-tailed for modelling real data.

The corresponding probability density function (pdf) of the OGE-G family can be obtained from equation (3.1.2) by taking the derivative of the CDF with respect to x as follows:

Recall that

$$f(x) = f(x,\alpha,\beta,\omega) = \frac{dF(x,\alpha,\beta,\omega)}{dx} \quad (3.1.4)$$

where

$$F(x) = \left(1 - e^{-\alpha\frac{G(x,\omega)}{\bar{G}(x,\omega)}}\right)^{\beta}$$

Let

$$y = 1 - e^{-\alpha\frac{G(x,\omega)}{\bar{G}(x,\omega)}} \quad (3.1.5)$$

Then, we can rewrite equation (3.1.2) as

$$F(x) = y^{\beta} \quad (3.1.6)$$

Therefore from equation (3.1.4), we have

$$f(x) = \frac{dF(x)}{dy} = \frac{d(y^{\beta})}{dy} = \beta y^{\beta-1} \quad (3.1.7)$$

Differentiating equation (3.1.5) with respect to x lead to

$$\frac{dy}{dx} = \frac{d}{dx}\left(1 - e^{-\alpha\frac{G(x,\omega)}{\bar{G}(x,\omega)}}\right) = \alpha e^{-\alpha\frac{G(x,\omega)}{\bar{G}(x,\omega)}} \times \frac{d}{dx}\left(\frac{G(x,\omega)}{\bar{G}(x,\omega)}\right) \quad (3.1.8)$$

but

$$\frac{d}{dx}\left(\frac{G(x,\omega)}{\bar{G}(x,\omega)}\right)=\frac{d}{dx}\left(\frac{G(x,\omega)}{1-G(x,\omega)}\right)$$

$$\frac{\left(1-G(x,\omega)\right)g(x,\omega)+G(x,\omega)g(x,\omega)}{\bar{G}(x,\omega)^2}=\frac{g(x,\omega)}{\bar{G}(x,\omega)^2} \tag{3.1.9}$$

substituting equation (3.1.9) into (3.1.8) gives:

$$\frac{dy}{dx}=\frac{\alpha g(x;\omega)}{(\bar{G}(x;\omega))^2}e^{-\alpha\frac{G(x;\omega)}{\bar{G}(x;\omega)}} \tag{3.1.10}$$

Hence by equations (3.1.5), (3.1.7) and (3.1.10), we arrived at equation (3.1.11) below

$$f(x)=f(x;\alpha,\beta,\omega)=\frac{\alpha\beta g(x;\omega)}{(\bar{G}(x;\omega))^2}e^{-\alpha\frac{G(x;\omega)}{\bar{G}(x;\omega)}}\left(1-e^{-\alpha\frac{G(x;\omega)}{\bar{G}(x;\omega)}}\right)^{\beta-1} \tag{3.1.11}$$

Thus, the probability density function (pdf) corresponding to equation (3.1.2) is given in equation (3.1.11) above where $G(x;\omega)$ and $g(x;\omega)$ are the CDF and pdf of any baseline distribution andω is the vector of parameter of any baseline distribution. We can omit the dependence on the vector of parameter ω and simply write $G(x;\omega)=G(x)$. The major benefit of this model is to offer more flexibility to any univariate continuous baseline distribution and therefore it becomes suitable for analyzing right skewed lifetime datasets.

Thus, let $G(x)$ be any continuous baseline distribution, for each $G(x)$, we define the OGE-G distribution with two additional parameters α and β by equation (3.1.11).

A random variable X with pdf (3.1.11) is denoted by $X\sim OGE-G(\alpha,\beta,\boldsymbol{\omega})$. The additional parameters induced by the OGE generator above are to develop a more flexible distribution compare to its baseline distribution. This class of distribution becomes a special case of some other distributions. For $\beta=1$ the generator reduced to Odd

Exponential-G family (Bourguignon *et al.*, (2014)). When $G(x) = \frac{x}{1+x}$ it gives a Generalized Exponential distribution (Gupta and Kundu, 1999). Taking $\beta = 1$ and $G(x) = \frac{x^c}{1+x^c}$, it give Exponentiated Weibull distribution (Mudholkar and Srivastava, 1993). Also when $1 - e^{-\alpha x}$ it lead to Generalized Gompertz distribution (El-Gohary *et al.*, 2013)

3.2 Definition of the Proposed Distribution

A random variable X that follows inverse-exponential distribution also known as inverted exponential distribution has its cumulative distribution function (CDF) probability density function (pdf) respectively given by:

$$G(x) = e^{-\frac{\lambda}{x}} \qquad x, \lambda > 0 \tag{3.2.1}$$

$$g(x) = \frac{\lambda}{x^2} e^{-\frac{\lambda}{x}} \tag{3.2.2}$$

Taking the pdf and the CDF given in equation (3.2.1) and (3.2.2) of the inverse exponential distribution with scale parameter λ into equation (3.1.2) and (3.1.11), we defined new three parameter distribution called Odd Generalized Exponential-Inverse-Exponential distribution $(OGE - IED)$ having parameters α, β and λ and denoted as $OGE - IED(\varpi)$, where ϖ is a parameter vector defined by $\varpi = (\alpha, \beta, \lambda)'$. Let X be a random variable that follows $OGE - IED$ with vector of parameter $\varpi = (\alpha, \beta, \lambda)'$, then the cumulative distribution function (CDF) and probability density function (pdf) from equation (3.1.2) and (3.1.11) are respectively given as:

$$F(x;\varpi)=\left(1-e^{-\alpha\left(\frac{e^{-\frac{\lambda}{x}}}{1-e^{-\frac{\lambda}{x}}}\right)}\right)^{\beta}=\left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{\beta} \quad (3.2.3)$$

and

$$f(x)=\frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}}}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2}e^{-\alpha\left(\frac{e^{-\frac{\lambda}{x}}}{1-e^{-\frac{\lambda}{x}}}\right)}\left(1-e^{-\alpha\left(\frac{e^{-\frac{\lambda}{x}}}{1-e^{-\frac{\lambda}{x}}}\right)}\right)^{\beta-1} \quad (3.2.4)$$

Equation (3.2.4) can further be rewrite as:

$$f(x)=\frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}}}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2}e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{\beta-1} \quad (3.2.5)$$

where $X>0$, $\alpha>0,\beta>0$ are shape parameters and λ is a scale respectively

Special case:

This distribution becomes a special case of Odd Exponential-Inverse-Exponential distribution for $\beta=1$.

3.2.1 Model Validity Check

Let the random variable X follows an$OGE-IE(\varpi)$ distribution, then the probability model $f(x)$ in equation (3.2.4) is referred to a valid probability density function (pdf) of X if

$$\int_0^{\infty} f(x)dx=1 \quad (3.2.6)$$

Proof:

Substituting equation (3.2.4) into (3.2.6), we have

$$\int_0^{\infty} \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}}}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2} e^{-\alpha\left(\frac{e^{-\frac{\lambda}{x}}}{1-e^{-\frac{\lambda}{x}}}\right)} \left(1-e^{-\alpha\left(\frac{e^{-\frac{\lambda}{x}}}{1-e^{-\frac{\lambda}{x}}}\right)}\right)^{\beta-1} dx \qquad (3.2.7)$$

Let

$$v = 1 - e^{-\frac{\lambda}{x}} \qquad (3.2.8)$$

Making x subject, we get

$$x = -\frac{\lambda}{\ln(1-v)} \qquad (3.2.9)$$

Therefore, by substituting the lower and upper limit of x into equation (3.2.8) the range of v is obtained as follows:

As $x \to \infty, v \to 0$ and as $x \to 0, v \to 1$

Thus, the range of v is between 0 and 1. That is $0 \leq v \leq 1$.

