ON STABILITY AND PROPOSITIONAL FUNCTIONS IN THE THEORY OF GENERALIZED DYNAMICAL SYSTEMS

BY

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DECLARATION

I, ABDULRAHEEM ABDULRAZAQ, declare that this thesis is written by me and it is a record of my own research work. It has never been presented in any form for a higher degree in any Institution or University.

All informations derived from the published and unpublished work of others have been aknowledged in the text and in the references.

CERTIFICATION

This thesis entitled "ON STABILITY AND PROPOSITIONAL FUNCTIONS IN THE THEORY OF GENERALIZED DYNAMICAL SYSTEMS" by Abdulraheem Abdulrazaq meets the regulation governing the award of the degree of MASTER OF SCIENCE IN MATHEMATICS of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This thesis is dedicated to:

(i) The memory of my late father. May the Almighty Allah grant him eternal peace.

(ii) my mother for her effort towards my successes. May the Almighty Allah grant her long life and good health.

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ABSTRACTS

In this thesis we study some stability concepts for generalized pseudo-dynamical systems under different considerations as well as some semi-stability of motions and limit sets in semi—systems, and dynamical semi—systems.

In chapter one, we give the general introduction to the concept of stability in the theory of generalized dynamical systems.

In chapter two, starting from stability of sets we present a stability concept for generalized pseudo—dynamical systems. The Lyapunov function and its bearing on stability is also discussed. Every Lyapunov function must be sufficient for the stability of the set (trajectory *or* motion) for which it is constructed.

In chapter three, we introduce a propositional function and give a definition of the stability of sets in terms of the propositional function. Based on this we are able to form a link among various types of stability concepts in the theory of pseudo—dynamical systems.

We also touch on the proofs of some theorems which hinge on the construction of Lyapunov functions for some types of stability and some of their applications.

In chapter four, limiting ourselves to semi-systems and dynamical semi—systems on metric spaces, we study regular dependence of limit sets on points moving semi—stable in semi—systems and dynamical systems.

Certain results on limit sets are presented for dynamical semi—systems, which enable us to study some connections between semi-stability conditions for motions and semi-continuity of limit sets, as well as semi—stability of motions and non—emptiness of some limit sets.

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CHAPTER I

PRELIMINARIES

1.1 INTRODUCTION

The theory of dynamical systems has as its background the qualitative theory of autonomous differential equations with unique solutions. The generalization of the theory to systems without uniqueness has been developed and is known as generalized dynamical systems.

The theory of dynamical semi-systems is an aspect of the theory of dynamical systems taken to the right or to the left. Pseudo-dynamical systems and pseudo-dynamical semi-systems are generalizations of dynamical systems and dynamical semi-systems respectively.

The theory of stability is a special aspect of the classical or generalized dynamical systems which began with respect to solutions of differential equations and then gave birth to dynamical systems and their generalizations.

The classicial theory of stability began with Lyapunov, Poincare and Laplace, and is now treated in general versions as stability of sets. This paves way for the formulation of the concept of stability in different forms for dynamical systems and their generalizations.

The basic feature of the theory of stability by

A.M. Lyapunov seeks to characterize stability and asymptotic stability of a given set or motion in terms of a non-negative scalar function defined on a neighbourhood of a given set and decreasing along its trajectories.

Properties of stability could be some consequences of properties of limit sets or other properties of motions for large time. In particular, regular dependence of limit sets on given points could be established if motions of the points are regular. Therefore, it is pertinent to mention that there is a connection between semi-continuity and semi-stability of some mappings satisfying some basic conditions.

1.1.1 Dynamical Systems and generalized dynamical Systems

The study of differential equations has made use of abstract mathematics to a large extent. For instance, Lie invented groups named after him to solve problems arising in differential equations. Fourier invented series to study heat equations. So also did Poincare when he invented algebraic topology for the same purpose, and thereby introduced the qualitative study of solution curves of differential equations. By drawing an anology between solution curves of differential equations and algebraic equations, Poincare' systematically studied the relationship between all the solutions of differential equations.

This abstract idea made it possible to define a

dynamical system on a lspace X as a triple (X,R;π) comprising a topological space X called the state or phase space, map π called the dynamics or flow or phase map and time space R such that

(1.1.2) $\pi: R \times X \longrightarrow X$

satisfying the following axioms;

(a) $\pi(o,x) = x$ For all $x \in X$,

(1.1.3) (b) $\pi(t,\pi(s,x)) = \pi(t+s,x)$ for $t,s \in R$, $x \in X$,

(c) π is continuous

Conditions (1.1.3)(a) and (1.1.3)(b) are called the identity and group axioms respectively and condition (1.1.3)(c) is the continuity axiom.

In order to define a dynamical system on Euclidean n - space $X = R^n$, consider the autonomous differential equation

(1.1.4) $\frac{dx}{dt} = x = f(x)\ ,\ x(o) = x_o$

where $f: R^n \longrightarrow R^n$ is continuous. Moreover, assume that for each $x \in R^n$ a unique solution $\varphi(t,x)$ of (1.1.4) exists and satisfies $\varphi(o,x) = x$. Uniqueness of solutions implies $\varphi(t, \varphi(s,x)) = \varphi(t+s,x)$ for $t,s \in R$. Considering φ as a function from $R \times R^n \longrightarrow R^n$, φ is continuous. Then the map $\pi: R \times R^n \longrightarrow R^n$ such that $\pi(t,x) = \varphi(t,x)$ defines a dynamical system on R^n.

Specifically, let $f(x) = x$. Then

(1.1.5) $x' = x\ ,\ x(o) = x_o$.

The solution of the differential equation (1.1.5) is given by

$$\varphi(t,x) = x_o e^t$$

The map $\varphi(t,x) = \pi(t,x) = x_o e^t$ is continuous. Hence the triple $(R^n, R; \pi)$ is a dynamical system since in addition to the continuity of the map π,

(i) $\pi(o,x) = x_o e^o = x_o$ for every $x_o \in R^n$,

(iii) $\pi(t, \pi(s,x)) = \pi(t, x_o e^s)$

$$= x_o e^s . e^t$$

$$= x_o e^{s+t}$$

Also $\pi(t+s,x) = x_o e^{s+t}$

That is all the anxioms in (1.1.3) are satisfied.

The next example is a dynamical system defined by the solution curves of a system of differential equations in plane and given by

$$(1.1.6) \qquad x' = y \quad , x(o) = x_o$$

$$y' = -x \quad , y(o) = y_o$$

The solution of the system (1.1.6) is given by

$$(1.1.7) \qquad \begin{aligned} x(t) &= y_o \sin t + x_o \cos t \\ y(t) &= y_o \cos t - x_o \sin t \end{aligned}$$

By introducing the natural projection

(1.1.8) $p: R^3 \ni (t,x,y) \longrightarrow (x,y) \in R^2$

we have that

$$p(t,x,y) = (x,y) = (y_o \sin t + x_o \cos t,\ y_o \cos t - x_o \sin t)$$

where p, being a projection map is continuous. By putting $p(t,x,y) = \pi(t,(x,y))$ we define the continuous map

$$\pi: R \times R^2 \ni (t, (x,y)) \longrightarrow \pi(t, (x,y)) \in R^2$$

Now we show that π satisfies axioms (1.1.3)(a) and (1.1.3)(b) also.

(i) $\pi(o,(x,y)) = (x_o,y_o)$ for all pair $(x,y)\in R^2$

(ii) $\pi(t,\pi(u,(x,y))=\pi(t,y_o \sin u + x_o \cos u,\ y_o \cos u - x_o \sin u)$

Putting $y_o \sin u + x_o \cos u = a$

$y_o \cos u - x_o \sin u = b.$

$\pi(t,(a,b)) = (b\sin t + a\cos t,\ b\cos t - a\sin t)$

$= [(y_o \cos u - x_o \sin u)\sin t + (y_o \sin u + x_o \cos u)\cos t,$

$(y_o \cos u - x_o \sin u)\cos t - (y_o \sin u + x_o \cos u)\sin t]$

$= [(y_o \cos u \sin t + y_o \sin u \cos t + x_o \cos u \cos t - x_o \sin u \sin t,$

$(y_o \cos u \cos t - y_o \sin u \sin t - x_o \sin u \cos t - x_o \cos u \sin t]$

$= (y_o \sin(u+t) + x_o \cos(u+t), y_o \cos(u+t) - x_o \sin(u+t))$

And

$\pi(u+t,\ (x,y)) = (y_o \sin(u+t) + x_o \cos(u+t),\ y_o \cos(u+t) - x_o \sin(u+t))$

These prove that $(R^2, R; \pi)$ is a dynamical system.

Let us investigate the axioms (1.1.3) on the solution curve of the non-autonomous differential equation

(1.1.9) $\quad x' = 2tx \quad , \ x(o) = x_o$

the solution of which is given by

(1.1.10) $\quad \varphi(t,x) = x_o\, e^{t^2}$

By putting $\varphi(t,x) = \pi(t,x)$ we have

(i) $\pi(o,x) = x_o$ for every $x\in R^n$

(ii) $\pi(t,\ \pi(s,x)) = \pi(t,\ x_o e^{s^2})$

$= x_o e^{s^2}.e^{t^2}$

$= x_o e^{s^2+t^2}$

And $\pi(t+s,x) = x_o\, e^{(t+s)^2}$

That is $\pi(t,\pi(s,x)) \neq \pi(t+s,x)$. Hence the group axiom

does not hold. This implies that the non-autonomous differential equation (1.1.9) is not a dynamical system.

In relation to the state space and time space of a dynamical system we shall obtain generalizations of dynamical systems.

A general dynamical system is defined if the state space is a topological space and an arbitrary abelian group G instead of the set R of reals. A general dynamical system whose time space is a topological group is called a continuous dynamical systems or a topological transformation group. So also do time spaces which are cyclic groups and partially ordered groups convert a general dynamical system into discrete and partially ordered dynamical systems respectively.

A system in which the state space is an arbitrary set X gives rise to generalized pseudo-dynamical systems. Here the continuity of the phase map does not hold.

1.2 THEORETICAL JUSTIFICATIONS OF THE CONCEPT OF STABILITY

1.2.1 The Concept of Stability

Mathematical equations that describe physical phenomena are, in most cases, ordinary differential equations of the form.

1.2.2 $X^{1} = f(t,x)$

with the initial conditions

1.2.3 $x(t_o) = x_o$

Since the initial conditions, which often result from all types of measurements, may have errors, it is important to know the extent to which small disturbances in the initial condition (1.2.3) affect the desired behaviour of the solution of the relation (1.2.2). The problem of investigating the conditions that will not allow the solutions to remarkably deviate from the desired behaviour is known as stability theory.

Stability theory, in a strict sense, is concerned with the existence of a solution x(t) of (1.2.2) for all time t>0 and its behaviour as t -->00. That is to say stability theory seeks to determine whether every solution $\bar{x}$ (t) which are sufficiently close to x(t) at time $t = t_o$ remains close to it at a larger time. There are many properties which may be used to describe such a behaviour.

The classical theory of stability with respect to the solutions of differential equations is due to Lyapunov, when he published in his work on the stability of motions in 1892, and considered stability in two different ways; the first of which is known as stability in Lyapunov sense and the second is called Lyapunov direct method.

1.2.4 Stability in the sense of Lyapunov

Let f(t,x) be continuous in a set S and let S be

open at the right of time t_o. A solution $x(t)$ of (1.2.3) at time t which has an initial state $x(t_o) = x_o$ is stable in Lyapunov sense at x_o if given a real positive constant ϵ, there is $\delta(\epsilon)$ such that if $\|x_o\| < \delta$ then $|x(t,x_o,t_o)| < \epsilon$ for all $t > t_o$.

In geometrical terms, this means that x_o is stable if it is possible to find a hyperspherical region $R(\delta)$ such that all trajectories starting within $R(\delta)$ remain within a hyperspherical region $R(\epsilon)$ for all time. In this case, x_o is called on equilibrium point.

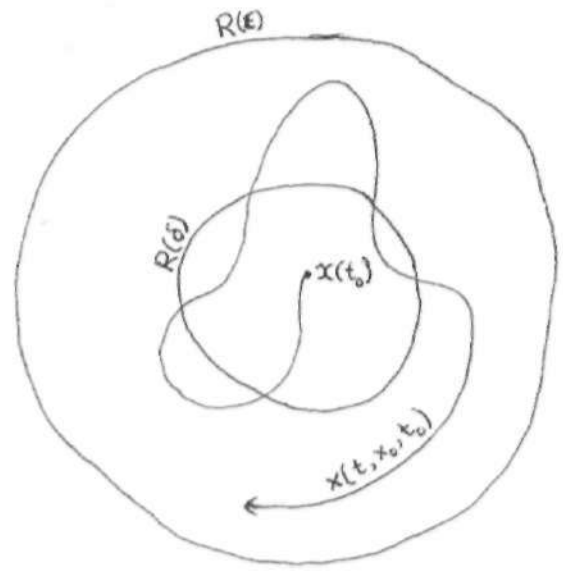


Fig. 1.1

The point x_o is said to be asymptotically stable in Lyapunov sense if it is stable and there exists a positive real number δ such that if $\|x_o\| \leq \delta$ then $x(t,x_o,t_o) \dashrightarrow 0$ as $t \dashrightarrow \infty$.

Geometrically, this means

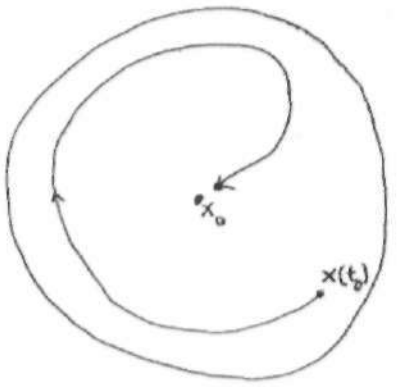


Fig.1.2

The stability in the Lyapunov sense involve the knowledge of an explicit solution of (1.2.2). Also they are somewhat weak in the sense that they do not allow stability to be presented under small disturbances. Hence they are limited in application when investigating the behaviours of physical systems.

1.2.5 Lyapunov's Direct Method

In order to remedy the inadequacy of the Lyapunov first method, we shall introduce a completely different technique to determine the stability behaviour of solutions of (1.2.2), and is known as Lyapunov's direct method. The major advantage of this method over the other is that the stability in the large can be obtained without prior knowledge of the solution of (1.2.2). Also the stability of non-linear state equations can be investigated since solutions of non-linear differential equations are very rarely obtainable.

Just as we have stated earlier in section one,

Lyapunov's direct method is characterized by the construction of scalar functions known as Lyapunov functions.

1.2.6 The Lyapunov function

'Let us consider the autonomous form of equation

(1.2.2)

1.2.7 $x^{I} = f(x)$

1.2.8 $x(o) = 0$

Here the origin is an isolated critical point, and f is smooth enough to ensure the existence and uniqueness of the solutions of (1.2.7).

Let us consider a function V(x), real and continuous in some neighbourhood of the points $x = (x_1, x_2, \ldots, x_n)$. And in view of stating basic stability theories of Lyapunov we shall introduce the concepts of definite and semi-definite of the scalar function V(x).

Definition 1.2.9

A function V(x) is said to be positive (negative) semi-definite if $V \geqslant 0$ ($V \leq 0$) in some finite set S and $V(0) = 0$.

Definition 1.2.10

A function V(x) is said to be positive (negative) definite if $V > 0$ ($V < 0$) in some finite set S and $V(0) = 0$.

Definition 1.2.11

Let W(x) be a scalar function which is independent of time t. A function V(t,x) is said to be positive (negative) definite if there exists a positive definite function
W(x) such that V > W (- V > W) in some set S.

Having introduced the concepts of definite and semi-definite of the state variables, we shall now define the Lyapunov functions.
Given a system $\frac{dx^2}{dt^2} + \frac{dx}{dt} + c = 0$ we can obtain

a system of two first order differential equations from it as follows:

$$\frac{dx_1}{dt} = F(x_1,x_2)$$

1.2.12

$$\frac{dx_2}{dt} = G(x_1,x_2)$$

Let $V(x_1, x_2)$ be a function defined in the region where the system (1.2.12) is defined and connected with the system in the following manner;

$$dV = \frac{\partial V}{\partial x_1} dx_1 + \frac{\partial V}{\partial x_2} dx_2$$

so that

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial V}{\partial x_2} \frac{dx_2}{dt}$$

$$= \frac{\partial V}{\partial x_1} F(x_1,x_2) + \frac{\partial V}{\partial x_2} G(x_1, x_2)$$

then $V(x_1, x_2)$ will be a Lyapunov function if

(a) V is positive definite

(b) $\frac{dV}{dt}$ is negative semi-definite, and

(c) the first partial derivatives of V are continuous in the region.

1.2.13 The statements of basic Lyapunov theorems are given in section three (under general definitions).

Although Lyapunov, who introduced this method in 1892, used the method to establish simple stability concepts, his basic ideas have been improved upon and extensively applied to entirely new problems in physics and engineering.

The basic theories have been, in recent years, put into forms which have been proved as both necessary and sufficient conditions for stability. The problem of determining a Lyapunov function for a system for which the origin is known to be stable is called inverse problem.

Once a Lyapunov function has been found in some region around the origin, it becomes possible to test the stability or asymptotic stability or instability of the zero solution of a given system.

