AN EXAMINATION OF THE MOTION OF AN OBLATE INFINITESIMAL BODY IN THE GENERALIZED PHOTOGRAVITATIONAL ER3BP WITH CIRCUMBINARY DISC

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DEPARTMENT OF MATHEMATICS,

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OCTOBER, 2021

DECLARATION

I declare that the work in this dissertation titled **AN EXAMINATION OF THE MOTION OF AN OBLATE INFINITESIMAL BODY IN THE GENERALIZED PHOTOGRAVITATIONAL ER3BP WITH CIRCUMBINARY DISC** has been carried by me under the supervision of Dr. (Mrs.) A. Umar and Prof. J. Singh in the Department of Mathematics, Ahmadu Bello University, Zaria. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at any other institution.

Umar Bashir	_____	_____
Name of Student	Signature	Date

CERTIFICATION

This Dissertation titled **AN EXAMINATION OF THE MOTION OF AN OBLATE INFINITESIMAL BODY IN THE GENERALIZED PHOTOGRAVITATIONAL ER3BP WITH CIRCUMBINARY DISC** by Umar BASHIR meets the regulations governing the award of the degree of Master of Science of the Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

To my late mothers, diamonds in the rough

ABSTRACT

This study examined the motion of an oblate infinitesimal body under the influence of two finite luminous bodies together with the gravitational potential from a disc, within the framework of the elliptic restricted three body problem. The locations and stability of the equilibrium points with respect to the eccentricity, oblateness of the third body, radiation pressure parameters and gravitational potential from a disc were established. A close examination of μ_c, shows that all the parameters cause a reduction in the size of the region of stability confirming their destabilizing tendencies, while the gravitational potential from the circumbinary disc has a stabilizing effect. The study also, numerically highlights the influence of the aforementioned parameters on the locations/stability of the triangular and collinear points for the binary systems HD98800B and GJ630.1A. It is observed that the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c < \mu < \frac{1}{2}$, and in the case of collinear points we see that the equation determining the distance ratio among the three masses is an eleventh-order polynomial. The collinear points however remain unstable even under the effects of radiation pressure of both the primaries, oblateness of the third body and potential from a disc.

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LIST OF SYMBOLS

ξ, η *and* ζ	Are the Coordinates
μ	Mass ratio
μ_c	Critical value of the mass parameter
m_1	Mass of the bigger primary
m_2	Mass of the smaller primary
m_3	Mass of the infinitesimal body
Ω	Potential force function
A	Oblateness of the infinitesimal body
r_1	Distance from the third body to the bigger primary
r_2	Distance from the third body to the smaller primary
r_c	Radial distance of the infinitesimal body
M_b	Disc potential
T	Determines the density profile of the circular cluster of material points
$a = 1 - \alpha$	Semi-major axis of the system, where α is a small value less than 1 (i.e. $0 < \alpha << 1$)
e	Eccentricity of the orbits of the primaries

$q_i = 1 - \beta_i$	Radiation coefficient, where β is the ratio of the radiation pressure force to the gravitational force of the primaries, $\beta_i << 1$, ($i = 1,2$)
n	Mean motion

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

In a three-body problem, three bodies move in space under their mutual gravitational interactions as described by Newton's law of gravitation. Solutions of this problem require that future and past motions of the bodies be determined uniquely based solely on their present positions and velocities. In general, the motions of the bodies take place in three dimensions (3D), and there are no restrictions on their masses or on the initial conditions. Thus, this is referred to as the general three-body problem. At first glance, the difficulty of the problem is not obvious, especially when considering that the two-body problem has well-known closed form solutions given in terms of elementary functions.

Adding one extra body makes the problem too complicated to obtain similar types of solutions. In the past, many physicists, astronomers and mathematicians attempted unsuccessfully to find closed form solutions to the three-body problem. Such solutions do not exist because motions of the three bodies are in general unpredictable, which makes the three-body problem one of the most challenging problems in the history of science.

In celestial mechanics, the general three-body problem deals with gravitationally interacting astronomical bodies and intends to predict their motions. In our Solar System, the planets and asteroids move around the Sun, while the moons orbit their host planets, which in turn also move around the Sun. As typical examples of the three-body problem, consider the Sun-Planet-Planet, Sun-Planet-Moon, or Sun-Planet- Asteroid systems. The three-body problem representing the latter system can be significantly simplified because the mass of the asteroid is always negligible when compared to the mass of either the Sun or the Planet, which means that the gravitational influence of the Asteroid on the Planet

and the Sun can be omitted from the theory, If this condition is satisfied, then the general three-body problem becomes the restricted three-body problem, and there are two possibilities, namely, the two bodies with dominant masses move around their center of mass either along circular or elliptic orbits, which leads to the respective circular or elliptic restricted three-body problems. It is the circular restricted three-body problem (CR3BP) that has been the most extensively studied. The three-body problem has been studied for over three hundred years.

A brief historical overview of the most significant results that have contributed to the current developments in the 3BP can be divide into two parts: first, the time period from the publication of Newton's Principia in 1687, to the publication of Poincare's Les M'ethodes Nouvelles de la M'ecanique C'eleste published in 1892(volume 1 and 2) and in 1899 (volume3).

The three-body problem was formulated and studied by Newton in 1687 in his Principia, where he considered the motion of the Earth and the Moon around the Sun. In this problem, there are key elements such as the ratio of the Moon's mass, MM, to the Earth's mass, ME, that is MM/ME = 0.0123, which is a small but not negligible number. Another element is the tilt of the Moon's orbit to the Earth's orbit, about 5°. This makes the problem completely predictable despite the fact that the Earth's orbit around the Sun and the Moon's orbit around the Earth are almost perfect circles. Newton stated that the problem was very difficult to solve; however, he was able to obtain an approximate solution which agreed with observations to within 8%.

A special form of the general three-body problem was proposed by Euler in 1767. He considered three bodies of arbitrary (finite) masses and placed them along a straight line. Euler showed that the bodies would always stay on this line for suitable initial conditions, and that the line would rotate about the center of mass of the bodies, leading to periodic

motions of all three bodies along ellipses. Around the same time, Lagrange in 1772 found a second class of periodic orbits in the general three-body problem. He showed that if the bodies were positioned in such a way that they form a triangle of equal sides which would move along ellipses for certain initial conditions, preserving always their original configuration. The Euler and Lagrange solutions are now known as particular solutions to the general three-body problem.

Euler in 1767 was the first to formulate the CR3BP in a rotating (or synodic) coordinate system. This was an important development in the study of the three-body problem. Lagrange also studied the CR3BP and demonstrated that there were five equilibrium points (now known as the Lagrange points) at which the gravitational forces of the bodies balanced out; the Trojan asteroids discovered in 1906 along Jupiter's orbit occupy space close to two of the Lagrange points. Important contributions to the CR3BP were made by Jacobi in 1836; he used the synodic (rotating) coordinate system originally introduced by Euler in 1767 to demonstrate that there was an integral of motion, which is now named after him. The Jacobi integral was used by Hill in 1877and 1878 to determine the motion of an asteroid in the three-body problem and to introduce the so-called zero velocity curves (ZVC), which establish regions in space where the bodies are allowed to move. Hill considered a special case of the CR3BP in which two masses were much smaller than the first one (the problem is now known as the Hill problem), and in this way, he discovered a new class of periodic solutions. His main contribution was to present a new approach to solve the Sun-Earth-Moon three-body problem, After almost two hundred years since the original formulation of the problem by Newton 1687, Hill developed his lunar theory, with some modifications made by Brown in 1896.

In the second half of the nineteenth century, Poincare studied and advanced the solution of the Three-body problem. His monumental three-volume book Les M'ethodes

Nouvelles de la M'ecanique C'eleste, originally published in 1892-99, contains his most important contribution to the study of the CR3BP; the book was translated as New Methods of Celestial Mechanics and edited by L. D. Goroff in 1993. In this work, Poincare developed a number of new qualitative methods to solve differential equations and used them to identify and study possible periodic orbits, while demonstrating the non integrability of the system of equations describing the three-body problem. Poincare's new methods allowed him to identify the unpredictability of the problem, and to discover the first manifestation of a new phenomenon, which is now commonly known as chaos, Poincare submitted some of those results for the King Oscar II Birthday Competition and he was awarded the prize (Musielak & Quarles, 2015).

1.2 Statement of the Problem

The orbits of most celestial bodies are elliptic, which has prompted many researches (Kumar and Choudry, 1990; Markellos *et al*., 1992; Zimovschikov and Tkhai, 2004; Ammar, 2008; Kumar and Ishwar, 2009, 2011). Recently, Singh and Umar (2012) have investigated the motion of a dust particle, in an orbit with dark oblate, degenerate primary and a stellar secondary companion; around triangular equilibrium points when both oblate and luminous primaries move around their common center of mass respectively, in the framework of the ER3BP.

There are ring-type discs of dust particles in the stellar systems which are regarded as young analogues of the Kuiper belt (Singh and Taura, 2014). Of an observed 69 A3-F8 main sequence binary star systems, nearly 60% showed dust rings surrounding binary stars. This inspired many scientists to study the CR3BP by taking into consideration the additional gravitational potential from the disc (Pitjeva & Pitjev 2014, Kushvah (2008), Singh & Taura (2013, 2014a, 2014b), lorio (2007, 2012)). In their work Yeh and Jiang (2006) have included the gravitational potential from the belt in their studies of the

CR3BP. Also, Singh and Taura (2013) examined the combined effect of radiation and oblateness of both primaries, together with additional gravitational potential from the disc on the motion of an infinitesimal body in the CR3BP. They, Singh and Taura (2013) studied the effects of oblateness up to J4 of the less massive primary and gravitational potential from a belt on the linear stability of triangular equilibrium points in the photogravitational CR3BP, They found that the gravitational potential from the disc makes the structure of the CR3BP quite different such that new libration points exist under certain conditions. Also, the orbital motion of a test particle around a primary is greatly affected in the presence of a massive disc (lorio, 2007).

Hence this research examines the dynamics of an oblate circumbinary planet in the vicinity of luminous primaries moving in elliptic orbits about their common barycenter, surrounded by ring-type discs. Thus, apart from the Newtonian gravitational attraction between the primaries, the effects of the luminosity of both primaries, oblateness of the circumbinary planet and eccentricity of the orbits of the primaries, an additional gravitational potential from the disc has been considered in this study.

1.3 Aim and Objectives

The aim of this research is to determine the motion of an oblate infinitesimal body in the generalized photogravitational ER3BP with a circumbinary disc when both primaries are sources of radiation. This is achieved through the following objectives:

i. Determining the effect of the parameters on the locations and stability of the triangular equilibrium points
ii. Determining the effect of the parameters on the locations and stability of collinear equilibrium points

iii. Applying the results to obtain numerically the triangular points and determine the stability of the oblate circumbinary planet in the neighborhood of HD 98800 B and GJ630.1A

1.4 Significance of the Study

In recent studies, many perturbing forces such as radiation, variation of the masses, Poynting-Robertson drag (P-R drag), oblateness and triaxiality, have been included in the study of the circular or elliptic R3BP.The orbits of the celestial bodies are mostly elliptic not circular. It is therefore more accurate to study the ER3BP, because it has a significant effect which the CR3BP neglects.

1.5 Justification of the Study

The human civilization is no longer bound to its environment via land and water, but has evolved to a space-faring civilization. So the expansion into space, in terms of artificial satellites, requires the formulation of new models that include the effects of some of the perturbing forces on the satellite. Motion in the elliptic restricted three-body problem is one of such models which consider the effects of radiation of the primaries, an oblate infinitesimal body and gravitational potential from a disc. Thus, this model will be very helpful in the study of binary stars.

1.6 Research Methodology

Following the work of Singh and Umar (2012), in which they studied the ER3BP under oblate and radiating primaries in barycentric, pulsating and non-dimensional co-ordinates, and the work of Singh and Taura (2014), in which they studied the motion of an infinitesimal mass in the framework of the restricted three-body problem (R3BP) under the assumption that the primaries of the system are radiating-oblate spheroids, enclosed by a circular cluster of material points. This study has adopted the equations of motion of

the infinitesimal particle with respect to the radiating primaries, to locate the equilibrium points and study their linear stability when the third body is an oblate circumbinary planet, and the primaries are surrounded by a circumbinary disc.

Therefore, the effects of an oblate circumbinary planet, and the potential from a disc on the binary star HD 98800 B and GJ630.1 A are established. Lastly, using the software packages for numerical computations and plotting of graphs where necessary, the research shows both numerically and graphically the effects of the perturbing forces of oblateness, radiation pressures, and gravitational potential from the circular disc, on the equilibrium points around the triangular points/stability of the problem.

1.7 Theoretical Frame Work

1.7.1 Kepler's Laws of Planetary Motion

The orbit of a planet is the path it follows around the sun. The orbit of every planet is an ellipse with the sun occupying one focus of the ellipse. We owe the discovery of the three laws of planetary motion to Johannes Kepler (1571-1630). He formulated these laws from a study of the mass of observational data on the planets positions collected by Tucho Brahe (1546-1601). The three laws are: -

i. The orbit of each planet is an ellipse with the sun at one focus. This law tells us what the shapes of the planetary orbits are and gives the position of the sun within them,

ii. For any planet, the rate of description of area by the radius vector joining planet to Sun is constant. This indicates that a planet moves in its orbit about the sun at a varying angular velocity.

iii. The cubes of the semi major axes of the planetary orbits are proportional to the squares of the planets period of revolution.

1.7.2 Three-Body Problem

If three particles (of finite masses) are moving under their mutual gravitational attraction and the positions and velocities of these particles are initially given, then we can predict their future motion. This is called the three-body problem (3BP).

If one of the masses of these three particles is negligible, then the 3BP reduces to a problem which is known as the restricted three-body problem (R3BP).

The smallest mass is called infinitesimal mass, third body or test particle and the two massive bodies are called finite bodies or primaries.

If the motion of the infinitesimal mass is determined when the finite bodies move in ellipses about their common center of mass, then the problem is called the elliptical restricted three body problem (ER3BP).

If the motion of the infinitesimal mass is obtained when the finite bodies move in circles about their common center of mass, then the problem is termed as the circular restricted three-body problem (CR3BP).

1.7.3 Radiation and Radiation Pressure

Radiation pressure act as an orbital perturbation and affects the orbits and trajectories of small bodies, all space crafts and all natural bodies (comets, asteroids, dust grains, gas molecules) can cause dust grains to either leave the Solar system or spiral into the Sun. Because of the many importance of radiation pressure, several researchers have included radiation pressure force of either one or both primaries in the study of the R3BP, They include among others, Chernikov (1970), Bhatnagar and Chawla (1979), Singh and Ishwar (1999), AbdulRaheem and Singh (2006, 2008), Singh and Leke (2010), Singh and Umar (2012), Singh and Haruna (2014a, 2014b).

The radiation pressure force F_p changes with distance by the same law as the gravitational force F_g and acts opposite to it. It is possible that the force will lead to a reduction of the

effective mass of the particle. Since this reduction depends on the properties of the particles, it is acceptable to speak about a reduced mass, thus, the resultant force *F* on the particle is

$$F = (F_g - F_p) = F_g\left(1 - \frac{F_p}{F_g}\right) = qF_g$$

Where, $q = 1 - \frac{F_p}{F_g}$ is the mass radiation factor. We denote the radiation factor as q such that $0 < 1 - q << 1$**.**

1.7.4 Oblateness

The degree of flattening of a celestial object, such as a planet, from a true spherical form, largely as a result of rotation is called the oblateness of the celestial object. It is numerically equal to the difference between the equatorial and polar diameter of the object divided by the eccentric diameter.

The rotation of a star produces an equatorial bulge due to centrifugal force; as a result, the stars are often oblate in shape. Neutron stars and black dwarfs which is a result of the cooling of white dwarfs are also oblate due to their rapid spinning after formation, For example, the Earth, Jupiter, Saturn and stars (Archerner, Antares and Altair) have the shape of an oblate spheroid.

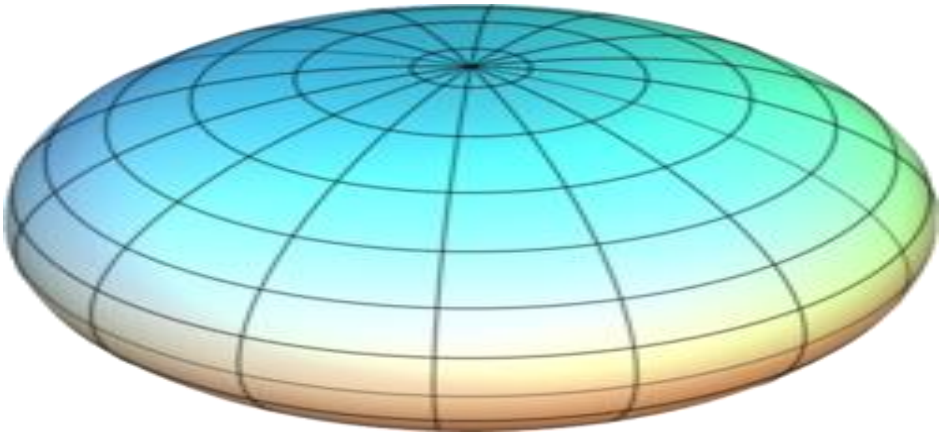


Figure 1. 1: The shape of an oblate spheroid

1.7.5 Eccentricity

The orbital eccentricity of an astronomical body is the amount by which its orbit deviates from a perfect circle and is usually denoted by e. Where for $e < 1$ the orbit is elliptic; for $e = 0$, the orbit is circle; for $e = 1$ the orbit is parabolic and for $e > 1$ the orbit is hyperbolic.

1.7.6 Synodic or Rotating Co-ordinate System

If the system of coordinates is such that the *X -Y* plane rotates in the positive direction with an angular velocity equal to that of the common velocity of one primary with respect to the other keeping the origin fixed, then the coordinate system is known as synodic co-ordinate system. The synodic co-ordinate system is accelerated, since it is rotating with a velocity.

1.7.7 Equilibrium Point and Stability of Equilibrium Point

A particle is said to be at equilibrium point if it is stationary at that point or it is in a steady state. Mathematically, if a dynamical system is in state of equilibrium it remains in that state as $t \to \infty$. Precisely, we can say that, a system of ordinary differential equation $\dot{x} = X(x)$, which possess a point $x = x_{eq}$ .Where $x = (x_1, x_2, ..., x_n)$ and the component of n-vector $X = (X_1, X_2, ..., X_n)$.*A* Solution *x(t)* is called the equilibrium point where $x(t_o) = x_{eq}$. This point is obtained from equation $X(x) = 0$, that is, $x = 0$ is the equilibrium point.

A solution $x = x_{eq}$ or a point 'x_{eq}' is said to be stable if for a given $\varepsilon > 0$ there exist a $\delta(x) > 0$ such that when the disturbances satisfy $\left|x(t_0) - x_{eq}\right| \le \delta$, then for all $t > t_0$, $\left|x(t) - x_{eq}\right| \le \varepsilon$. Otherwise, the equilibrium point $x = x_{eq}$ is unstable.