Then, taking the derivative of (3.2.8), lead to

$$dv = -\frac{\lambda}{x^2} e^{-\frac{\lambda}{x}} dx$$

which implies that,

$$dx = -\frac{dv}{\frac{\lambda}{x^2} e^{-\frac{\lambda}{x}}} \qquad (3.2.10)$$

Substituting equations (3.2.8), (3.2.9) and (3.2.10) into equation (3.2.7) above, we have

$$\int_0^{\infty} f(x)dx = -\alpha\beta \int_0^1 \frac{1}{v^2} e^{-\frac{\alpha(1-v)}{v}} \left(1-e^{-\frac{\alpha(1-v)}{v}}\right)^{\beta-1} dv \qquad (3.2.11)$$

Now, let

$$u = 1 - e^{-\frac{\alpha(1-v)}{v}} \qquad (3.2.12)$$

Implies

$$v = \frac{1}{\left(1-\frac{1}{\alpha}ln(1-u)\right)} \tag{3.2.13}$$

Thus, the range of u is also between 0 and 1. That is $0 \leq u \leq 1$

then,

$$\frac{du}{dv} = -e^{-\frac{\alpha(1-v)}{v}} \times -\alpha \frac{d}{dv}\left(\frac{1-v}{v}\right) \tag{3.2.14}$$

And using quotient rule,

$$\frac{d}{dv}\left(\frac{1-v}{v}\right) = \frac{-v-(1-v)}{v^2} = -\frac{1}{v^2}$$

Therefore,

$$\frac{du}{dv} = -\frac{\alpha}{v^2}e^{-\frac{\alpha(1-v)}{v}} \tag{3.2.15}$$

Implies,

$$dv = -\frac{du}{\frac{\alpha}{v^2}e^{-\frac{\alpha(1-v)}{v}}} \tag{3.2.16}$$

Substituting for equation (3.2.12) and (3.2.16) into (3.2.11) leads to

$$\int_0^1 f(x)dx = -\alpha\beta \int_0^1 \frac{1}{v^2} e^{-\frac{\alpha(1-v)}{v}} u^{\beta-1} \times -\frac{du}{\frac{\alpha}{v^2}e^{-\frac{\alpha(1-v)}{v}}} = \beta \int_0^1 u^{\beta-1}du \tag{3.2.17}$$

Therefore,

$$\beta \int_0^1 u^{\beta-1}du = u^{\beta}\Big|_0^1 = 1-0 = 1 \tag{3.2.18}$$

Hence, equation (3.2.18) shows that the probability model in equation (3.2.4) is a valid probability density function.

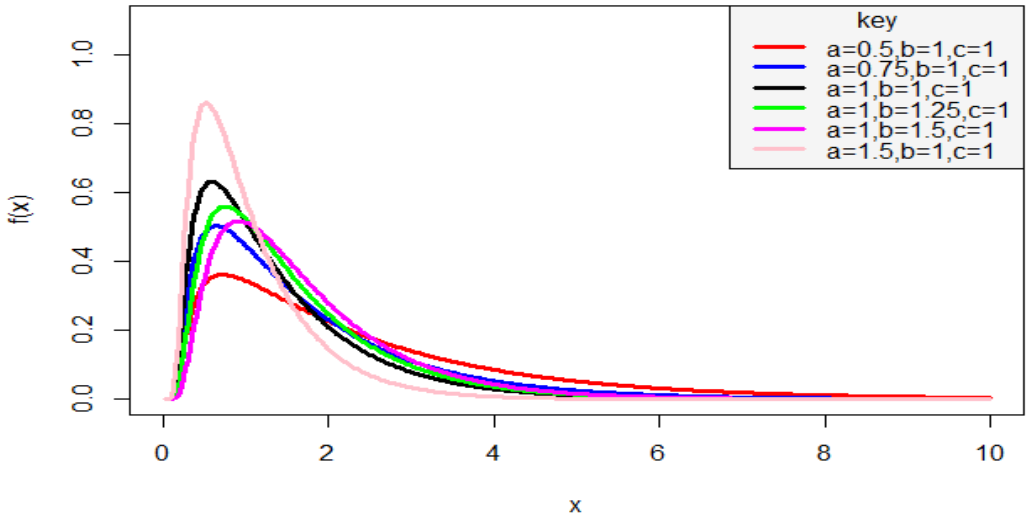


Figure 3.1:pdf plot of OGE-IED

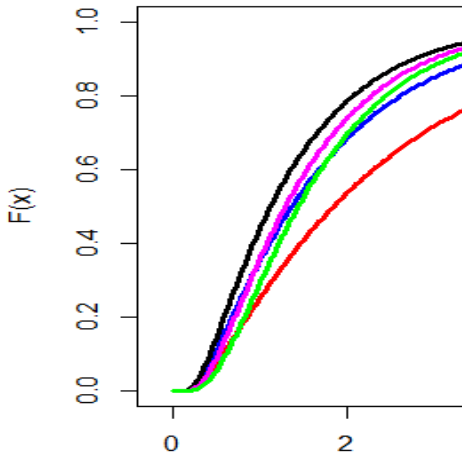


Figure 3.1: CDF plot of OGE-IED

The plots of the pdf and CDF of the OGE-IED for some selected parameter values are displayed in figures **3.1** and **3.2** above. The pdf plot shows that the OGE-IEDis skewed to the right (positively skewed) and hence can be used to modelling positively skewed dataset. Also, the CDF has an increasing upward trend that approaches one as the value of the random variable X increases. This has also confirmed the validity of the *pdf*.

3.3 Statistical Properties

3.3.1 Asymptotic Behaviour of the OGE-IED

Here, we tried to observe the behavior of the proposed probability model given in equation (3.2.4) as $x \to 0$ and as $x \to \infty$. This includes considering $\lim_{x\to 0} f(x)$ and $\lim_{x\to \infty} f(x)$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}}}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2} e^{-\alpha e^{\frac{\lambda}{x}-1^{-1}}} \left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}}\right)^{\beta-1}$$

$$= \lim_{x\to 0} e^{-\frac{\lambda}{x}} \times \lim_{x\to 0} \frac{\alpha\beta\lambda}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2} e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}} \left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}}\right)^{\beta-1}$$

$$= 0 \times \lim_{x\to 0} \frac{\alpha\beta\lambda}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2} e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}} \left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}}\right)^{\beta-1} = 0 \qquad (3.3.1)$$

Similarly,

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}}}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2} e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}} \left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}}\right)^{\beta-1}$$

$$= \lim_{x\to\infty} \frac{1}{x^2} \times \lim_{x\to\infty} \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}}}{\left(1-e^{-\frac{\lambda}{x}}\right)^2} e^{-\alpha\left(e^{-\frac{\lambda}{x}-1}\right)^{-1}} \left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}}\right)^{\beta-1}$$

$$= 0 \times \lim_{x\to 0} \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}}}{\left(1-e^{-\frac{\lambda}{x}}\right)^2} e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}} \left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}}\right)^{\beta-1} = 0 \qquad (3.3.2)$$

Hence, equation (3.3.1) and (3.3.2) indicates that $OGE-IED$ has at least one mode.