An important question, therefore, is the

procedure to be adopted in selecting or constructing a Lyapunov function as it is sometimes very difficult to find a proper Lyapunov function for a given system. But in general, there is no satisfactory technique that provides the answer, particularly for non-linear non-autonomous systems.

However, Lyapunov's direct method is widely recognized as an excellent tool not only in the study of differential equations but also in the theory of control systems, theory of dynamical systems and so on.

The following concepts are associated with dynamical systems;

Let $(X, G; \pi)$ be a dynamical system. The mapping (π_t), $t \in G$ defined by

(1.2.14) $\pi_t : X \ni x \longrightarrow \pi_t(x) := \pi(t,x) \in X$

for every $t \in G$, is called the translation given by the time t.

And the mapping (π^x), $x \in X$ defined by

(1.2.15) $\pi^x : G \ni t \longrightarrow \pi^x(t) := \pi(t,x) \in X$

for every $x \in X$

is called the motion of the point x, or motion through the point x.

1.3.0 GENERAL DEFINITIONS

In this section we give general definitions that are basic to the study.

Definition 1.3.1

(1) Let X be a non-empty set, (G,+) an abelian semi-group with the neutral element o, and π a mapping

π: GxX ----> X

The triple (X, G; π) is called a semi-system if and only if.

(a) π (O.x) = x for every x$\in$X

(b) π(t, π(s,x)) = π(t+s,x) for t, s$\in$G, x$\in$X

(2) The triple (X,G;π) is called a dynamical semi-system if and only if (1.3.1)(a) and (1.31(b) hold and (c) π is continuous.

Definition 1.3.2

(1) Let X be a non-empty set, (G,+) a group and π as given in defintion (1.3.1). The triple (X,G; π) is called a dynamical system if and only if relations (1.3.1)(a) (1.3.1)(b) and (1.3.1) (c) hold.

(2) The classical theory of dynamical systems was developed with respect to the case X = R^n, G = R (that is Euclidean real space and real line respectively). The corresponding dynamical semi-stystems (That is G = R = (t$\in$R: t $\geq$ 0) are called semi-dynamical systems.

Definition 1.3.3

Let X be a non-empty set, (G,+) an abelian semi-group having the neutral element O and A a subset of G x X such that

1.3.4) $\{0\} \times X \subset A$

and finally, let λ be a mapping defined by

(1.3.5) $\lambda : (t,x) \in A \longrightarrow \lambda(t,x) \in P(X)$

where P(X) *denote the collection of all non-empty* subsets of the set X (also called the power set of the space X).

For $x \in X$, we put

(1.3.6) $I_x := (t \in G: (t,x) \in A)$

I_x is non-empty since $0 \in I_x$.

In view of the relations (1.3.4) to (1.3.6) we say that the quadruple $(X, G, A; \lambda)$ is a generalized local pseudo-dynamical system (shortly; GLPD - system) if and only if

(a) $\lambda(0,x) = (x)$ for every $x \in X$

(b) if $t \in I_x$, $y \in \lambda(t,x)$, $S \in I_y$ then $t + s$ ($= s+t$) $\in I_x$, and

(c) $\lambda(s,y) \subset \lambda(t+s,x)$ for every $x \in X$, $t \in I_x$, $y \in \lambda(t,x)$ $s \in I_y$

Definition 1.3.7

Let $(X,G,A;\lambda)$ be a GLPD - system. We put

(a) $\lambda_t(x) = \lambda(t,x): (t,x) \in A$

(b) $\lambda(x) = \bigcup \{\lambda(t,x): (t \in I_x)$ and $x \in X\}$

(c) $\lambda(B) = \bigcup \{\lambda(x): x \in B$ and $B \subset X\}$

$\lambda(x)$ is called the trajectory of the point x, and

Definition 1.3.13

A sufficient condition for the origin to be asymptotically stable is that there exists a scalar function V in some region R satisfying the conditions (1.3.12) (a) (1.3.12)(c) and (1.3.12)(d), and

(e) $\frac{dv}{dt}$ is negative definite.

Definition 1.3.14

A sufficient condition for the orgin to be globally aymptotically stable is that there exists a scalar function

V in some region R satisfying

(a) $V(x) > 0$ in the entire space

(b) $\frac{dv}{dt} < 0$ in the entire space

(c) $V(x) < \infty$ if $\|x\| < \infty$

(d) $V(x) \longrightarrow \infty$ if $\|x\| \longrightarrow \infty$

(1.3.14) is still valid if (b) is weakened.

Definition 1.3.15

An equilibrium point is unstable if there exists a function V(x) such that

(a) there exists at least one point in R in which $V(x) > 0$

(b) $\frac{dv}{dt} > 0$ in R.

Definition 1.3.16

Let M be a Hausdorff space and G a topological group such that to each $g \in G$ is associated a

CHAPTER 2

STABILITY THEORY IN GENERALIZED PSEUDO-DYNAMICAL SYSTEMS

In this chapter we discuss some ideas of stability concepts for generalized pseudo-dynamical systems. In doing this we limit ourselves to simple cases only.

In section 1, we give two important definitions of stability of sets, and thereafter put the two definitions together as a necessary and sufficient conditions in a therem, the proof of which is produced. In section 2 we discuss Lyapunov function as a sufficient condition for the stability of the set which we consider; while in section 3 we discuss the inverse problem of those discussed in section 2.

2.1 The following assumptions are pertinent.

Let us consider a GLPD - system $(X, G, A; \lambda)$ fixed. Let Ω be a subfamily of $P(X)$, β a normal mapping and M a non-empty subset of X.

Definition 2.1.1

We say that M fulfils the condition $S(\Omega,\beta;\lambda)$ or $M \in S(\Omega,\beta;\lambda)$ or that M is (Ω,β) - stable with respect to λ if and only if for every $Q \in \Omega$ and every $x \in M$ there exists a set $B \in \beta(x)$ such that

2.1.2) $y \in B \implies \lambda(y) \subset Q$

Definition 2.1.3

If $\mathcal{A}$ and $\mathcal{B}$ are two subfamilies of P(X), then we define $\mathcal{A} \succ \mathcal{B}$ as follows;

(2.1.4) $\mathcal{A} \succ \mathcal{B}$:= (for every $B \in \mathcal{B}$ there is $A \in \mathcal{A}$ such that

$A \subset B$)

Remark 2.1.5

$A \succ B$ is known as set inclusion property and is read as "B is stronger than A".

Definition 2.1.6

We say that M fulfils the condition L $(\Omega, \beta; \lambda)$ or $M \in L$ $(\Omega, \beta; \lambda)$ if and only if there is a family $\mathcal{A} \in P(X)$ such that

(2.1.7) $\beta(y) \succ \mathcal{A}$ for every $y \in M$ (that is for every $A \in \mathcal{A}$ there is a set $B \in \beta(y)$ such that $B \subset A$)

(2.1.8) $A \succ \Omega$ (that is $A \subset Q$ for $A \in \mathcal{A}$ and $Q \in \Omega$)

(2.1.9) every set $A \in \mathcal{A}$ is λ- invariant.

Remark 2.1.10

Definitions 2.1.1) and (2.1.6) agree in the sense that they both restrict the trajectories contained in them to small neighbourhoods. In other words, the two stability concepts are the stability in the small or local stability.

The compatibility of definitions (2.1.1) and (2.1.6) gives rise to the following theorem.

Theorem 2.1.11

For every subfamily Ω of P(X) and every normal

map β, the conditions S $(\Omega, \beta; \lambda)$ and L $(\Omega, \beta; \lambda)$ are equivalent.

Proof

Suppose that $M \in L(\Omega, \beta; \lambda)$. That is to say M fulfils the condition $L(\Omega, \beta; \lambda)$. Then by the relation (2.1.7) there exists a set $B \in \beta(y)$ for every $y \in M$ and $A \in \mathcal{A}$ such that $B \subset A$.

Also by the relation (2.1.8) there exists $A \in \mathcal{A}$ and $Q \in \Omega$ such that $A \subset Q$. Hence $B \subset A$ and $A \subset Q$ implies $B \subset Q$.

The next step is to show that if $Z \in B$, then $\lambda(z) \subset Q$. Suppose $Z \in B \subset A$. Since relation (2.1.9) implies that A is invariant, then $\lambda(z) \subset B \subset A$. This is $\lambda(z) \subset Q$. Which implies that M is (Ω, β) - stable with respect to λ.

Now assume that $M \in S(\Omega, \beta; \lambda)$. For any subset D of X we define

(2.1.12) $E(D) = (z \in X: \lambda(z) \subset D)$

and

(2.1.13) $\mathcal{E} = (E(Q): Q \in \Omega)$

We shall show that by putting $\mathcal{A} = \mathcal{E}$ we shall obtain a family fulfilling the conditions (2.1.7), (2.1.8) and (2.1.9).

Let $x \in M$ and $Q \in \Omega$ be given. Since $M \in S(\Omega, \beta; \lambda)$ we can find $B \in \beta(x)$ such that $y \in B \Longrightarrow \lambda(y) \subset Q$, which then means that B is invariant and belongs to E(Q). Thus relation (2.1.7) is satisfied.

where

(2.2.5) $A_\eta = (y \in W: V(y) \leq \eta)$

fulfils the conditions (2.1.7), (2.1.8) and ~~(2.2.6)~~ (2.2.6) for every $y \in W$ and $t \in I_y$ the inequality

$V(w) \leq V(y)$ holds true for every $w \in \lambda(t,y)$.

Remark 2.2.7

The map V in definition (2.2.2) is a Lyapunov function as it agrees with the followings;

(a) For any $y \in W$, $V(y)$ is continous on W.

(b) By construction, V is a scalar - valued function, and partial ordering of T gurantees the positive definiteness of V.

(c) P ensures that V has a positive minimum on W.

That is $V(\lambda(t_0,x)) \leq V(y)$ for $t > t_0$. Then

$V(\lambda(t_0 x)) - V(y) < 0$

so that $V(x) \leq 0$. In other words V decreases along its trajectories.

The following theorem emphasises the view on the Lyapunov direct method. That is, knowing the Lyapunov function for a set in a region, the stability property of the set can be described.

Theorem 2.2.8

If a set $M \subset X$, $M \neq \emptyset$, is such that there exists a Lyapunov $(W, P; \Omega, \beta)$ - function for M (with some invariant subset W of the space X and some partial ordered space T, where $P = T \setminus (\inf T)$ or $P=T$) then

M fulfils the condition S (Ω, β; λ).

Proof

Let V be a Lyapunov (W, P; Ω, β) - function for M. This means that there exists a family $\mathcal{A} = (A_\eta : \eta \in P)$ such that for every $A_\eta \in \mathcal{A}$ there exists $B \in \beta(y)$ and $B \subset A$. Also for every $\Omega \in \Omega$ there exists $A_\eta \in \mathcal{A}$ such that $A_\eta < \Omega$.

Next we show that every $A_\eta \in \mathcal{A}$ is invariant. Let $\mathcal{A}$ be defined as above. Then by relation (2.2.6) let z be an arbitrary point of A_η. For every $t \in I_z$ and every $w \in \lambda(t,z)$ we have $V(w) \leq V(z) \leq \eta$. Hence $w \in A_\eta$. Then $\lambda(t,z) \subset A_\eta$ for every $t \in I_z$ and consequently $\lambda(z) \subset A_\eta$. This means that A_η is invariant. Hence M fulfils the condition L (Ω, β, λ). And by Theorem (2.1.11), M fulfils the condition S (Ω, β, λ) which is what we set out to prove.

Remark 2.2.9

In theorem (2.1.11) it is not possible to describe the stability of the set M unless we know the nature and scope of the trajectories of the points in M. While for the set M in theorem (2.2.8), the existence of Lyapunov function in any region in M is enough to describe the stability of the set M.

It is evident from the poof of theorem (2.2.8) that if $\phi \neq M \subset X$, is such that there exists a Lyapunov (W, P; Ω, β) - function for M, then M fulfils the condition L (Ω, β; λ) as well.

Remarks 2.2.10

Every Lyapunov function constructed must be such that it satisfies the conditions that make it sufficient for the stability of the set for which it is constructed.

In classical dynamical systems theory the phase group is the set of reals, which is also endowed with a natural ordering relation. This provides us with the concepts of positive semi-trajectory and so on. But for generalized dynamical systems the phase group is topological transformation group where natural ordering relation is not available. Therefore, there is a problem in direct lifting of results from dynamical systems theory to topological dynamics. Hence we begin to examine partial ordering in transformation groups.

The partial ordering considered in theorem (2.2.8) enables us to define our Lyapunov function V, and the removal of infimum of T is to make way for the extension of the real line.

2.3 In this section we discuss the converse of the main issue discussed in the last section (that is section 2) and the following assumptions are pertinent.

2.3.1 Let the partial ordered space T be such that for every S, a subset of T, $S \neq \phi$, there exists infS and

put as in the previous section

$$P = \begin{cases} T \setminus (\inf T) & \text{if } \inf T \text{ exists} \\ T & \text{if } \inf T \text{ does not exist} \end{cases}$$

Further we assume the following;

2.3.2 For each $\lambda \in P$, there exists $\mu \in P$ such that $\mu < \lambda$; for every non-empty subset S of T and every $p \in P$ such that $\inf S < p$, there exists $\eta \in S$ such that $\eta \leq p$.

2.3.3 Suppose that $\Omega = (\Omega_\eta)_{\eta \in P}$ and if $\eta, \lambda \in P$ and $\eta \leq \lambda$, then $\Omega_\eta \subset \Omega_\lambda$.

Furthermore, let β be a normal mapping and M a non-empty subset of X.

Theorem 2.3.4

Under the assumptions (2.3.1) to (2.3.3), if M is (Ω, β) - stable with respect to λ, there exists an invariant set W such that for every $x \in M$ there exists $B \in \beta(x)$ contained in W, and there exists a Lyapunov $(W, P; \Omega, \beta)$ - function for M.

Proof

Let us consider $\sigma \in P$ arbitrarily fixed such that $\Omega = (\Omega_\eta), \eta \in P$. Let us put

(2.3.5) $W = (x \in X: \lambda(x) < \Omega_\sigma)$

We show that for every $x \in M$ there exists $B \in \beta(x)$ contained in W, and W is invariant.

Let $x \in M$ be fixed. Since $M \in S(\Omega, \beta; \lambda)$ there is a set $B \in \beta(x)$ such that $y \in B \Longrightarrow \lambda(y) < \Omega$. This means that $B \subset W$, and in particular $M \subset W$.

If $x \in W$ and $t \in I_x$, then for every $w \in \lambda(t,x)$ we have $\lambda(s,w)$ for $s \in I_w$. This means that for every $w \in \lambda(t,x)$ it implies that

$\lambda(w) \subset \lambda(x) \subset Q_\sigma$; thus $w \in W$.

But this means finally that $\lambda(x) \subset W$ and this completes the proof of the invariance of W.

Now we construct the Lyapunov function based on definition (2.2.2). Let us put for $x \in W$;

(2.3.6) $V(x) = \inf \{\eta \in P: \lambda(x) \subset Q_\eta\}$

and

$A_\eta = \{y \in W : V(y) \leq \eta\}$

Then we show that V(x) is a Lyapunov function. We achieve this by showing that the family $\mathcal{A} = (A_\eta)$ fulfils the conditions (2.1.7), (2.1.8) and (2.2.6).

Let Q be fixed. Consider $\eta^\circ \in P$ such that $\eta^\circ < \eta$. We show that $A_{\eta^\circ} \subset Q_\eta$ and this establishes the relation (2.1.8).

If $y \in A_{\eta^\circ}$, then

$V(y) = \inf \{\mu \in P: \lambda(y) \subset Q_\mu\} \leq \eta^\circ$. We can choose $\sigma \in P$ such that $\sigma \leq \eta$ (see condtion (2.3.2), and $\lambda(y) \subset Q_\sigma$. This, however, gives $y \in \lambda(y) \subset Q_\sigma \subset Q_\eta$, and then $y \in Q_\eta$. That is $A_{\eta^\circ} \subset Q_\eta$.

Now we prove that V(x) satisfies the relation (2.1.7).

Let $y \in M$ and A be fixed. We consider

$\mu = \inf (\eta, \sigma)$

and the set $A\mu$ is contained in A_η. By virtue of the assumption (2.3.2) it follows that $\mu \in P$. For μ and y

we can choose $B \in \beta(y)$ such that $\lambda(x) \subset Q\mu$ for $x \in B$. Since $Q\mu \subset Q_\sigma$, then $\lambda(x) \subset Q\mu$, for $x \in B$, implies that $B \subset W$ and then the values of $V(x)$ are well defined for $x \in B$. Moreover we have $V(x) \leq \mu$ for $x \in B$. The above condition implies that $B \subset A\mu \subset A_\eta$. In this way relation (2.1.7) has been verified.

In order to prove relation (2.2.6), we consider $y \in W$, $t \in I_y$ and $w \in \lambda(t,y)$; we have

$V(x) = \inf (\eta \in P: \lambda(w) \subset Q_\eta) \leq \inf \{ \eta \in P: \lambda(y) \subset Q_\eta = V(y)$. That is $V(x) \leq V(y)$, and the proof is finished.

Remarks 2.3.7

The partially ordered space $(T, \leq)$ with respect to the relation "$\leq$" by implication is a total ordering. As a matter of fact, the assumption (2.3.1) which was assumed in theorem (2.3.4) is totally ordered.