Mathematically for asymptotic stability, $\lim_{t \to \infty} x(t) = x_{eq}$

1.7.8 Stability of a Linear System

If the system of differential equation $\dot{x} = X(t)$ can be written in the form $\dot{x} = Ax + f(x)$, then the linearized system of equation is $\dot{x} = Ax$. Where *A* is a constant matrix and *f(x)* is a vector function such that $\frac{|f(x)|}{|x|} \to \infty$ for all $t \geq t_0$

The stability condition for $t \geq t_0$ of the linearized system can be stated as:

i. For complex root of characteristic equation of *A,* we have
   - When the characteristic roots all have negative and also when some of the roots are multiple, then the equilibrium point are stable.
   - When some or all the characteristic roots have positive real part, the equilibrium points are unstable. This is also valid for multiple roots.

ii. For pure imaginary roots, the motion is oscillatory and the solution is stable though not asymptotically stable. If these are multiple roots the solution contains mixed (period and secular) terms and the equilibrium point is unstable.

iii. If the roots are real and all negative, the solution is stable; if, however, any of the roots are positive the point is unstable. These statements are also true for multiple roots.

1.7.9 Binary Star System

A binary star is a star system having two stars orbiting around their common center of mass. Systems of three, four, or even more stars are called multiple star systems. These systems, especially when more distant, often appear to the unaided eye as a single point of light, and are then revealed as double (or more) by other means. Over the last two

centuries, researches have suggests that half or more of visible stars are part of multiple star systems (Filippenko, 2007).

In fact, most stars exist in binary systems. The Sun, with its collection of planets, is an exception. Stars in binaries revolve around their common center of mass (which itself moves through space). Examples of binary systems include among many, CM Draconis, Sirius, Achird, Luyten 726-8, Kruger 60, Alpha Centauri AB, Xi Bootis and Gamma Leporis, etc.

1.7.10 Circumbinary Planets

A planet orbiting two stars instead of one is called a circumbinary planet. Because of the short orbits of some binary stars, the only way for planets to form is by forming outside the orbit of the two stars. The system PSR B1620-26, is being orbited by the first confirmed circumbinary extrasolar planet, containing a millisecond pulsar and a white dwarf and is located in the globular cluster M4. In 1993 the existence of the third body was first reported, and was suggested to be a planet based on 5 years of observational data. In 2003 the planet was characterized as being 2.5 times the mass of Jupiter in a low eccentricity orbit with a semi major axis of 23 AU,

1.7.11 Circumbinary Disc

Circumbinary disc that may indicate processes of planet formation have been found around several stars, and are in fact common around binaries with separations less than 3 AU (Trilling et al. 2007). One notable example is in the HD 98800 system, which comprises two pairs of binary stars separated by around 34 AU (Fig 1.2). The binary subsystem HD 98800 B, which consists of two stars of 0.70 and 0.58 solar masses in a highly eccentric orbit with semi major axis 0.983 AU, is surrounded by a complex dust disc that is being warped by the gravitational effects of the mutually-inclined and eccentric stellar orbits (Akeson et al. 2007; Verrier and Evans 2008).

1.7.12 Protoplanetary Disc

A protoplanetary disc is a rotating circumstellar disc of dense gas and dust surrounding a young newly formed star, a T Tauri star, or Herbig Ae/Be star. The protoplanetary disc may also be considered as an accretion disc for the star itself, because gasses or other material may be falling from the inner edge of the disc onto the surface of the star. But this process should not be confused with the accretion process thought to build up the planets themselves. Externally illuminated photo-evaporating protoplanetary discs are called proplyds. Protoplanetary disks have been observed around several young stars in our galaxy. Recent observations by the Hubble Space Telescope have shown proplyds and planetary discs to be forming within the Orion Nebula. Protoplanetary disc are thought to be thin structures, with a typical vertical height much smaller than the radius, and a typical mass much smaller than the Central young star (Armitage, 2011).

1.7.13 Debris Disc

Gas (poor disc) of circumstellar dust have been found around many nearby stars-most of which have ages in the range of ~10 million years (e.g. Beta Pictoris, 51 Ophiuchi) to billions of years (e.g. Tau Ceti). These systems are usually referred to as "debris discs". Given the older ages of these stars, and the short lifetimes of micrometer-sized dust grains around stars due to Poynting Robertson drag, collisions, and radiation pressure (typically hundreds to thousands of years), it is thought that this dust is from the collisions of planetesimals (e.g. asteroids, comets). Hence the debris discs around these examples (e.g. Vega, Alphecca, Fomalhaut, etc.) are probably not truly "protoplanetary", but represent a later stage of disc evolution, where extrasolar analogs of the asteroid belt and Kuiper belt are home to dust-generating collisions between planetesimals.

1.7.14 Asteroid belt

The asteroid belt is the circumstellar disc in the Solar System located roughly between the orbits of the planets Mars and Jupiter. It is occupied by numerous irregularly shaped bodies called asteroids or minor planets. The asteroid belt is also termed the main asteroid belt or main belt to distinguish it from other asteroid populations in the Solar System such as near-Earth asteroids and Trojan asteroids, about half the mass of the belt is contained in the four largest asteroids: Ceres,Vesta, Pallas, and Hygiea as observed by Matt, W. (2015). The total mass of the asteroid belt is approximately 4% that of the Moon, or 22% that of Pluto, and roughly twice that of Pluto's moon Charon (whose diameter is 1200 km).

1.7.15 Asteroid with Disc

Hitherto, rings have been found exclusively around the four giant planets in the Solar System as stated by Tiscarena, (2013). Rings are natural laboratories in which to study dynamical processes analogous to those that take place during the formation of planetary systems and galaxies. Their presence also tells us about the origin and evolution of the body they encircle. Observations of a multichord stellar occultation that revealed the presence of a ring system around Chariklo (10199), which is a Centaur-that is, one of a class of small objects orbiting where also stipulated by Fornasier, et al (2013), primarily between Jupiter and Neptune-with an equivalent radius of 124+9 kilometers. There are two dense rings, with respective widths of about 7 and 3 kilometers, optical depths of 0.4 and 0.06, and orbital radii of 391 and 405 kilometers. The present orientation of the ring is consistent with an edge-on geometry in 2008, which provides a simple explanation for the dimming as explained by Belskaya, (2010) of the Chariklo system between1997 and 2008, and for the gradual disappearance of ice and other absorption features in its spectrum over the same period as forwarded by Guilbert, A. et al. (2009). This implies

that the rings are partly composed of water ice. They may be the remnants of a debris disc, possibly confined by embedded, kilometer-sized satellites. Chariklo is the largest known Centaur orbiting in a region between Saturn and Uranus, and has an orbital eccentricity of 0.175 and a semi major axis of 15.8 astronomical units (1 AU is the Earth-Sun distance). It may be a former trans-Neptunian object that has been recently (less than lOMyr ago) scattered by gravitational perturbations from Uranus as indicated by Horner, et al (2004),

No clear detection of Chariklo's rotation has been made so far. Its surface is very dark with a geometric albedo of 0.035 ± 0.011 as pointed out by Fornasier, et al (2013)., and it is subject to long-term spectral and photometric variabilities (Belskaya, 2010). Other evidence supporting the interpretation of a flat circular ring system around Chariklo are given by Braga- Ribas et al, (2014).

1.7.16 CM Draconis

CM Draconis (GJ 630.1A) is an eclipsing binary system approximately 47 light-years away (Van, 1995) in the constellation of Draco (the Dragon). The system consists of two nearly identical red dwarf stars located in the constellation Draco. The two stars orbit each other with a period of 1.27 days with a separation of 2.7 million kilometers (0.018 AU). Along with two stars in the triple system KOI 126, the stars in CM Draconis are the lightest stars with precisely measured masses and radii. Consequently, the system plays an important role in testing stellar structure models for very low mass stars (Morales et al. 2009). These comparisons find that models under predict the stellar radii by approximately 5%. This is attributed to consequences of the stars' strong magnetic activity. According to the system's entry in the Combined General catalogue of variable stars, at least one of the components is a flare star and at least one is a BY Draconis variable. The white dwarf star GJ 630.1B, located 25.7 arc seconds away shares the same

proper motion as the CM Draconis stars and is thus a true companion star of the system (Deeg et al, 2008), Given the system's distance of 47 light years, this corresponds to a separation of at least 370 astronomical units between CM Draconis and GJ 630.1B (Gliese et al, 1991).

1.7.17 HD 98800

HD 98800, also catalogued as TV Crateris (TV Crt), is a quadruple star system approximately 150 light-years away in the constellation of Crater (the cup). The system is located within the TW Hydrae association. The system consists of "HD 98800 A" and "HD 98800 B" each of which contains two stars.

In 2007, a debris disc was discovered orbiting "HD 98800 B" consisting of two rings which indicates there may be an extrasolar planet orbiting within a distance of 1.5 to 2 astronomical units. The system consists of four T Tauri stars that are located within the TW Hydrae association. Within the system, there are two separate systems of binary stars, each system (separately catalogued as "HD 98800 A" and "HD 98800B") consists of two stars. Although the four stars are gravitationally bound, the distance separating the two binary pairs is about 50 astronomical units (somewhat more than the average distance between the Sun and Pluto). Not much is known about each individual star, except that they are basically solar twin stars.



Figure1. 2: Artists impression of HD 98800.

1.7.18 Exoplanets

Outside of the Solar System there exist planets called exoplanets or extrasolar planets. In 1988 the first scientific detection of an exoplanet was made. However, the first accurate detection was made in 1992; since then, there have been 3,608 exoplanets in 2,702 planetary systems and 610 multiple planetary systems confirmed detections as of 1st May, 2017, (Schneider, 2014). Interest has increased in the search for extraterrestrial life since the discovery of exoplanets. There is keen interest in planets that orbit in a star's habitable zone, since it is possible for liquid water, a basic condition for life on Earth, to exist on the surface. A wide range of other factors in determining the suitability of a planet for hosting life was considered in the study of planetary habitability (Overbye, 2015). On average, there is at least one planet per star (Cassan, *el al.* 2012). One in five Sun-like stars has an "Earth-sized" planet in the habitable zone (Petigura, et al 2013).

Majority of known exoplanets orbit stars roughly similar to the Sun, i.e. main-sequence stars of spectral categories F, G, or K. Lower-mass stars (red dwarfs, of spectral category M) are less likely to have planets massive enough to be detected by the radial-velocity method (Gumming, et al 2008). Despite this, several tens of planets around red dwarfs have been discovered by the Kepler spacecraft, which uses the transit method to detect smaller planets.

Stars with a higher metallicity than the Sun are more likely to have planets, especially giant planets, than stars with lower metallicity (Wang and Fischer, 2014).

Some planets orbit one member of a binary star system (Schwarz, 2016), and several circumbinary planets have been discovered which orbit around both members of binary star. A few planets in triple star systems are known and one in the quadruple system Kepler-64 (Schwarz, 2016).

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

The R3BP has been treated from many angles, from the effects of some parameters relating to the shape and nature of orbits of the primaries and of the third body, to perturbing forces of Coriolis, centrifugal, radiation pressure and the effect of the gravitational potential from the disc. We take a brief of the R3BP.

2.2 The Restricted Three-Body Problem

In celestial mechanics, the general three-body problem encompasses gravitationally interacting astronomical bodies and intends to predict their motions. In our Solar System, the planets and asteroids move around the Sun, while the moons orbit their host planets, which in turn also move around the Sun. As typical examples of the three body problem, we may consider the Sun-planet-planet, Sun-planet-moon, or Sun planet asteroid systems. The three-body problem representing the latter system can be significantly simplified because the mass of the asteroid is always negligible when compared to the mass of either the Sun or the planet, which means that the gravitational influence of the asteroid on the planet and the Sun can be omitted from the theory. If this condition is satisfied, then the general three-body problem becomes the restricted three-body problem, and there are two possibilities, namely, the two bodies with dominant masses move around their center of mass either along circular or elliptic orbits which leads to the respective circular or elliptic restricted three-body problems. It is an old problem, which follows from the two body problem solved by Isaac Newton in his *Principia* in 1687. Almost 100 years later in 1772, Euler proposed the restricted three body problem. However, at about the same time,

the first libration solutions L_4 and L_5 of the restricted three body problem were discovered by J.L Lagrange and later, the collinear points L_1, L_2 and L_3 by L. Euler (Szebehely, 1967).

2.3 The Elliptic Restricted Three-Body Problem

Similar to the CR3BP, the elliptic restricted three-body problem (ER3BP) requires that the two objects (called the primaries) have their masses significantly larger than the third one ($M_1 >> M_3$ and $M_2 >> M_3$), and that the third mass has no gravitational influence on the primaries. However, the main difference between the CR3BP and ER3BP is that, in the latter, motions of the primaries are along elliptical orbits around their center of mass, which has important implications for the mathematical description of the system as there is not a Jacobi's constant in the ER3BP (Szebehely, 1967, Marchal, 1990, Palaci´an et al., 2006).

When the orbits of the primaries are elliptic, a non-uniformly rotating-pulsating coordinate system is commonly used. This new coordinate system has the felicitous property that, the positions of the primaries are fixed; however, the Hamiltonian is explicitly time-dependent (Szebehely 1967a). Such an oscillating coordinate system has been introduced by using the variable distance between the primaries as a unit of length of the system by which distances are divided. This has motivated quite a number of researches (Szebehely 1967a, b; Zimovshchikov & Tkhai 2004; Narayan et al. 2015; Singh & Umar 2012a, b, 2013, 2014, 2015 and Umar & Singh 2015} both with variable and constant coefficients. In a recent study with variable coefficients, Narayan et al. 2015 investigated the effects of triaxiality and radiation pressures of both primaries on the triangular points of the ER3BP. They concluded that any increase in these factors has a destabilizing effect on the orbits of the third body of negligible mass.

2.4 Related Literatures

The motion of an infinitesimal body in the generalized, restricted three-body problem, have been presented by Singh and Taura (Singh and Taura, 2013). It was found that, in addition to the usual five equilibrium points, there appear two new collinear points L_{nl}, L_{n2} due to the potential from the belt, and in the presence of all these perturbations, the equilibrium points L_1, L_3 come nearer to the primaries; while L_2, L_4, L_5, L_{nl} move towards the less massive primary and L_{n2} moves away from it.

The dynamics of galactic systems with central binary black holes was studied by modifying the restricted three body problem, in which a galactic potential was added as an external potential (Jiang and Yeh, 2014). Narayan and Shrivastava (2013) discussed the effects of oblateness and the photogravitational effect of the primaries on the location and the stability of the triangular equilibrium points in the elliptical restricted three-body problem (ER3BP). The orbital motion of a test particle around a primary is greatly affected in the presence of a massive belt (lorio, 2007, 2012). Quite recently, a bi-dimensional ring model for the minor asteroids of our solar system has just been implemented and modeled in the latest EPM2013 planetary ephemerides (Pitjeva and Pitjev, 2014). Kushvah (2008) studied analytically and numerically the effects of radiation pressure of the more massive primary, oblateness of less massive primary and gravitational potential from the belt on the linear stability of libration points in the R3BP. While, Singh and Taura (2013) examined the combined effect of radiation and oblateness of both primaries, together with additional gravitational potential from the belt on the motion of an infinitesimal body in the CR3BP. Singh and Taura (2014a) studied the effects of oblateness up to J4 of the less massive primary and gravitational potential from a belt on the linear stability of triangular equilibrium points in the photogravitational CR3BP. While, Singh and Taura (2014b) looked at the effects of triaxiality of the more

massive primary, oblateness up to the zonal harmonic J4 of the smaller primary and gravitational potential from a belt on the linear stability of the triangular libration points in the CR3B.

Later, Abouelmagd (2013) examined the effects of photogravitational force and oblateness in the perturbed restricted three-body problem. Singh and Taura (2013) studied the combined effect of electromagnetic radiation and oblateness up to *J2* of both primaries, together with additional gravitational potential from the circumbinary belt on the motion of an infinitesimal body in the binary stellar systems within the framework of CR3BP.

The oblateness of a body can produce perturbations-deviations from the two-body motion. This justifies the inclusion of oblateness of a body in the study of CR3BP (Vidyakin 1974; SubbaRao and Sharma 1975,1976; Sharma 1987; Kalvouridis 1993; Singh and Ishwar 1999; Kushvah 2008; Abouelmagd 2012; Singh and Taura 2013; Singh and Umar 2013).

In the study of stability of equilibrium points, Szebehely (1967) studied the stability of the equilibrium points of the R3BP. He established in the linear sense that, the collinear points L_1, L_2 and L_3 are unstable for any value of the mass ratio μ, while the triangular points L_4 and L_5 are stable for $0 < \mu < \mu_c = 0.08352$.....Also; Simmons *et al.* (1985) established the complete solution of the R3BP and discussed the existence and linear stability of the equilibrium points for all the values of radiation pressure of both luminous bodies. They show that, the inner Lagrange point, L_1, can be stable, but only when both large masses are luminous. Further, Kumar & Choudhry (1986) studied the existence of libration points for the generalized photogravitational R3BP by considering the infinitesimal mass as an oblate spheroid and both finite masses as sources of radiation.

They found that, there was a possibility of nine libration points, in which three are collinear, four are coplanar and two are triangular.

Subsequently, Singh & Ishwar (1999) studied the effect of oblateness and radiation pressure forces of the primaries on the locations and the linear stability of the triangular points in the R3BP. They found that, these points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c < \mu < \frac{1}{2}$, and μ_c depends on the radiating and oblateness coefficients.

Singh & Umar (2013b, 2014a) investigated the effects of the luminosity and oblateness of both primary bodies and the effect of the triaxiality of the bigger primary on the collinear libration points of the binary systems Achird, Luyten 726-8, Kruger 60, Alpha Centauri AB and Xi Bootis, and on binary pulsars respectively, moving in elliptic orbits around their common center of mass. They found that, the collinear points are affected by the eccentricity, oblateness, radiation and triaxiality factors; they however remain unstable.

The existence of two families of periodic motion near the Lagrangian solutions in the planar CR3BP was proved for arbitrary values of the parameter μ by Charlier (1899). In 1967 Sharma established the existence of periodic orbits of the first kind through analytic continuation in the Delaney's canonical variables. Also, Sharma (1981) extended his study by utilizing secular perturbing potentials due to oblateness to establish the existence of periodic orbits of the second kind through analytic continuation using the Delaney's canonical variable in the planar R3BP when the more massive primary is an oblate spheroid with its equatorial plane coincident with the plane of motion. Furthermore, Ahmed (1985) established the existence of periodic orbits of the first kind for the photogravitational CR3BP. Moreover, Sarris (1989) studied the families of symmetric-periodic orbits in the 3-dimensional elliptic problem with a variation of the mass ratio μ and the eccentricity.