3.3.2 Quantile Function and Median

The quantile function of the $OGE-IED(\varpi)$ is simply the solution of the following equation with respect to $x_u, 0<u<1$

$$u = F(x;\varpi) = \left(1-e^{-\alpha\left(\frac{e^{-\frac{\lambda}{x}}}{1-e^{-\frac{\lambda}{x}}}\right)}\right)^{\beta} \qquad (3.3.3)$$

Multiplying both sides by the inverse of the power by $\frac{1}{\beta}$, gives

$$u^{\frac{1}{\beta}} = 1-e^{-\alpha\left(\frac{e^{-\frac{\lambda}{x}}}{1-e^{-\frac{\lambda}{x}}}\right)} \qquad (3.3.4)$$

Then from equation (3.3.4), we get

$$1-u^{\frac{1}{\beta}} = e^{-\alpha\left(\frac{e^{-\frac{\lambda}{x}}}{1-e^{-\frac{\lambda}{x}}}\right)} \qquad (3.3.5)$$

Taking natural log of both side, lead to

$$-\frac{1}{\alpha} ln\left(1-u^{\frac{1}{\beta}}\right) = \left(\frac{e^{-\frac{\lambda}{x}}}{1-e^{-\frac{\lambda}{x}}}\right) \qquad (3.3.6)$$

Since, $\left(\frac{e^{-\frac{\lambda}{x}}}{1-e^{-\frac{\lambda}{x}}}\right) = \left(\frac{1}{e^{\frac{\lambda}{x}}-1}\right)$, then rewriting equation (3.3.6), we have

$$-\frac{1}{\alpha} ln\left(1-u^{\frac{1}{\beta}}\right)=\left(\frac{1}{e^{\frac{\lambda}{x}}-1}\right) \tag{3.3.7}$$

Equation (3.3.7) implies

$$e^{\frac{\lambda}{x}}=1-\frac{\alpha}{ln\left(1-u^{\frac{1}{\beta}}\right)} \tag{3.3.8}$$

taking natural log of both sides of equation (3.3.8), lead to

$$\frac{\lambda}{x}=ln\left(1-\frac{\alpha}{ln\left(1-u^{\frac{1}{\beta}}\right)}\right) \tag{3.3.9}$$

Thus,

$$x_u=\frac{\lambda}{ln\left(1-\frac{\alpha}{ln\left(1-u^{\frac{1}{\beta}}\right)}\right)} \tag{3.3.10}$$

Hence, equation (3.3.10) gives the quantile function of the OGE-IED and it is refered to as the median of the distribution when u is set to be 0.5. That is, the median is given by the following relation.

$$x_u=\frac{\lambda}{ln\left(1-\frac{\alpha}{ln\left(1-(0.5)^{\frac{1}{\beta}}\right)}\right)} \tag{3.3.11}$$

3.3.3 Skewness and kurtosis based on quantiles

Skewness measures the degree of the long tail while kurtosis is a measure of the degree of tail heaviness. Based on quantile function $Q(.)$. Galtons (1883) and Moors (1988) defined the skewness and kurtosis respectively as

$$Sk=\frac{Q(3/4)-2Q(1/2)+Q(1/4)}{Q(3/4)-Q(1/4)} \tag{3.3.12}$$

$$K = \frac{Q(7/8)-Q(5/8)-Q(3/8)+Q(1/8)}{Q(6/8)-Q(2/8)} \quad (3.3.13)$$

3.3.4 Moments

Moments are used to study some of the most important features and characteristics of a random variable such as mean (central tendency measure), variance (dispersion measure), skewness (SK) and kurtosis (KU). It is stated by the theorem below.

By definition the r^{th} moment of the random variable $X \sim OGE - IED(\varpi)$ is derived as follows:

The r^{th} moment of a random variable X with pdf $f(x)$ is defined by

$$\mu_r' = \int_0^\infty x^r f(x) dx \quad (3.3.14)$$

Substituting $f(x)$, as in equation (3.2.5) into equation (3.2.14), we obtain

$$\mu_r' = \int_0^\infty x^r \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}}}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2} e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}} \left(1 - e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{\beta-1} dx \quad (3.3.15)$$

Since $0 < \left(1 - e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{\beta-1} < 1$, then using binomial expansion, we obtain

$$\left(1 - e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{\beta-1} = \sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i e^{-i\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}} \quad (3.3.16)$$

Substituting equation (3.3.16) into equation (3.3.15), we have

$$\mu_r' = \sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i \int_0^\infty x^{r-2} \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}}}{\left(1-e^{-\frac{\lambda}{x}}\right)^2} e^{-\alpha(i+1)\left(e^{\frac{\lambda}{x}}-1\right)^{-1}} dx \quad (3.3.17)$$

Also by Maclaurin series expansion

$$e^{-\alpha(i+1)\left(e^{\frac{\lambda}{x}}-1\right)^{-1}} = e^{-\alpha(i+1)\left(\frac{e^{-\frac{\lambda}{x}}}{1-e^{-\frac{\lambda}{x}}}\right)} = \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^j (i+1)^j \left(\frac{e^{-\frac{\lambda}{x}}}{1-e^{-\frac{\lambda}{x}}}\right)^j}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^j (i+1)^j \left(e^{-\frac{\lambda}{x}}\right)^j}{j!\left(1-e^{-\frac{\lambda}{x}}\right)^j} \tag{3.3.18}$$

Next, equation (3.3.18) is substituted into (3.3.17) and simplified to have (3.3.19) below

$$\mu_r' = \sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i \int_0^{\infty} x^{r-2} \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}}}{\left(1-e^{-\frac{\lambda}{x}}\right)^2} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^j (i+1)^j \left(e^{-\frac{\lambda}{x}}\right)^j}{j!\left(1-e^{-\frac{\lambda}{x}}\right)^j} dx$$

$$= \beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\beta-1}{i} \frac{(-1)^{i+j} \alpha^{j+1} (i+1)^j}{j!} \int_0^{\infty} x^{r-2} \frac{e^{-\frac{\lambda(1+i)}{x}}}{\left(1-e^{-\frac{\lambda}{x}}\right)^{j+2}} dx$$

$$= \beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\beta-1}{i} \frac{(-1)^{i+j} \alpha^{j+1} (i+1)^j}{j!} \int_0^{\infty} x^{r-2} e^{-\frac{\lambda(1+i)}{x}} \left(1-e^{-\frac{\lambda}{x}}\right)^{-(j+2)} dx \tag{3.3.19}$$

Also applying binomial expansion to $\left(1-e^{-\frac{\lambda}{x}}\right)^{-(j+2)}$ resulted to

$$\left(1-e^{-\frac{\lambda}{x}}\right)^{-(j+2)} = \sum_{k=0}^{\infty} \binom{j+2}{k} e^{-\frac{k\lambda}{x}} \tag{3.3.20}$$

Then substituting equation (3.3.20) into (3.3.19), gives

$$\mu_r' = \beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\beta-1}{i} \frac{(-1)^{i+j} \alpha^{j+1} (i+1)^j}{j!} \int_0^{\infty} x^{r-2} e^{-\frac{\lambda(1+i)}{x}} \sum_{k=0}^{\infty} \binom{j+2}{k} e^{-\frac{k\lambda}{x}} dx$$

$$= \beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{j+2}{k} \binom{\beta-1}{i} \frac{(-1)^{i+j} \alpha^{j+1} (i+1)^j}{j!} \int_0^{\infty} x^{r-2} e^{-\frac{\lambda(1+i+k)}{x}} dx$$

(3.3.21)

Let

$$m = \frac{\lambda(1+i+k)}{x} \tag{3.3.22}$$

Implies

as $x \to 0, m \to \infty$ and as $x \to \infty, m \to 0$, that is $m > 0$

And

$$x = \frac{\lambda(1+i+k)}{m} \tag{3.3.23}$$

Then

$$dx = -\frac{\lambda(1+i+k)dm}{m^2} \tag{3.3.24}$$

Substituting equations (3.3.22), (3.3.23) and (3.3.24) into (3.3.21), gives

$$\mu_r' = \beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{j+2}{k}\binom{\beta-1}{i} \frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j}{j!} \times$$

$$\int_{\infty}^{0} \left(\frac{\lambda(1+i+k)}{m}\right)^{r-2} e^{-m} \times -\frac{\lambda(1+i+k)dm}{m^2} \tag{3.3.25}$$