Remark 2.3.8

Fixing a point $x \in M$ in the proof of theorem (2.3.4) is a matter of convenience since β, being a normal map means that every element $x \in M$ belongs to each $B \in \beta(x)$.

Remarks 2.3.9

In the definition (2.2.2), $A_\eta = \{y \in W; V(y) \leq \eta\}$ for $\eta \in P$ and $P = T \setminus (\inf T)$. But in theorem (2.3.4) the partially ordered space T is such that every subset S of T, $S \neq \emptyset$, has an infimum. Hence $V(x) = \inf (\eta \in P: \lambda(x) \subset Q_\eta)$.

Conclusion 2.3.10

Based on the definitions of $S(\Omega,\beta;\lambda)$ and $L(\Omega,\beta;\lambda)$, and suitable assumptions we have shown that some set M satisfies the two stability concepts.

Under new assumptions we have shown that the two stabilities are equivalent.

Based on the definition of a Lyapunov function used in this thesis we have shown that some set M fulfils condition $(S(\Omega,\beta;\lambda)$. On the other hand where a set M is given to satisfy the condition $S(\Omega,\beta;\lambda)$ and adding other suitable conditions we have shown that there exists a Lyapunov $(W,P;\Omega,\beta)$ function for set M.

CHAPTER 3

QUANTIFIERS IN GENERALIZED LOCAL PSEUDO-DYNAMICAL SYSTEMS

In this chapter we continue with our discussion on the stability concepts for generalized pseudo-dynamical systems. However, much as the main issue is still on the stability of generalized pseudo-dynamical systems the approach in this chapter is different from that of the last chapter.

In section one we give a general definition of the stability of sets in terms of a propositional function, and then we consider some types of stability entrenched in the general definition.

In section two, we construct a sub family of the power set of the space X, modify our propositional function and define conditions equivalent to the definition of general stability formula mentioned in section one. In other words, we generalize the classical Lyapunov stability which we define in section one using the propositional functions for the set S.

In section three we present a general stability of the Lyapunov type by constructing Lyapunov functions in the theorems to be stated. Thereafter we give an application of the theorems in section four.

3.1 Let $(X, G, A; \lambda)$ be a generalized local pseudo-dynamical system, and

(3.1.1) $\beta: X \longrightarrow P(P(X))$

a mapping on X. Let $\Omega \subset P(X)$, where $\Omega \neq \emptyset$ and $M \in P(X)$.

Remarks 3.1.2

If X is a metric space and $M \in P(X)$, then we may put

(3.1.3) $\beta: X \ni x \longrightarrow (K(x,\epsilon): \epsilon>0) \in P(P(x))$

where $K(X,\epsilon)$ is an open neighbourhood of x and

(3.1.4) $\Omega = (K(M,\epsilon): \epsilon>0)$

where

(3.1.5) $K(B,\epsilon) = (y \in X: p(y, B) < \epsilon)$

and

(3.1.6) $K(x,\epsilon) = K(x), \epsilon)$

for every $x \in X$ and $B \subset X$

The open neighbourhood $K(x,\epsilon)$ and open set $K(M,\epsilon)$ are illustrated in figures 3.1 and 3.2 respectively;

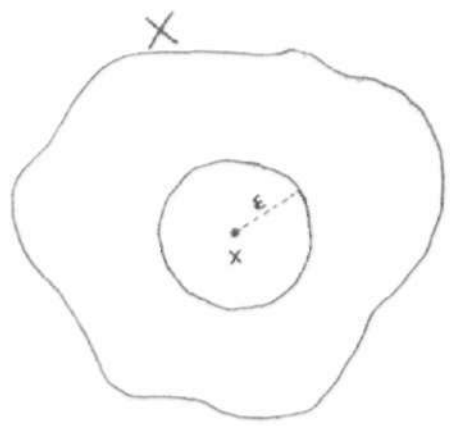


Fig 3.1

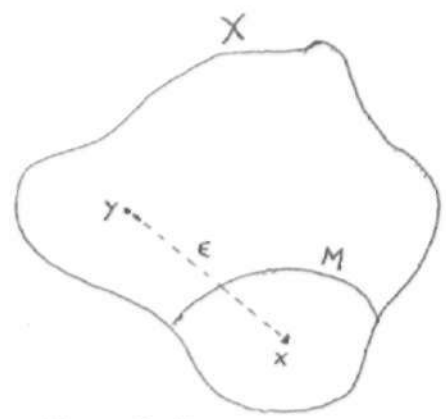


fig. 3.2

Then we put

(3.1.7) $\bar{\beta} = \bigcup(\beta(x): x \in X)$

The general definition of the stability of the set X to be given in terms of a propositional function, and the operations on the propositional function are presented in table 3.3 below.

The table consists of the variables, their domains and quantifiers. We denote each variable by a Greek letter and the corresponding quantifiers (existential and universal by the corresponding roman lower and upper letters respectively.

Variable	Domain	Existential Quantifier	Universal Quantifier
ω	Ω	o	O
ζ	M	z	Z
δ	β	d	D
η	X	h	H
∇	G	n	N
α	A	a	A

Table 3.3

We read the existential quantifier O as

"there exists $w \in \Omega$"

and the universal quantifier D is read a

"for every $\delta \in \beta$"

The expression

(3.1.8) $\delta\in\beta(\int)\underset{\delta}{\supset}[\eta\in\delta\underset{\eta}{\supset}(\forall\in I_\eta\underset{\forall}{\supset}\lambda(\lambda_\forall(\eta))\ \subset w))$,

which will be denoted by Φ is a propositional function in the five variables w, $\int$,δ,η and $\forall$.

Let i stand for any of the variables in table 3.3. Then the symbol $\underset{i}{\supset}$ in the formula (3.1.8) denotes $\wedge$ (= conjunction) if it is within the scope of the corresponding existential quantifier and denotes ===> (=implication) otherwise.

With the above explanation, formular (3.1.8) is understood to mean

(3.1.9) $\delta\in\beta(\int)\Longrightarrow[\eta\in\delta\wedge(\forall\in I_\eta\Longrightarrow\lambda(\lambda_\forall(\eta)\subset w)]$

In view of what is stated in (3.1.9) any sequence of letters from the third and fourth columns of table 3.3 forms a word. Example of such words are oZ, dD, OdZHN, and so on. If a word contains only one letter from each row, we say that the word is feasible. So, words like OZDHn and DhZOn are feasible words that can be formed from table 3.3. The class of feasible words is denoted by $\mathcal{S}$.

Let $S\in\mathcal{S}$ and let us replace the letters of S by the quantifiers. If we put this sequence of five quantifiers before the formular Φ we obtain a formular without the free variables, and will be denoted by $S\Phi$.

In view of the above explanations we now give the general stability formula:

Definition 3.1.10

The set M is (S,Ω,β) - stable if and only if the formula $S\Phi$ is true.

3.1.11 Streamlining defintion (3.1.10) with definition (2.1.1),definition (3.1.10) can be restated as:

The set M is (S,Ω,β) - stable if and only if for every $x\in M$ and every $w\in(K(M,\epsilon) : \epsilon>o) = \Omega$ there exists open neighbourhood $K(x,\delta)$ such that $y\in K(x,\delta)$ and $t\in I_y$ such that $\lambda(\lambda_y(t)) \subset w$.

Definition (3.1.10) combines different types of stability. We give some example. In the examples, X is a metric space, $(X, R_* A;\lambda)$ is GLPD - system and $\Omega=\{K(M,r);\ r>o\}$

For examples (3.1.14) and (3.1.15) we put

(3.1.12) $\beta: X \ni x \longrightarrow (K(x,r): r > 0)) \in P(P(X))$

and in example (3.1.18) we put

(3.1.13). $\beta: X \ni x \longrightarrow (K(M,r): r > o) \in P(P(X))$.

Example 3.1.14

The classical definition of stability states that a set M is Lyapunov stable or stable in Lyapunov sense if and only if for every $x\in M$ and $\epsilon > o$ there exists $\delta>0$ such that $\lambda(y) \subset K(M,\epsilon)$ for every $y\in K(x,\delta)$.

Using the quantifiers as given in table 3.3 above the classical Lyapunov stability of the set M is equvalent to the $(ZOdHN,\Omega,\beta)$ - stability of the set M. And this is illustrated in figure (3.4) below;

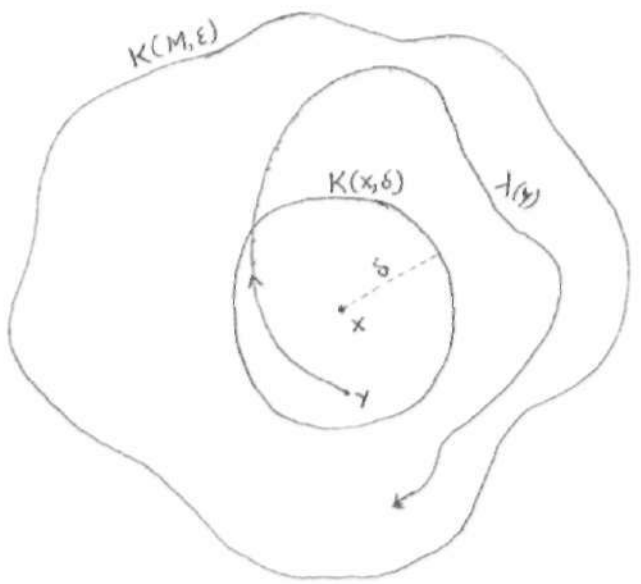


Fig 3.4

Example 3.1.15

Let us consider the weak stability of the set M. A set M is weakly stable if and only if for every $x \in M$ and $\epsilon > 0$ there exists $\delta > 0$ such that for every $y \in K(x,\delta)$ $\lambda(\lambda_s(y)) \subset K(M,\epsilon)$ for the same $s \in I_y$.

By using the quantifiers in table (3.3) the above stability is the $(ZOdHn, \Omega, \beta)$ - stability of the set M. The idea is shown in figure (3.5) below.

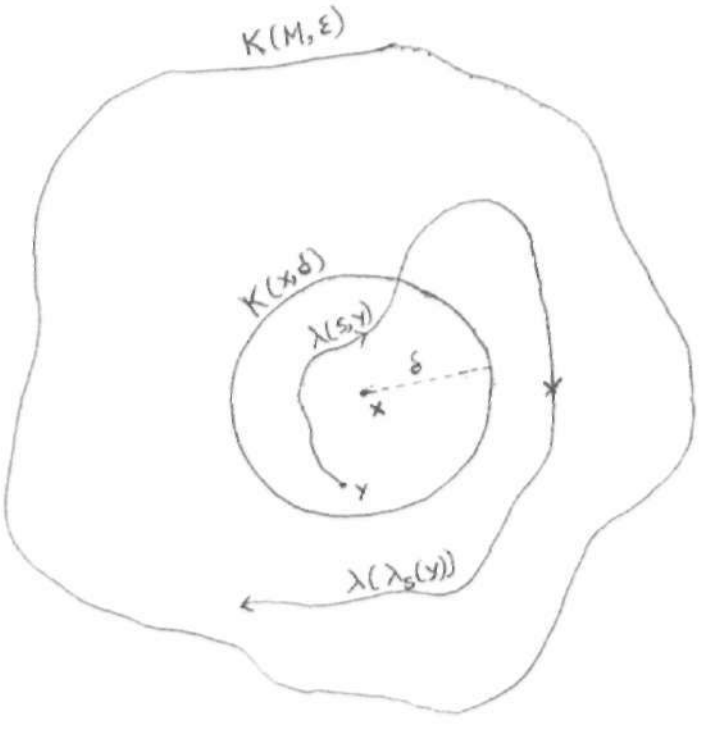


Fig 3.5

Before we give the third example we state the following definitions.

Definition 3.1.16

A point $y \in X$ is called a positive (or Omega) limit point of the point $x \in X$ if there exist a sequence of time (t_n), $t_n \in R_+$, with $t_n \longrightarrow +\infty$ and $(t_n, x) \longrightarrow y$. The set of all positive (Omega) limit points of a point $x \in X$ is called the positive (Omega) limit set of x and is denoted by $\Lambda^+(x)$.

Negative (alpha) limit point is defined similarly with t_n ---> - o0 and (x, t_n) ---> y. And the negative limit set of a point x is denoted by $\Lambda^-(x)$.

Definition 3.1.17

A(M) is called region of attraction for the set M if and only if for every $x \in X$, $\Lambda^+(x) \neq \emptyset$ and $\Lambda^+(x) \subset M$. The set M is called an attractor if there is an $\epsilon > o$ such that for all $x \in K(M,\epsilon)$, $\Lambda^+(x) \neq \emptyset$ and $\Lambda^+(x) \subset M$.

Example 3.1.18

We suppose that the mapping $\lambda = \pi$ gives the global pseudo-dynamical system (X, R_*, π). We say that the set M is an attractor if and only if there exists $\delta > o$ such that $\lim_{t \to o0} p(\pi(t,y),M) = 0$ for every $y \in K(x,\delta)$.

By applying the quantifiers in table (3.3), the above is equivalent to $(\exists d H O n, \Omega, \beta)$ - stability of M. The concept is shown in figure (3.6) below;

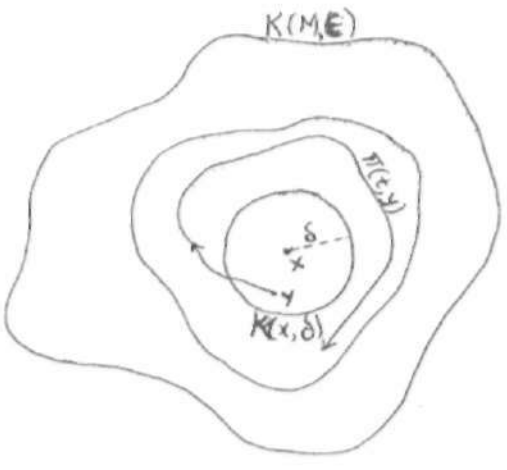


Fig 3.6

Remarks 3.1.19

In the example (3.1.15) if we put the quantifier "there exists $s \in R_*$" in different places we obtain different types of Lyapunov stability. For example, if $s \in I_z$ where $z \in M$, $z \notin K(x,\varepsilon)$ such that $\lambda(\lambda_s(y) \subset K(M,\epsilon)$ we obtain a different stability concept.

3.2 In view of generalizing the classical Lyapunov stability for sets in our generalized pseudo-dynamical systems in terms of the word S we add one variable α, the domain of which is $\mathcal{A} \subset P(X)$, $\mathcal{A} \neq \emptyset$, a and A are the corresponding existential and universal quantifiers respectively.

Let us introduce the words S_1 and S_2 associated with the word S. In order to make the word S_1 we put a in place of o and A replaces O in the word S. The word S_2 is made in the following manner;

(3.2.1) $S_2 = \begin{cases} Oa \text{ if O occurs in the word S} \\ Ao \text{ if o occurs in the word S} \end{cases}$

As an illustration of S_1 and S_2 consider the word S = ZOdHN.

Associated with S are S_1 = ZAdHN and S_2 = Oa.

Let us take the formulas P_1, P_2, P_3;

$$P_1: \delta \in \beta(\xi) \underset{\delta}{\supset} [\eta \in \beta \underset{\eta}{\supset} (\nu \in I_\eta \underset{\nu}{\supset} \lambda_\nu(\eta) \subset \alpha)]$$

(3.2.2) $P_2: \alpha \subset w$

$P_3: \forall \alpha \in A$, α is invariant.

Then, in accordance with our convention, the propositional function P_1 has the feasible word S_1 before it, and P_2 has S_2 in front of it while P_3 remains the same. Then we have the following;

$$\begin{array}{ll} & (a)\ \ S_1P_1 \\ (3.2.3) & (b)\ \ S_2P_2 \\ & (c)\ \ P_3 \end{array}$$

We state and prove the main theorem of this section. The theorem considers relation (3.2.3) to be equivalent to the (S,Ω,β) - stability of the set M. But first we prove the following lemma.

Lemma 3.2.4

Suppose that there exists a family $\mathcal{C} \subset P(X) \cup \{\phi\}$ = (Y: Y⊂X) such that all the assumptions of relation (3.2.3) are true, then there exists a family $\mathcal{A} \subset P(X)$, $\mathcal{A} \neq \phi$, such that the relation (3.2.3) is also true.

Proof

Let us assume that "0" occurs in the word S. Then S_2 = 0a. From the assumptions of the Lemma the domains of the variables are not empty. (refer to table (3.3) and note that $\alpha \in \mathcal{A}$).

Since $\mathcal{C}$ fulfils the conditions of (3.2.3) we have from (3.2.3) (a) that $\alpha \neq \phi$ for every $\alpha \in \mathcal{C}$. Also by virtue of (3.2.3) (b) the family $\mathcal{C}$ is not empty so that we can put $\mathcal{A} = \mathcal{C}$ in this case. Condition

(3.2.3)(c) follows from (3.2.3)(a).

Now suppose that "o" occurs in the word S so that S_2 = Ao. Then for every $\alpha \in \mathcal{A}$ there exists w, and by virtue of (3.2.3)(a), $\alpha \neq \phi$ and $\alpha \in \mathcal{B}$. By putting $\mathcal{A} = (\alpha \in \mathcal{B} : \alpha \neq \phi)$, (3.2.3)(b) implies that A is not empty and $\mathcal{A} \subset P(X)$. This implies that $\mathcal{A} \subset \mathcal{B}$. And lastly (3.2.3)(c) follows from (3.2.3) (a).