In extension, Sharma et al. (2001) generalize their work which dealt with the stationary solutions of the planar R3BP when the primaries are triaxial rigid bodies by further considering the primaries as sources of radiation. They found that, the state of stability of the triangular and collinear points remain unchanged. They also found that, the triangular points have long and short periodic elliptic orbits in the same range of the mass parameter μ.While AbdulRaheem & Singh (2008) studied the combined effects of perturbations, radiation and oblateness on the periodic orbits in the restricted three-body problem and found that, the frequencies, eccentricities and axes of the long and short periods are affected by the perturbing forces of the primaries. Then, Mital et al, (2009) studied periodic orbits in the photogravitational R3BP with the smaller primary an oblate body and determined periodic orbits for different values of the mass parameter μ, energy constant h, and oblateness factor. Subsequently, Dutt & Sharma (2010) studied the location and stability of periodic orbits and quasi-periodic motion for the Earth-moon system and they identified periodic solutions, quasi-periodic and chaotic region in CR3BP. Furthermore, Safiya & Sharma (2012) studied the effects of oblateness of Saturn on the regions of quasi-periodic motion around both primaries in the Saturn-Titan system, and combined effects of oblateness and radiation on periodic orbits in the circular framework of the R3BP respectively.

In the elliptic frame of the R3BP Umar & Singh (2014) studied the periods, eccentricities and axes around $L_{4,5}$ under radiating and oblate primaries. They found that, the expressions for the frequencies of the long and short periods around the triangular points with their orientations, eccentricities, semi-major and semi-minor axes are influenced by the eccentricities of the orbits of the primaries, radiation pressure, semi-major axis and oblateness. And in the absence of perturbing forces, their results are in accordance with Szebehely (1967a). Recently, Singh & Umar (2015) examined the effect of the oblateness

of an artificial satellite on the orbits around the triangular points of the Earth-moon system in the axisymmetric ER3BP. They concluded that, the frequencies of the long and short periods around the triangular points with their orientations, eccentricity, semi-major and semi-minor axes are influenced by the eccentricity of the orbits, oblateness and semi-major axis of the primaries and of the third body as well.

The present study, examines the equations of motions of the elliptic restricted three body problem (ER3BP) when both primaries are sources of radiation and the third body is an oblate circumbinary planet, surrounded by a circumbinary disc.

CHAPTER THREE

LOCATIONS AND LINEAR STABILITY OF EQUILIBRIUM POINTS

3.1 Introduction

In this chapter, the equations of motion of two finite bodies which are sources of radiation and a third body (an oblate circumbinary exoplanet) surrounded by gravitational potential from a disc are presented in a synodic pulsating dimensionless coordinates. The locations and linear stability of the equilibrium points are thereafter obtained.

3.2 Equations of Motion

We present the equations of motion of an infinitesimal oblate circumbinary body in the ER3BP with two finite bodies which are sources of radiation in a dimensionless-pulsating coordinate system (ξ,η,ζ) as adopted from (Singh and Umar 2012) and an additional gravitational potential from a disc from (Singh and Taura, 2014) as:

$$\begin{aligned}\ddot{\xi}-2\dot{\eta}&=\Omega_{\xi}\\ \ddot{\eta}+2\dot{\xi}&=\Omega_{\eta}\\ \ddot{\zeta}&=\Omega_{\zeta}\end{aligned} \tag{3.1}$$

where the force function is

$$\Omega=\frac{1}{\left(1-e^{2}\right)^{\frac{1}{2}}}\left[\frac{\xi^{2}+\eta^{2}}{2}+\frac{1}{n^{2}}\left\{\frac{(1-\mu)q_{1}}{r_{1}}+\frac{\mu q_{2}}{r_{2}}+\frac{(1-\mu)A}{2r_{1}^{3}}+\frac{\mu A}{2r_{2}^{3}}+\frac{M_{b}}{\left(r^{2}+T^{2}\right)^{\frac{1}{2}}}\right\}\right] \tag{3.2}$$

$$\begin{aligned}r_{1}^{2}&=(\xi+\mu)^{2}+\eta^{2}+\zeta^{2}\\ r_{2}^{2}&=(\xi+\mu-1)^{2}+\eta^{2}+\zeta^{2}\end{aligned} \tag{3.3}$$

and the mean motion

$$n^2 = \frac{1}{a}\left(1+\frac{3}{2}e^2+\frac{2M_b r_c}{\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right) \tag{3.4}$$

where *n, a, e, A* are the mean motion, semi-major axis, eccentricity of the orbits of the primaries, oblateness of the third body respectively; $\frac{M_b}{\left(r^2+T^2\right)^{\frac{1}{2}}}$ is the potential due to the circular cluster of material points, r is the radial distance of the infinitesimal body and is given by $r^2=\xi^2+\eta^2$, $T = b + d$, b and d are parameters which determine the density profile of the circular cluster of material points. The parameter b controls the flatness of the profile and is known as the flatness parameter. The parameter d controls the size of the core of the density profile and is called the core parameter. When $b = d = 0$, the potential equals to the one by a point mass, r_c is the radial distance of the infinitesimal body in the classical restricted 3BP (Singh and Taura, 2013), and is given as $r_c^2 = 1-\mu+\mu^2$

3.3 Variational Equation

Let (ξ_0,η_0) be the co-ordinates of an equilibrium point under consideration; and (α,β) be the small displacements in the co-ordinates of the equilibrium point. Then the displacement, velocity and acceleration of the infinitesimal particle are:

$$\begin{aligned} \xi &= \xi_0+\alpha, & \eta &= \eta_0+\beta \\ \dot{\xi} &= \dot{\alpha} & \dot{\eta} &= \dot{\beta} \\ \ddot{\xi} &= \ddot{\alpha} & \ddot{\eta} &= \ddot{\beta} \end{aligned} \tag{3.5}$$

Making use of equation (3.5) in equation (3.1), we obtain

$$\begin{aligned} \Omega_\xi &= \Omega_\xi(\xi,\eta) = \Omega_\xi(\xi_0+\alpha,\eta_0+\beta) \\ \Omega_\eta &= \Omega_\eta(\xi,\eta) = \Omega_\eta(\xi_0+\alpha,\eta_0+\beta) \end{aligned} \tag{3.6}$$

Expanding (3.6) using Taylor's expansion, and considering only linear terms in α and β we have:

$$\begin{aligned}\Omega_\xi &= \Omega_\xi(\xi,\eta) = \Omega_\xi(\xi_0,+\eta_0) + \alpha\Omega^0_{\xi\xi},+\beta\Omega^o_{\xi\eta}\\ \Omega_\eta &= \Omega_\eta(\xi,\eta) = \Omega_\eta(\xi_0,+\eta_0) + \alpha\Omega^0_{\eta\xi},+\beta\Omega^o_{\eta\eta}\end{aligned} \tag{3.7}$$

where the super script 0 denotes that the partial derivatives are to be evaluated at the equilibrium point (ξ_0,η_0)

At equilibrium point $\Omega_\xi(\xi_0,\eta_0) = \Omega_\eta(\xi_0,\eta_0) = 0$

hence equations (3.7) becomes

$$\begin{aligned}\Omega_\xi &= \alpha\Omega^0_{\xi\xi} + \beta\Omega^o_{\xi\eta}\\ \Omega_\eta &= \alpha\Omega^0_{\eta\xi} + \beta\Omega^o_{\eta\eta}\end{aligned} \tag{3.8}$$

Substituting equations (3.5) and (3.8) in the equations of motion (3.1), we obtain the variational equations of motion as:

$$\begin{aligned}\ddot{\alpha} - 2\dot{\beta} &= \alpha\Omega^0_{\xi\xi} + \beta\Omega^o_{\xi\eta}\\ \ddot{\beta} + 2\dot{\alpha} &= \alpha\Omega^0_{\eta\xi} + \beta\Omega^o_{\eta\eta}\end{aligned} \tag{3.9}$$

3.4 Characteristic Equation

Let us take trial solutions for the variational equations (3.9) as:

$\alpha = Ae^{\lambda t}$ and $\beta = Be^{\lambda t}$

(3.10)

where *A, B and* λ are parameters to be determined,

now

$\dot{\alpha} = A\lambda e^{\lambda t}$ and $\dot{\beta} = B\lambda e^{\lambda t}$

$\ddot{\alpha} = A\lambda^2 e^{\lambda t}$ and $\ddot{\beta} = B\lambda^2 e^{\lambda t}$ (3.11)

Substituting equations (3.10) and (3. 11) in equations (3.9), we have

$$A\lambda^2 e^{\lambda t} - 2B\lambda e^{\lambda t} = Ae^{\lambda t}\Omega^0_{\xi\xi} + Be^{\lambda t}\Omega^o_{\xi\eta}$$
$$B\lambda^2 e^{\lambda t} + 2A\lambda e^{\lambda t} = Ae^{\lambda t}\Omega^0_{\eta\xi} + Be^{\lambda t}\Omega^o_{\eta\eta}$$

Dividing *by* $e^{\lambda t}$, we have

$$A\lambda^2 - 2B\lambda = A\Omega^0_{\xi\xi} + B\Omega^o_{\xi\eta}$$
$$B\lambda^2 + 2A\lambda = A\Omega^0_{\eta\xi} + B\Omega^o_{\eta\eta}$$

or

$$A(\lambda^2 - \Omega^0_{\xi\xi}) + B(-2\lambda - \Omega^o_{\xi\eta}) = 0$$
$$B(\lambda^2 - \Omega^o_{\eta\eta}) + A(2\lambda - \Omega^0_{\eta\xi}) = 0$$

These will be a non-trivial solution for *A and B* if

$$\begin{vmatrix} \lambda^2 - \Omega^0_{\xi\xi} & -2\lambda - \Omega^0_{\xi\eta} \\ 2\lambda - \Omega^0_{\eta\xi} & \lambda^2 - \Omega^0_{\eta\eta} \end{vmatrix} = 0$$

Expanding the determinant yields

$$\lambda^4 - (\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} - 4)\lambda^2 + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - (\Omega^0_{\xi\eta})^2 = 0 \qquad (3.12)$$

Equation (3.12) constitutes the characteristic equation of the system.

3.5 Jacobian Integral

Multiplying the above equations (3.1) by $2\dot{\xi}, 2\dot{\eta}$ *and* $2\dot{\zeta}$ respectively, we obtain

$$2\dot{\xi}\ddot{\xi} - 4\dot{\xi}\dot{\eta} = 2\dot{\xi}\frac{\partial\Omega}{\partial\xi}$$
$$2\dot{\eta}\ddot{\eta} + 4\dot{\eta}\dot{\xi} = 2\dot{\eta}\frac{\partial\Omega}{\partial\eta}$$
$$2\dot{\zeta}\ddot{\zeta} = 2\dot{\zeta}\frac{\partial\Omega}{\partial\zeta}$$

Adding these equations together, we get

$$2\dot{\xi}\ddot{\xi} + 2\dot{\eta}\ddot{\eta} + 2\dot{\zeta}\ddot{\zeta} = 2(\xi\frac{\partial\Omega}{\partial\xi} + \dot{\eta}\frac{\partial\Omega}{\partial\eta} + \dot{\zeta}\frac{\partial\Omega}{\partial\zeta})$$

This can be written as:

$$\frac{d}{dt}\left(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2\right)=2\frac{d\Omega}{dt}$$

Integrating we have

$$\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2=2\Omega-C \tag{3.13}$$

where C is the constant of integration, and is known as the Jacobian constant. Equation (3.13) is called the Jacobian Integral.

3.6 Locations of the Equilibrium Points

These are points at which velocity and acceleration of an infinitesimal mass are zero i.e.

$\Omega_\xi=\Omega_\eta=0$

3.6.1 Locations of triangular equilibrium points

The stationary solutions of the ER3BP can be found by setting $\dot{\xi}=\dot{\eta}=\dot{\zeta}=\ddot{\xi}=\ddot{\eta}=\ddot{\zeta}=0$ in the equations of motion (3.1). They are the solutions of the equations, $\Omega_\xi=\Omega_\eta=\Omega_\zeta=0$ that is,

$$\Omega_\xi=\frac{1}{\left(1-e^2\right)^{\frac{1}{2}}}\left[\xi-\frac{1}{n^2}\left\{\frac{(1-\mu)(\xi+\mu)q_1}{r_1^3}+\frac{\mu(\xi+\mu-1)q_2}{r_2^3}+\frac{3(1-\mu)(\xi+\mu)A}{2r_1^5}\right.\right.$$
$$\left.\left.+\frac{3\mu(\xi+\mu-1)A}{2r_2^5}+\frac{M_b\xi}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right\}\right]=0$$

Since $\Omega_\xi=0$,

$$\Rightarrow n^2\xi-\frac{(1-\mu)(\xi+\mu)q_1}{r_1^3}-\frac{\mu(\xi+\mu-1)q_2}{r_2^3}-\frac{3(1-\mu)(\xi+\mu)A}{2r_1^5}-\frac{3\mu(\xi+\mu-1)A}{2r_2^5}-\frac{M_b\xi}{\left(r^2+T^2\right)^{\frac{3}{2}}}=0. \tag{3.14}$$

the triangular points denoted by *L_4 and* L_5 are the solutions of equation (3.14) when $\eta\neq0$, from equation (3.14) by expanding and collecting terms in ξ, we have

$$\xi\left[n^2-\frac{(1-\mu)q_1}{r_1^3}-\frac{\mu q_2}{r_2^3}-\frac{3(1-\mu)A}{2r_1^5}-\frac{3\mu A}{2r_2^5}-\frac{M_b}{(r^2+T^2)^{\frac{3}{2}}}\right]-\frac{\mu(1-\mu)q_1}{r_1^3}+\frac{\mu(1-\mu)q_2}{r_2^3}$$
$$-\frac{3\mu(1-\mu)A}{2r_1^5}+\frac{3\mu(1-\mu)A}{2r_2^5}=0$$

(3.14a)

also,

$$\Omega_\eta=\frac{1}{\left(1-e^2\right)^{\frac{1}{2}}}\left[\eta-\frac{1}{n^2}\left\{\frac{(1-\mu)\eta q_1}{r_1^3}+\frac{\mu\eta q_2}{r_2^3}+\frac{3(1-\mu)\eta A}{2r_1^5}+\frac{3\mu\eta A}{2r_2^5}+\frac{M_b\eta}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right\}\right]=0$$

Since $\Omega_\eta=0$,

$$\Rightarrow n^2\eta-\frac{(1-\mu)\eta q_1}{r_1^3}-\frac{\mu\eta q_2}{r_2^3}-\frac{3(1-\mu)\eta A}{2r_1^5}-\frac{3\mu\eta A}{2r_2^5}-\frac{M_b\eta}{\left(r^2+T^2\right)^{\frac{3}{2}}}=0 \qquad (3.15)$$

From equation (3.15) when $\eta\neq 0$ substituting in equation (3.14a), we obtain

$$n^2-\frac{(1-\mu)q_1}{r_1^3}-\frac{\mu q_2}{r_2^3}-\frac{3(1-\mu)A}{2r_1^5}-\frac{3\mu A}{2r_2^5}-\frac{M_b}{(r^2+T^2)^{\frac{3}{2}}}=0$$

also,

$$\Omega_\zeta=-\frac{1}{\left(1-e^2\right)^{\frac{1}{2}}}\left[\frac{1}{n^2}\left\{\frac{(1-\mu)\zeta q_1}{r_1^3}-\frac{\mu\zeta q_2}{r_2^3}-\frac{3(1-\mu)\zeta A}{2r_1^5}-\frac{3\mu\zeta A}{2r_2^5}\right\}\right]=0$$

Since $\Omega_\zeta=0$,

$$\Rightarrow -\zeta\left\{\frac{(1-\mu)q_1}{r_1^3}-\frac{\mu q_2}{r_2^3}-\frac{3(1-\mu)A}{2r_1^5}-\frac{3\mu A}{2r_2^5}\right\}$$

From above equation when $\zeta=0$

$$\frac{(1-\mu)q_1}{r_1^3}-\frac{\mu q_2}{r_2^3}-\frac{3(1-\mu)A}{2r_1^5}-\frac{3\mu A}{2r_2^5}\neq 0 \qquad (3.16)$$

This indicate that the equilibrium points exist in the $\xi\eta$-plane in which the motion of the finite masses occur i.e. the motion of the infinitesimal mass occurs in the orbital plane. In the absence of oblateness and potential from the circular cluster of disc, we obtain

$$n^2 - \frac{q_1}{r_1^3} = 0, \text{ and } n^2 - \frac{q_1}{r_2^3} = 0. \tag{3.17}$$

Making use of equation (3.17), we have

$$\begin{aligned} r_1 &= \left(\frac{q_1}{n^2}\right)^{\frac{1}{3}} \\ r_2 &= \left(\frac{q_2}{n^2}\right)^{\frac{1}{3}} \end{aligned} \tag{3.17a}$$

Considering the oblateness and potential from the disc, the above equation (3.17a) will change slightly by $\varepsilon_i (i = 1,2)$ *say,* where $\varepsilon_i << 1$ and depends upon oblateness and potential from the disc, we have

$$\begin{aligned} r_1 &= (aq_1)^{\frac{1}{3}}\left(1 - \frac{1}{2}e^2 - \frac{2M_b r_c}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}}\right) + \varepsilon_1 \\ r_2 &= (aq_2)^{\frac{1}{3}}\left(1 - \frac{1}{2}e^2 - \frac{2M_b r_c}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}}\right) + \varepsilon_2 \end{aligned} \tag{3.18}$$

where ε_1 *and* ε_2 are obtained on substituting $r_i (i = 1,2)$ and the value of the mean motion n^2 into equation (3.14)

$$\begin{aligned} \varepsilon_1 &= \frac{A}{2a^{\frac{1}{3}}q_1^{\frac{4}{3}}} + \frac{M_b a^{\frac{4}{3}} q_1^{\frac{1}{3}}}{3\left(r^2 + T^2\right)^{\frac{3}{2}}} \\ \varepsilon_2 &= \frac{A}{2a^{\frac{1}{3}}q_2^{\frac{4}{3}}} + \frac{M_b a^{\frac{4}{3}} q_2^{\frac{1}{3}}}{3\left(r^2 + T^2\right)^{\frac{3}{2}}} \end{aligned} \tag{3.19}$$

Putting equation (3.19) into equation (3.18) result in

$$r_1 = (aq_1)^{\frac{1}{3}}\left(1-\frac{1}{2}e^2+\frac{A}{2a^{\frac{2}{3}}q_1^{\frac{5}{3}}}-\frac{2M_b r_c}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}+\frac{M_b a}{3\left(r^2+T^2\right)^{\frac{3}{2}}}\right)$$

$$r_2 = (aq_2)^{\frac{1}{3}}\left(1-\frac{1}{2}e^2+\frac{A}{2a^{\frac{2}{3}}q_2^{\frac{5}{3}}}-\frac{2M_b r_c}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}+\frac{M_b a}{3\left(r^2+T^2\right)^{\frac{3}{2}}}\right) \qquad (3.20)$$