Simplifying equation (3.3.25), lead to

$$\mu_r' =$$

$$-\beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{j+2}{k}\binom{\beta-1}{i} \frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j\lambda^r(1+i+k)^{r-1}}{j!} \times$$

$$\int_{\infty}^{0} m^{-r}e^{-m}\,dm = \beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{j+2}{k}\binom{\beta-1}{i} \frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j\lambda^r(1+i+k)^{r-1}}{j!} \times$$

$$\int_{0}^{\infty} m^{-r}e^{-m}\,dm \tag{3.3.26}$$

By incomplete gamma function

$$\int_{0}^{\infty} m^{-r}e^{-m}\,dm = \Gamma(1-r) \tag{3.3.27}$$

Therefore, we finally substitute equation (3.3.27) into (3.3.26) to have

$$\mu_r' = \beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{j+2}{k}\binom{\beta-1}{i}\frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j\lambda^r(1+i+k)^{r-1}\Gamma(1-r)}{j!} \tag{3.3.28}$$

This complete the proof.

Thus, the theorem is stated as:

Theorem: The r^{th} moment of the random variable $X \sim OGE-IED(\varpi)$ is given by;

$$\mu_r' = \beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{j+2}{k}\binom{\beta-1}{i}\frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j\lambda^r(1+i+k)^{r-1}\Gamma(1-r)}{j!} \tag{3.3.29}$$

For $r = 1, 2, 3, ...$

Equation (3.3.28) prove ther^{th} moment of the OGE-IED. We can observe that the gamma function, $\Gamma(1-r)$ can only return a value when $0 < r < 1$. Therefore, the moment only exist if $r < 1$, that is, the first, second and other higher order moments do not exist. So, the mean, variance, coefficient of kurtosis as well as coefficient of skeweness cannot be determine from the r^{th} moment.

3.3.5 Moment Generating Function (MGF)

The moment generating function of the random variable X that follows $OGE-IED(\varpi)$ having probability density function, $f(x)$ given in equation (3.2.4) is obtained using the definition:

$$M_x(t) = E(e^{tx}) = \int_o^{\infty} e^{tx} f(x)dx \tag{3.3.30}$$

This can be derived by replacing x^r with e^{tx} in equation (3.3.21). That is

$$M_x(t) = E(e^{tx}) = \int_o^{\infty} e^{tx} f(x)dx =$$

$$= \beta\lambda \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{j+2}{k} \binom{\beta-1}{i} \frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j}{j!} \int_0^{\infty} e^{tx} x^{-2} e^{-\frac{\lambda(1+i+k)}{x}} dx$$

(3.3.31)

Expanding e^{tx} using Maclaurin series expansion, that is

$$e^{tx} = \sum_{l=0}^{\infty} \frac{t^l x^l}{l!} \tag{3.3.32}$$

By substituting equation (3.3.32) into (3.3.31), we arrived at

$$M_x(t) = \beta\lambda \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{j+2}{k} \binom{\beta-1}{i} \frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j t^l}{j!l!} \int_0^{\infty} x^{l-2} e^{-\frac{\lambda(1+i+k)}{x}} dx$$

(3.3.33)

Next, we let

$$m = \frac{\lambda(1+i+k)}{x} \tag{3.3.34}$$

then

as $x \to 0, m \to \infty$ and as $x \to \infty, m \to 0$, that is $m > 0$

and

$$x = \frac{\lambda(1+i+k)}{m} \tag{3.3.35}$$

Then

$$dx = -\frac{\lambda(1+i+k)dm}{m^2} \tag{3.3.36}$$

Substituting equations (3.3.34),(3.3.35) and (3.3.36) into equation (3.3.33), gives

$$M_x(t) = \beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{j+2}{k}\binom{\beta-1}{i}\frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j t^l}{j!\,l!}\times$$

$$\int_{\infty}^{0}\left(\frac{\lambda(1+i+k)}{m}\right)^{l-2} e^{-m} \times -\frac{\lambda(1+i+k)dm}{m^2} \qquad (3.3.37)$$

Simplifying equation (3.3.37) above, we have

$$M_x(t) = -\beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{j+2}{k}\binom{\beta-1}{i}\frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j t^l \lambda^l (1+i+k)^{l-1}}{j!\,l!}$$

$$\times \int_{\infty}^{0} m^{-l} e^{-m}\, dm$$

$$= \beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{j+2}{k}\binom{\beta-1}{i}\frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j t^l \lambda^l (1+i+k)^{l-1}}{j!\,l!}$$

$$\times \int_{0}^{\infty} m^{-l} e^{-m}\, dm \qquad (3.3.38)$$

By incomplete gamma function

$$\int_{0}^{\infty} m^{-l} e^{-m}\, dm = \Gamma(1-l) \qquad (3.3.39)$$

Therefore, substituting equation (3.3.39) into (3.3.38), results to

$$M_x(t) = \beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{j+2}{k}\binom{\beta-1}{i}$$

$$\times \frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j t^l \lambda^l (1+i+k)^{l-1}\Gamma(1-l)}{j!\,l!} \qquad (3.3.40)$$

This complete the proof.

Theorem: the MGF of the random variable $X \sim OGE - IED(\varpi)$ is given by;

$$M_x(t) = \beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{j+2}{k}\binom{\beta-1}{i}\times$$

$$\frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j t^l \lambda^l (1+i+k)^{l-1}\Gamma(1-l)}{j!l!} \qquad (3.3.41)$$

Equation (3.3.41) give the prove of the OGE-IED moment generating function $(MGF)\}$. It can be observed that the gamma function, $\Gamma(1-l)$ can only return a reasonable value when $l = 0$, but for $l > 0$, the (MGF) will tend to infinity (∞) .

3.4 Reliability Analysis

Here, we present the reliability function and hazard function of our proposed distribution Odd Generalized Exponential-Inverse-Exponential distribution (OGE-IED).

3.4.1 Survival function

The Odd Generalized Exponential Inverse Exponential distribution can be a useful characterization of failure time of a given system because of the analytical structure. The reliability function $S(x)$, which is the probability of an item not failing prior to some time x, is defined by

$$S(x) = P(X > 0) = \int_x^{\infty} f(u)du = 1 - F(x), \;\; x > 0 \qquad (3.4.1)$$

The survival (reliability) function of an Odd Generalized Exponential Inverse Exponential distribution is given by

$$S_{OGE-IED}(x) = 1 - \left(1 - e^{-\alpha\frac{G(x,\omega)}{\overline{G}(x,\omega)}}\right)^{\beta} = 1 - \left(1 - e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{\beta} \qquad (3.4.2)$$

A plot for the survival function of the Odd Generalized Exponential-Inverse-Exponential distribution (OGE-IED) at different parameter values is as shown in **Figure 3.3** below. It

can be seen from the graph in **Figure 3.3**, that the value of the survival function equals one at initial time or early age and it decreases as X increases and tends to zero as X increases.