Theorem 3.2.5

Let $M \in P(X)$. Then M is (S, Ω, β) - stable if and only if there exists a non-empty subfamily $\mathcal{A} \subset P(X)$ such that the relation (3.2.3) is true.

Proof

First assume that there exists a family $\mathcal{A} \subset P(X)$, $\mathcal{A} \neq \phi$, which fulfils the relation (3.2.3). Let "O" occur in the word S so that S_2 = Oa. Then for every $w \in \Omega$ there exists $\alpha \in \mathcal{A}$. Since the quantifiers in front of O are in S and S_1, we make the corresponding choices up to the position of O in the formula $S\Phi$ as in (3.2.3)(a). That is, in place of A in the word S_1 we put O. By virtue of condition (3.2.3)(b) we have that $\alpha \in w$.

Also by condition (3.2.3)(c) we have that w is invariant. So that when we put the above in (3.2.3)(a) we obtain the formula $S\Phi$, which is (S, Ω, β) - stability of M.

Next we check whether the choice we made is

correct. Let us consider the formula

P_1: $\delta \in \beta(\xi) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} (\nu \in I_\eta \underset{\nu}{\supset} \lambda_\nu(\eta) \subset \alpha)]$

We may write $\lambda_\nu(\eta) \subset \alpha$ in the following form;

$\forall\ y \in \lambda_\nu(\eta),\ y \in \alpha$

condition (3.2.3)(c) implies that α is invariant and then we have $\forall\ y \in \lambda_\nu(\eta),\ \lambda(y) \subset \alpha$ or $\lambda(\lambda_\nu(\eta)) \subset \alpha$

And as $\alpha \subset w$, we have $\lambda(\lambda_\nu(\eta) \subset w$. Thus we obtain Φ, where

$\Phi = \delta \in \beta(\xi) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} (\nu \in I_\eta \underset{\nu}{\supset} \lambda(\lambda_\nu(\eta)) \subset \omega)]$

and then we have $S\Phi$, which proves the case "0" occurs in S.

Now let us suppose that "o" occurs in the word S. Then S_2 = Ao. This means that for every $\alpha \in \mathbb{A}$ there exists $w \in \Omega$, and by condition (3.2.3)(b) $\alpha \in W$. The invariance of α means that w is also invariant. Since the quantifiers are the same in the words S and S_1 up to the position of o, we replace a in S_1 by o so that we obtain S. And by putting the above in S_1P_1 we obtain the formula Φ which is (S, Ω, β) - stability.

In order to check whether the choice is correct, consider the formula

P_1: $\delta \in \beta(\xi) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} (\nu \in I_\eta \underset{\nu}{\supset} \lambda_\nu(\eta) \subset \alpha)]$

And we can write $\lambda_\nu(\eta) \subset \alpha$ as

$y \in \lambda_\nu(\eta),\ y \in \alpha$

By invariance of α, we have $\lambda(y) \subset \alpha$ so that

$\lambda(\lambda_\nu(\eta)) \subset \alpha$.

As $\alpha \subset w$, we have

$\Phi = \delta \in \beta(\xi) \underset{\delta}{\supset} [\eta \in \delta \underset{\eta}{\supset} (\nu \in I_\eta \underset{\nu}{\supset} \lambda(\lambda_\nu(\eta)) \subset \omega)]$

And so $S\Phi$ is true. Hence, the existence of the family $\mathcal{A}$ satisfying the relation (3.2.3) is sufficient for the (S,Ω,β) - stability of M.

Now let us assume that M is (S,Ω,β) - stable. We define the family $\mathcal{C}$ as follows

$$E(w) = (y\in X: \lambda(y) \subset w) \qquad (3.2.6)$$

and $\mathcal{C} = (E(w): w\in\Omega)$

We show that $\mathcal{C}$ fulfils the conditions in (3.2.3) and the proof of theorem will be completed by applying Lemma (3.2.4).

We have from $S\Phi$ that

$$\Phi = \delta\in\beta(\xi) \underset{\delta}{\supseteq} [\eta\in\delta \underset{\eta}{\supseteq} (v\in I_\eta \underset{v}{\supseteq} \lambda(\lambda_v(\eta)) \subset w)]$$

and we may write $\lambda(\lambda_v(\eta)) \subset w$ in the form

$$\forall \quad y\in\lambda_v(\eta),\ y\in E(w)$$

That is $\underset{v}{\lambda(\eta)} \subset E(w)$, since w is invariant.

By putting the corresponding quantifier of letter w in S, that is by putting "a" in place of "o" and "A" in place of "O" in S we obtain

$$S_1\{\delta\in\beta(\xi) \underset{\delta}{\supseteq} [\eta\in\delta \underset{\eta}{\supseteq} (v\in I_\eta \underset{v}{\supseteq} \lambda_v(\eta)\subset\alpha)]\} = S_1P_1$$

where the quantifier A gurantees the existence of α. Thus the family $\mathcal{C}$ fulfils (3.2.3) (a).

Next, we want to show that $\mathcal{C}$ satisfies (3.2.3)(b). This is achieved by showing that $E(w) \subset w$ for every $w\in\Omega$. Let $y\in E(w)$ be fixed. Then $\lambda(y)\subset w$ and by invariance of w, $y\in w$. Hence $y\in E(w)$ implies $y\in w$ proves that $E(w) \subset w$, thereby proving (3.2.3)(b).

Now we show that $\mathcal{b}$ fulfils (3.2.3)(c). With the understanding that $\alpha = E(w)$ for the same w, let $\alpha \in \mathcal{b}$ be fixed. Let $y \in \alpha$, $s \in I_y$, $z \in \lambda_s(y)$ and $t \in I_z$. Then we have that $\lambda_t(z) \subset w$. Thus $\lambda_s(y) \subset \alpha$. This proves that α is invariant and thus $\mathcal{b}$ satisfies (3.2.3)(c). By applying theorem (3.2.4) we have that the theorem is true.

Remark 3.2.7

In the proof of theorem (3.2.5) it does not matter if we did not fix a point $y \in E(w)$ since by the relation (3.2.6) we are interested in those points $y \in X$ whose trajectories belong to w.

Remark 3.2.8

The function λ occurs in (3.2.3)(a) and (3.2.3)(c). However, λ may be omitted in certain feasible words in (3.2.3) (a). For example, the feasible word in which the universal quantifier N is in the last position belongs to this group. That is the trajectory of the point $\lambda_\nu(\eta)$ at the time ν is removed.

Definition 3.2.9

Let the class of feasible words in which the function λ is omitted be denoted by $\bar{\mathcal{B}}$, and the class of feasible words in which the letter N is omitted by $\bar{S}$. Then we may write the condition of the (S, Ω, β) - stability of the set M in the simple form:

(3.2.10) $\bar{S}\ (\delta \in \beta(\xi) \supset_{\delta}\ (\eta \in \delta \supset_{\eta} \lambda(\eta) \subset w)] = \bar{S}\ \bar{\Phi}$

The following corollary is immediate from theorem (3.2.5) and remark (3.2.8).

Corollary 3.2.11

If $\bar{S} \in \bar{\beta}$, then the set M is (S, Ω, β) - stable if and only if there exists a family $\mathcal{A} \subset P(X)$ such that

$$\text{(a) } \bar{S}_1\{\delta \in \beta(\xi) \underset{\delta}{\supset} (\eta \in \delta \underset{\eta}{\supset} \eta \in \alpha)\}$$

(3.2.12) (b) $\bar{S}_2\ P_2$

(c) P_3

hold.

Proof

First suppose that M is $(\bar{S}, \Omega, \beta)$ - stable. Then the relation (3.2.10) is true. We show that there exists a family $\mathcal{A} \subset P(X)$, $\mathcal{A} \neq \emptyset$, such that the relation (3.2.12) is true.

Consider the family $\mathcal{C} \subset P(X)$ such that;

$$E(w) = (y \in X : \lambda(y) \subset w \in \Omega) \qquad (3.2.13)$$

and $$\mathcal{C} = (E(w) : w \in \Omega)$$

By applying the relation (3.2.13) in the last term of the relation (3.2.10), that is $\lambda(\eta) \subset w$ for every $\eta \in X$, we have that $\eta \in E(w)$. And by putting the corresponding quantifier of w in the relation (3.2.10) we have

$$\bar{S}_1\ (\delta \in \bar{\beta}(\xi) \underset{\delta}{\supset}\ (\eta \in \delta \supset \eta \in \alpha)]$$

That is $\alpha = E(w)$ and $\mathcal{C}$ satisfies the condition (3.2.12)(a).

To show that $\mathcal{C}$ fulfils (3.2.12)(b) we show that $\alpha \subset w$. Let $y \in \alpha$ be fixed. Then $\lambda(y) \subset w$ implies $y \in w$ and

hence $\alpha \subset w$.

In order to prove relation (3.2.12)(c) we show that α is invariant. Let $y \in \alpha$, $s \in I_y$, $z \in \lambda_s(y)$ and $t \in I_z$, then $\lambda_t(z) < \lambda_{t+s}(y)$. Since $y \in \alpha$, then $\lambda_t(z) \subset w$. Thus $\lambda_s(y) \subset \alpha$, and that proves that α is invariant. Then by lemma (3.2.4) we have that the sufficient condition is true.

Now assume that there exists a family $\mathcal{A} \subset P(X)$, $\mathcal{A} \neq \emptyset$, such that the relation (3.2.12) is true. We show that the formula (3.2.10) holds.

Consider O occur in $\bar{S}$. By replacing the quantifier A in $\bar{S}_1$ by the quantifier O we shall obtain the word $\bar{S}$. Next we consider an arbitrary $w \in \Omega$, and relation (3.2.12)(b) implies that for each $\alpha \in \mathcal{A}$ we have that $\alpha \subset w$. Hence we obtain the formula

$$\bar{S} \quad \{\delta \in \beta(\xi) \underset{\delta}{\supset} \quad (\eta \in \delta \underset{\eta}{\supset} \eta \in w)]$$

By condition (3.2.12)(c), α is invariant so that for each $\eta \in \alpha$, $\lambda(\eta) < \alpha < w$. Thus $\lambda(\eta) < w$. With this inserted in the above formula we have the relation (3.2.10).

Now let "o" occur in the word $\bar{S}$ we replace the existential quantifier a in $\bar{S}_1$ by the existential quantifier o to obtain the word $\bar{S}$. Further, by choosing arbitrary $w \in \Omega$ and applying (3.2.12)(b) and (3.2.12)(c), just as was done for the case "O" occuring in $\bar{S}$, we have the formula (3.2.10). Hence M

is $(\bar{S},\Omega,\beta)$ - stable.

Remark 3.2.14

The function λ does not occur in (3.2.12)(a), and by omitting the universal quantifier N in the formula (3.2.10) the corollary (3.2.11) thus presents an equivalent general formula of stability for the set M in smaller region to that presented in theorem (3.2.5).

3.3 In order to begin our discussions on general stability of the Lyapunov type (or in the sense of Lyapunov) in this section, the following assumptions are pertinent.

Let $(T,\leq)$ be a partially ordered space. We denote

$$(3.3.1)\quad P = \begin{cases} T, & \text{if } \inf T \text{ does not exist} \\ T \setminus (\inf T), & \text{if there exist } \inf T. \end{cases}$$

Further, let W be a non-empty invariant subset of X.

Definition 3.3.2

A function

$$(3.3.3)\quad V: W \longrightarrow T$$

is called the (S,Ω,β) - Lyapunov function for the set M if and only if the family $\mathcal{A} = \{A_\xi\}_{\xi \in P}$, where

$$A_\xi = (y \in W: V(y) \leq \xi)$$

fulfils the conditions

$$(3.3.4)\quad \begin{aligned} &(a)\ S_1\ P_1 \\ &(b)\ S_2\ P_2 \\ &(c)\ \forall\, y \in W\ \forall\, z \in \lambda(y),\ z \in W \wedge V(z) \leq V(y) \end{aligned}$$

Remark 3.3.5

Definitions (3.3.2) and (2.2.2) compare favourable except that conditions (3.3.4)(a) and (3.3.4)(b) are defined in terms of quantifiers, and set inclusion property was used in definition (2.2.2).

Remarks 3.3.6

Suppose that $\bar{S} \in \bar{\&}$. Then the function V is a Lyapunov function for the set M if and only if the family $\mathcal{A} = (A_\xi)_{\xi \in P}$ fulfils the conditions (3.2.12)(a), (3.2.12)(b) and (3.3.4)(c). This is possible because the statements of corollary (3.2.11) have been justified.

Now we shall state and prove some theorems that are formulated on the basis of the existence of Lyapunov functions.

Theorem 3.3.7

If there exists the (S, Ω, β) - Lyapunov function for the set M, then M is (S, Ω, β) - stable.

Proof

Suppose that V is the (S, Ω, β) - Lyapunov function for the set M. Then there exists a family $\mathcal{A} = (A_\xi)_{\xi \in P}$ such that the relation (3.3.4) holds. Having established conditions (3.3.4)(a) and (3.3.4)(b) in section two, the next step is to show that each $A_\xi \in \mathcal{A}$ is invariant.

Let $A_\xi \in \mathcal{A}$, where $\xi \in P$ is arbitrary. Let $y \in A_\xi$ and $s \in I_y$. Then condition (3.3.4)(c) implies that $\lambda_s(y) \subset A_\xi$, and hence A_ξ is invariant. Relation (3.2.3) is

thus established. Therefore, by theorem (3.2.5), set M is (S,Ω,β) - stable.

Remarks 3.3.8

In order to formulate the inverse theorem we make the following additional assumptions;

Let Y be a non-empty subset of T such that infY exists, and

$$\forall\ \tau_1 \in P\ \exists\ \tau_2 \in P, \tau_2 < \tau_1, \text{if } 0 \text{ occurs in } S$$

(3.3.9) $$\forall\ \tau_1 \in P\ \exists\ \tau_2 \in P, \tau_2 > \tau_1, \text{if } o \text{ occurs in } S$$

$$\forall\ Y \subset T\ \forall\ \xi \in P,\ (\inf Y < \xi \Longrightarrow \exists\ \bar{\xi} \in P : \bar{\xi} \leq \xi\ )$$

Theorem 3.3.10

If the family Ω is in the form $\Omega = (w_\xi)_{\xi \in P}$, where $w_{\xi_1} \subset w_{\xi_2}$ for every $\xi_1 \leq \xi_2;\ \xi_1,\ \xi_2 \in P$, and the set M is (S,Ω,β) - stable, then there is a set $W \neq \emptyset$ and (S,Ω,β) - Lyapunov function for the set M.

Proof We define

$$W = (y \in X : R(y) \neq \emptyset)$$

$$V(y) = (\inf R(y) : y \in W) \qquad (3.3.11)$$

where $R(y) = (\xi \in P : \lambda(y) \subset w_\xi\ )$

By the (S,Ω,β) - stability of the set M, M must contain some trajectory. Hence $W \neq \emptyset$.

In order to prove the existence of (S,Ω,β) - Lyapunov function for the set M, consider the family $\{A_\xi\}, \xi \in P$, where $A_\xi = \{\xi \in W : V(y) \leq \xi\}$. We shall show that the family $(A_\xi), \xi \in P$ satisfies the relation (3.3.4) above.

By virtue of the (S,Ω,β) - stability of M we have

(3.3.12) $S^1\{\xi \in \beta(\xi) \supseteq_{\delta} [\eta \in \delta \supseteq_{\eta} (\nu \in I_\eta \supseteq_{\nu} \lambda(\lambda_\nu(\eta)) \subset \omega_\xi)]\}$ ~~$\leftarrow w \rightarrow$~~ where S^1 is the word obtained from the word S when the quantifier of the letter ξ replaces the quantifier of the letter w. Hence we have

(3.3.13) $S^1\{(\delta \in \beta(\xi) \supseteq_{\delta} [\eta \in \delta \supseteq_{\eta} (\nu \in I_\eta \supseteq_{\nu} \lambda_\nu(\eta) \subset (y \in X : \xi \in R(y))]\}$

Let us justify the last term of the relation (3.3.13).

$\lambda(\lambda_\nu(\eta)) \subset w_\xi$ can be written as

$\forall\ y \in \lambda(\lambda_\nu(\eta)),\ \lambda(y) \subset w_\xi$ and

$\xi \in \lambda_\nu(\eta) \Longrightarrow \xi \in R(y)$

That is $\lambda_\nu(\eta) \subset R(y)$. And since $\{y \in X : \xi \in R(y)\} \subset A_\xi$ which is the quantifier of R(y). Thus condition (3.3.4)(a) is satisfied.

$\{A_\xi\}_{\xi \in P}$ will satisfy condition (3.3.4)(b) if we show the implication $\xi_1 < \xi_2 \Longrightarrow A_{\xi_1} \subset w_{\xi_2}$

Let $y \in A_{\xi_1}$. Then $y \in A_{\xi_1} \Longrightarrow V(y) \leq \xi_1 \Longrightarrow \inf R(y) \leq \xi_1 \Longrightarrow \inf R(y) \leq \xi_2$. Then by relation (3.3.9) there exists $\xi_3 \in R(y)$ such that $\xi_3 < \xi_2$ and the following sequence of implications

$\xi_3 \in R(y) \Longrightarrow \lambda(y) \subset w_{\xi_3} \Longrightarrow y \in w_{\xi_3} \Longrightarrow y \in w_{\xi_2}$. That is $y \in A_{\xi_1} \Longrightarrow y \in w_{\xi_2}$ and hence $A_{\xi_1} \subset w_{\xi_2}$. Thus $\{A_\xi\}$, $\xi \in P$ satisfies (3.3.4)(b).