Now since $r^2 = \xi^2 + \eta^2 \Rightarrow r_c^2 = 1-\mu+\mu^2$

therefore,

$$r_1^2 = (aq_1)^{\frac{2}{3}}\left(1-\frac{1}{2}e^2+\frac{A}{2a^{\frac{2}{3}}q_1^{\frac{5}{3}}}-\frac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)$$

$$r_2^2 = (aq_2)^{\frac{2}{3}}\left(1-\frac{1}{2}e^2+\frac{A}{2a^{\frac{2}{3}}q_2^{\frac{5}{3}}}-\frac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right) \qquad (3.21)$$

Substituting the values of r_1^2 and r_2^2 into

$$\xi = \frac{1}{2}-\mu+\frac{r_1^2+r_2^2}{2} \quad and \quad \eta^2 = r_1^2-(\xi^2+\mu)^2 \qquad (3.22)$$

we obtain

$$\xi = \frac{1}{2}-\mu+\frac{1}{2}\left[\begin{array}{l}(aq_1)^{\frac{2}{3}}\left(1-e^2+Aa^{-\frac{1}{2}}q_1^{-\frac{5}{3}}-\dfrac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)\\ -(aq_2)^{\frac{2}{3}}\left(1-e^2+Aa^{-\frac{1}{2}}q_2^{-\frac{5}{3}}-\dfrac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)\end{array}\right]$$

$$\eta = \pm\left[(aq_1)^{\frac{2}{3}}\left(1-e^2+Aa^{-\frac{1}{2}}q_1^{-\frac{5}{3}}-\frac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)\right.$$

$$\left.-\frac{1}{4}\left[\begin{array}{l}1+2(aq_1)^{\frac{2}{3}}\left(1-e^2+Aa^{-\frac{1}{2}}q_1^{-\frac{5}{3}}-\frac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)\\ -2(aq_2)^{\frac{2}{3}}\left(1-e^2+Aa^{-\frac{1}{2}}q_2^{-\frac{5}{3}}-\frac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)\end{array}\right]\right]^{\frac{1}{2}} \tag{3.23}$$

Equations (3.23) are the locations of the triangular points (corresponding to $L_{4,5}$) denoted by $(\xi,\pm\eta)$

3.6.2 Locations of the Collinear Equilibrium Points

To obtain the collinear points, we obtain the first derivative of equation (3.2) with respect to ξ,η *and* ζ respectively, and equate them to zero. That is $\Omega_\xi=\Omega_\eta=\Omega_\zeta=0$. Since the collinear point lies only, on the ξ*-axis*, it implies that $\eta=\zeta=0$ on the system i.e.

$$\Omega_\xi=\frac{1}{\left(1-e^2\right)^{\frac{1}{2}}}\left[\xi-\frac{1}{n^2}\left\{\frac{(1-\mu)(\xi+\mu)q_1}{r_1^3}+\frac{\mu(\xi+\mu-1)q_2}{r_2^3}+\frac{3(1-\mu)(\xi+\mu)A}{2r_1^5}\right.\right.$$

$$\left.\left.+\frac{3\mu(\xi+\mu-1)A}{2r_2^5}+\frac{M_b\xi}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right\}\right]=0$$

$$\Omega_\eta=\frac{1}{\left(1-e^2\right)^{\frac{1}{2}}}\left[\eta-\frac{1}{n^2}\left\{\frac{(1-\mu)\eta q_1}{r_1^3}+\frac{\mu\eta q_2}{r_2^3}+\frac{3(1-\mu)\eta A}{2r_1^5}+\frac{3\mu\eta A}{2r_2^5}+\frac{M_b\eta}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right\}\right]=0$$

$$\Omega_\zeta=-\frac{1}{\left(1-e^2\right)^{\frac{1}{2}}}\left[\frac{1}{n^2}\left\{\frac{(1-\mu)\zeta q_1}{r_1^3}-\frac{\mu\zeta q_2}{r_2^3}-\frac{3(1-\mu)\zeta A}{2r_1^5}-\frac{3\mu\zeta A}{2r_2^5}\right\}\right]=0 \tag{3.24}$$

Since $\eta = \zeta = 0$, we have $|r_1| = |\xi + \mu|$ and $|r_2| = |\xi + \mu - 1|$, the first equation of (3.24), reduces to

$$n^2\xi - \frac{(1-\mu)(\xi+\mu)q_1}{r_1^3} - \frac{\mu(\xi+\mu-1)q_2}{r_2^3} - \frac{3(1-\mu)(\xi+\mu)A}{2r_1^5} - \frac{3\mu(\xi+\mu-1)A}{2r_2^5} - \frac{M_b\xi}{\left(\xi^2+T^2\right)^{\frac{3}{2}}} = 0$$

(3.25)

Converting equation (3.25) to modulus form we have,

$$n^2\xi - \frac{(1-\mu)(\xi+\mu)q_1}{|\xi-\xi_1|^3} - \frac{\mu(\xi+\mu-1)q_2}{|\xi-\xi_2|^3} - \frac{3(1-\mu)(\xi+\mu)A}{2|\xi-\xi_1|^5} - \frac{3\mu(\xi+\mu-1)A}{2|\xi-\xi_2|^5} - \frac{M_b\xi}{\left|\xi^2+T^2\right|^{\frac{3}{2}}} = 0$$

(3.25a)

And equation (3.3) becomes

$$\begin{aligned} r_1^2 &= (\xi+\mu)^2 \\ r_2^2 &= (\xi+\mu-1)^2 \end{aligned} \quad (3.26)$$

Solving equation (3.25a) using $\xi_1 = -\mu$ and $\xi_2 = 1-\mu$ $\xi_1 = -\mu$ we obtain

$$\begin{aligned} &2n^2\xi(\xi-\xi_1)^4(\xi-\xi_2)^4K - 2(1-\mu)(\xi-\xi_1)^2(\xi-\xi_2)^4Kq_1 - 2\mu(1-\mu)(\xi-\xi_1)^4(\xi-\xi_2)^2Kq_2 \\ &-3\mu(1-\mu)(\xi-\xi_2)^4KA - 3\mu(\xi-\xi_1)^4KA - 2M_b\xi(\xi-\xi_1)^4(\xi-\xi_2)^4 = 0 \end{aligned} \quad (3.27)$$

The orbital plane on the ξ-axis is divided into three parts; $\xi > \xi_2, \xi_1 < \xi < \xi_2$ and $\xi_1 > \xi$ with respect to the primaries, which corresponds to the collinear points $L_i\,(i = 1,2,3)$.

Case 1: Position of $L_1(\xi > \xi_2)$

Let the collinear point L_1 be on the right-hand side of the smaller primary at a distance h from it on the ξ-axis.

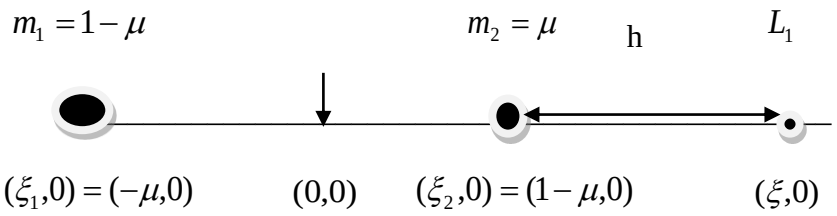


Figure 3. 1: Position of the collinear equilibrium point L_1

In the interval $(\xi > \xi_2)$, we let $\xi - \xi_2 = h,\ \ \xi - \xi_1 = 1 + h \Rightarrow \xi = 1 + h - \mu$

Since the distance between the primaries is unity, substituting all the above values in equation (3 .25a) yields

$$\begin{aligned} &2n^2(1+h-\mu)(1+h)^4h^4K - 2(1-\mu)(1+h)^2h^4Kq_1 - 2\mu(1+h)^4h^2Kq_2 \\ &-3(1-\mu)h^4KA - 3\mu(1+h)^4KA - 2M_b(1+h-\mu)(1+h)^4h^4 = 0 \end{aligned} \tag{3.28}$$

Hence, expanding equation (3.28) yields

$$\begin{aligned} &2h^9(n^2K - M_b) + 2(n^2(5-\mu)K - (5-\mu)M_b)h^8 + 4(n^2(5-2\mu)K - (5-2\mu)M_b)h^7 \\ &+ 2(2n^2(5-3\mu)K - 2(5-3\mu)M_b - (1-\mu)Kq_1 - \mu Kq_2)h^6 + 2(n^2(5-4\mu)K \\ &-(5-4\mu)M_b - 2(1-\mu)Kq_1 - 4\mu Kq_2)h^5 + (2(n^2(1-\mu)K - (1-\mu)M_b - (1-\mu)Kq_1 - 6\mu Kq_2) \\ &-3(1+2\mu)AK)h^4 - 4\mu(2q_2 + 3A)Kh^3 - 2\mu(q_2 + 9A)Kh^2 - 12\mu AKh - 3\mu AK = 0 \end{aligned} \tag{3.29}$$

Cases 2: Position of $L_2(\xi_1 < \xi < \xi_2)$

Let the collinear point L_2 be on the left-hand side of the smaller primary at a distance h from it on the ξ-axis.

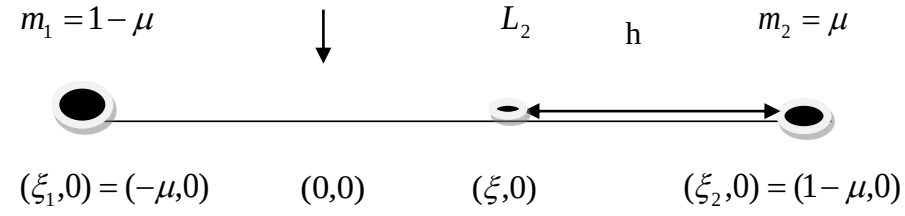


Figure 3. 2**:** Position of the collinear equilibrium point L_2

In the interval $L_2(\xi_1 < \xi < \xi_2)$, we let

$\xi_2 - \xi = h,\ \ \xi - \xi_2 = -h,$ and $\xi - \xi_1 = 1 - h \Rightarrow \xi = 1 - h - \mu$

$r_1 = |\xi + \mu| = 1 - h,$ and $r_2 = |\xi + \mu - 1| = h$ (3.30)

Using equation (3.30) in equation (3.25a) gives

$$2n^2(1-h-\mu)(1-h)^4h^4K-2(1-\mu)(1-h)^2h^4Kq_1+2\mu(1-h)^4h^2Kq_2 \\ -3(1-\mu)h^4KA+3\mu(1-h)^4KA-2M_b(1-h-\mu)(1-h)^4h^4=0 \tag{3.31}$$

Hence, expanding equation (3.31) yields

$$-2h^9(n^2K-M_b)+2(n^2(5-\mu)K-(5-\mu)M_b)h^8-4(n^2(5-2\mu)K-(5-2\mu)M_b)h^7 \\ +2(2n^2(5-3\mu)K-2(5-3\mu)M_b+(1-\mu)Kq_1-\mu Kq_2)h^6-2(n^2(5-4\mu)K \\ -(5-4\mu)M_b-2(1-\mu)Kq_1-4\mu Kq_2)h^5+(2(n^2(1-\mu)K-(1-\mu)M_b-(1-\mu)Kq_1-6\mu Kq_2) \\ -3(1+2\mu)AK)h^4+4\mu(2q_2+3A)Kh^3-2\mu(q_2+9A)Kh^2+12\mu AKh-3\mu AK=0 \tag{3.32}$$

Case 3: Position of L_3 ($\xi_1>\xi$)

Let the collinear point L_3 be on the left-hand side of the bigger primary at a distance 1-h from it on the ξ-axis.

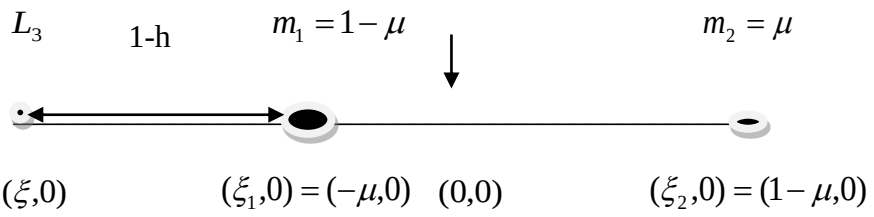


Figure 3. 3: Position of the collinear equilibrium point L_3

Lastly, in the interval $L_3\,(\xi_1>\xi)$

we let $\xi_1-\xi=1-h,\ \Rightarrow \xi=h-\mu-1$

$\xi-\xi_1=-(1-h)$, and $\xi_2-\xi=2-h\Rightarrow\xi-\xi_2=-(2-h)$

$$\Rightarrow r_1=|\xi+\mu|=h-1 \text{ and } r_2=|\xi+\mu-1|=h-2 \tag{3.33}$$

using equation (3.33) in equation (3.25a) gives

$$2n^2(h-\mu-1)(1-h)^4(2-h)^4K-2(1-\mu)(1-h)^2(2-h)^4Kq_1-2\mu(1-h)^4(2-h)^2Kq_2 \\ -3(1-\mu)(2-h)^4KA-3\mu(1-h)^4KA-2M_b(h-\mu-1)(1-h)^4(2-h)^4=0 \tag{3.34}$$

Hence, expanding equation (3.34) gives

$$2h^9(n^2K-M_b)-2(n^2(13+\mu)K-(13+\mu)M_b)h^8+4(n^2(37+6\mu)K-(37+6\mu)M_b)h^7$$
$$-2(2n^2(121+31\mu)K-2(121+31\mu)M_b+(1-\mu)Kq_1-\mu Kq_2)h^6+2(n^2(500+180\mu)K$$
$$-(500+180\mu)M_b+2(5-5\mu)Kq_1+8\mu Kq_2)h^5-(2(n^2(672+320\mu)K-(672+320\mu)M_b+$$
$$(40-40\mu)Kq_1+26\mu Kq_2)+3AK)h^4+(2(n^2(576+352\mu)K-(576+352\mu)M_b+16(5-5\mu)Kq_1$$
$$+88\mu Kq_2)+12(2-\mu)AK))h^3-(2(n^2(288+224\mu)K-(288+224\mu)M_b+16(5-5\mu)Kq_1+40Kq_2)$$
$$+18(4-3\mu)AK)h^2+(2(n^2(80+64\mu)K-(80+64\mu)M_b+32Kq_1+16\mu Kq_2)+96(1-\mu)AK+$$
$$12\mu AKq_2)h-32(\mu-32M_b)+32\mu Kq_1-8\mu Kq_2+45\mu AK-32(n^2K-M_b)-32Kq_1-48AK=0 \quad (3.35)$$

Where $K=(\xi^2+T^2)^{\frac{3}{2}}$

Equations (3.29), (3.32) and (3.35) are eleventh degree equations and these equations by Descartes rule of sign have one or more real root, in each case there is only one physically acceptable root which corresponds to one of the three collinear points $L_i(i=1,2,3)$

3.7 Linear Stability of the equilibrium points

Denoting the positions of equilibrium points by (ξ_0,η_0) and displacements α and β we obtain equations leading to the derivation of equations in $(\Omega^0_{\xi\xi})$, $(\Omega^0_{\eta\eta})$ and $(\Omega^0_{\eta\eta})^2$.

These derivations are given below

$$\Rightarrow\left(1-e^2\right)^{-\frac{1}{2}}=\left(1+\frac{1}{2}e^2\right)$$

$$\Rightarrow\frac{1}{n^2}=a\left(1-\frac{3}{2}e^2-\frac{2M_br_c}{\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)$$

$$\Rightarrow\frac{1}{r_1^3}=(aq_1)^{-1}\left(1+\frac{3}{2}e^2-\frac{3}{2}Aa^{-\frac{2}{3}}q_1^{-\frac{5}{3}}+\frac{M_b(2r_c-a)}{\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)$$

$$\Rightarrow\frac{1}{r_1^5}=(aq_1)^{-\frac{5}{3}}\left(1+\frac{5}{2}e^2-\frac{5}{2}Aa^{-\frac{2}{3}}q_1^{-\frac{5}{3}}+\frac{5M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)$$

$$\Rightarrow \left(\frac{1}{r_1^5}\right)^2 = (aq_1)^{-\frac{10}{3}}\left(1+5e^2-5Aa^{-\frac{2}{3}}q_1^{-\frac{5}{3}}+\frac{10M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)$$

$$\Rightarrow \frac{1}{r_1^7} = (aq_1)^{-\frac{7}{3}}\left(1+\frac{7}{2}e^2-\frac{7}{2}Aa^{-\frac{2}{3}}q_1^{-\frac{5}{3}}+\frac{7M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)$$

$$\Rightarrow \frac{1}{r_2^3} = (aq_2)^{-1}\left(1+\frac{3}{2}e^2-\frac{3}{2}Aa^{-\frac{2}{3}}q_2^{-\frac{5}{3}}+\frac{M_b(2r_c-a)}{\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)$$

$$\Rightarrow \frac{1}{r_2^5} = (aq_2)^{-\frac{5}{3}}\left(1+\frac{5}{2}e^2-\frac{5}{2}Aa^{-\frac{2}{3}}q_2^{-\frac{5}{3}}+\frac{5M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)$$

$$\Rightarrow \left(\frac{1}{r_2^5}\right)^2 = (aq_2)^{-\frac{10}{3}}\left(1+5e^2-5Aa^{-\frac{2}{3}}q_2^{-\frac{5}{3}}+\frac{10M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)$$

$$\Rightarrow \frac{1}{r_2^7} = (aq_2)^{-\frac{7}{3}}\left(1+\frac{7}{2}e^2-\frac{7}{2}Aa^{-\frac{2}{3}}q_2^{-\frac{5}{3}}+\frac{7M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)$$

$$\Rightarrow \left(\frac{1}{n^2}\right)^2 = a^2\left(1-3e^2-\frac{4M_b r_c}{\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right)$$

$$\Rightarrow (\xi+\mu) = \frac{1}{2}+\frac{1}{2}(aq_1)^{\frac{2}{3}}-\frac{1}{2}(aq_1)^{\frac{2}{3}}e^2+\frac{1}{2}Aq_1^{-1}-\frac{1}{2}Aq_2^{-1}-\frac{1}{2}(aq_2)^{\frac{2}{3}}+\frac{1}{2}(aq_2)^{\frac{2}{3}}e^2$$

$$-\frac{M_b(2r_c-a)(aq_1)^{\frac{2}{3}}}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}+\frac{M_b(2r_c-a)(aq_2)^{\frac{2}{3}}}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}$$

$$\Rightarrow (\xi+\mu-1) = -\frac{1}{2}+\frac{1}{2}(aq_1)^{\frac{2}{3}}-\frac{1}{2}(aq_1)^{\frac{2}{3}}e^2+\frac{1}{2}Aq_1^{-1}-\frac{1}{2}Aq_2^{-1}-\frac{1}{2}(aq_2)^{\frac{2}{3}}+\frac{1}{2}(aq_2)^{\frac{2}{3}}e^2$$

$$-\frac{M_b(2r_c-a)(aq_1)^{\frac{2}{3}}}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}+\frac{M_b(2r_c-a)(aq_2)^{\frac{2}{3}}}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}$$

$$\Rightarrow (\xi+\mu)(\xi+\mu-1) = -\frac{1}{4}\Bigg[1-\left(aq_1\right)^{\frac{4}{3}}-\left(aq_2\right)^{\frac{4}{3}}+2\left(aq_1\right)^{\frac{4}{3}}e^2+2\left(aq_2\right)^{\frac{4}{3}}e^2-2Aa^{\frac{2}{3}}q_1^{-\frac{1}{3}}-2Aa^{\frac{2}{3}}q_1^{-\frac{1}{3}}$$

$$+2a^{\frac{4}{3}}\left(q_1q_2\right)^{\frac{2}{3}}-4a^{\frac{4}{3}}\left(q_1q_2\right)^{\frac{2}{3}}e^2+2Aa^{\frac{2}{3}}q_1^{\frac{2}{3}}q_2^{-1}+2Aa^{\frac{2}{3}}q_1^{-1}q_2^{-\frac{2}{3}}-\frac{4M_b(2r_c-a)(aq_1)^{\frac{4}{3}}}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}$$

$$+\frac{4M_b(2r_c-a)(aq_2)^{\frac{4}{3}}}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}-\frac{8M_b(2r_c-a)a^{\frac{4}{3}}(q_1q_2)^{\frac{2}{3}}}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\Bigg]$$

$$\Rightarrow (\xi+\mu)^2 = \frac{1}{4}\Bigg[1+2\left(aq_1\right)^{\frac{2}{3}}\left\{1-e^2+Aa^{-\frac{2}{3}}q_1^{-\frac{5}{3}}-\frac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right\}$$

$$-2\left(aq_2\right)^{\frac{2}{3}}\left\{1-e^2+Aa^{-\frac{2}{3}}q_2^{-\frac{5}{3}}-\frac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right\}\Bigg]$$

$$\Rightarrow (\xi+\mu-1)^2 = \frac{1}{4}\Bigg[1-2\left(aq_1\right)^{\frac{2}{3}}\left\{1-e^2+Aa^{-\frac{2}{3}}q_1^{-\frac{5}{3}}-\frac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right\}$$

$$+2\left(aq_2\right)^{\frac{2}{3}}\left\{1-e^2+Aa^{-\frac{2}{3}}q_2^{-\frac{5}{3}}-\frac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right\}\Bigg]$$

$$\Rightarrow \xi^2 = \frac{1}{4}\left[(1-2\mu)^2 + 2(aq_1)^{\frac{2}{3}}\left\{1-e^2 + Aa^{-\frac{2}{3}}q_1^{-\frac{5}{3}} - \frac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right\}\right.$$

$$\left. -2(aq_2)^{\frac{2}{3}}\left\{1-e^2 + Aa^{-\frac{2}{3}}q_2^{-\frac{5}{3}} - \frac{2M_b(2r_c-a)}{3\left(r_c^2+T^2\right)^{\frac{3}{2}}}\right\}\right]$$