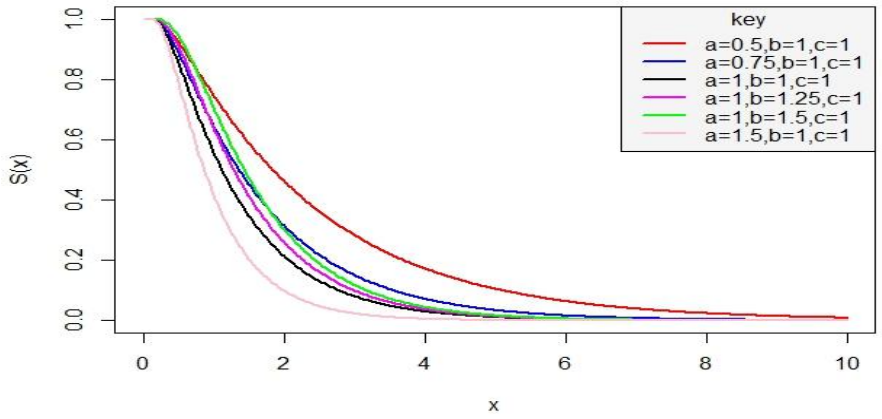


Figure 2.4: Survival function plot of OGE-IED

3.4.2 Hazard function

The other characteristic of interest of a random variable is the hazard function otherwise known as the hazard rate, failure rate or force of mortality defined by

$$H(x) = pr\left(\frac{failure}{survived\ to\ the\ time}\right) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}$$

which is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time x. The hazard function for an Odd Generalized Exponential-Inverse-Exponential random variable is given by

$$H_{OGE-IED}(x) = \frac{f(x)}{S(x)} = \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}} e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}} \left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{\beta-1}}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2 \left(1-\left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{\beta}\right)} \quad (3.4.3)$$

Figure 3.4 below, provides some possible curves for the hazard rate at various values of the model parameters. From the graph, it is observed that the value of the hazard function tends to be invariant when X increases. This means that the OGE-IED may be appropriate in modelling time or age-dependent events, where risk or hazard tends to be constant with increases in time.

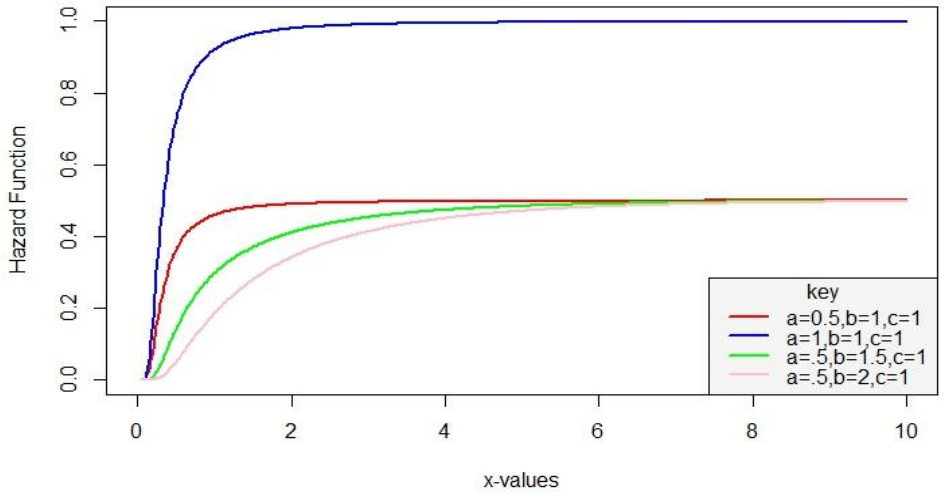


Figure 3.4: Hazard function plot of OGE-IED

3.5 Order Statistics

Order statistics have been used in a wide range of problems including robust statistical estimation and detection of outliers, characterization of probability distributions and goodness of fit tests, entropy estimation, analyses of censored samples, reliability

analysis, quality control and strength of materials. In this section, we derive the closed form expressions for the pdf of the i^{th} order statistic of the Odd Generalized Exponential- Inverse-Exponential distribution.

Suppose $x_1, \cdots x_n$ is a random sample from the OGE-IED and let $x_{1:n} < \cdots < x_{n:n}$ denote the corresponding order statistic obtained from this sample. The pdf, $f_{i:n}(x)$ of the i^{th} order statistic can be obtained by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \qquad (3.5.1)$$

Using binomial expansion, we have;

$$[1-F(x)]^{n-i} = \sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} [F(x)]^j \qquad (3.5.2)$$

Therefore, equation (3.5.1) can be rewritten as

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} f(x)[F(x)]^{j+i-1}$$

$$= \sum_{j=0}^{n-i}\frac{(-1)^j n!}{(i-1)!(n-i-j)!j!} f(x)F(x)^{j+i-1} \qquad (3.5.3)$$

Using (3.2.3) and (3.2.4), the *pdf* of the i^{th} order statistic $X_{i:n}$, can be expressed from (3.5.3) as

$$f_{i:n}(x) = \sum_{j=0}^{n-i}\frac{(-1)^j n!}{(i-1)!\,(n-i-j)!\,j!} f(x)F(x)^{j+i-1}$$

$$= \sum_{j=0}^{n-i} \frac{(-1)^j n!}{(i-1)!(n-i-j)!j!} \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}} e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}} \left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{\beta-1}}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2} \left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{\beta(i+j-1)} \quad (3.5.4)$$

Equation (3.5.4) can further be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \frac{(-1)^j n!}{(i-1)!(n-i-j)!j!} \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}} e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}} \left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{\beta(i+j)-1}}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2} \quad (3.5.5)$$

Hence, the pdf of the minimum order statistic $X_{1:n}$ and maximum order statistic $X_{n:n}$ of the Odd Generalized Exponential-Inverse-Exponential distribution are respectively given by

$$f_{1:n}(x) = \frac{n!}{(n-1)!} f(x)[1-F(x)]^{n-i} = nf(x)[1-F(x)]^{n-1} =$$

$$\sum_{j=0}^{\infty} \binom{n-1}{j} (-1)^j f(x)F(x)^j \quad (3.5.6)$$

Substituting equations (3.2.3) and (3.2.4) into (3.5.6), we have

$$f_{1:n}(x) = \sum_{j=0}^{\infty} \binom{n-1}{j} (-1)^j \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}} e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}} \left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{\beta(1+j)-1}}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2} \quad (3.5.7)$$

And for the maximum, equation (3.5.1) reduces to

$$f_{n:n}(x) = \frac{n!}{(n-1)!} f(x)F(x)^{n-1} = nf(x)F(x)^{n-1} \quad (3.5.8)$$

Therefore, substituting equations (3.2.3) and (3.2.4) into (3.5.8), lead to

$$f_{n:n}(x) = \frac{n\alpha\beta\lambda e^{-\frac{\lambda}{x}} e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}} \left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right)^{n\beta-1}}{x^2\left(1-e^{-\frac{\lambda}{x}}\right)^2} \quad (3.5.9)$$

3.6 Estimation of Parameters of the OGE-IED

We make use of the method of maximum likelihood estimation (MLE) to estimate the parameters of the OGE-IED. Let $x_1, \cdots x_n$be a random sample of size n from the OGE-IED with parameter vector $\varpi = (\alpha, \beta, \lambda)'$. Then log-likelihood function for ϖ is given by;

$$l(\varpi) = logL(x_1, \cdots x_n \backslash \varpi) = log \prod_{i=1}^{n} f(x_i \backslash \varpi)$$

$$== nln\alpha + nln\beta + nln\lambda - \lambda \sum_{i=1}^{n} \frac{1}{x_i} + 2\sum_{i=1}^{n} \frac{1}{x_i} - 2\sum_{i=1}^{n} ln\left(1-e^{\frac{-\lambda}{x}}\right) - \alpha \sum_{i=1}^{n} \left(1-e^{\frac{-\lambda}{x}}\right)$$

$$+(\beta-1)\sum_{i=1}^{n} ln\left(1-e^{\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-1}}\right) \quad (3.6.1)$$

This log-likelihood function will result to a system of non-linear likelihood equations which cannot be solved analytically due to its complexity but it can be maximized numerically by using global optimization methods available with software's like SAS, R, and Mathematica or by solving the non-linear likelihood equations obtained by differentiating (3.6.1).