Now we shall show that $\{A_\xi\}_{\xi \in P}$ satisfies (3.3.4)(c). Let $y \in W$, $s \in Iy$ and $z \in \lambda_s(y)$. Then we show that for $z \in W$, $V(z) \leq V(y)$.

Let $\xi \in R(y)$ and $t \in I_z$, then $t+S \in I_y$ so that $\lambda_{s+t}(y) \subset w_\xi$. Also $\lambda_t(z) \subset \lambda_{t+s}(y)$. Thus inf $R(z) \leq$ inf$R(y)$ hence $R(y) \subset R(z)$. The proof of the theorem is finished.

3.4 In this section we give some applications of theorem (3.3.7) and theorem (3.3.10). The applictations will be by way of constructing the Lyapunov function for each of the three examples (3.1.14), (3.1.15) and (3.1.18). Here we assume, in each example, that $T=R_*$

3.4.1 The classical Lyapunov stability of the set M considered in example (3.1.14) is equivalent to the existence of a non-empty set W and a function V: W ---> R_* such that

(a) $\forall x \in M \ \forall \epsilon > 0 \ \exists \delta > 0 \ \forall y \in K(x,\delta), y \in W \wedge V(y) \leq \epsilon$

(b) $\forall \epsilon > 0 \ \exists \delta > 0$: $\{y \in W \wedge V(y) \leq \delta \Longrightarrow \rho(M,y) < \epsilon\}$

(c) $\forall y \in W \ \forall z \in \lambda(y), z \in W \wedge V(z) \leq V(y)$

3.4.2 The weak stability of the set M considered in example (3.1.15) is equivalent to the existence of a non-empty set W and the mapping V: W ---> R_* such that

(a) $\forall x \in M \ \forall \epsilon > 0 \ \exists \delta > 0 \ \forall y \in K(x,\delta) \ \exists S \in I_y$ such that $\{z \in \lambda_S(y) \Longrightarrow z \in W \wedge V(z) \leq \epsilon\}$

(b) $\forall \epsilon > 0 \ \exists \delta > 0$: $\{y \in W \wedge V(z) \leq \delta \Longrightarrow \rho(M,y) < \epsilon\}$

(c) $\forall y \in W \ \forall z \in \lambda(y), z \in W \wedge V(z) \leq V(y)$.

3.4.3 In the example (3.1.18) it was considered that the set M is an attractor if and only if there exists $\delta>0$ such that $\lim_{t\to\infty} p(\pi(t,y), M) = 0$ for every $y\in K(x,\delta)$. This condition is equivalent to the $(ZdHOn, \Omega, \beta)$ - stability of the set M.

By using theorems (3.3.7) and (3.3.10) we observe that those conditions are equivalent to the existence of a non-empty set W and a function $V:W \longrightarrow R_*$ such that

(a) $\delta>0\ \forall\ y\in K(x,\delta),\ \lim_{t\to\infty} V(\pi_t(y)) = 0$

(b) $\forall\ \epsilon>0\ \exists\ \delta>0$: $\{y\in W \wedge V(y) \leq \delta \Longrightarrow p(M,y) < \epsilon\}$

(c) $\forall\ y\in W\ \ \forall\ z\in\pi(y),\ z\in W \wedge V(z) \leq V(y)$.

Remarks 3.4.4

It could be seen easily from the above that remark (3.3.6) was used in relations (3.4.1) and (3.4.3) while definition (3.1.10) was used in the relation (3.4.2). These show the roles that the feasible words $\bar{S}$ and S have played in the three examples.

In other words, the feaible word $\bar{S}$ manifests in relations (3.4.1) and (3.4.3) and the feasible word S manifests in the relation (3.4.2). The implication is that the three examples have shown that, using theorem (3.3.7) and theorem (3.3.10) one can easily construct corresponding Lypunov function for every feasible word.

In order to give one more applications of these

theorems we shall first consider the following definitions of an attractor.

Definition 3.4.5

Let M be a non-empty sub-set of X, and Ω, Δ non-empty subfamilies of P(X) such that, for every $Q\in\Omega$ and $L\in\Delta$, $M \subset Q\cap L$. We say that M is an (Ω, Δ) - attractor if and only if there exists $Q\in\Omega$ such that for every $x\in Q$, $L\in\Delta$ there exists $s\in I_x$ such that $\pi(\pi_s(x))\subset L$.

Further, we give a set of definitions based on the ideas that we are going to use in the proof of the next theorem.

Definition 3.4.6

We call a subfamily $\mathcal{B}$ of P(X), $\mathcal{B}\neq\emptyset$, the basis of X if and only if for every A, $B\in\mathcal{B}$ there exists $C\in\mathcal{B}$ such that $C \subset A\cap B$.

Definition 3.4.7

A point $y_0\in X$ is said to be the limit point of basis $\mathcal{B}$ of X if $y_0\in \bigcap\{\overline{B}: B\in\mathcal{B}\}$

Definition 3.4.8

A non-empty set X is compact if and only if every basis of X has a limit point.

The following theorem gives a sufficient condition for M to be an attractor.

Theorem 3.4.9

Let us suppose that X is a topological space, the maps

(3.4.10) π_g: X∋x --> π(g,x) ∈X

are continuous for every g∈G.

Let W be a non-empty invariant compact subset of the space X such that W ⊃ Q for every Q∈Ω. Let us assume that every L∈Δ is open. If there exists a continuous function ϕ:W --> R_* such that

(i) ϕ(x) = 0 if and only if x∈M,

(3.4.11) (ii) ϕ(π_t(x) ≤ϕ(x) for every x∈W and t∈I_x,

(iii) for every x∈ W\M there exists s∈I_x such that ϕ(π_s(x) ≤ ϕ(x),

then M is an (Ω, Δ) - attractor.

Proof

By putting Δ = $\tilde{\Omega}$ and Ω = $\tilde{\beta}$(x) for every x∈X, example (3.1.18) asserts that M is an (Ω , Δ) - attractor if and only if it is (ZdHOn, $\tilde{\Omega}$,$\tilde{\beta}$) - stable.

Therefore theorem (3.3.7) is sufficient to show that the function ϕ: W ---> P_* is a Lyapunov function and fulfils the following conditions:

(a) ∃ Q∈Ω ∀ x∈Q ∀ ∈>0 ∃ s∈I_x: π_s(x) ∈W ∧ϕ(π_s(x))≤∈.

(b) ∀ L∈ Δ ∃δ>0: (x∈W, ϕ(x)≤δ==> x∈Δ)

(c) ∀ y∈W ∀ z∈π(y), z∈W∧ ϕ(z) ≤ϕ(y)

We prove that above conditions one after the other. First let us prove (c). z∈π(y) for every y∈W implies that there exists t_o∈I_z such that z = π_{to}(y) for every z∈W. By applying the map ϕ on z, we have

$\phi(z) = \phi(\pi_{to}(y) \leq \phi(y)$ for every $y \in W$.

Hence (c) is immediate from the second condition of relation (3.4.11).

To show (a), we know from the hypothesis that there exists $\Omega \in \Omega$ such that $\Omega \subset W$. We put

(3.4.12) $\lambda(x) = \inf \{\phi(\pi_s(x): s \in I_x\}$

for every $x \in \Omega$. Then the condition (a) can be formulated as:

$\forall\, x \in \Omega\ \forall\, \epsilon > 0\ \exists\, s \in I_x$: $\pi_s(x) \in W \wedge \phi(\pi_s(x)) \leq \epsilon \Longrightarrow \lambda(x) = 0$

That is for all $x \in \Omega$,

(3.4.13) $\lambda(x) = o$

Indeed, let us assume the contrary. That is, there exists $x_o \in \Omega$ such that $\lambda = \lambda(x_o) > 0$. Thus $(\phi^{-1}([\lambda, \mu]) \cap \pi(x_o)]\ \mu > \lambda$, is basis on W. Since W is compact, there exists a point $y_o \in W$ such that

$y_o \in \phi^{-1}([\lambda, \mu]) \cap \overline{\pi(x_o)}$ for every $\mu > \lambda$.

Since $\lambda < \mu$ we have $\phi(y_o) = \lambda$.

And by the first of relations (3.4.11) we have $y_o \notin M$.

From the third condition of relation (3.4.11) we have that there exists $s \in I_{y_o}$: $\phi(\pi_s(y_o)) < \phi(y_o) = \lambda\ \forall\ y_o \in W \setminus M$.

On the other hand $\pi(x_o) \subset (z \in W:\ \phi(z) \geq \lambda)$, and because of continuity of ϕ, we have $\overline{\pi(x_o)} \subset \{z \in W:$ $\phi(z) \geq \lambda)$. By the continuity of the map (3.4.10), the set $\overline{\pi(x_o)}$ is invariant. Hence, there exists $y_o \in \overline{\pi(x_o)}$ such that

$\pi_s(y_o) \subset \overline{\pi(xo)} \subset \{z \in W:\ \phi(z) \geq \lambda\}$ and $(\pi_s(y_o)) \geq \lambda$.

This is a contradiction. Therefore, $\forall\ x \in \Omega$, $\lambda(x) = 0$ and (a) is proved.

Now we shall prove (b). Again let us assume the contrary. That is, there exists $L \in \Delta$ such that for every $\delta>0$ there exists $x \in W \setminus L$ such that $\phi(x) \leq \delta$. Then the family $\{\phi^{-1}([0,\delta]) \cap (W \setminus L)\}_{\delta>0}$ is a basis of closed sets on W. Since W is compact, there exists a point y_0 such that $y_0 \in \phi^{-1}([0,\delta)] \cap (W \setminus L)$. Then by the first condition of the relation (3.4.11), $\phi(y_0)=0$ if and only if $y_0 \in W \setminus L$. This is a contradiction because $M \subset L$. Thus the proof of the theorem is completed.

Remark 3.4.14

If $x \in M$ contrary to what is stated in the relation (3.4.11)(iii), the theorem will not hold in general. The theorem will, however, hold only if the trajectory is periodic.

Remark 3.4.15

Theorem (3.4.9) is the generalization of the classical theorem on the asymptotic stability of compact sets. Thus, we obtain the next corollary on asymptotic stability in Euclidean space.

Definition 3.4.16

The set M is a symptotically stable if it is stable and an attractor.

Corollary 3.4.17

Let us assume the continuous map

(3.4.18) $\pi: R_+ \times R^n \longrightarrow R^n$

defines the pseudo-dynamical system on R^n, and M is a non-empty subset of R^n. Let $W \supset M$ be an invariant and

compact set such that $p(\partial W, M)>0$. If there exists a continuous function $\phi:W \longrightarrow R_{\#}$ such that

(i) $\phi(x) = 0$ if and only if $x \in M$,

(ii) $\phi(\pi_t(x) < \phi(x)$ for every $t>0$, $x \in W$

then the set M is asymptotically stable.

Proof

In order to show that M is asymptotically stable, we show that M is stable and is an attractor.

First we show that M is an attractor. Since W is invariant and compact, it follows that the boundary, ∂W, of W is also invariant. Then there exists some point $x \in \partial W$, $x \notin M$, such that $\pi_s(x) \subset \partial W$ for $s \in R_{\#}$. Hence the condition three of the relation (3.4.11) is satisfied and so by theorem (3.4.9) M is an attractor.

Next is to show that M is stable. By virtue of the fact that the function ϕ is a Lyapunov function, we immediately apply theorem (3.3.7) to establish that M is stable. Hence we have shown that M is asymptotically stable.

Conclusion 3.4.19

There is a close relationship between $S(\Omega,\beta;\lambda)$ stability discussed in chapter 2 and that of $(S\Omega,\beta)$ - stability we mentioned in this chapter. The later is more or less an extension of the former.

The stability concepts we discussed in this chapter is hinged on a propositional function defined in this chapter. And with the adjustment of the

feasible words we have proved theorems on different types of stability.

Furthermore under close observation and based on relation (3.2.3) we have shown that the existence of (S, Ω, β) - Lyapunov function ushers in (S,Ω,β) - stability of the set M coupled with some suitable assumptions.

And the conditions satisfied by the $S,\Omega,\beta)$ - Lyapunov function in this chapter has some parallel bearing with that discussed in chapter 2.

Theorem (3.3.7) cannot be considered as a converse of theorem (3.3.10) since in the later there are additional conditions that should be satisfied by the family $\Omega=(w_\zeta)_{\zeta\in P}$ which are not part of the assumptions for the former theorem.

We have shown that M is an attractor for every $x\in M\subset X$, where X is a topological space, on the condition that there exists a function φ satisfying suitable conditions similar to Lyapunov functions which, with additional assumptions we have shown that M is asymptotically stable.

We also gave some applications of theorems (3.3.7) and (3.3.10) via examples listed.

CHAPTER 4

SEMI-STABILITY AND LIMIT SETS IN SEMI-SYSTEMS

In this chapter we investigate the regular dependence of limit sets on points moving semi-stable in semi-systems and dynamical semi-systems in metric spaces.

In section one, we discuss the idea of limit sets and generalizations of semi-stability of motions in semi-systems in metric spaces.

In section two, we focus our attention on some regularity conditions for set-valued mapings as well as certain compactness conditions for metric spaces. Thereafter, we discuss some connections between semi-stability conditions for motions and semi-continuity properties of some maps in section three.

Section four is an extension of section three, and the idea that we investigate is the weak semi-stability of motions and non-emptiness of some limit sets. Finally in section five, we make some observations about the regularity of the map we consider in section three.

4.1 Consider a semi-system $(X,R_{+};\pi)$ fixed, where (X,ρ) is a metric space.

With reference to definition (3.1.16), we have that for $x\in X$, $\Lambda^{+}(x)$ and $\Lambda^{-}(x)$ are positive and negative limit sets of the point x respectively. We

have the following defintions of the same concept as well.

Definition 4.1.1

Let X be a topological space but not a metric space. Then for $x \in X$

(4.1.2) $\Lambda^+(x) = \bigcap \{ \overline{\pi_t(\pi_+(x))}: t \in R \}$

(4.1.3) $\Lambda^-(x) = \bigcap \{ \overline{\pi_t(\pi_-(x))}; t \in R \}$ and

(4.1.4) $\Lambda(x) = \Lambda^+(x) \cup \Lambda^-(x)$

Remarks 4.1.5

If X is a metric space. Then we have for a system (X,R,π) that

(4.1.6) $\Lambda^+(x) = \Lambda(x)$

So that in view of (4.1.6), results obtained for limit sets in semi-systems can be modified in such a natural way that parallel theorems for positive limit sets in systems are obtained.

Relations (4.1.2) and (4.1.3) imply that for $x \in X$, the limit set of x is closed. Therefore we present a direct proof of the closedness of $\Lambda(x)$.

Theorem 4.1.7

For every $x \in X$, the set $\Lambda(x)$ defined by (4.1.4) is closed.

Proof:

Consider a sequence (y_m) such that $y_m \in \Lambda(x)$, $(m=1,2,...)$, and let $y_m \longrightarrow y$ as $m \longrightarrow \infty$. We show that $y \in \Lambda(x)$.

For every $m \in N$ there is a sequence $(t^m_k) \subset R_*$ such that $t^m_k \longrightarrow \infty$ and $\pi(t^m_k, x) \longrightarrow y$ as $k \longrightarrow \infty$ for every m. Without loss of generality we can assume that for each m there is a natural number k_m such that $p(\pi(t^m_{k_m}, x), y_m) < \frac{1}{m}$ and $t^m_k \longrightarrow \infty$ as $m \longrightarrow \infty$

Then we have from triangle inequality that

$p(\pi(t^m_{k_m}, x), y) \leq p(y, y_m) + p(y_m, \pi(t^m_{k_m}, x)) = p(y, y_m) + \frac{1}{m}$. And as $m \longrightarrow \infty$ $\pi(t^m_{k_m}, x) \longrightarrow y \in \Lambda(x)$.

Next we discuss some stability - like properties of motions. We begin by defining the stability of the motion π^x in Lyapunov sense.

Definition 4.1.8

Let $x \in X$ be given. The motion π^x is said to be stable in the sense of Lyapunov if given $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that

(4.1.9) $p(\pi(t,y), \pi(t,x)) < \epsilon$

for every $y \in B(x, \delta)$ and $t \in R$.

Definition 4.1.10

Let α be a non-negative real number and let $x \in X$ be given. The motion π^x is said to be α - semi-stable if and only if for every $\epsilon > 0$ there exists $\delta > 0$ and $u \in R_*$ such that

(4.1.11) $p(\pi(t+u,x), \pi(t+u,y)) < \alpha + \epsilon$

for every $y \in B(x, \delta)$, $t \geq 0$.

Remarks 4.1.11

The α - semi-stability of π^x defined above is a special case of (Ω,β) - stability considered in chapter two. The introduction of α in the relation (4.1.11) makes the definition (4.1.10) to be a generalization of the classical stability due to Lyapunov.