3.7.1 Stability of the Triangular Equilibrium Points

A body is in stable equilibrium if after a slight displacement, it returns to its original position, while a body is said to be unstable if after slight displacement causes the body to move to a new position as time elapses and does not return to its state.

In evaluation of the second partial derivatives of equation (3.2), in the case of triangular points, we have:

$$\Omega^0_{\xi\xi} = \left(1-e^2\right)^{-\frac{1}{2}}\left[\frac{3(1-\mu)}{4(aq_1)^{\frac{2}{3}}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)q_2^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}} + \frac{3\mu}{4(aq_2)^{\frac{2}{3}}} - \frac{3\mu q_1^{\frac{2}{3}}}{2q_2^{\frac{2}{3}}} + \frac{3\mu}{2} + A\left\{\frac{3(1-\mu)}{2a^{\frac{2}{3}}q_1^{\frac{5}{3}}}\right.\right.$$

$$\left. - \frac{3(1-\mu)}{2(aq_1)^{\frac{2}{3}}q_2} - \frac{3\mu}{2(aq_2)^{\frac{2}{3}}q_1} + \frac{3\mu}{2a^{\frac{2}{3}}q_2^{\frac{5}{3}}}\right\} + e^2\left\{\frac{3(1-\mu)}{4(aq_1)^{\frac{2}{3}}} + \frac{3\mu}{4(aq_2)^{\frac{2}{3}}}\right\} - \frac{3M_b a}{2(r_c^2+T^2)^{\frac{3}{2}}}$$

$$+ \frac{M_b(4r_c-5a)(aq_1)^{-\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}} - \frac{\mu M_b(4r_c-5a)(aq_1)^{-\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}} + \frac{3M_b a q_1^{-\frac{2}{3}}q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} - \frac{3\mu M_b a q_1^{-\frac{2}{3}}q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}}$$

$$+ \frac{3\mu M_b a q_1^{\frac{2}{3}}q_2^{-\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} + \frac{\mu M_b(4r_c-5a)(aq_2)^{-\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}} + \frac{3M_b a}{(r_c^2+T^2)^{\frac{5}{2}}}\left\{\frac{1}{4} - \mu + \mu^2 + \frac{1}{2}(aq_1)^{\frac{2}{3}} - \frac{1}{2}(aq_2)^{\frac{2}{3}}\right.$$

$$\left.\left. - \mu(aq_1)^{\frac{2}{3}} + \mu(aq_2)^{\frac{2}{3}} + \frac{1}{4}(aq_1)^{\frac{4}{3}} + \frac{1}{4}(aq_2)^{\frac{4}{3}} - \frac{1}{2}a^{\frac{4}{3}}q_1^{\frac{2}{3}}q_2^{\frac{2}{3}}\right\}\right]$$

$$\Omega_{\eta\eta}^{0}=\left(1-e^{2}\right)^{-\frac{1}{2}}\left[-\frac{3(1-\mu)}{4(aq_1)^{\frac{2}{3}}}+\frac{3(1-\mu)}{2}-\frac{3(1-\mu)q_2^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}}-\frac{3\mu}{4(aq_2)^{\frac{2}{3}}}+\frac{3\mu q_1^{\frac{2}{3}}}{2q_2^{\frac{2}{3}}}+\frac{3\mu}{2}+A\left\{\frac{3(1-\mu)}{2a^{\frac{2}{3}}q_1^{\frac{5}{3}}}\right.\right.$$

$$\left.+\frac{3(1-\mu)}{2(aq_1)^{\frac{2}{3}}q_2}+\frac{3\mu}{2(aq_2)^{\frac{2}{3}}q_1}+\frac{3\mu}{2a^{\frac{2}{3}}q_2^{\frac{5}{3}}}\right\}+e^{2}\left\{-\frac{3(1-\mu)}{4(aq_1)^{\frac{2}{3}}}-\frac{3\mu}{4(aq_2)^{\frac{2}{3}}}\right\}-\frac{3M_b a}{2(r_c^2+T^2)^{\frac{3}{2}}}$$

$$-\frac{M_b(4r_c-5a)(aq_1)^{-\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}}+\frac{\mu M_b(4r_c-5a)(aq_1)^{-\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}}-\frac{3M_b aq_1^{-\frac{2}{3}}q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}}+\frac{3\mu M_b aq_1^{-\frac{2}{3}}q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}}$$

$$-\frac{3\mu M_b aq_1^{\frac{2}{3}}q_2^{-\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}}-\frac{\mu M_b(4r_c-5a)(aq_2)^{-\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}}+\frac{3M_b a}{(r_c^2+T^2)^{\frac{5}{2}}}\left\{\frac{1}{2}(aq_1)^{\frac{2}{3}}+\frac{1}{2}(aq_2)^{\frac{2}{3}}-\frac{1}{4}\right.$$

$$\left.\left.-\frac{1}{4}(aq_1)^{\frac{4}{3}}-\frac{1}{4}(aq_2)^{\frac{4}{3}}+\frac{1}{2}a^{\frac{4}{3}}q_1^{\frac{2}{3}}q_2^{\frac{2}{3}}\right\}\right]$$

$$\Omega_{\xi\eta}^{0}=\left(1-e^{2}\right)^{-\frac{1}{2}}\left[\frac{3(1-2\mu)}{2}+\frac{3(1-\mu)}{2(aq_1)^{\frac{2}{3}}}-\frac{3(1-\mu)q_2^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}}-\frac{3\mu}{2(aq_2)^{\frac{2}{3}}}+\frac{3\mu q_1^{\frac{2}{3}}}{2q_2^{\frac{2}{3}}}+A\left\{\frac{3(1-\mu)}{2a^{\frac{2}{3}}q_1^{\frac{5}{3}}}\right.\right.$$

$$\left.-\frac{3(1-\mu)}{2(aq_1)^{\frac{2}{3}}q_2}+\frac{3\mu}{2(aq_2)^{\frac{2}{3}}q_1}-\frac{3\mu}{2a^{\frac{2}{3}}q_2^{\frac{5}{3}}}\right\}+e^{2}\left\{\frac{3(1-\mu)}{2(aq_1)^{\frac{2}{3}}}-\frac{3\mu}{2(aq_2)^{\frac{2}{3}}}\right\}-\frac{3M_b a}{2(r_c^2+T^2)^{\frac{3}{2}}}-\frac{3\mu M_b a}{(r_c^2+T^2)^{\frac{3}{2}}}$$

$$+\frac{M_b(4r_c-5a)(aq_1)^{-\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}}-\frac{\mu M_b(4r_c-5a)(aq_1)^{-\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}}+\frac{3M_b aq_1^{-\frac{2}{3}}q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}}-\frac{3\mu M_b aq_1^{-\frac{2}{3}}q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}}$$

$$-\frac{3\mu M_b aq_1^{\frac{2}{3}}q_2^{-\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}}-\frac{\mu M_b(4r_c-5a)(aq_2)^{-\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}}+\frac{3M_b a}{(r_c^2+T^2)^{\frac{5}{2}}}\left\{\frac{1}{2}-\mu+\frac{1}{2}(aq_1)^{\frac{2}{3}}-\frac{1}{2}(aq_2)^{\frac{2}{3}}\right\}\Bigg]$$

$$\left(\Omega_{\xi\eta}\right)^{02}=\left(1-e^2\right)^{-\frac{1}{2}}\left[\frac{9\left(1-2\mu+2\mu^2\right)}{8}-\frac{9\left(1-2\mu+\mu^2\right)}{16(aq_1)^{\frac{4}{3}}}-\frac{9\left(1-10\mu+15\mu^2\right)(aq_1)^{\frac{4}{3}}}{16}\right.$$

$$+\frac{9\left(1-3\mu+2\mu^2\right)(aq_2)^{\frac{2}{3}}}{4}-\frac{9\left(6-20\mu+15\mu^2\right)(aq_2)^{\frac{4}{3}}}{16}+\frac{9\left(1-2\mu+\mu^2\right)q_2^{\frac{2}{3}}}{4q_1^{\frac{2}{3}}}-\frac{27\left(1-2\mu+\mu^2\right)q_2^{\frac{4}{3}}}{8q_1^{\frac{4}{3}}}$$

$$+\frac{9\left(1-2\mu+\mu^2\right)q_2^{\frac{2}{3}}}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}}-\frac{9\left(2-5\mu+3\mu^2\right)a^{\frac{2}{3}}q_2^{\frac{4}{3}}}{4q_1^{\frac{2}{3}}}+\frac{9\left(1-2\mu+\mu^2\right)a^{\frac{2}{3}}q_2^{\,2}}{4q_1^{\frac{4}{3}}}+\frac{9\left(2-5\mu+10\mu^2\right)a^{\frac{4}{3}}q_1^{\frac{2}{3}}q_2^{\frac{2}{3}}}{8}$$

$$+\frac{9\left(2-5\mu+3\mu^2\right)a^{\frac{4}{3}}q_2^{\,2}}{8q_1^{\frac{2}{3}}}-\frac{9\left(1-2\mu+\mu^2\right)a^{\frac{4}{3}}q_2^{\frac{8}{3}}}{16q_1^{\frac{4}{3}}}-\frac{9\mu(1-2\mu)(aq_1)^{\frac{2}{3}}}{4}-\frac{9\mu(1-2\mu)}{4(aq_1)^{\frac{2}{3}}}-\frac{9\mu(1-2\mu)}{4(aq_2)^{\frac{2}{3}}}$$

$$+\frac{9\mu(1-2\mu)}{8a^{\frac{4}{3}}(q_1q_2)^{\frac{2}{3}}}+\frac{9\mu(2-3\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{8q_2^{\frac{2}{3}}}-\frac{9\mu(2-3\mu)a^{\frac{4}{3}}q_1^{\,2}}{8q_2^{\frac{2}{3}}}-\frac{27\mu^2a^{\frac{2}{3}}q_2^{\frac{4}{3}}}{4q_1^{\frac{2}{3}}}+\frac{9\mu^2q_1^{\frac{2}{3}}}{4q_2^{\frac{2}{3}}}-\frac{27\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4q_2^{\frac{2}{3}}}$$

$$+\frac{9\mu^2q_1^{\frac{2}{3}}}{4a^{\frac{2}{3}}q_2^{\frac{4}{3}}}-\frac{9\mu^2}{16a^{\frac{4}{3}}q_2^{\frac{4}{3}}}-\frac{27\mu^2q_1^{\frac{4}{3}}}{8q_2^{\frac{4}{3}}}+\frac{9\mu^2a^{\frac{2}{3}}q_1^{\,2}}{4q_2^{\frac{4}{3}}}-\frac{9\mu^2a^{\frac{4}{3}}q_1^{\frac{8}{3}}}{16q_2^{\frac{4}{3}}}+e^2\left\{\frac{9(2-7\mu+3\mu^2)a^{\frac{2}{3}}q_2^{\frac{4}{3}}}{4q_1^{\frac{2}{3}}}\right.$$

$$-\frac{9(1-3\mu+2\mu^2)a^{\frac{2}{3}}q_2^{\frac{4}{3}}}{4}-\frac{9(1-2\mu+\mu^2)a^{\frac{2}{3}}q_2^{\,2}}{4q_1^{\frac{4}{3}}}+\frac{9(1-5\mu+15\mu^2)a^{\frac{4}{3}}q_1^{\frac{4}{3}}}{8}-\frac{9(1-2\mu+\mu^2)}{8a^{\frac{4}{3}}q_1^{\frac{4}{3}}}$$

$$+\frac{9(1-2\mu+\mu^2)q_2^{\frac{2}{3}}}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}}+\frac{9(6-20\mu+15\mu^2)a^{\frac{4}{3}}q_2^{\frac{4}{3}}}{8}-\frac{9(1-5\mu+5\mu^2)a^{\frac{4}{3}}q_1^{\frac{2}{3}}q_2^{\frac{2}{3}}}{2}-\frac{9(2-5\mu+3\mu^2)a^{\frac{4}{3}}q_2^{\,2}}{4q_1^{\frac{2}{3}}}$$

$$+\frac{9(1-2\mu+\mu^2)a^{\frac{4}{3}}q_2^{\frac{8}{3}}}{4q_1^{\frac{4}{3}}}+\frac{9\mu(1-2\mu)a^{\frac{2}{3}}q_2^{\frac{2}{3}}}{4}-\frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}q_1^{\frac{2}{3}}}-\frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}q_2^{\frac{2}{3}}}+\frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}q_1^{\frac{2}{3}}q_2^{\frac{2}{3}}}+\frac{9\mu(1-3\mu)a^{\frac{4}{3}}q_1^{\,2}}{4q_2^{\frac{2}{3}}}$$

$$\left.\left.-\frac{9\mu(1-3\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4q_2^{\frac{2}{3}}}-\frac{9\mu a^{\frac{2}{3}}q_1^{\frac{2}{3}}q_2^{\frac{4}{3}}}{2}+\frac{9\mu^2q_1^{\frac{2}{3}}}{4a^{\frac{2}{3}}q_2^{\frac{4}{3}}}-\frac{9\mu^2a^{\frac{2}{3}}q_1^{\,2}}{4q_2^{\frac{4}{3}}}+\frac{9\mu^2a^{\frac{4}{3}}q_1^{\frac{8}{3}}}{8q_2^{\frac{4}{3}}}-\frac{9\mu^2}{4a^{\frac{4}{3}}q_2^{\frac{4}{3}}}\right\}\right.$$

$$+A\left\{\frac{9(1-2\mu+\mu^2)}{4a^{\frac{4}{3}}q_1^{\frac{4}{3}}q_2}+\frac{9(1-2\mu+\mu^2)}{4a^{\frac{2}{3}}q_1^{\frac{5}{3}}}-\frac{9(3-6\mu+3\mu^2)}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}q_1^{\frac{1}{3}}}+\frac{9(1-2\mu+\mu^2)}{4a^{\frac{2}{3}}q_1^{\frac{2}{3}}q_2}+\frac{9(1-2\mu+\mu^2)q_2^{\frac{2}{3}}}{4a^{\frac{2}{3}}q_1^{\frac{7}{3}}}\right.$$

$$-\frac{9(4-11\mu+7\mu^2)}{4q_1^{\frac{2}{3}}q_2^{\frac{1}{3}}}+\frac{9(1-4\mu+3\mu^2)}{4q_2}+\frac{9(2-5\mu+3\mu^2)q_2^{\frac{2}{3}}}{4q_1^{\frac{5}{3}}}+\frac{9(3-6\mu+3\mu^2)q_2^{\frac{1}{3}}}{4q_1^{\frac{4}{3}}}-\frac{9(1-2\mu+\mu^2)q_2^{\frac{4}{3}}}{2q_1^{\frac{7}{3}}}$$

$$-\frac{9(1-8\mu+13\mu^2)a^{\frac{2}{3}}}{4q_1^{\frac{1}{3}}}+\frac{9(1-8\mu+5\mu^2)a^{\frac{2}{3}}q_2^{\frac{1}{3}}}{4q_2}+\frac{9(3-12\mu+10\mu^2)a^{\frac{2}{3}}q_2^{\frac{2}{3}}}{4q_1}+\frac{9(3-8\mu+5\mu^2)a^{\frac{2}{3}}q_2^{\frac{1}{3}}}{4q_1^{\frac{2}{3}}}$$

$$-\frac{9(3-12\mu+7\mu^2)a^{\frac{2}{3}}}{4q_2^{\frac{1}{3}}}-\frac{9(3-8\mu+5\mu^2)a^{\frac{2}{3}}q_2^{\frac{4}{3}}}{4q_1^{\frac{5}{3}}}-\frac{9(1-2\mu+\mu^2)a^{\frac{2}{3}}q_1^{\frac{1}{3}}}{4q_2}+\frac{9(1-2\mu+\mu^2)a^{\frac{2}{3}}q_2^{\ 2}}{4q_1^{\frac{7}{3}}}-\frac{9\mu(2-3\mu)}{4q_1}$$

$$-\frac{9\mu(1-\mu)}{4a^{\frac{4}{3}}q_1^{\frac{5}{3}}q_2^{\frac{2}{3}}}-\frac{9\mu(1-\mu)}{4a^{\frac{4}{3}}q_1^{\frac{2}{3}}q_2^{\frac{5}{3}}}+\frac{9\mu(3-7\mu)}{4q_1^{\frac{1}{3}}q_2^{\frac{2}{3}}}-\frac{9\mu(1-\mu)q_1^{\frac{2}{3}}}{4q_2^{\frac{5}{3}}}-\frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{1}{3}}}{4q_2^{\frac{2}{3}}}+\frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4q_2^{\frac{5}{3}}}+\frac{9\mu^2}{4a^{\frac{2}{3}}q_1q_2^{\frac{2}{3}}}$$