Meanwhile, differentiating $l(\varpi)$ partially with respect to each parameter α, β and λ and setting the results equal to zero gives the maximum likelihood estimates of the respective parameters. The partial derivatives of $l(\varpi)$ with respect to each parameter are respectively given by:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(1 - e^{\frac{-\lambda}{x_i}}\right) + (\beta - 1)\sum_{i=1}^{n}\frac{\left(e^{\frac{\lambda}{x_i}-1}\right)^{-1} e^{-\alpha\left(e^{\frac{\lambda}{x_i}-1}\right)^{-1}}}{\left(1-e^{-\alpha\left(e^{\frac{\lambda}{x_i}-1}\right)^{-1}}\right)} \tag{3.6.2}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} ln\left(1 - e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}}\right) \tag{3.6.3}$$

and

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\frac{1}{x_i} - 2\sum_{i=1}^{n}\frac{e^{\frac{-\lambda}{x_i}}}{x_i\left(1 - e^{\frac{-\lambda}{x_i}}\right)}$$

$$-\alpha\sum_{i=1}^{n}\frac{e^{\frac{-\lambda}{x}}}{x_i} - \alpha(\beta - 1)\sum_{i=1}^{n}\frac{e^{\frac{-\lambda}{x_i}}\left(e^{\frac{\lambda}{x_i}-1}\right)^{-2} e^{-\alpha\left(e^{\frac{\lambda}{x_i}-1}\right)^{-1}}}{x_i\left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}}\right)} \tag{3.6.4}$$

Maximization of equation (3.6.1) can be performed by using well established routines like the nlm routine or optimize in the R statistical package. Setting these equations to zero and solving them simultaneously yields the maximum likelihood estimate (MLE) $\widehat{\varpi} = \left(\hat{\alpha}, \hat{\beta}, \hat{\lambda}\right)'$ of$(\alpha, \beta, \lambda)'$.

Note: In this work, we employed Newton-Raphson maximization in R statistical package to estimate the parameters of the proposed distribution using the method of maximum likelihood.

CHAPTER FOUR: ANALYSIS AND DISCUSSION

4.1 Introduction

In this chapter, we illustrated the performance of the Odd Generalized Exponential-Inverse-Exponential distribution by fitting real dataset to the model and its baseline model (Inverse-exponential distribution) together with some of the baseline generalizations.

4.2 Real data application

Reliability data-set is used to assess the performance of the OGE-IED among some classical and recent continuous distributions based on a set of goodness-of-fit tests. The compared distributions are:

Inverse-Exponential distribution (IED)

The pdf of the ***IED*** distribution is given as;

$$f(x) = \frac{\lambda}{x^2} e^{-\frac{\lambda}{x}} \tag{4.2.1}$$

Transmuted Inverse Exponential Distribution (TIED)

The pdf of the ***TIED*** distribution is given as;

$$f(x) = \frac{\lambda}{x^2} e^{-\frac{\lambda}{x}} \left(1 + \alpha - 2\alpha e^{-\frac{\lambda}{x}}\right) \tag{4.2.2}$$

Generalized Inverse Exponential Distribution (GIED)

The pdf of the ***GIED*** distribution is given as;

$$f(x) = \frac{\alpha\lambda}{x^2} e^{-\frac{\lambda}{x}} \left(1 - e^{-\frac{\lambda}{x}}\right)^{\alpha-1} \tag{4.2.3}$$

Exponentiated Generalized Inverse Weibull Distribution (EGIWD)

The pdf of the ***EGIWD*** distribution is given as;

$$f(x) = \alpha\beta\theta\lambda^{\theta} e^{-\left(\frac{\lambda}{x}\right)^{\theta}} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^{\theta}}\right)^{\alpha-1} \left[1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)^{\theta}}\right)^{\alpha}\right]^{\beta-1} \quad x, \alpha, \beta, \theta, \lambda > 0 \tag{4.2.4}$$

Kumaraswamy Inverse-Exponential Distribution (KI-ED)

The pdf of the ***KI-ED*** distribution is given as;

$$f(x) = \frac{ab\lambda}{x^2} e^{-\frac{a\lambda}{x}} \left(1 - e^{-\frac{a\lambda}{x}}\right)^{b-1} \quad x, a, b, \lambda > 0 \tag{4.2.5}$$

Odd Generalized Exponential Inverse-Exponential Distribution (OGE-IED)

The pdf of the ***OGE-IED*** distribution is given as;

$$f(x) = \frac{\alpha\beta\lambda e^{-\frac{\lambda}{x}}}{x^2\left(1 - e^{-\frac{\lambda}{x}}\right)^2} e^{-\alpha\left(\frac{e^{-\frac{\lambda}{x}}}{1 - e^{-\frac{\lambda}{x}}}\right)} \left(1 - e^{-\alpha\left(\frac{e^{-\frac{\lambda}{x}}}{1 - e^{-\frac{\lambda}{x}}}\right)}\right)^{\beta-1} \tag{4.2.6}$$

4.2.1 Dataset

The data used is obtained from Lee and Wang, 2003, and widely reported in many literatures. It represents the Remission times (in months) of 128 patients of Bladder

cancer and it is used by Elbatal and Muhammed (2014), Dar *et al*., (2017) and many more. The dataset can be found in **Table** 1 below.

Table 1: Remission times (in months) of 128 patients of Bladder cancer (Lee and Wang, 2003)

0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69

Table 2: Summary of the dataset from Lee and Wang, (2003)

Parameters	N	Q_1	Median	Q_1	Mean	Variance	Skewness	Kurtosis
Value	128	3.348	6.395	11.840	9.366	110.425	3.286569	18.48308

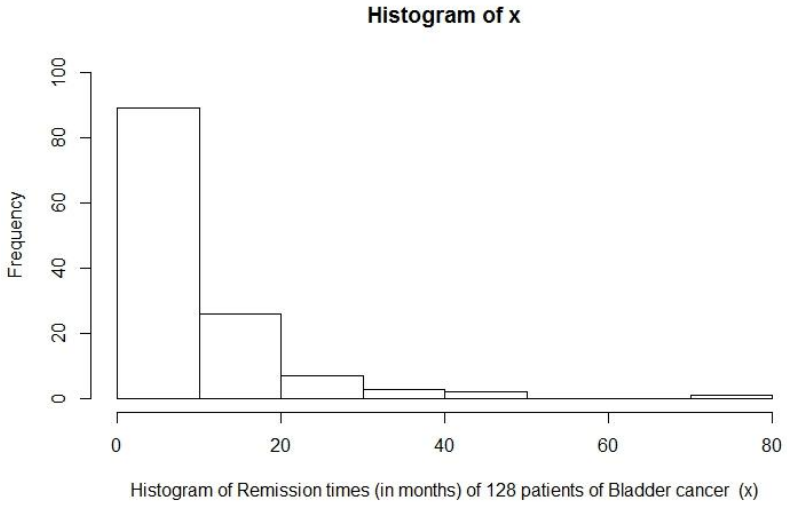


Figure 3: Histogram of Remission times (in months) of 128 patients of Bladder cancer

Both the summary and the histograms for the dataset described that the dataset is right skewed and therefore suitable for skewed distributions which the OGE-IED is not an exception.

In comparing the performance of our proposed distribution with other distribution, the following criteria are considered: the Akaike Information Criterion (***AIC)***, Consistent Akaike Information Criterion (***CAIC***) and Bayesian Information Criterion (***BIC***). Below are their respected statistics:

$$AIC = -2L + 2k \qquad (4.2.7)$$

$$BIC = -2L + klog\,(n) \qquad (4.2.8)$$

and

$$CAIC = -2L + \frac{2kn}{n-k-1} \qquad (4.2.9)$$

where L denotes the log-likelihood function evaluated using the MLE's, k is the number of model parameters and n is the sample size.