If α is removed and u is made to be zero, the motion π^x which is 0 - semi-stable is called semi-stable.

Theorem 4.1.12

Let $\alpha \geq 0$ be given and $x \in X$. Then for the motion π^x the following three conditions are equivalent:

(a) π^x is α - semi-stable,

(b) for every sequence (u_n) of elements of R_+ with $u_n \longrightarrow \infty$, for $\epsilon > 0$ there exists $\delta > 0$ and $k \in N$ such that $p(\pi(u_n,y), \pi(u_n,x)) < \alpha + \epsilon$ for every $y \in B(x,\delta)$ and $n \geq k$, (c) if $\{s_n\}$ is a sequence of elements of R_+, $s_n \longrightarrow \infty$ and $\{x_n\}$ is a sequence of elements of X such that $x_n \longrightarrow x$ then limsup $P(\pi(s_n,x_n), \pi(s_n,x)) \leq \alpha$.

Proof

Assume(a) is true. Let (u_n) be such that $u_n \longrightarrow \infty$ and $\epsilon > 0$ given. There are $\delta > 0$ and $u \in R_+$ such that $P(\pi(t+u,x), \pi(t+u,y)) < \alpha + \epsilon$ for $y \in B(x,\delta)$ and $t \geq 0$. Then there is $k \in N$ such that

$$p(\pi(u_n, x), \pi(u_n, y)) < \alpha + \epsilon$$

(4.1.13) for $y \in B(x,\delta)$ and $n \geq k$. Which is (b).

Now assume(b) and let $(s_n),(x_n)$ be such that $s_n \dashrightarrow 00$ and $x_n \dashrightarrow x$). Let $\epsilon>0$ be given. We can find $\delta>0$ and $k\in N$ such that $y\in B(x,\delta)$, and $n\geq k$ implies that

$$\rho(\pi(s_n,x)\ \pi(s_n, y)) < \alpha + \epsilon$$

By taking m so large that $m\geq k$, and $\rho(x,x_n) <\delta$ we have $x_n \in B(x,\delta)$ for $n\geq m$. Then $\rho(\pi(s_n, x_n), \pi(s_n, x)) <\alpha+\epsilon$. Thus for $n\geq m$.

(4.1.14) Limsup $\rho(\pi(s_n,x_n), \pi(s_n,z_n)) \leq \alpha+\epsilon$

This is true for every $\epsilon>0$ and then finally Limsup $\rho(\pi(s_n,x_n), \pi(s_n,x)) \leq \alpha$

Now let (c) be satisfied and suppose that π^x *is not* α - semi-stable. Then there is $\epsilon^1>0$ such that for every $\delta>0$ and $u\in R_*$, there exist $y\in B(x,\delta)$ and $s\geq 0$ such that $\rho(\pi(s+u,x), \pi(s+u,y)) \geq \alpha +\epsilon^1$.

Then there exists sequences $\{x_n\} \subset X$ and $\{s_n\} \subset R_*$ such that $x_n \dashrightarrow x$, $s_n \dashrightarrow 00$ and

(4.1.15) $\rho(\pi(s_n,x_n), \pi(s_n,x)), \geq \alpha+\epsilon^1$ for n=1,2....

This is a contradiction. Hence π^x is α - semi-stable.

Remarks 4.1.16

The stability of the motion π^x considered is for all time. Hence we cannot assume the sequence (u_n) of elements of R_* to be convergent.

Also the sequence (x_n) of element of X is assumed convergent to the limit x for $n\geq m$, where m is sufficiently large. Therefore by setting

lim sup $\{x_n\}$ = sup $\{x\in X: x$ is a limit point of $(x_n)\}$

$n\dashrightarrow 00$ we have the relation (4.1.14).

4.2 We being this section by defining the upper semi continuity of a set valued map.

Definition 4.2.1

Let CL(X) denote the family of all close subsets of the space X, and F a map F: X ∋ x---> CL(X). Then F is said to be upper semi-continuous at a given point x∈X if and only if

$$(4.2.2)\quad \left\{ \begin{array}{l} x_n \in X \quad x_n \dashrightarrow x \\ y_n \in F(x_n),\ y_n \dashrightarrow y \end{array} \right\} \qquad \Longrightarrow \quad y \in F(x)$$

Remark 4.2.3

By applying the usual notation

$$(4.2.4)\quad p(z, A) := \inf \{p(z,w): w \in A\}$$

for z∈X and A⊂X, $A=\bar{A}$ and admitting the convention

$$(4.2.5)\quad p(z,\emptyset) = \infty$$

We can replace in the relation

(4.2.2) y∈ F(x) by

$$(4.2.6)\quad p(y, F(x)) = 0.$$

The ideas in remark (4.2.3) suggest that we can have the following generalization of upper semi-continuity of the set valued map F.

Definition 4.2.7

Let F be a mapping from X into CL(X), α a non-negative real number and x∈X be given. The mapping F is said to be α - upper semi-continuous at the point x if and only if

$$(4.2.8)\quad \left.\begin{cases} x_n \in X,\ x_n \dashrightarrow x \\ y_n \in F(x_n),\ y_n \dashrightarrow y \end{cases}\right\} \quad \Longrightarrow P(y,F(x)) \leq \alpha$$

Remark 4.2.9

Usually we suppose that $F(x) \neq \emptyset$ for $x \in X$. Although it is not assumed but it is possible to have, in particular, $F(z) = \emptyset$ for $z \in B(x,\delta)$ with some $\delta > 0$. In such a case F is α-upper semi-continuous at x for every $\alpha \geq 0$.

For $\alpha = 0$ we shall have the trivial upper semi-continuity of F at x. This is true for the case $F(z) \neq \emptyset$ also.

Remark 4.2.10

In this study we shall discuss F as the mapping $F:\ x \dashrightarrow \Lambda(x)$, where $\Lambda(x) \neq \emptyset$. And we shall observe that semi-stability of motion π^X implies that in a neighbourhood of a point x, all points have the same property.

Now we define some compactness conditions which are transitions between local compactness and compactness. The idea will be applied in some theorems in the subsequent sections.

Definition 4.2.11

Let $\alpha \geq 0$ be given. We say that X satisfies at a point y the condition Comp(α) if and only if there is a positive number β such that the ball $B(y,\alpha+\beta)$ is relatively compact, that is if and only if the

closure of $B(y,\alpha+\beta)$ is compact.

Definition 4.2.12

If $\alpha>0$, then X satisfies the condition Comp[α] at the point y if and only if the ball $B(y,\alpha)$ is relatively compact. That is, if and only if $\overline{B(y,\alpha)}$ is compact.

Remark 4.2.13

Local compactness is equivalent to the existence of a function $x \longrightarrow \alpha(x)$ such that the space X satisfies at every point x the condition comp($\alpha(x)$).

It is clear that every compact space X satisfies the conditions Comp(α) and comp[α] for certain fixed, α as this cannot be assured by local compactness. In particular, the assumption that X satisfies at every point y the condition Comp(α), where α is the same for all y, is essentially stronger than local compactness.

4.3 The mapping

(4.3.1) $X \ni x \longrightarrow \Lambda(x) \in CL(X)$

for each $x \in X$, defined in the last section will be of paramount importance to our discussion in this section.

Theorem 4.3.2

Let $\alpha \geq 0$ and $x \in X$ be given. Suppose that the motion π^X is α- semi-stable. Assume that (x_n) and (y_n) are sequences of X such that

(4.3.3) x_n ---> x as n ---> 00

(4.3.4) y_n ---> y as n ---> 00

(4.3.5) $y_n \in \bigwedge(x_n)$ for every n.

Then there exists a sequence (s_n) of elements of R_* such that

(4.3.6) s_n ---> 00 as n ---> 00

and

(4.3.7) Lim sup $p(\pi(s_n,x), y) \leq \alpha$

Proof

Relation (4.3.5) means that for every $m \in N$ there is a sequence $\{t^m_k\}$, k= 1,2...., of elements of R_* such that

(4.3.8) t^m_k --> 00 as k ---> 00 (m = 1,2,...)

and

(4.3.9) $\pi(t^m_k,x)$ ---> y_m as k--->00 (m=1,2,...)

From relation (4.3.3), we have

(4.3.10) $p(x_m,x)$ --> 0 as m ---> 00

So that by the relation (4.3.9) we have

(4.3.11) $p(\pi(t^m_k,x_m), y))$ -->0 as k ---> 00

(m=1,2,..).

Also from the relation (4.3.4) we get

(4.3.12) P (y_m,y) ---> 0 as m --> oo

By virtue of relation (4.3.8) we can find strictly increasing sequence of positive integers (k_m) such that

(4.3.13) $m \leq t^m_{k_m}$ for every m.

From relations (4.3.11) and (4.3.13) it follows

that for every $m \in N$ there is $p(m) \in N$ such that

(4.3.14) $p(\pi(t^m_{k_{p(m)}}, x_m), y_m) < \frac{1}{m}$, $(m=1,2,\ldots)$

Without loss of generality we may assume that

(4.3.15) $m \leq p(m) \leq p(m+1)$ $(m=1,2,\ldots.)$

This gives in particular

(4.3.16) $p(m) \longrightarrow \infty$ as $m \longrightarrow \infty$

Now put

(4.3.17) $s_m := t^m_{k_{p(m)}}$ $(m=1,2,\ldots)$

Then we have

(4.3.18) $s_m \longrightarrow \infty$ as $m \longrightarrow \infty$.

And that proves the relation (4.3.6).

Now to prove the relation (4.3.7). By applying relation (4.3.18) in (4.3.14) we have

(4.3.19) $p(\pi(s_m, x_m), y_m) < \frac{1}{m}$ for every m

and as $m \longrightarrow \infty$ we have

(4.3.20) $p(\pi(s_m, x_m), y_m) \longrightarrow 0$

Therefore using triangle inequality we get

(4.3.21) $p(\pi(s_m, x), y) \leq p(y_m, y)$

$+ p(y_m, \pi(s_m, x_m)) + p(\pi(s_m, x_m), \pi(s_m, x))$ for every m.

Then by virtue of relations (4.3.12), (4.3.20) and the assumption that π^x is α - semi-stable we obtain

Lim sup $p(\pi(s_m, x), y) \leq \alpha$.

Remark 4.3.22

By our constructions, we have for each m, $k \longrightarrow \infty$, $x_m \longrightarrow x$ and $\pi(t^m_k, x) \longrightarrow y_m$. Also for each m, $p(\pi(t^m_{kp(m)}, x_m) \leq \frac{1}{m}$.

The implication is that there are two indices m and k such that m -->00 and k-->00 at the same time. To avoid this we have to define the relation (4.3.17).

We shall now state the theorem which touches on upper semi-continuity of the mapping (4.3.1) as sufficient for its semi-stability.

Theorem 4.3.23

If π^X is semi-stable then the mapping (4.3.1) is upper semi-continuous at the point x.

Proof

Let π^X be semi-stable and (x_n), (y_n) be sequences of elements of X such that the conditions (4.3.3) and (4.3.5) hold. Then by theorem (4.3.2) we can find a sequence (s_n) of real non-negative numbers such that s_n ---> 00 and

(4.3.24) Lim sup $p(\pi(s_n,x), y) \leq 0$

Inequality (4.3.24) means that

(4.3.25) lim supp $(\pi(s_n, x) ,y) = 0$ and then $y \in \Lambda(x)$.

Remark 4.3.26

We got the relation (4.3.24) for the case $\alpha = 0$, and the relation (4.3.24) gives rise to the relation (4.3.25) because in (4.3.24) we have a sequence of numbers in which the least upper bound is the same as the upper bound.

Remark 4.3.27

The ideas in theorem (4.3.2) are generalizations of theorem (4.3.23). But the basic idea is that α -

semi-stable motion has a limit set.

In view of remark (4.1.5) we shall state the following corollary.

Corollary 4.3.28

If π^X is stable then the mapping (4.3.1) is upper semi-continuous at x.

Proof

By definition (4.1.10) we have that π^X is stable when we put both u and α to be zero, and $t \in R$.

Also, since X is a metric space, remark (4.1.5) implies that $\Lambda^+(x) = \Lambda(x)$. Then straightforward application of theorem (4.3.23) completes the proof.

We shall now make use of the compactness conditions in definitions (4.2.11) and (4.2.12) respectively to establish the connection between α - semi-stability of π^X and α - semi-continuity of the mapping (4.3.1).

Theorem 4.3.29

Suppose that X satisfies at every point $x \in X$, the condition comp(α), and if π^X is α - semi-stable then the mapping (4.3.1) is α - upper semi - continuous.

Proof

Suppose that the sequences (x_n) and (y_n) are such that the conditions (4.3.3), 4.3.4) and (4.3.5) hold. Let $\eta > 0$ be such that the ball $B(y, \alpha+\eta)$ is relatively compact. Such η exists because X satisfies

comp(α) at every point.

Let (s_n) be a sequence of non-negative real numbers such that $s_n \longrightarrow 0$ and is such that the relation (4.3.7) holds true. There is $k \in N$ such that

(4.3.30) $p(\pi(s_n,x), y) < \alpha+\eta$ for $n>k$.

Since $B(y, \alpha+\eta)$ is compact, we can assume without loss of generality that the sequence of points $(\pi(s_n,x))$ is convergent to some y^*. By virtue of relation (4.3.7) we have that

(4.3.31) $p(y, y^*) \leq \alpha$.

On the other hand we have

(4.3.32) $y^* \in \bigwedge(x)$

But relations (4.3.31) and (4.3.32) imply that $p(y, \bigwedge(x)) \leq \alpha$ and this implies that the mapping (4.3.1) is α - upper semi-continuous at x.

Remark 4.3.33

Since for each $x \in X$ such that X satisfies comp(α), the sequence $(\pi(s_n,x))$ is convergent, relation (4.3.7) means that $\inf_k (\sup_{n \geq k} p(\pi(s_n, x))) \leq \alpha$. Therefore, by compactiness of $B(y, \alpha+\eta)$, for each $y \in X$, any pair of limit points y_1 and y_2 satisfy $p(y_1,y_2) \leq \alpha$.

In remark (4.2.13) we stated that every compact space X satisfies the conditions comp(α) and comp[α] for all $\alpha \geq 0$. This fact will be needed in the proof of the following corollary.

Corollary 4.3.34

If X is compact and π^X is α - semi-stable, then the mapping (4.3.1) is α- upper semi-continuous at x.

Proof

X being compact means X satisfies Comp(α). Therefore, a straightforward application of theorem (4.3.29) completes the proof.

So far we have applied the condition comp(α) only. We shall now apply comp[α] in the next theorem.

Theorem 4.3.35

Suppose that X satisfies at every point the condition Comp [α]. If there is $\beta<\alpha$ such that $\Phi\pi^X$ is β - semi- stable, then the mapping (4.3.1) is α- upper semi-continuous at x.

Proof

Let (x_n) and (y_n) be sequences such that assumptions (4.3.3) - (4.3.5) hold, and let (s_n) be such that the relation (4.3.6) is satisfied. Let

(4.3.36) Lim sup $p(\pi(s_n,x), y) \leq \beta$

Take $\eta = \frac{1}{2}(\alpha-\beta)$ and find $k\in N$ such that

(4.3.37) $p(\pi(s_n,x),y) < \beta+\eta$ for $n\geq k$.

Since $B(y,\alpha)$ is compact, relation (4.3.37) means that $\pi(s_n,x) \in \overline{B(y,\alpha)}$ for $n\geq k$.

Therefore without loss of generality we may assume that the sequence $(\pi(s_n,x)$ is convergent to some $y^*\in \Lambda(x)$. And by the relation (4.3.36) we obtain $p(y,\Lambda(x))$

$\leq \alpha$. That is $\bigwedge(x)$ is α- upper semi-continuous at x.

Remark 4.3.38

The condition comp[α] is equivalent to local compactness. That is every point $x \in X$ has a neighbourhood whose closure is compact.

4.4 Before discussing the main results in this section we shall give the following definition of stability of the motion π^X.

Definition 4.4.1

Let $\alpha \geq 0$ and $x \in X$ be given. The motion π^X is said to be weakly α- semi-stable if and only if for every $\epsilon > 0$ there is $\delta > 0$ such that for every $z \in B(x,\delta)$ there is $u \in R_+$ for which $\rho(\pi(t+u,x), \pi(t+u,z)) < \alpha + \epsilon$ For $t \geq 0$.

Remark 4.4.2

Here, one observes that α- semi - stability of π^X implies the weak α- semi - stability of π^X. If $\alpha = 0$, the motion π^X becomes weak semi-stable.

We shall find out whether the limit set of a neighbouring point to a point $x \in X$ and whose limit set is not empty, and such that π^X is weakly semi-stable, is empty or not. This will be taken care of in the following theorem.

Theorem 4.4.3

Assume that $x \in X$ is such that $\bigwedge(x) \neq \emptyset$ and suppose that X satisfies Comp (α) at every point. If π^X is weakly α- semi-stable then there is $r > 0$ such that $\bigwedge(z) \neq \emptyset$ for every $z \in B(x,r)$.

Proof

Let us take some fixed point $y \in \Lambda(x)$ and $\gamma > 0$ such that the ball $B(y, \alpha+\gamma)$ is relatively compact. Let (s_n) be a sequence of non-negative numbers such that $s_n \longrightarrow \infty$ and $\pi(s_n, x) \longrightarrow y$ as $n \longrightarrow \infty$

Then we have

(4.4.4) $\quad p(\pi(s_n, x), y) < \alpha + \frac{1}{2}\gamma$

for n sufficiently large.