$$+\frac{9\mu^2}{4a^{\frac{2}{3}}q_2^{\frac{5}{3}}}-\frac{117\mu^2a^{\frac{2}{3}}}{4q_1^{\frac{1}{3}}}+\frac{45\mu^2a^{\frac{2}{3}}q_2^{\frac{2}{3}}}{2q_1}-\frac{45\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2q_2^{\frac{5}{3}}}+\frac{9\mu^2}{2a^{\frac{4}{3}}q_1q_2^{\frac{4}{3}}}-\frac{27\mu^2}{2a^{\frac{2}{3}}q_1^{\frac{2}{3}}q_2^{\frac{4}{3}}}+\frac{9\mu^2q_1^{\frac{2}{3}}}{2a^{\frac{2}{3}}q_2^{\frac{7}{3}}}+\frac{27\mu^2q_1^{\frac{1}{3}}}{2q_2^{\frac{4}{3}}}-\frac{9\mu^2q_1^{\frac{4}{3}}}{q_2^{\frac{7}{3}}}$$

$$\left.-\frac{9\mu^2a^{\frac{2}{3}}q_1}{2q_2^{\frac{4}{3}}}\right\}-\frac{9M_ba}{4(r_c^2+T^2)^{\frac{3}{2}}}-\frac{4M_ba(a^{\frac{2}{3}}q_2^{\frac{2}{3}})}{(r_c^2+T^2)^{\frac{3}{2}}}-\frac{3M_b(4r_c+a)q_2^{\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{2}{3}}}+\frac{6M_b(r_c+a)a^{\frac{2}{3}}q_2^{\frac{4}{3}}}{(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{2}{3}}}+\frac{3M_b(4r_c+13a)q_2^{\frac{4}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{4}{3}}}$$

$$+\frac{M_b(12r_c-23a)q_2^{\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{2}{3}}q_1^{\frac{4}{3}}}-\frac{3M_b(r_c+a)a^{\frac{2}{3}}q_2^2}{(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{4}{3}}}+\frac{3M_b(4r_c+a)a^{\frac{4}{3}}q_1^{\frac{4}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}}+\frac{3M_b(12r_c+3a)a^{\frac{4}{3}}q_2^{\frac{4}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}}$$

$$-\frac{3M_b(8r_c-a)a^{\frac{4}{3}}q_1^{\frac{2}{3}}q_2^{\frac{2}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}}-\frac{3M_b(4r_c+a)a^{\frac{4}{3}}q_2^2}{2(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{2}{3}}}+\frac{3M_b(4r_c+a)a^{\frac{4}{3}}q_2^{\frac{8}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{4}{3}}}-\frac{3M_b(4r_c-5a)a^{\frac{4}{3}}q_1^{\frac{4}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}}$$

$$+\mu\left\{\frac{9M_ba}{2(r_c^2+T^2)^{\frac{3}{2}}}+\frac{27M_ba(aq_1)^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}q_2^{\frac{4}{3}}}-\frac{27M_baq_2^{\frac{4}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{4}{3}}}+\frac{9M_baq_2^{\frac{2}{3}}}{(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{2}{3}}}-\frac{3M_b(r_c-2a)}{(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{2}{3}}q_1^{\frac{4}{3}}}\right.$$

$$\left.-\frac{6M_b(r_c-2a)q_2^{\frac{2}{3}}}{(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{2}{3}}q_1^{\frac{4}{3}}}-\frac{15M_b(r_c+a)a^{\frac{2}{3}}q_2^{\frac{4}{3}}}{(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{2}{3}}}+\frac{9M_b(r_c+a)a^{\frac{2}{3}}q_2^{\frac{2}{3}}}{(r_c^2+T^2)^{\frac{3}{2}}}+\frac{6M_b(r_c+a)a^{\frac{2}{3}}q_2^2}{(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{4}{3}}}-\frac{3M_b(r_c-2a)}{(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{2}{3}}q_2^{\frac{2}{3}}}\right.$$

$$-\frac{15M_b(4r_c+a)a^{\frac{4}{3}}q_1^{\frac{4}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}}-\frac{15M_b(4r_c+a)a^{\frac{4}{3}}q_2^{\frac{4}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}}+\frac{15M_b(4r_c+a)a^{\frac{4}{3}}q_1^{\frac{2}{3}}q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}}+\frac{15M_b(4r_c+a)a^{\frac{4}{3}}q_2^2}{4(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{2}{3}}}$$

$$-\frac{3M_b(r_c+a)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{(r_c^2+T^2)^{\frac{3}{2}}q_2^{\frac{2}{3}}}+\frac{3M_b(4r_c+a)a^{\frac{2}{3}}q_1^2}{4(r_c^2+T^2)^{\frac{3}{2}}q_2^{\frac{2}{3}}}-\frac{3M_b(4r_c+a)a^{\frac{4}{3}}q_2^{\frac{8}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{4}{3}}}+\frac{3M_b(r_c+a)a^{\frac{2}{3}}q_1^{\frac{2}{3}}}{(r_c^2+T^2)^{\frac{3}{2}}}+\frac{3M_b(4r_c-5a)}{4(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{4}{3}}q_1^{\frac{4}{3}}}$$

$$-\frac{3M_b(4r_c-5a)}{2(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{2}{3}}q_2^{\frac{2}{3}}}+\frac{3M_b(4r_c-5a)}{4(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{4}{3}}q_1^{\frac{2}{3}}q_2^{\frac{2}{3}}}\Bigg\}+\mu^2\Bigg\{\frac{27M_baq_2^{\frac{4}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{4}{3}}}-\frac{9M_baq_2^{\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{2}{3}}}-\frac{27M_baq_1^{\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}q_2^{\frac{2}{3}}}$$

$$-\frac{9M_ba}{2(r_c^2+T^2)^{\frac{3}{2}}}+\frac{3M_b(r_c-2a)q_2^{\frac{2}{3}}}{(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{2}{3}}q_1^{\frac{4}{3}}}-\frac{3M_b(16r_c+7a)a^{\frac{2}{3}}q_2^{\frac{2}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}}-\frac{3M_b(r_c+a)a^{\frac{2}{3}}q_2^2}{(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{4}{3}}}+\frac{3M_b(16r_c+25a)a^{\frac{2}{3}}q_2^{\frac{4}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{2}{3}}}$$

$$+\frac{3M_b(r_c-2a)q_1^{\frac{2}{3}}}{(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{2}{3}}q_2^{\frac{4}{3}}}+\frac{3M_b(16r_c-13a)}{8(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{2}{3}}q_2^{\frac{2}{3}}}+\frac{3M_b(24r_c-27a)a^{\frac{2}{3}}q_1^{\frac{2}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}q_2^{\frac{4}{3}}}-\frac{3M_b(16r_c+25a)a^{\frac{2}{3}}q_1^{\frac{2}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}}$$

$$+\frac{3M_b(4r_c+16a)q_1^{\frac{4}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}q_2^{\frac{4}{3}}}-\frac{3M_b(r_c+a)a^{\frac{2}{3}}q_1^2}{(r_c^2+T^2)^{\frac{3}{2}}q_2^{\frac{4}{3}}}-\frac{3M_b(12r_c+3a)a^{\frac{4}{3}}q_1^2}{4(r_c^2+T^2)^{\frac{3}{2}}q_2^{\frac{2}{3}}}-\frac{30M_b(4r_c+a)a^{\frac{4}{3}}q_1^{\frac{2}{3}}q_2^{\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}}$$

$$+\frac{3M_b(60r_c-15a)a^{\frac{4}{3}}q_1^{\frac{4}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}}+\frac{3M_b(4r_c-a)a^{\frac{4}{3}}q_1^{\frac{8}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}q_2^{\frac{4}{3}}}+\frac{3M_b(60r_c+15a)a^{\frac{4}{3}}q_2^{\frac{4}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}}+\frac{3M_b(4r_c+a)a^{\frac{4}{3}}q_2^{\frac{8}{3}}}{8(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{4}{3}}}$$

$$-\frac{9M_b(4r_c+a)a^{\frac{4}{3}}q_2^2}{4(r_c^2+T^2)^{\frac{3}{2}}q_1^{\frac{2}{3}}}+\frac{3M_b(8r_c-19a)}{8(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{2}{3}}q_1^{\frac{2}{3}}}-\frac{3M_b(4r_c-5a)}{8(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{4}{3}}q_1^{\frac{4}{3}}}-\frac{3M_b(4r_c-5a)}{8(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{4}{3}}q_2^{\frac{4}{3}}}-\frac{3M_b(4r_c-5a)}{4(r_c^2+T^2)^{\frac{3}{2}}a^{\frac{4}{3}}q_1^{\frac{2}{3}}q_2^{\frac{2}{3}}}\Bigg\}\Bigg] \tag{3.36}$$

Substituting these values in the characteristic equations (3.12) and restricting ourselves only to the linear terms in e^2, a, A, q_1, q_2 *and* M_b where for simplicity we let, $a=1-\alpha$ and $q_i=1-\beta_i$ where $i=1,2$ we obtain,

$$4(\lambda^2)^2+4(4-3\psi_1)\lambda^2+27\mu(1-\mu)+4\psi_2=0 \tag{3.37}$$

where,

$$\psi_1=(1-e^2)^{-\frac{1}{2}}\left(1+A-\frac{M_b}{(r_c^2+T^2)^{\frac{3}{2}}}+\frac{M_br_c^2}{(r_c^2+T^2)^{\frac{5}{2}}}\right)$$

and

$$\psi_2 = 3\mu(1-\mu)\alpha + 9\mu(1-\mu)A + \frac{45\mu(1-\mu)}{4}e^2 + \frac{3\mu(1-\mu)}{2}(\beta_1+\beta_2) + \frac{3\mu M_b(1-\mu)(4r_c-11)}{2(r_c^2+T^2)^{\frac{3}{2}}}$$
$$+ \frac{27\mu M_b(1-\mu)}{4(r_c^2+T^2)^{\frac{5}{2}}} \tag{3.38}$$

Equation (3.37) is a quadratic equation in λ^2 which yields

$$\lambda^2 = \frac{-4(4-3\psi_1) \pm \left[(4-3\psi_1)^2 - 27\mu(1-\mu) - 4\psi_2\right]^{\frac{1}{2}}}{2}$$

For stable motion, we require λ to be purely imaginary, this implies that, the motion of the infinitesimal must be bounded and periodic, such that $\lambda^2 < 0$. This implies that

$$3\psi_1 - 4 \le 0$$

and the discriminant

$$\Delta = (4-3\psi_1)^2 - 27\mu(1-\mu) - 4\psi_2 \tag{3.39}$$

This yield

$$0 < e \le \left[1 - \frac{9}{16}\left(1 + A - \frac{M_b}{(r_c^2+T^2)^{\frac{3}{2}}} + \frac{M_b r_c^2}{(r_c^2+T^2)^{\frac{5}{2}}}\right)^2\right]^{\frac{1}{2}} \tag{3.40}$$

When $A = M_b = 0$ equation (3.40) becomes

$$0 < e \le \frac{\sqrt{7}}{4} \tag{3.41}$$

The characteristics roots will be either real or complex conjugate if equation (3.38) is not satisfied. In case of complex roots, the positive real part indicates instability of the investigated triangular points.

Now, making use of equations (3.38) in equation (3.39), we have

$$\Delta = \left(27 + 12\alpha + 36A + 45e^2 + 6\beta_1 + 6\beta_2 + \frac{6M_b(4r_c - 11)}{(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{27M_b}{(r_c^2 + T^2)^{\frac{5}{2}}} \right) \mu^2$$
$$- \left(27 + 12\alpha + 36A + 45e^2 + 6\beta_1 + 6\beta_2 + \frac{6M_b(4r_c - 11)}{(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{27M_b}{(r_c^2 + T^2)^{\frac{5}{2}}} \right) \mu \qquad (3.42)$$
$$+ 1 - 3e^2 - 6A + \frac{6M_b}{(r_c^2 + T^2)^{\frac{3}{2}}} - \frac{6M_b r_c^2}{(r_c^2 + T^2)^{\frac{5}{2}}}$$

If $\Delta = 0$, then

$$\mu_c = \frac{1}{2}\left(1 - \frac{\sqrt{69}}{9}\right) - \frac{4}{27\sqrt{69}}\alpha - \frac{22}{9\sqrt{69}}A - \frac{14}{9\sqrt{69}}e^2 - \frac{2}{27\sqrt{69}}(\beta_1 + \beta_2)$$
$$+ \left[\frac{(76 - 8r_c)(r_c^2 + T^2)^{\frac{5}{2}} - 9(1 + 6r_c^2)(r_c^2 + T^2)^{\frac{3}{2}}}{27\sqrt{69}(r_c^2 + T^2)^4} \right] M_b \qquad (3.43)$$

Equations (3.40) and (3.42) are the necessary conditions for the stability of the triangular points. The critical value μ_c given by equation (3.43) is the solution of the quadratic equation $\Delta = 0$. It represents the effects of the various parameters involved on the size of the region of Stability. A close examination shows that they all cause a reduction in the size of the region confirming their destabilizing tendencies, while the potential from the circumbinary disc has a stabilizing effect.

3.7.2 Stability of the Collinear Equilibrium Points

To study the stability of the collinear equilibrium points $L_i (i = 1,2,3)$, we consider the characteristic equation (3.12) of the system given by:

$$\lambda^4 - (\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} - 4)\lambda^2 + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - (\Omega^0_{\xi\eta})^2 = 0$$

Here, we obtain the points corresponding to the collinear by taking the second partial derivatives of equation (3.2), with respect to the (ξ, η, ζ) with $\eta = \zeta = 0$. Thus, we have

$$\Omega^0_{\xi\xi} = \left(1-e^2\right)^{-\frac{1}{2}}\left[1+\frac{2}{n^2}\left\{\frac{(1-\mu)q_1}{|\xi+\mu|^3}+\frac{\mu q_2}{|\xi+\mu-1|^3}+\frac{3(1-\mu)A}{|\xi+\mu|^5}+\frac{3\mu A}{|\xi+\mu-1|^5}\right.\right.$$

$$\left.\left.+\left(\frac{3\xi^2(\xi^2+T^2)^{\frac{3}{2}}-(\xi^2+T^2)^{\frac{5}{2}}}{2(\xi^2+T^2)^4}\right)M_b\right\}\right]$$

$$\Omega^0_{\eta\eta} = \left(1-e^2\right)^{-\frac{1}{2}}\left[1-\frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{|\xi+\mu|^3}+\frac{\mu q_2}{|\xi+\mu-1|^3}+\frac{3(1-\mu)A}{2|\xi+\mu|^5}+\frac{3\mu A}{2|\xi+\mu-1|^5}+\frac{M_b}{(\xi^2+T^2)^{\frac{3}{2}}}\right\}\right]$$

$$\Omega^0_{\zeta\zeta} = 0 \tag{3.44}$$

Since all the parameters in $\Omega^0_{\xi\xi}$ are positive and $\Omega^0_{\xi\xi}$ contains only positive terms, it implies that $\Omega^0_{\zeta\zeta} > 0$ (3.45)

3.7.2.1 Stability of $L_1(\xi > \xi_2)$

In the first interval, we have; $r_1 = (\xi+\mu) \Rightarrow \xi = (r_1-\mu)$, and $r_2 = (\xi+\mu-1)$

(3.46) substituting equation (3.46) in equation (3.25) yields

$$\frac{(1-\mu)q_1}{r_1^2} = n^2\xi - \frac{\mu q_2}{r_2^2} - \frac{3(1-\mu)A}{2r_1^4} - \frac{3\mu A}{2r_2^4} + \frac{M_b\xi}{(\xi^2+T^2)^{\frac{3}{2}}} \tag{3.47}$$

Substituting equation (3.47) in the second equation of (3.44) gives

$$\Omega^0_{\eta\eta} = \left(1-e^2\right)^{-\frac{1}{2}}\left[1-\frac{1}{n^2}\left\{\frac{1}{r_1}\left(-n^2\xi-\frac{\mu q_2}{r_2^2}-\frac{3(1-\mu)A}{2r_1^4}-\frac{3\mu A}{2r_2^4}+\frac{M_b\xi}{(\xi^2+T^2)^{\frac{3}{2}}}\right)\right.\right.$$

$$\left.\left.+\frac{\mu q_2}{r_2^3}+\frac{3(1-\mu)A}{2r_1^5}+\frac{3\mu A}{2r_2^5}+\frac{M_b}{(\xi^2+T^2)^{\frac{3}{2}}}\right\}\right] \tag{3.48}$$

This implies that

$$\Omega_{\eta\eta}^{0}=\left(1-e^{2}\right)^{-\frac{1}{2}}\left[\frac{\mu}{r_1}+\frac{1}{n^{2}}\left\{\frac{\mu q_2}{r_2^{2}}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+\frac{3\mu A}{2r_2^{4}}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{2M_b r_1-\mu M_b}{r_1\left[(r_1-\mu)^{2}+T^{2}\right]^{\frac{3}{2}}}\right\}\right] \tag{3.49}$$

Thus, $\Omega_{\eta\eta}<0$, Since $\mu<\frac{1}{2}$ and $A, M_b<<1,\ \ r_1>1,\ \ r_2<1$ clearly $\Omega_{\xi\xi}>0,\ \ \Omega_{\eta\eta}<0$

and $\Omega_{\xi\eta}=0$

3.7.2.2 Stability of $L_2(\xi_1<\xi<\xi_2)$

In the second interval, we have; $r_1=(\xi+\mu)\Rightarrow\xi=r_1-\mu$ and $r_2=-(\xi+\mu-1)$ (3.50)

Substituting equation (3.50) in the equation (3.25) yields

$$\frac{(1-\mu)q_1}{r_1^{2}}=n^{2}\xi+\frac{\mu q_2}{r_2^{2}}-\frac{3(1-\mu)A}{2r_1^{4}}+\frac{3\mu A}{2r_2^{4}}-\frac{M_b\xi}{(\xi^{2}+T^{2})^{\frac{3}{2}}}=0 \tag{3.51}$$

Substituting equation (3.51) in the second equation of (3.44) gives

$$\Omega_{\eta\eta}^{0}=\left(1-e^{2}\right)^{-\frac{1}{2}}\left[1-\frac{1}{n^{2}}\left\{\frac{1}{r}\left(n^{2}\xi+\frac{\mu q_2}{r_2^{2}}-\frac{3(1-\mu)A}{2r_1^{4}}+\frac{3\mu A}{2r_2^{4}}-\frac{M_b\xi}{(\xi^{2}+T^{2})^{\frac{3}{2}}}\right)\right.\right. \tag{3.52}$$

$$\left.\left.+\frac{\mu q_2}{r_2^{3}}+\frac{3(1-\mu)A}{2r_1^{5}}+\frac{3\mu A}{2r_2^{5}}+\frac{M_b}{(\xi^{2}+T^{2})^{\frac{3}{2}}}\right\}\right]$$

this Implies that

$$\Omega_{\eta\eta}^{0}=\left(1-e^{2}\right)^{-\frac{1}{2}}\left[\frac{\mu}{r_1}-\frac{1}{n^{2}}\left\{\frac{\mu q_2}{r_2^{2}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+\frac{3\mu A}{2r_2^{4}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+\frac{\mu M_b}{r_1\left[(r_1-\mu))^{2}+T^{2}\right]^{\frac{3}{2}}}\right\}\right] \tag{3.53}$$