Note: The model with the lowest values for these statistics is chosen as the best model to fit the dataset.

Table 3: MLE's of the parameters

The Model	MLE's of the parameter(s)
$\boldsymbol{IED(\lambda)}$	$\hat{\lambda} = 2.4847$
$\boldsymbol{TIED(\lambda, \alpha)}$	$\hat{\lambda} = 1.67708, \hat{\alpha} = -0.85950$
$\boldsymbol{GIED(\lambda, \alpha)}$	$\hat{\lambda} = 1.99445, \hat{\alpha} = 0.74629$
$\boldsymbol{EGIWD(\lambda, \alpha, \beta, \theta)}$	$\hat{\lambda} = 1.006, \hat{\alpha} = 2.000, \hat{\beta} = 1.050, \hat{\theta} = 0.500$
$\boldsymbol{KIED(\lambda, \alpha, \beta)}$	$\hat{\lambda} = 1.4167, \hat{\alpha} = 1.40776, \hat{\beta} = 0.74630$
$\boldsymbol{OGE - IED(\lambda, \alpha, \beta)}$	$\hat{\lambda} = 0.09390, \hat{\alpha} = 0.01121, \hat{\beta} = 1.17986$

The values of log-likelihood functions (L), Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), and the Consistent Akaike Information Criteria (CAIC) are given in **Table 4** for the models in order to verify which of the distributions best fit the dataset.

Table 4: Model Performance Comparisons

The Model	Measures			
	L	AIC	$CAIC$	BIC
$IED\ \lambda$	-460.3823	922.7646	926.958	926.0862
$TIED\ \lambda, \alpha$	-442.7767	889.5534	891.7469	891.8750
$GIED\ \lambda, \alpha$	-457.2024	918.4048	920.5980	920.7264
$EGIWD\ \lambda, \alpha, \theta, \beta$	-488.0460	984.0920	982.2855	995.5000
$KIED\ \lambda, a, b$	-457.2024	918.4048	920.5980	920.7264

$OGE-IED\ \lambda,\alpha,\beta$	**-412.9466**	**831.8932**	**832.0867**	**832.2148**

4.3 Results discussion

Table 3.3 shows parameter MLEs to each one of the fitted distributions for the data set. While **Table 3.4** indicates the corresponding values of***L***, ***AIC***, ***CAIC*** and ***BIC*** for each model. It is enough from the table to show that Odd Generalized Exponential Inverse-Exponential distribution (OGE-IED) has better performance than the baseline distribution and some of its extensions. Therefore it could be selected as the best model compared to the other five (5) distributions involved in the performance comparisons for fitting the same data set. Since this data is highly skewed, we can equally say that this distribution is an appropriate model for analyzing right skewed distributed data sets.

For the dataset, we estimated the parameters of each distribution by the maximum-likelihood method, and we used the log-likelihood L to obtain the values of Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC). The results obtained are presented in **Tables 3.3** and **3.4**. As we can see, the smallest values of the AIC, BIC and CAIC are obtained for the OGE-IED. Hence, we conclude that the OGE-IED provides the best fit among the compared distributions.

CHAPTER FIVE: SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

In this dissertation, we introduced a new three parameter probability model named the Odd Generalized Exponential-Inverse-Exponential distribution (OGE-IED), which is a hybridization of Generalized Exponential distribution with Inverse-Exponential distribution by adding skewness to the classical Inverse-Exponential distribution. An obvious reason for generalizing a classical distribution is the fact that the generalization provides more flexibility to the new distribution which will enable it to capture more information contained in the real life data that cannot be assessed by the classical distribution. Some of its statistical properties like its asymptotic behaviour, moments, moment generating function, quantile function, reliability analysis and order statistics are studied. The parameter estimates has been determined by the method of maximum likelihood. The usefulness of the OGE-IED has been illustrated by an application to real

data set. The new distribution provides better fits compared to the baseline distribution and four of its generalizations. However, the results showed that the new distribution is more flexible and appropriate for modelling right skewed data sets.

5.2 Conclusion

A new three parameter probability model named the Odd Generalized Exponential-Inverse-Exponential distribution (OGE-IED) was developed. Some statistical properties of the proposed distribution have been studied appropriately and derived explicit expressions for its moments, moment generating function, quantile function, median, survival function, hazard function, ordered statistics and limiting behaviour. Some plots of the distribution revealed that the distribution is also positively skewed and has only one mode. The model parameters were estimated using the method of maximum likelihood. The implications of the plots for the survival and hazard functions indicate that the OGE-IED would be appropriate in modelling time-dependent events, where survival and failure rate decreases with time.

5.3 Recommendations

Based on the findings of this research, (i) the proposed distribution should be used for modelling lifetime events if the data set in question is right skewed (ii) this distribution can be used confidently in modelling time-dependent events, systems, components or random variables.

5.4 Contribution to Knowledge

A three parameter distribution which is useful for modelling right skewed data sets better than some existing compound inverse-exponential based probability distributions has been proposed.

5.5 Areas for Further Research

Based on the review in this area of research, interested researchers can look at estimation of confidence intervals of the proposed distribution parameters. Also, researchers can estimate the parameters of the new distribution using Bayesian approaches for the purpose of theoretical comparison and validation of methodology.

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APPENDIX

The following are the R-codes of the plotted pdf, CDF, survival and hazard functions respectively

R-code for pdf plot

```
> c1<-function(x, a, b, c){(a*b*c*exp(-(c/x))*exp(-a*(1/(exp(c/x)-1)))*(1-exp(-a*(1/(exp(c/x)-1))))^(b-1))/(x^2*(1-exp(-(c/x)))^2)}

c2<-c1(x, a=1,b=0.5,c=1)

plot(x,c2,type="n",xlab = "x",ylab = "f(x)")

> lines(x,c1(x,a=0.5,b=1,c=1),col="red",lwd=3,lty=1)

> lines(x,c1(x,a=0.75,b=1,c=1),col="blue",lwd=3,lty=1)

> lines(x,c1(x,a=1,b=1,c=1),col="black",lwd=3,lty=1)

> lines(x,c1(x,a=1,b=1.25,c=1),col="green",lwd=3,lty=1)
```

```
> lines(x,c1(x,a=1,b=1.5,c=1),col="magenta",lwd=3,lty=1)

> lines(x,c1(x,a=1.5,b=1,c=1),col="pink",lwd=3,lty=1)

>legend("topright",cex=1, title="key",
c("a=0.5,b=1,c=1","a=0.75,b=1,c=1","a=1,b=1,c=1","a=1,b=1.25,c=1","a=1,b=1.5,c=1","a=1.5,b=1
,c=1"),horiz=F, lty=c(1,1,1,1,1,1), lwd=c(2,2,2,2,2,2),
col=c("red","blue","black","green","magenta","pink"), bg="grey96")
```

R-code for CDF plot

```
c3<-function(x, a, b, c){(1-exp(-a*(1/(exp(c/x)-1))))^b}

> c2<-c3(x,a=1,b=0.5,c=1)

> plot(x,c2,type="n",xlab = "x",ylab = "F(x)")

> lines(x,c3(x,a=0.5,b=1,c=1),col="red",lwd=3,lty=1)

> lines(x,c3(x,a=0.75,b=1,c=1),col="blue",lwd=3,lty=1)

> lines(x,c3(x,a=1,b=1,c=1),col="black",lwd=3,lty=1)

> lines(x,c3(x,a=1,b=1.25,c=1),col="magenta",lwd=3,lty=1)

> lines(x,c3(x,a=1,b=1.5,c=1),col="green",lwd=3,lty=1)

>legend("bottomright",cex=1, title="key",
c("a=0.5,b=1,c=1","a=0.75,b=1,c=1","a=1,b=1,c=1","a=1,b=1.25,c=1","a=1,b=1.5,c=1"),horiz=F,
lty=c(1,1,1,1,1),lwd=c(3,3,3,3,3), col=c("red","blue","black","magenta","green","pink"),
bg="grey96")
```