On the other hand, by weak α- semi- stability of π^X we can find $r \geq 0$ such that for every $z \in B(x,r)$ there is $u \in R_\pi$ for which

$p(\pi(t+u,x), \pi(t+u,z)) < \alpha + \frac{1}{2}\gamma$ for $t \geq 0$

So for $z \in B(x,r)$ and n sufficiently large we have by relation (4.1.12)(b) and remark (4.4.2) that

(4.4.5) $p(\pi(s_n, x), \pi(s_n, z)) < \alpha + \frac{1}{2}\gamma$.

From implications (4.4.4) and (4.4.5) we get that $\pi(s_n, z) \in B(y, \alpha+\gamma)$ for $z \in B(x,r)$ and n sufficiently large. Therefore we may assume without loss of generality the sequence $\{\pi(s_n, z)\}$ is convergent and the limit belongs to $\Lambda(z)$. So $\Lambda(z) \neq \emptyset$, and this completes the proof.

We shall state the corollary to the above theorem.

Corollary 4.4.6

If π^X α- semi-stable, $\Lambda(x) \neq \emptyset$ and X satisfies comp(α) at every point of $\Lambda(x)$, then there is $r > 0$ such that $\Lambda(z) \neq \emptyset$ for every $z \in B(x,r)$.

Proof

By applying remark (4.4.2) the statement of the corollary will become the statement of theorem (4.4.3) and the proof is implied.

So far in this section we have been making use of the conditions comp(α) in our discussions with regard to the connections between weak stability and the emptiness of some limit sets. However, the assertion will still hold if we assume a weaker compactiness condition for the space X. Hence have the following theorem, which does not make use of weak α-semi-stability condition but of weak semi-stability.

Theorem 4.4.7

Assume that X is locally compact and suppose that $x \in X$ is such that $\bigwedge(x) \neq \emptyset$. If π^X is weak semi-stable then there is $r>0$ such that $\bigwedge(z) \neq \emptyset$ for $z \in B(x,r)$.

Proof

Let X be locally compact and (x_n) and (y_n) be sequences such that such that x_n ----> x and y_n ---> y. Since $\bigwedge(x) \neq \emptyset$, we consider some fixed $y \in \bigwedge(x)$ and $\delta>0$ such that $B(y,\delta)$ is compact.

Let (s_n) be a sequence of non-negative numbers such that s_n --> OO and $\pi(s_n,x)$ -->y as n---> OO. Then there is a number $k \in N$ such that

(4.4.8) $p(\pi(s_n,x), y) < \epsilon$ for $n \geq k$.

By weak semi-stability of π^X, we can find $r \geq 0$ such that for every $z \in B(x,r)$ there exists $u \in R_+$ for which

$p(\pi(t+u,x), \pi(t+u,z)) < \epsilon$ for $t \geq 0$.

Therefore by relation (4.1.12)(b) and remark (4.4.2) when $\alpha = 0$, we have

(4.4.9) $P(\pi(s_n,x), \pi(s_n,z)) < \epsilon$

for $z \in B(x,r)$ and $n \geq k$.

Then relations (4.4.8) and (4.4.9) imply that $\pi(s_n, z) \in B(x,\delta)$ and without loss of generality we may assume that $(\pi(s_n,x))$ converges, and the limit belongs to $\Lambda(z)$. So $\Lambda(z) \neq \phi$.

Remark 4.4.10

By remark (4.3.38) in the place of local compactness, X can be assumed to satisfy the condition Comp [α].

We state the following corollary to the above theorem.

Corollary 4.3.11

If X is locally compact, π^X is semi-stable and $\Lambda(x) \neq \phi$, then there exists $r>0$ such that $\Lambda(z) \neq \phi$ for $z \in B(x,r)$.

Proof

By making use of the fact that semi-stability implies weak semi-stability we shall obtain the statement of theorem (4.4.7), and the proof the corollary is implied.

4.5 In this section we shall make some observations about the regularity of the mapping (4.3.1) if π is continuous. We begin with the following defintion.

every n, and $\pi(t^n_k, x_n) \dashrightarrow y_n$ as $k \dashrightarrow \infty$ for every n.

We have from relation (4.5.7) that $p(x_n, x) \dashrightarrow 0$ as $n \dashrightarrow \infty$, and from relation (4.5.8) that $p(\pi(t^n_k, x_n), y) \dashrightarrow 0$ as $k \dashrightarrow \infty$. By relation (4.5.10) we can find a strictly increasing sequence of positive integers (k_n) such that $n \leq t^n_{k_n}$ for every n. Then it follows that for every $n \in N$ there is $p(n) \in N$ such that $p(\pi(t^n_{k_{p(n)}}, x_n) < \frac{1}{n}$, $n=1,2,\ldots n$

Without loss of generality we may assume that $n \leq p(n) \leq p(n+1)$ for $n = 1,2,\ldots$.

This gives $p(n) \dashrightarrow \infty$ $n \dashrightarrow \infty$. Now put

$s_n := t^n_{k_{p(n)}}$, $n = 1,2,\ldots$. We have

(4.5.11) $s_n \dashrightarrow \infty$ as $n \dashrightarrow \infty$.

and $p(\pi(s_n, x_n), y_n) < \frac{1}{n}$ for every n.

This gives

(4.5.12) $p(\pi(s_n, x_n), y_n) \dashrightarrow 0$ as $n \dashrightarrow \infty$.

And by the fact that $p(y, y_n) \dashrightarrow 0$ as $n \dashrightarrow \infty$ we get from relation (4.5.12) that

(4.5.13) $p(\pi(s_n, x_n), y) \dashrightarrow 0$.

Now by relations (4.5.5) and (4.5.6), let $\epsilon > 0$ be arbitiarly fixed, and let $\delta > 0$ and $u \in R_*$ be chosen in such a way that if $z \in B(x, \delta)$ then

(4.5.14) $p(\pi(t,z), \pi(x)) < \frac{1}{2}\epsilon$ for $t \geq u$.

Now let $n^* \in N$ be such that

(4.5.15) $p(x_n, x) < \delta$ for $n > n^*$,

(4.5.16) $u < s_n$ for $n > n^*$

and

(4.5.17) $p(\pi(s_n, x_n), y) < \frac{1}{2}\epsilon$ for $n > n^*$

By making use of implications (4.5.15), (4.5.16) and (4.5.14) we obtain

(4.5.18) $p(\pi(s_n,x_n), \pi(x)) < \frac{1}{2}\epsilon$ for $n>n^*$, and there exists $w\in\pi(x)$ such that

(4.5.19) $p(\pi(s_{n^*+1}, x_{n^*+1}), w) < \frac{1}{2}\epsilon$

From triangle inequality we have

$p(y,w) \leq p(y,\pi(s_n,x_n)) + p(\pi(s_n,x_n), \pi(s_n{}^*{}_{+1}, x_{n^*+1})) + p(\pi(s_{n^*+1}, x_{n^*+1}), w) = \frac{1}{2}\epsilon+\frac{1}{2}\epsilon$

That is $p(y,w) < \epsilon$. This implies

(4.5.20) $B(y,w) \cap \pi(x) \neq \phi$.

Since implication (4.5.20) holds true for every $\epsilon>0$, we have the assertion that $y\in\overline{\pi(x)}$.

The second lemma depends on the motion through the point x.

Lemma 4.5.21

Let $x\in X$ be such that π^X is continuous. If (x_n) and (y_n) satisfy relations (4.5.7) and (4.5.8), and

(4.5.22) $y_n\in\overline{\pi(x_n)}$,

then $y\in\overline{\pi(x_n)}$

Proof

For every n there is a sequence (t_n) such that $y_n = \pi(t_n,x)$. There are two possibilities;

(1) (t_n) contains a subsequence (t_{p_n}) convergent to some $t^* \in R_*$

(2) (t_n) contains a subsequence (t_{q_n}) tending to

infinity.

In the first case we have, by relation (4.5.8), and continuity of π^X that

$$y = \lim_{n\to\infty} \pi(t_{p_n}, x) = \pi(t^*, x) \in \pi(x)$$

In the second case we obtain $y = \lim (t_{q_n}, x) \in \Lambda(x)$

The two cases imply that, everywhere,

(4.5.23) $y \in \pi(x) \cup \Lambda(x)$ or $y \in \overline{\pi(x)}$

Remark 4.5.24

The continuity of π^X and the statement of Lemma (4.5.21) can be connected with definition (4.2.7) in the following manner.

By continuity of π^X we have that $\pi(x)$ is continuous, hence $\lim \pi(x_n) = \pi(x)$. Then $y \in \overline{\pi(x)}$ can be replaced by $p(y, \pi(x)) = 0$ (or that there exists $\{x_n\}$ in $\pi(x)$ such that $x_n \dashrightarrow y$.

Since $\overline{\pi(x)} = \pi(x) \cup \Lambda(x)$, we can have $\lim_{n\to\infty} \pi(t_n, x) = y \in \Lambda(x)$ so that $p(y, \Lambda(x)) = 0$.

Therefore we can make the following definition:

The mapping $x \dashrightarrow \Lambda(x)$ is said to be upper semi-continuous at the point x if and only if $x_n \in X$, $x_n \dashrightarrow x$, $y_n \in \Lambda(x_n)$, $y_n \dashrightarrow y$ and $p(y, \Lambda(x)) \dashrightarrow 0$

We have been considering the semi-stability of the motion π^X. The semi-stability of sets will now be considered in its own merit. Before then, we give the following relevant definition.

Definition 4.5.25

A set M is semi-stable if and only if for every $x\in M$ and every $\epsilon>0$ there are $\delta>0$ and $u\in R_+$ such that if $z\in B(x,\delta)$ then $p(\pi(t,z), M) < \epsilon$ for $t\geq u$.

Remark 4.5.26

If $x\in\pi(x)$, condition (4.5.5) is satisfied if the trajectory $\pi(x)$ is semi-stable in the sense of definition (4.5.25). The converse is not true in general.

In view of definition (4.5.25) and remark (4.5.26) we state the following propostion.

Proposition 4.5.27

If for every t the mapping π_t is open then for $M = \pi(x)$, relation (4.5.5) implies definition (4.5.25).

Proof

Assume that the condition (4.5.5) is true. Since for each $x\in X$, $\pi(t,x) \in \pi(x)$, it follows that $\pi(x)=(\pi_t(x):t\in R)$ forms on open subset of X. Therefore by setting $M = \pi(x)$ we have $p(\pi(t,x),M) <\epsilon$ for $t\geq u$.which is definition (4.5.25).

We state the following corollary.

Corollary 4.5.28

If (X, R; π) is a system such that for every t the mapping π_t is continuous then;

(4.5.29) relation (4.5.5) is fulfilled for $\pi(x)$ =

$\pi_+(x)$ implies

definition (4.5.25) is fulfilled for M = $\pi_+(x)$.

Proof

By definition, $\pi_+(x) = (\pi(t,x): t\in R_*)$. Then $u\in R_*$ and $p(\pi(t,z), \pi_+(x)) < \epsilon$ for $t\geq u$ and $z\in B(x,\delta)$ holds, which is what relation (4.5.5) stipulates. Therefore by proposition (4.5.27), definition (4.5.25) is satisfied for $M = \pi_+(x)$.

Remark 4.5.30

By what we established in remark (4.1.5), the two conditions of relation (4.5.29) hold true if $(X,R;\pi)$ is a dynamical system.

The following proposition will be needed in the proof of the next theorem that we shall state.

Proposition 4.5.31

If $(r_n) \subset R_*$ with $r_n \dashrightarrow \infty$, then there exist $\beta>0$ and $n^*\in N$ such that $\pi(r_n,y) \notin \overline{B(y,\beta)}$ for $n>n^*$.

Proof

Suppose the contrary. So for every $\beta>0$ there is a subsequence (r_{n_k}) of $\{r_n\}$ such that $\pi(r_{n_k},y) \in \overline{B(y,\beta)}$ for every k. Let us take $\beta = 1, \frac{1}{2}, 1/3,\ldots$. For every m we have a subsequence $(r_{n_{m,k}})$ of (r_n) such that

(4.5.32) $\pi(r_{n_{m,k}}, y) \in \overline{B\ (y, \frac{1}{m})}$, k, m = 1,2,...

Now take $w_1 := r_{n_{1,k}}$ and find w_2 among members of the sequence $(r_{n_{2,k}})$, k = 1,3...., in such a way that $w_1 < w_2$. Then we find w_3 among the members of

the sequence $(r_{n_{3,k}})$ k = 1,2..., in such a way that $w_2 < w_3$, and so on.

Having w_{p-1}, we can find w_p among the members of the sequence $\{r_{n_{p,k}}\}$ k=1,2..., in such a way that $w_{p-1} < w_p$. By so doing we shall obtain the ordering

$w_1 < w_2 < w_3 < \ldots < w_{p-1}, < w_p$.

By construction, as p -->00 we have w_p ---> 00. So that in general w_m --> 00 as m---> 00.

The closed ball $\overline{B(y,\frac{1}{m})}$ and the sequence w_p both depend on m, hence as m --> 00 the ball becomes smaller. This means that $\pi(r_{n_{m,k}}, y)$ cannot belong to the closed ball $\overline{B(y, \frac{1}{m})}$. This is a contradiction. Hence the proposition is implied.

Remark 4.5.33

For values of m and k, the subsequence $\{r_{n_{m,k}}\}$ can be shown in a matrix as given below:

$$\begin{bmatrix} r_{n_{1,1}} & r_{n_{1,2}} & \ldots & r_{n_{1,k}} \\ r_{n_{2,1}} & r_{n_{2,2}} & \ldots & r_{n_{2,k}} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n_{m,1}} & r_{n_{m,2}} & \ldots & r_{n_{k,k}} \end{bmatrix}$$

Since we are starting from the first entry $\{r_{n_{1,1}}\}$ for values of m and k, a possible selection of w_p's is the set of the diagonal elements of the matrix. The double sequence in the main diagonal will converge under the condition that the sequences forming each row and column converge respectively.

The following theorem combines lemma (5.4.3) with local compactness of X to investigate the regularity

of the mapping (4.3.1.) on the dynamical semi-system (X,R_*,π).

Theorem 4.5.34

Assume that X is locally compact and (X,R_*, π) is a dynamical semi-system. Let $x\in X$ be such that the condition (4.5.5) holds true, and morever,

(4.5.35) for every $\epsilon>0$ there is $\delta>0$ such that for every $z\in B(x,\delta)$ and every $t\in R_*$ there is $u\in R_*$ for which $p(\pi(t+u,x), \pi(t+u,z)) < \epsilon$.

If (x_n) is a sequence of elements of X such that $x_n \dashrightarrow x$ as $n \dashrightarrow \infty$

$\bigwedge(x_n) = (y_n)$ for every n,

and $y_n \dashrightarrow y$ as $n \dashrightarrow \infty$

then $y\in\bigwedge(x)$.

Proof

By virtue of lemma (4.5.3) we have that $y\in\pi(x)$, and so

(4.5.36) $y\in\pi(x) \cup \bigwedge(x)$.

We have to prove that $y\in\bigwedge(x)$. Assume the contrary. Then

(4.5.37) $y\in \pi(x)\setminus\bigwedge(x)$

Since $y\in\pi(x)$, there is $s\in R_*$ such that

(4.5.38) $y = \pi(s,x)$

From the scheme of proof of theorem (4.3.2) we recall the followings;

(4.5.39) the statement that $y_n\in\bigwedge(x_n)$ means for every $m\in N$ there is a sequence (t^m_k), $k=1,2,\ldots$, such that

$t^m_k \dashrightarrow \infty$ as $m \dashrightarrow \infty$ for every m.

(4.5.40) from the sequence (t^m_k), $k = 1,2,....,$ we constructed a sequence (s_n) such that for $n \to \infty$ $s_n \to \infty$, $\pi(s_n x_n) \to y$ and $p(\pi(s_n,x_n), y) \to 0$.

Let $\gamma>0$ be such that $B(y,\gamma)$ is relatively compact. By proposition (4.5.31), let $\beta>0$ and n_1 be such that

(4.5.41) $\pi(s_n,y) \notin \overline{B(y,\beta)}$ for $n>n_1$

Let us put

(4.5.42) $\eta := \min(\frac{1}{2}\beta,\gamma)$

By virtue of implication (4.5.35) we have that

(4.5.43) there is $\delta>0$ (chosen for $\epsilon = \frac{1}{2}\beta$) such that for every $z\in B(x,\delta)$ and $t\in R_*$ there is $u\in R_*$ for which

$p(\pi(t+u,x), \pi(t+u,z)) < \frac{1}{2}\beta$.

Now choose n_2 such that

(4.5.44) $x_n \in B(x,\delta)$ for $n>n_2$.

Since (y_n) tends to y, there is n_3 such that

(4.5.45) $y_n \in B(y,\eta)$ for $n>n_3$.