$\Omega_{\eta\eta}<0,\ \ \sin ce\ \ \mu<\frac{1}{2}\ \ and\ \ A, M_b<<1,\ \ r_1>1,\ \ r_2<1,\ \ clearly\ \ \Omega_{\xi\xi}>0,\ \ \Omega_{\eta\eta}<0\ \ and\ \ \Omega_{\xi\eta}=0$

3.7.2.3 Stability of L_3 ($\xi_1 > \xi$)

In the last interval, we have $r_1 = -`(\xi + \mu) \Rightarrow \xi = -(r_1 + \mu)$ and $r_2 = -(\xi + \mu - 1)$ (3.54)

Substituting equation (3.54) in the equation (3.25) yields

$$\frac{(1-\mu)q_1}{r_1^2} = -n^2\xi - \frac{\mu q_2}{r_2^2} - \frac{3(1-\mu)A}{2r_1^4} - \frac{3\mu A}{2r_2^4} + \frac{M_b\xi}{(\xi^2+T^2)^{\frac{3}{2}}} = 0 \tag{3.55}$$

Substituting equation (3.55) in the second equation of (3.44) gives

$$\Omega_{\eta\eta}^0 = \left(1-e^2\right)^{-\frac{1}{2}}\left[1 - \frac{1}{n^2}\left\{\frac{1}{r}\left(-n^2\xi - \frac{\mu q_2}{r_2^2} - \frac{3(1-\mu)A}{2r_1^4} - \frac{3\mu A}{2r_2^4} + \frac{M_b\xi}{(\xi^2+T^2)^{\frac{3}{2}}}\right)\right.\right. \tag{3.56}$$

$$\left.\left. + \frac{\mu q_2}{r_2^3} + \frac{3(1-\mu)A}{2r_1^5} + \frac{3\mu A}{2r_2^5} + \frac{M_b}{(\xi^2+T^2)^{\frac{3}{2}}}\right\}\right]$$

this Implies that

$$\Omega_{\eta\eta}^0 = \left(1-e^2\right)^{-\frac{1}{2}}\left[\frac{\mu}{r_1} + \frac{1}{n^2}\left\{\frac{\mu q_2}{r_2^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{3\mu A}{2r_2^4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \frac{\mu M_b}{r_1\left[(-\mu - r_1))^2 + T^2\right]^{\frac{3}{2}}}\right\}\right] \tag{3.57}$$

$\Omega_{\eta\eta} < 0$ Since $\mu < \frac{1}{2}$ and $A, M_b << 1$, $r_1 > 1$, $r_2 < 1$ clearly $\Omega_{\xi\xi} > 0$, $\Omega_{\eta\eta} < 0$ and

$\Omega_{\xi\eta} = 0$ since $\Omega_{\xi\xi}^0\Omega_{\eta\eta}^0 - (\Omega_{\xi\eta}^0)^2 < 0$, its characteristics roots can be expressed as:

$$\lambda_{1,3} = -\frac{\left[-4 + \Omega_{\xi\xi}^0 + \Omega_{\eta\eta}^0 - \sqrt{16 - 8\Omega_{\eta\eta}^0 + (\Omega_{\eta\eta}^0)^2 + 4(\Omega_{\xi\eta}^0)^2 - 8\Omega_{\xi\xi}^0 - 2\Omega_{\xi\xi}^0\Omega_{\eta\eta}^0 + (\Omega_{\xi\xi}^0)^2}\right]^{\frac{1}{2}}}{\sqrt{2}}$$

$$\lambda_{2,4} = \frac{\left[-4 + \Omega_{\xi\xi}^0 + \Omega_{\eta\eta}^0 + \sqrt{16 - 8\Omega_{\eta\eta}^0 + (\Omega_{\eta\eta}^0)^2 + 4(\Omega_{\xi\eta}^0)^2 - 8\Omega_{\xi\xi}^0 - 2\Omega_{\xi\xi}^0\Omega_{\eta\eta}^0 + (\Omega_{\xi\xi}^0)^2}\right]^{\frac{1}{2}}}{\sqrt{2}}$$

(3.58)

Hence, we conclude that, the collinear equilibrium points are unstable due to the nature of the characteristic roots of equations (3.58).

CHAPTER FOUR

NUMERICAL COMPUTATIONS

4.1 Triangular Equilibrium Points

The triangular points are obtained numerically for the binary systems HD 98800 B and GJ 630.1A. These are all binary systems with elliptic orbits, radiating primaries and circumbinary disc. Table 4.1 presents the numerical data of the systems.

Table 4. 1: Relevant Numerical Data

Binary system	Masses (M_0)		Luminosity (L_0)		Eccentricity
	M_1	M_2	L_1	L_2	(e)
HD 98800 B	0.70	0.58	0.33	0.17	0.5
GJ 630. 1A	1.35	0.34	0.0052	0.0049	0.0054

The mass reduction factors q_1 and q_2 are computed, taking K=l, on the basis of Stefan-Boltzmann's law, $q_i = 1 - \frac{A_i K_i L_i}{r_i \rho_i M_i}, (i = 1,2)$, where M and L are the mass and luminosity of a star respectively; r and ρ are the radius and density of a moving body; K is the radiation pressure efficiency factor of a star; $A = \frac{3}{16\pi CG}$ is a constant. In the CGS system, $A = 2.9838 \times 10^{-5}$

In order to show the effects of the various parameters on the positions of triangular points $L_{4,5}$ we use equations (3.23) in the following cases, and for stability we use equation (3.43)

1. Absence of eccentricity, semi-major axis, radiation, and potential from the belt (classical case).
2. Radiation of the bigger primary only

3. Radiation of the smaller primary only,
4. Eccentricity only
5. Potential from the disc only.
6. Oblateness of the third body
7. Both radiating, elliptic and gravitational potential from the disc
8. Presence of eccentricity, semi-major axis, radiation of the primaries, oblateness of the third body and potential from the disc

The circular case *($e = M_b = 0$, and $a = q_1 = q_2 = 1$)* are first shown and for simplicity we set T=0.01.

Table 4. 2: Locations and Stability of Triangular Equilibrium Points

Binary System	mass ratio μ	perturbations q_1	q_2	e	a	A	M_b	location and stability of triangular points ξ	$\pm\eta$	μ_c
HD	0.453125	1	1	0	1	0	0	0.046875	0.866025	0.0385209
	0.99903	1	0	1	0	0	0.0465516	0.860960	0.0385122
	1	0.99941	0	1	0	0	0.0470717	0.864591	0.0385156
	1	1	0.5	1	0	0	0.046875	0.707107	-0.0082958
	1	1	0	1	0	0.01	0.046875	0.861682	0.0353836
	1	1	0	1	0.01	0	0.046875	0.87178	0.0355781
	0.99903	0.99941	0.5	1	0	0.01	0.0467809	0.701505	-0114458
	0.99903	0.99941	0.5	1.471	0.01	0.01	0.046755	0.85250	-0.005988
GJ	0.201183	1	1	0	1	0	0	0.298817	0.866025	0.0385209
	0.99995	1	0	1	0	0	0.29880	0.86016	0.0385205
	1	0.9995	0	1	0	0	0.298834	0.86016	0.0385205
	1	1	0.0054	1	0	0	0.298817	0.86009	0.0385154
	1	1	0	1	0	0.01	0.298817	0.86168	0.0353836
	1	1	0	1	0.01	0	0.298817	0.87178	0.0355781

0.99995	0.99995	0.0054	1	0	0.01	0.298817	0.861814	0.0359029

We see that the positions of the triangular points are affected by the radiation of the primaries and the potential from the circumbinary disc such that they do not form equilateral triangles with the primaries. We also observe that the triangular points come nearer to the primaries in the presence of all these perturbations (Table (4.2) case 8).

These positions of triangular points with respect to the eccentricity, semi-major axis, oblateness of the infinitesimal body, the potential from the disc can be better seen by using the program in appendix 1

Table 4. 3: Effect of eccentricity on $L_{4,5}$ in the binary system HD98800B

e	ξ	$\pm\eta$
0.1	0.0467152	1.018340
0.3	0.0467283	0.966228
0.5	0.0467546	0.852500
0.7	0.0467939	0.645365
0.9	0.0468463	0.053052

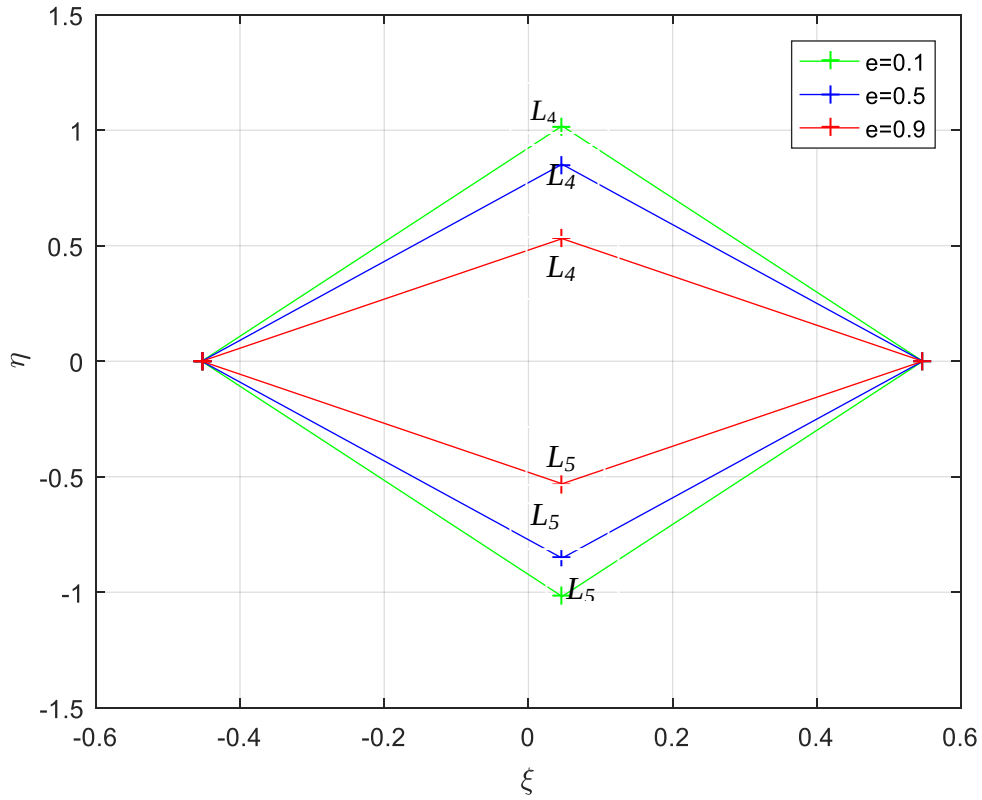


Figure 4. 1: Effect of eccentricity on $L_{4,5}$ in the binary system HD98800B

Table 4. 4: Effect of eccentricity on $L_{4,5}$ in the binary system GJ 630.1A

e	ξ	$\pm\eta$
0.00001	0.298817	1.49146
0.01	0.298817	1.49138
0.1	0.298817	1.48333
0.3	0.298817	1.41653
0.5	0.298817	1.27245

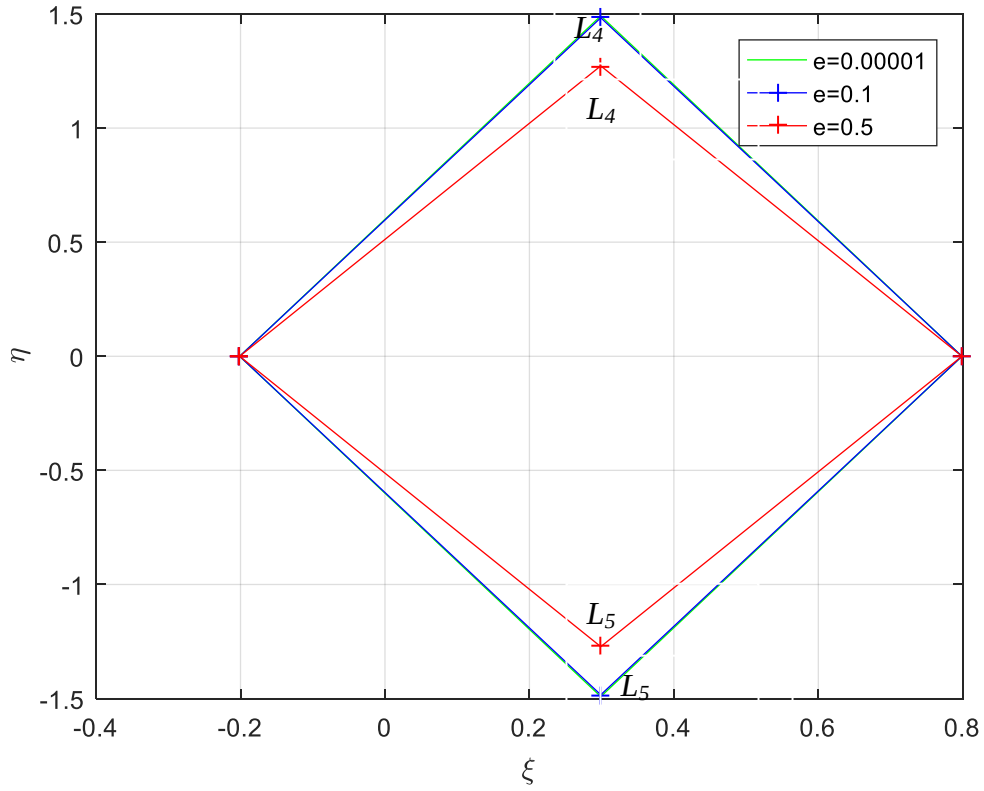


Figure 4. 2: Effect of eccentricity on $L_{4,5}$ in the binary system GJ 630.1A

Table 4. 5: Effect of semi-major axis on $L_{4,5}$ in the binary system HD98800B

a	ξ	±η
0.95	0.0467828	0.708602
0.85	0.0467645	0.805021
0.75	0.0467529	0.860137
0.65	0.0467418	0.910042
0.55	0.046731	0.955855

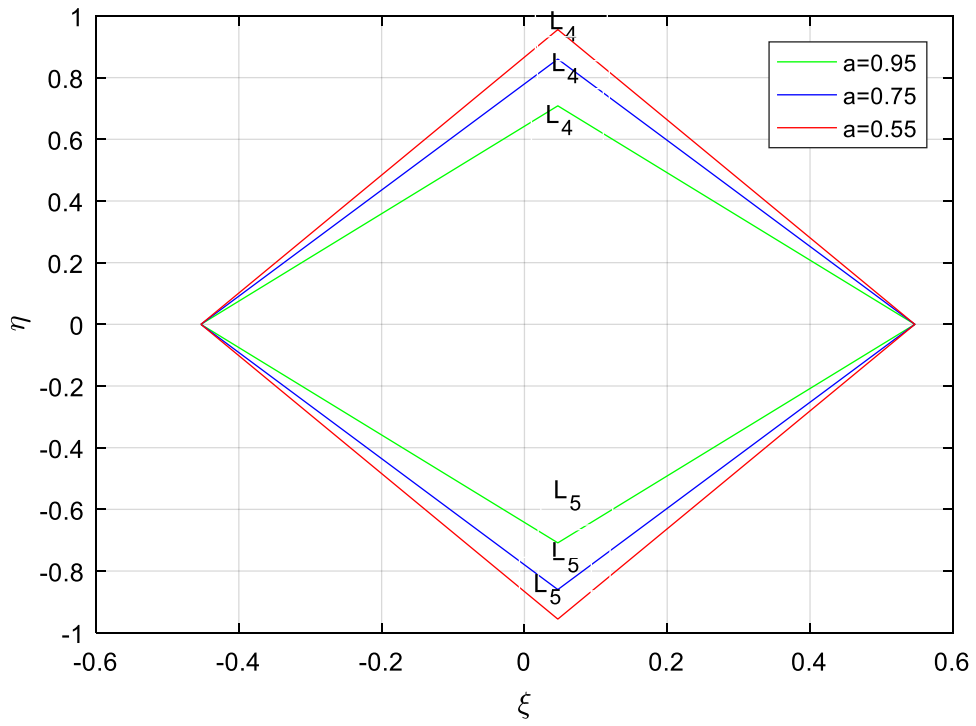


Figure 4. 3: Effect of semi-major axis on $L_{4,5}$ in the binary system HD98800B

Table 4. 6: Effect of semi-major axis on $L_{4,5}$ in the binary system GJ 630.1A

a	ξ	±η
0.9	0.298817	0.617702
0.7	0.298817	1.03306
0.5	0.298817	1.44939
0.3	0.298817	1.59731
0.1	0.298817	1.72569

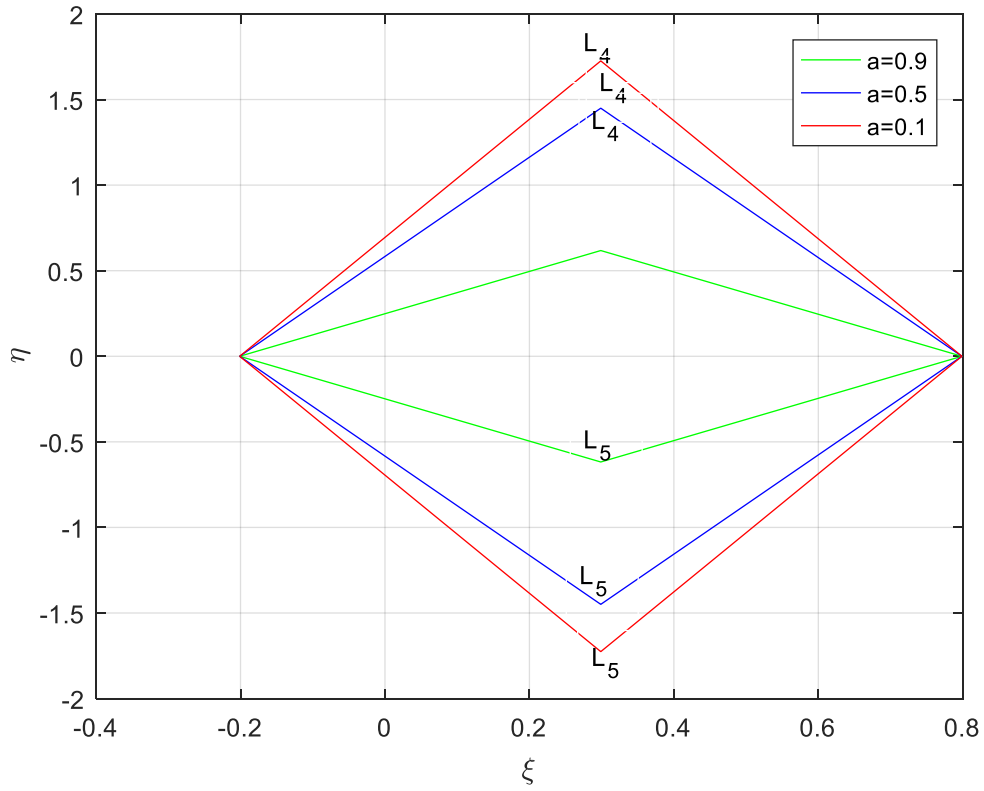


Figure 4. 4: Effect of semi-major axis on $L_{4,5}$ in the binary system GJ 630.1A