R-code for Survival function plot

```
c4<-function(x, a, b, c){(1-(1-exp(-a*(1/(exp(c/x)-1))))^b)}

> c2<-c4(x,a=1,b=0.5,c=1)

> plot(x,c2, main="Survival of OGE-IED",type="n",xlab = "x(Time)",ylab = "H(x)")

> lines(x,c4(x,a=0.5,b=1,c=1),col="red",lwd=2,lty=1)

> lines(x,c4(x,a=0.75,b=1,c=1),col="blue",lwd=2,lty=1)

> lines(x,c4(x,a=1,b=1,c=1),col="black",lwd=2,lty=1)
```

```
> lines(x,c4(x,a=1,b=1.25,c=1),col="magenta",lwd=2,lty=1)

legend("topright",cex=1, title="key",
c("a=0.5,b=1,c=1","a=0.75,b=1,c=1","a=1,b=1,c=1","a=1,b=1.25,c=1","a=1,b=1.5,c=1","a=1.5,b=1
,c=1"),horiz=F, lty=c(1,1,1,1),lwd=c(2,2,2,2),
col=c("red","blue","black","magenta","green","pink"), bg="grey96")
```

R-code for Hazard function plot

```
c4<-function(x, a, b, c){((a*b*c*exp(-(c/x))*exp(-a*(1/(exp(c/x)-1)))*(1-exp(-a*(1/(exp(c/x)-
1))))^(b-1))/(x^2*(1-exp(-(c/x)))^2))/(1-exp(-a*(1/(exp(c/x)-1))))^b}

c5<-function(x, a, b, c){(a*b*c*exp(-(c/x))*exp(-a*(1/(exp(c/x)-1)))*(1-exp(-a*(1/(exp(c/x)-
1))))^(b-1))/(x^2*(1-exp(-(c/x)))^2)/(1-(1-exp(-a*(1/(exp(c/x)-1))))^b)}

c2<-c5(x,a=1,b=0.5,c=1)

plot(x,c2,type="n",xlab = "x-values",ylab = "Hazard Function")

> lines(x,c5(x,a=0.5,b=1,c=1),col="red",lwd=2,lty=1)

> lines(x,c5(x,a=1,b=1,c=1),col="blue",lwd=2,lty=1)

> lines(x,c5(x,a=1.5,b=1,c=1),col="black",lwd=2,lty=1)

> lines(x,c5(x,a=.5,b=.5,c=1),col="magenta",lwd=2,lty=1)

> lines(x,c5(x,a=.5,b=1.5,c=1),col="green",lwd=2,lty=1)

> lines(x,c5(x,a=.5,b=2,c=1),col="pink",lwd=2,lty=1)

legend("topright",cex=1, title="key",
c("a=0.5,b=1,c=1","a=1,b=1,c=1","a=1.5,b=1,c=1","a=.5,b=.5,c=1","a=.5,b=1.5,c=1","a=.5,b=2,c=
1"),horiz=F, lty=c(1,1,1,1),lwd=c(2,2,2,2), col=c("red","blue","black","magenta","green","pink"),
bg="grey96")
```

The following are R-codes and results of the data analysis.

m

R-code for OGE-IED

```
> ab<-function(p) {n*log(p[1])+n*log(p[2])+n*log(p[3])-p[3]*sum(1/x)+sum(log(1/x^2))-
2*sum(log(1-exp(-(p[3]/x))))-p[1]*sum((exp(-(p[3]/x)))/(1-exp(-(p[3]/x))))+(p[2]-1)*sum(log(1-
exp(-p[1]*((exp(-(p[3]/x)))/(1-exp(-(p[3]/x)))))))}
```

```
> a<-maxLik(ab,start=c(1,1,1))
There were 50 or more warnings (use warnings() to see the first 50)
> summary(a)
```

Output:

```
Maximum Likelihood estimation
Newton-Raphson maximisation, 9 iterations
Return code 2: successive function values within tolerance limit
Log-Likelihood: -412.9466
3  free parameters
Estimates:
     Estimate Std. error t value  Pr(> t)
[1,]  0.01121    0.02373   0.472    0.637
[2,]  1.17986    0.16488   7.156 8.31e-13 ***
[3,]  0.09390    0.20159   0.466    0.641
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
--------------------------------------------
> AIC(a)
[1] 831.8932
```

R-code for GIED

```
> ac<-function(p){n*log(p[1])+n*log(p[2])-p[2]*sum(1/x)+(p[1]-1)*sum(log(1-exp(-
(p[2]/x))))+sum(log(1/x^2))}
> a<-maxLik(ac,start=c(1,1))
> summary(a)
```

Output:

Maximum Likelihood estimation

Newton-Raphson maximisation, 6 iterations

Return code 2: successive function values within tolerance limit

Log-Likelihood: -457.2024

2 free parameters

Estimates:

Estimate Std. error t value Pr(> t)

[1,] 0.74629 0.08834 8.448 < 2e-16 ***

[2,] 1.99445 0.27056 7.372 1.68e-13 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

> AIC(a)

[1] 918.4048

R-code for IED

> am<-function(p) n*log(p[1])+sum(log(1/x^2))-p[1]*sum(1/x)

> a<-maxLik(am,start=c(1))

> summary(a)

Output:

Maximum Likelihood estimation

Newton-Raphson maximisation, 5 iterations

Return code 2: successive function values within tolerance limit

Log-Likelihood: -460.3823

1 free parameters

Estimates:

Estimate Std. error t value Pr(> t)

[1,] 2.4847 0.2201 11.29 <2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

> AIC(a)

[1] 922.7646

R-code for KIED

> ad<-function(p) {n*log(p[1])+n*log(p[2])+n*log(p[3])-2*sum(log(x))-p[1]*p[3]*sum(1/x)+(p[2]-1)*sum(log(1-exp(-(p[1]*p[3]/x))))}

> a<-maxLik(ad,start=c(1,1,1))

There were 50 or more warnings (use warnings() to see the first 50)

> summary(a)

Output:

Maximum Likelihood estimation

Newton-Raphson maximisation, 5 iterations

Return code 2: successive function values within tolerance limit

Log-Likelihood: -457.2024

3 free parameters

Estimates:

Estimate Std. error t value Pr(> t)

[1,] 1.40776 4.25758 0.331 0.741

[2,] 0.74630 0.08829 8.452 <2e-16 ***

[3,] 1.41676 4.28468 0.331 0.741

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

> AIC(a)

[1] 920.4048

R-code for TIED

```
> ae<-function(p) {n*log(p[1])-2*sum(log(x))-p[1]*sum(1/x)+sum(log(1+p[2]-2*p[2]*exp(-(p[1]/x))))}
> a<-maxLik(ae,start=c(1,1))
> summary(a)
```

Output:

Maximum Likelihood estimation

Newton-Raphson maximisation, 9 iterations

Return code 1: gradient close to zero

Log-Likelihood: -442.7767

2 free parameters

Estimates:

```
     Estimate Std. error t value Pr(> t)
[1,]  1.67708    0.17246   9.725  <2e-16 ***
[2,] -0.85950    0.07725 -11.126  <2e-16 ***
---
```

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

> AIC(a)

[1] 889.5534