Now put

(4.5.46) $n^* := \max(n_1,n_2,n_3)$

and fix some $m=m^*$ greater than n^*. From relations (4.5.41) to (4.5.46) it follows that

(4.5.47) $\pi(s_n,y) \notin B(y,\beta)$ for $n>n^*$

(4.5.48) $x_{m^*} \in B(x,\delta)$

(4.5.49) $y_{m^*} \notin B(y,\eta)$ (since $m^*>n^*$)

According to implication (4.5.39) we have

$y^*_{m^*} = \lim \pi(t^{m^*}_k, x^*_{m^*})$, $t^{m^*}_k \to \infty$ as $k \to \infty$

and so there is k^* such that

(4.5.50) $\pi(t^{m^*}_k, x_{m^*}) \in B(y,\eta)$ for $k>k^*$ (recall $m^*>n^*$)

Now put

(4.5.51) $r_1 := t^{m*}_{k*+1}, m_1 = n^*+1$

and choose q_1 in such a way that

(4.5.52) $p(\pi(s_{m_1}+r_1+s+q_1, x_m), \pi(s_{m1}+r_1+s+q_1,x)) < \frac{1}{2}\beta$;

such a q_1 exists because of implication

(4.5.35) when applied to $\epsilon = \frac{1}{2}\beta$, and $t=s_{m_1}+r_1+s$, ($u=q_1$)

Now we take an integer $k_2>m_1$ such that

$t^{m*}_{k*+k_2}>q_1$;

Put

(4.5.53) $r_2 := t^{m*}_{k*+k_2}$

and find $m_2 > m_1$ such that

(4.5.54) $s_{m_2} > s_{m_1}+r_1+q_1$

We can find q_2 such that $q_2>q_1$ and

(4.5.55) $p(\pi(s_{m_2}+r_2+s+q_2,x_{m*}), \pi(s_{m_2}+r_2+s+q_2)) < \frac{1}{2}\beta$

Generally, if r_p, k_p, m_p and q_p are defined, we choose m_{p+1} in such a way that $m_{p+1} > m_p$ and $s_{m_{p+1}} > s_p+r_p+q_p$ and k_{p+1} such that $k_{p+1} > m_p$ and $t^{m*}_{k*+k_{p+1}} > q_p$ and define

$r_{p+1} := t^{m*}_{k*+k_{p+1}}$

and finally choose q_{p+1} such that $q_{p+1} > q_p$ and

$p(\pi(s_{m_{p+1}}+r_{p+1}+s+q_{p+1},x_{m*}), \pi(s_{m_{p+1}}+r_{p+1}+s+q_{p+1}, x)) < \frac{1}{2}\beta$.

Putting

(4.5.56) $w_p := s_{m_p} + r_p + s + q_p$,

We get a sequence (w_p) such that

(4.5.57) $w_p \dashrightarrow 00$ as $p \dashrightarrow 00$

(4.5.58) $p(\pi(w_p,x_{m*}), \pi(w_p, x)) < \frac{1}{2}\beta$, $p=1,2,...$

This implies that

(4.5.59) $\pi(w_p, x_{m*}) \notin \overline{B(y,\eta)}$ for p=1,2,....

On the other hand, putting for simplicity

(4.5.60) $v_p := t_p^{m*}$, p = 1,2,...

we obtain

(4.5.61) $v_p \dashrightarrow 00$ as $p \dashrightarrow 00$

and

(4.5.62) $\pi(v_p, x_{m*}) \in \overline{B(y,\eta)}$, p = 1,2,...

From relations (4.5.57) and (4.5.61) it follows that we may assume without loss of generality that

(4.5.63) $v_p \leq w_p < v_{p+1}$, p = 1,2,...

The mapping π is continous, so that because of relations (4.5.59), (4.5.62) and (4.5.63) we can find for every fixed p a real number c_p such that (4.5.64)

$w_p \leq c_p < v_{p+1}$

and

(4.5.65) $\pi(c_p, x_{m*}) \in \partial B(y,\eta)$.

It is clear that by relation (4.5.64), for $p \dashrightarrow 00$ we have (4.5.66) $c_p \dashrightarrow 00$.

And by relation (4.5.42) we have $\eta \leq \gamma$, and then $\partial B(y,\eta) \subset \overline{B(y,\gamma)}$. Therefore, by compactness of $\overline{B(y,\gamma)}$ we may assume without loss of generality that the sequence $\{\pi(c_p, x_{m*})\}$ is convergent to some $y^* \in \partial B(y,\eta)$. Relation (4.5.66) however, implies that $y^* \in \Lambda(x_{m*})$. So that by virtue of the assumption that $(y_n) = \Lambda(x_n)$ in the hypothesis,

(4.5.67) $y^* = y_{m*}$

This is a contradiction to the implication (4.5.49). Thus relation (4.5.37) cannot hold. Hence $y \in \bigwedge(x)$.

Remark 4.5.68

It is plausible to have represented

$r_{p+1} = t^{m*}_{k*} + k_{p+1}$

because as p -->00, k_p --> 00 and k_{p+1} --> 00 so that k^*+k_{p+1} --> 00. and by the implication (4.5.39) we have that $t^{m*}_{k*+k_{p+1}}$ --> 00 as p --> 00

Remark 4.5.69

In relation (4.5.59) we have that $\pi(w_p, x_{m*}) \notin \overline{B(y,\eta)}$ for p = 1,2,.... because by definition η : = min ($\frac{1}{2}\beta$; γ) and in remark (4.5.68) we have

$p(\pi(w_p, x_{m*}), \pi(v_p, x)) < \frac{1}{2}\beta$ for p = 1,2,...

Hence $\pi(w_p, x_{m*}) \notin B(y,\eta)$ for p=1,2,...

On the other hand we have in the relation (4.5.50) that $\pi(t^{m*}_k, x_{m*}) \in B(y,\eta)$ for $k>k^*$. So that for any k=p and $m^* > n^*$ $\pi(t^{m*}_p, x_{m*}) \in \overline{B(y,\eta)}$ for p = 1,2,....

Remark 4.5.70

However, if X is not locally compact, in theorem (4.5.34) the statement that $\bigwedge(x_n) = (y_n)$, for every n, may not be true and relation (4.5.47) will not hold. But conditions like those in lemma (4.5.21), implication (4.5.39) and relation (4.5.42) will be satisfied.

But if X is relatively compact at each point $x \in X$, the theorem will hold. This is because the closure of any neighbourhood relative to each point $x \cap X$ is compact.

As an analogy to theorem (4.5.34) we have the following theorem.

Theorem 4.5.71

Assume that X is locally compact, $(X, R_+; \pi)$ is a dynamical semi-system, and $x \cap X$ is such that the implication (4.5.5) and implication (4.5.35) are satisfied, and

(4.5.72) There is a neighbourhood V of x such that for every $z \in V$ the limit set $\Lambda(z)$ has exactly one element.

Then $\Lambda(x)$ has exactly one element and the mapping

(4.5.73) $V \ni y \longrightarrow L(y) \in X$,

where L(y) denotes the unique element of $\Lambda(y)$, is continuous at x.

Proof

Let CL(X) denote the class of all closed subsets of X. First we shall prove that the mapping

(4.5.74) $X \ni y \longrightarrow \Lambda(y) \in CL(X)$

is upper semi-continuous at x. Let (x_n) and (y_n) be such that $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $y_n \in \Lambda(x_n)$.

For n sufficiently large we have $\Lambda(x_n) = (y_n)$ for infinitely many n, OR $y_n \in \Lambda(x)$ for infinitely many n. (the second case appears when $x_n = x$ for infinitely

many n).

In the first case we shall apply theorem (4.5.34) for the subsequence (y_{n_k}) of those members of (y_n) such that $\bigwedge(x_n) = (y_n)$ for every n. So that $y_n \dashrightarrow y$ implies $y \in \bigwedge(x)$.

In the second case we shall obtain $y \in \bigwedge(x)$ directly since y is the limit of the sequence (y_n) which are elements of $\bigwedge(x)$. Thus $y \in \bigwedge(x)$ in each case.

Now we shall prove that $\bigwedge(x)$ has exactly one element. Since $\bigwedge(x)$ is non-empty it is enough to show that $\bigwedge(x)$ has at most one element.

Suppose the contrary, and let z,w be two distinct elements of $\bigwedge(x)$. We put $\alpha = p(z,w)$ and $z^0 = \alpha/3$. Take ε so small that the balls $B(z,\varepsilon)$ and $B(w,\varepsilon)$ are relatively compact. There are two sequences (t_n) and (s_n) such that $t_n \dashrightarrow 00$, $s_n \dashrightarrow 00$, $\pi(t_n,x) \dashrightarrow z$ and $\pi(s_n, x) \dashrightarrow w$. Then

(4.5.75) $p(\pi(t_n,x), z) < \varepsilon/2$ and $p(\pi(s_n, x), w) < \varepsilon/2$ for n sufficiently large.

On the other hand there is $\delta > 0$ such that for $y \in V \setminus \{x\}$, $p(y,x) < \delta$ and by implication (4.5.35) we have $p(\pi(t,x), \pi(t,y)) < \varepsilon/2$ for $t \geq 0$, and in particular for every n, $p(\pi(t_n, y), \pi(t_n,x)) < \varepsilon/2$ and $p(\pi(s_n,y), \pi(s_n,x)) < \varepsilon/2$. Thus by implication (4.5.75) and triangle inequality we have $p(\pi(t_n, y), z) < \varepsilon$ and $p(\pi(s_n, y), w) < \varepsilon$.

Since $\overline{B(z,\varepsilon)}$ and $\overline{B(w,\varepsilon)}$ are compact we may assume without loss of generality that the sequences $(\pi(t_n, y))$ and $(\pi(s_n, y))$ are convergent. The limits of these sequences are z and w respectively and belong to $\Lambda(x)$. Hence z=w. This is in view of the fact that X is a metric space and the limit if it exists is unique. This contradicts the fact that $p(z,w) > 0$. Hence $\Lambda(x)$ has exactly one element.

In order to finish the proof we observe that the upper semi-continuity of a set-valued mapping which takes values being one-element sets is equivalent to the usual continuity of the induced single-valued mapping. So the upper semi-continuity of the map (4.5.74) implies the continuity of the function (4.5.73). This completes the proof.

We state the next corollary.

Corollary 4.5.76

If X is locally compact, (X,R_+,π) is a dynamical semi-system, $\pi(x)$ is semi-stable, π^X is weakly semi-stable and relation (4.5.72) is satisfied. Then the mapping (4.5.74) is continuous at *x*.

Proof

To prove the corollary is suffices to show that the semi-stability of $\pi(x)$, for $x \in X$, is equivalent to relation (4.5.5) and π^X is weakly semi-stable corresponds to the implication (4.5.35).

In the sense of definition (4.5.25) and remark (4.5.26)

$\pi(x)$ is semi-stable if and only if for every $x \in \pi(x)$ and every $\varepsilon > 0$ there are $\delta > 0$ and $u \in R_+$ such that if $z \in B(x,\delta)$ then $p(\pi(t,z), \pi(x)) < \varepsilon$ for $t \geq u$, which is relation (4.5.5.)

Also π^X is weakly semi-stable if and only if for $\varepsilon > 0$ there is $\delta > 0$ such that for every $z \in B(x,\delta)$ there is $u \in R_+$ for which $p(\pi(t+u,x), \pi(t+u,z)) < \varepsilon$ for $t \geq 0$, which is implication (4.5.35).

Then by theorem (4.5.71), for $x \in X$, $\bigwedge(x)$ has exactly one element and the mapping (4.5.73) is continuous at the point x. Also, since the mapping (4.5.73) is a single - valued mapping induced by the mapping (4.5.74) then the continuity of the mapping (4.5.73) implies the continuity of the mapping (4.5.74).

Remark 4.5.77

In the proof of corallary (4.5.76) we showed that the semi-stability of $\pi(x)$ is equivalent to the implication (4.5.5) and the weak semi-stability of π^X is equivalent to implication (4.5.35).

But implication (4.5.5) does not imply implication (4.5.35) and vice versa. Also, semi-stability of $\pi(x)$ does not imply weak semi-stability of π^X and vice versa. However, semi-stability of $\pi(x)$ will be equivalent to the weak semi-stability of π^X if and only if $\pi(x)$ is positively stable in the sense of Lyapunov, the mapping π_t is open and π^X is weakly positively stable in the Lyapunov sense. It then means that implication (4.5.5) is

equivalent to implication (4.5.35).

We shall state and prove the following theorem as an analogy to lemma (4.5.3) and lemma (4.5.21).

Theorem 4.5.78

Assume that (X, R_*, π) is a dynamical semi-system, $x \in X$ is such that implication (4.5.5) is fulfilled. Then the mapping

(4.5.79) $X \ni y \dashrightarrow \overline{\pi(y)} \in CL(X)$

is upper semi-continuous.

Proof

Let (x_n) and (y_n) such that $x_n \dashrightarrow x$, $y_n \dashrightarrow y$ and $y_n \in \overline{\pi(x_n)}$ for every n. By lemma (4.5.3), there is a subsequence (y_{n_k}) of (y_n) such that $y_{n_k} \in \Lambda(x_n)$ and $y \in \overline{\pi(x)}$.

OR by lemma (4.5.21), there is a subsequence (y_{m_p}) of (y_n) such that $y_{m_p} \in \overline{\pi(x_{m_p})}$ and $y \in \overline{\pi(x)}$.

Therefore, $y \in \overline{\pi(x)}$ in both cases imply that the mapping (4.5.79) is upper semi - continuous.

Conclusion 4.5.80

This chapter is a further extension of the results obtained in the last two chapters in the sense that this chapter deals with the upper semi-continuity and the continuity of the limit set map. From definition the limit set of a point added with the tragectory passing through that point gives us the closure of the trajectory passing through that point.

Under suitable assumptions we have shown that if the motion through a point x is going to be semi-stable

with the map (4.3.1) being upper semi-continuous we have shown that the trajectory emanating from a point z in the neighbourhood of point x approaches the trajectory originating from point x.

In chapters 1 and 2 we talk of stability of motions (trajectories). In this chapter we discussed - semi-stability or weak- - semi- stability of motions or actions. Such theorems in this chapter would give similar results as we have done in chapters 1 and 2 if $\alpha = 0$.

Moreover, under suitable assumptions both on X and the trajectories emanating from the point $x \in X$ and motions through x we have shown that the map (4.3.1) is continuous.

CONCLUSIONS

In this thesis, we studied the generalization of stability concepts for generalized pseudo-dynamical systems. In this way some stability properties of solutions of ordinary differential equations without uniqueness may be generalized as well.

Other aspects of stability like semi-stability for generalized pseudo-dynamical systems may be investigated as well. For example, here is a direct generalization of the concept of semi-stability with respect to local pseudo-dynamical systems;

"A set M is said to be semi - (Ω, β) - stable with respect to λ if and only if

for each $x \in M$ and each $\theta \in \Omega$, there exists $u \in I_x$ and there exists $B \in \beta(x)$ such that $\lambda(y) \subset \theta$ for every $y \in \lambda(u,z)$ and every $z \in B$ such that $u \in I_z$."

We also studied some results that hold for semi-stability and limit sets for motions in metric spaces. The result can be generalized and extended for Hausdorff topological spaces satisfying the first axiom of countability.

REFERENCES

JOURNAL REFERENCES

1. Bathia N.P. et al (1965). On global weak attractors in dynamical systems. Journal of differential Equations and dynamical systems, Vol (1) 375-379.

2. Joseph Auslander (1969). On stability of closed sets in dynamical systems. Journal of Differential Equations and dynamical systems, Vol.(II), Springer-Verlag.

3. Roxin E. (1965). On generalized dynamical systems defined by contigent equations. Journal of Differential Equations, vol (1), 188-205.

4. Seibert P. (1968). A concept of stability in dynamical systems. Topological Dynamics, New York.

5. Seibert P. (1969). Relative stability and stability of closed sets. Journal of Differential Equations and dynamical Systems, Vol.(II), Springer-Verlag.

BOOK REFERENCES

6. Bachman G. and Narici L. (1966) Functional Analysis, Academic Press.

7. Bathia N.P. and Szego G.P. (1970). Stability theory of Dynamical Systems, Pringer-Verlag.

8. Cesari L. (1963). Asymptotic Behaviour and Stability Problems in Ordinary Differential Equations, 2nd Ed. Springer-Verlag.

9. Coddington E.A. and Levinson N. (1955). Theory of ordinary Differential Equations, McGraw-Hill, New York.

10. Eisenberg M. (1974). Topology, Holt, Rinehart and Winston.

11. Hurewicz W. (1958). Lectures on Ordinary Differential Equations, M.I.t. Massachusetts.

12. Kreyszig E. (1978). Introductory Functional analysis with applications, John Wiley.

13. Lynn A.S. and Seebach J.A. (1970). Counter Examples in Topology, Holt, Rinehart and Winston.

14. Nemytskii V.V. and Stepanov V.V. (1949). Qualitative theory of Differential Equations, Moscow.

15. Olubummo A. (1979). Introduction to Real Analysis, Heinemann Nigeria.

16. Rao R.M.M. (1981). Ordinary Differential Equations. Theory and applications, Edwin Arnold U.K. 190-253.

17. Sibirsky K.S. (1970). Introduction to topological Dynamics, Noordiholff, Lyden.

18..Simmons G.F.(1972). Differential Equations with Applications and Historical Notes, Tata McGraw New Delhi.

OTHER REFERENCES

19. Adewale M.O. (1989). Introduction to dynamical systems. Lecture Notes in M.Sc. Mathematics, Second semester A.B.U., Zaria.