Table 4. 7: Effect of oblateness of the third body on $L_{4,5}$ in the binary system HD98800B

A	ξ	±η
0	0.0467525	0.846217
0.01	0.0467546	0.852500
0.02	0.0467566	0.858737
0.03	0.0467582	0.866939
0.1	0.0467728	0.907089
0.2	0.0467931	0.964125

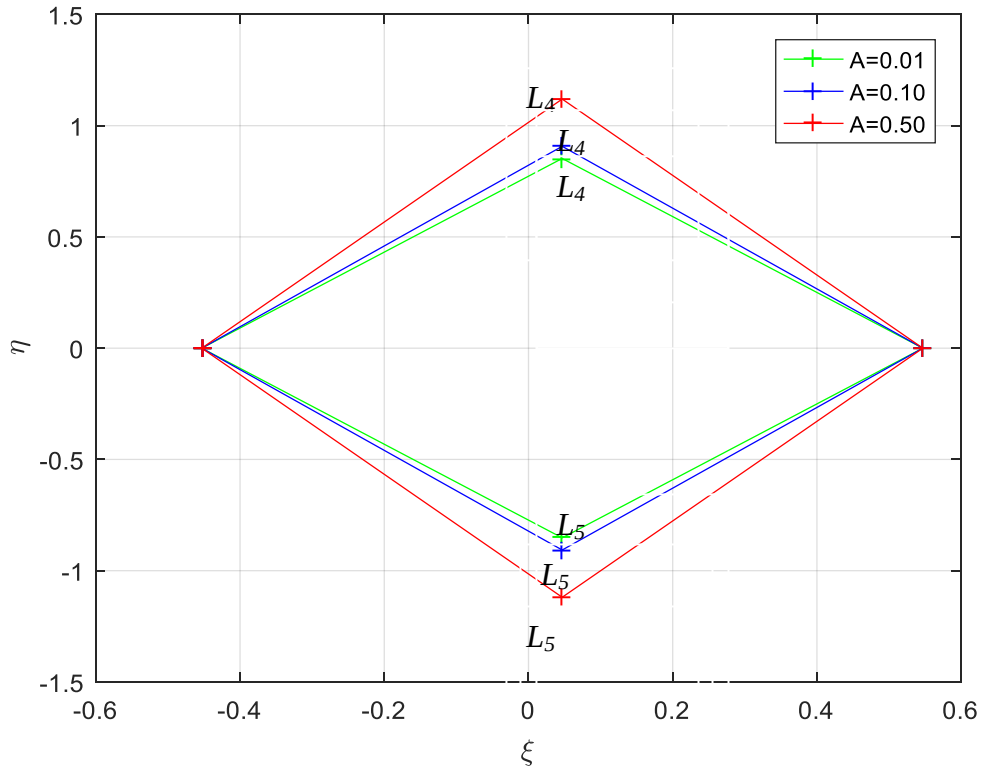


Figure 4. 5: Effect of oblateness of the third body on $L_{4,5}$ in the binary system HD98800B

Table 4. 8: Effect of oblateness of the third body on $L_{4,5}$ in the binary system GJ 630.1A

A	ξ	$\pm\eta$
0	0.298817	1.47353
0.01	0.298817	1.47776
0.02	0.298817	1.48197
0.03	0.298817	1.48617
0.1	0.298817	1.51527
0.2	0.298817	1.55589

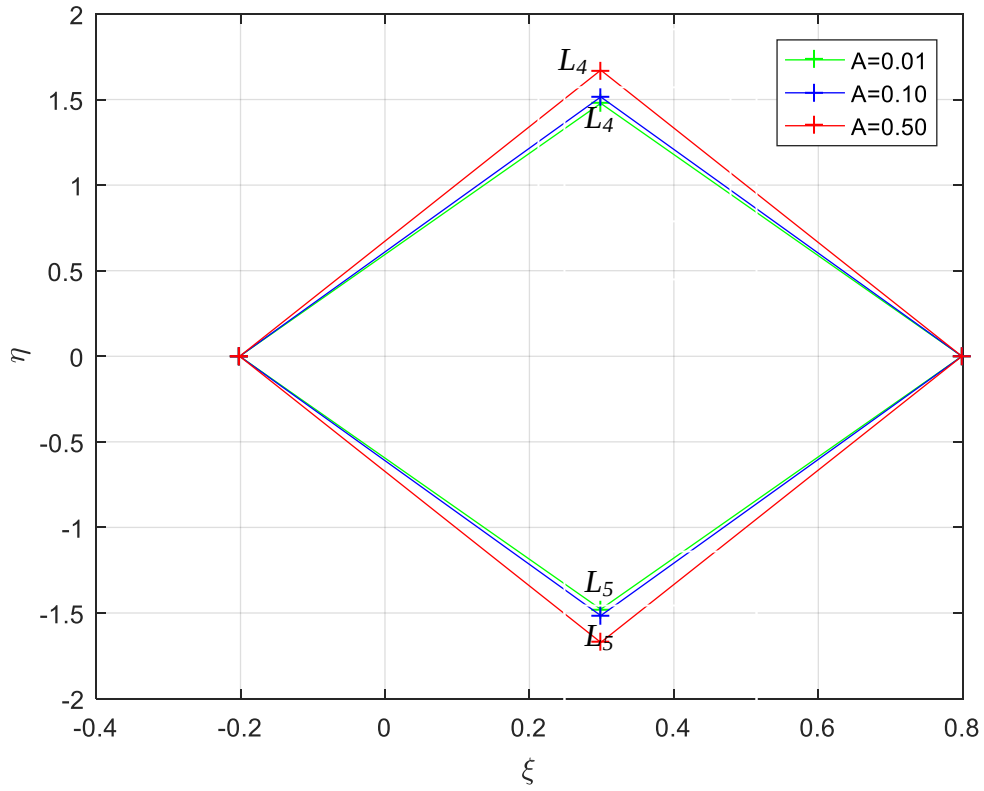


Figure 4. 6: Effect of oblateness of the third body on $L_{4,5}$ in the binary system GJ 630.1A

Table 4. 9: Effect of potential from the disc on $L_{4,5}$ in the binary system HD98800B

M_b	ξ	±η
0	0.0467521	0.848272
0.01	0.0467525	0.846217
0.02	0.0467530	0.844158
0.03	0.0467653	0.784297
0.1	0.0467565	0.827497
0.2	0.0467609	0.806186
0.4	0.0467697	0.761779

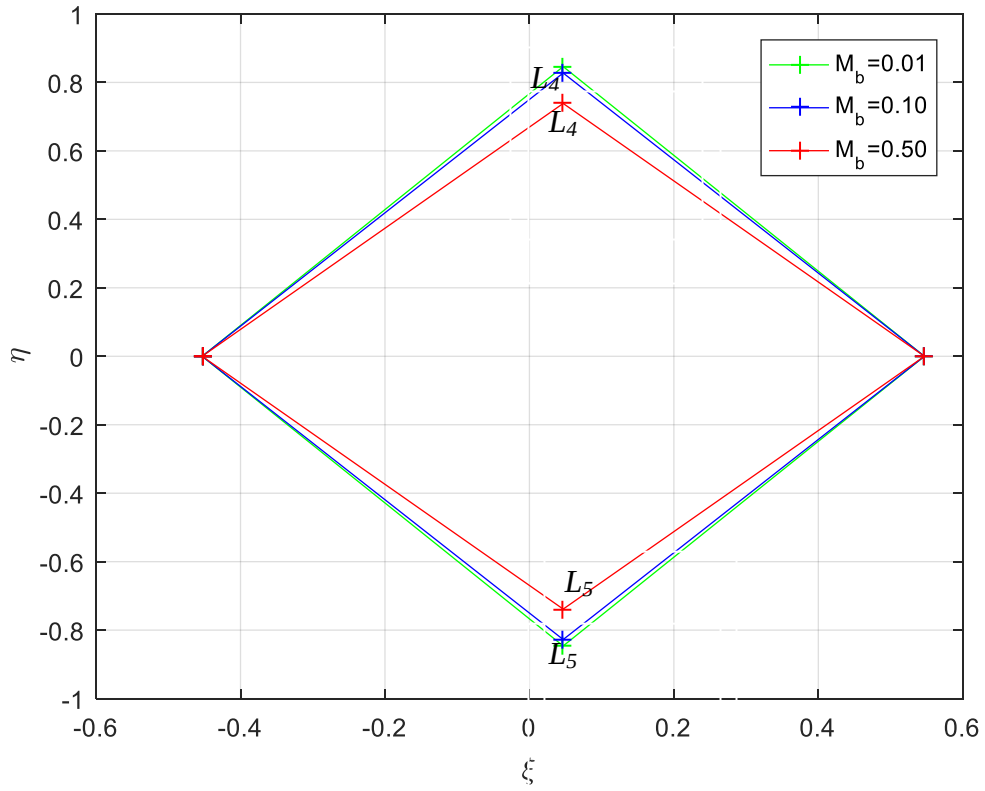


Figure 4. 7: Effect of potential from the disc on $L_{4,5}$ in the binary system HD98800B

Table 4. 10: Effect of potential from the disc on $L_{4,5}$ in the binary system GJ 630.1A

M_b	ξ	±η
0	0.298817	1.47353
0.01	0.298817	1.48725
0.02	0.298817	1.50085
0.03	0.298817	1.51433
0.1	0.298817	1.60549
0.2	0.298817	1.72741
0.4	0.298817	1.94848
0.5	0.298817	2.05010

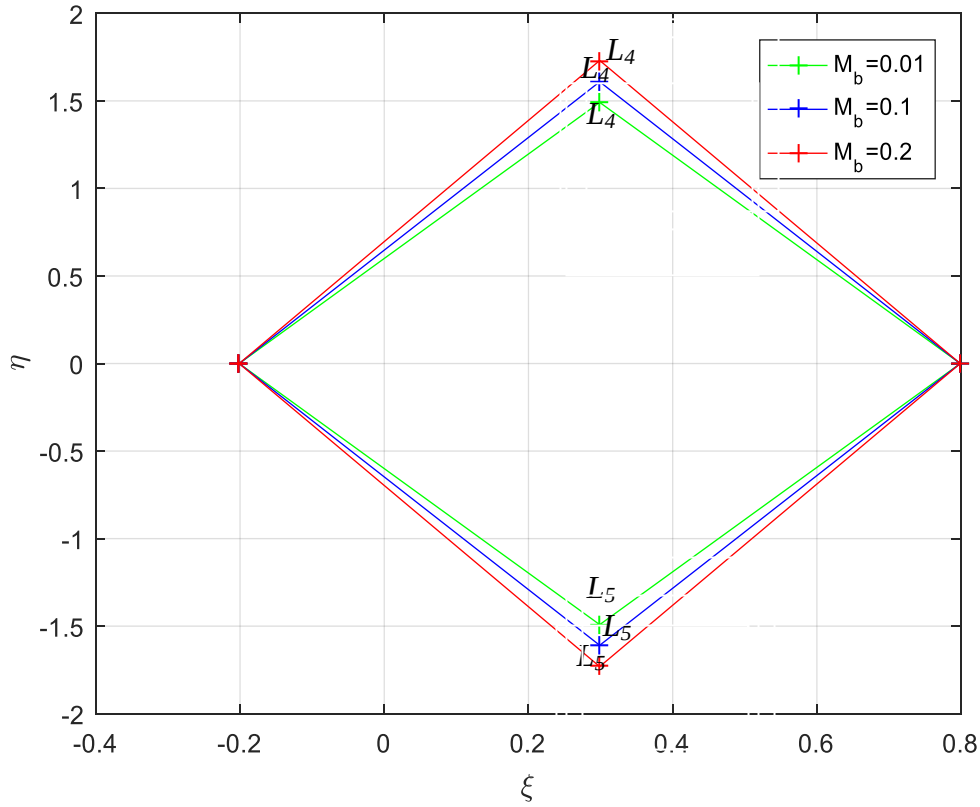


Figure 4. 8: Effect of potential from the disc on $L_{4,5}$ in the binary system GJ 630.1A

4.2 Stability of triangular equilibrium points

Equation (3.43) is used to obtain the effects of the perturbations mentioned in this dissertation on the stability of triangular points by using the program in appendix 2

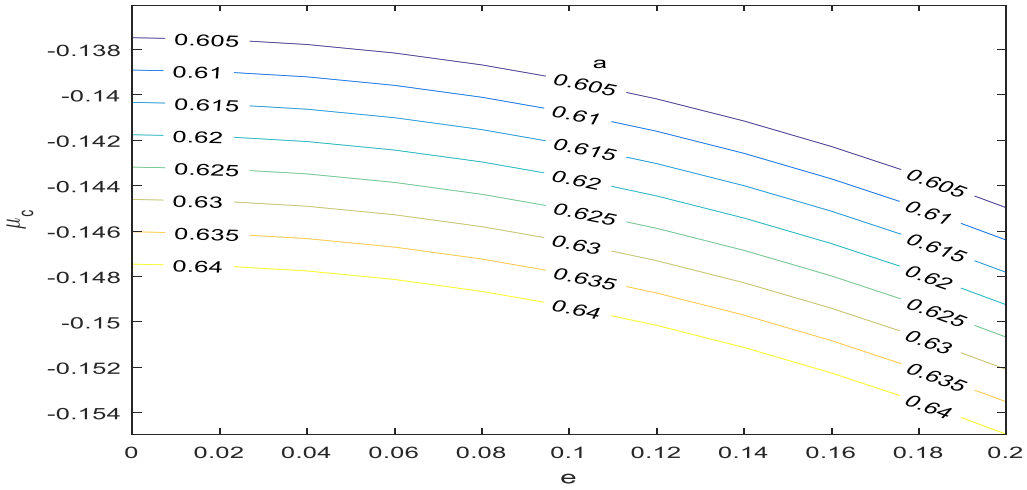


Figure 4. 9: Effect of semi-major axis on the stability $L_{4,5}$ in HD 98800 B

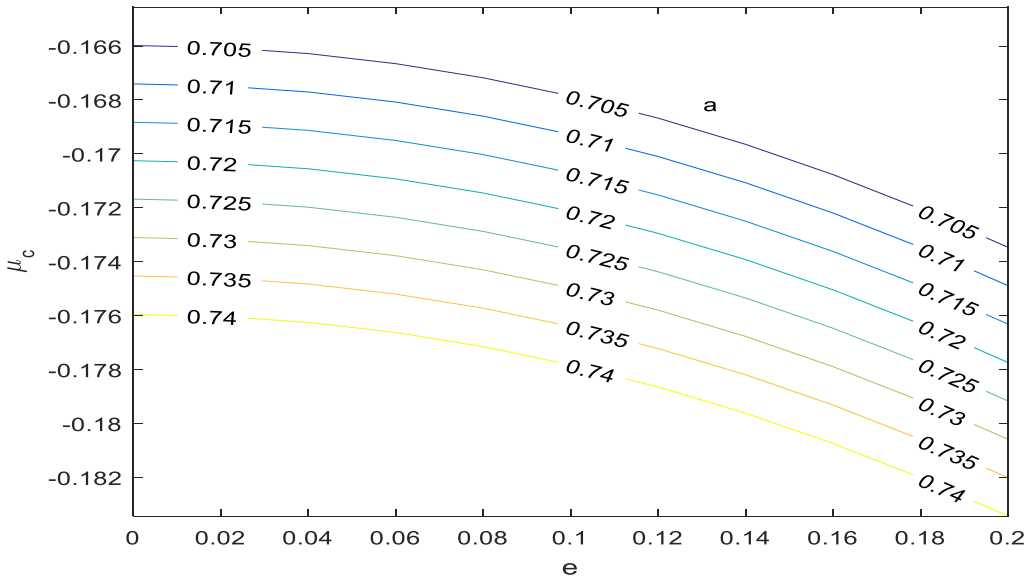


Figure 4. 10: Effect of semi-major axis on the stability $L_{4,5}$ in GJ 630.1A

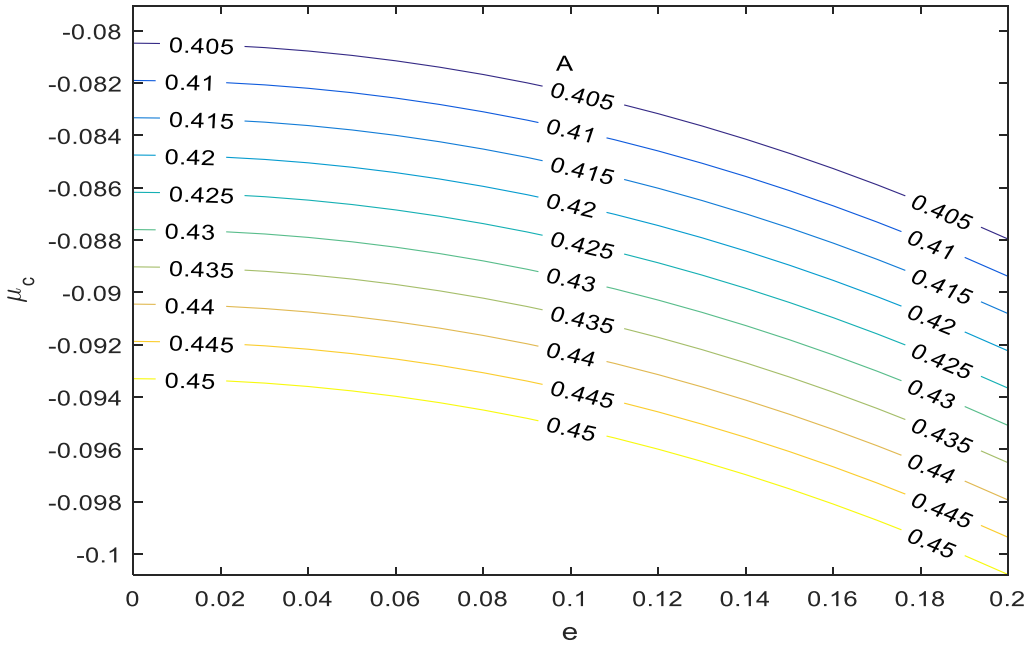


Figure 4. 11: Effect of oblateness on the stability $L_{4,5}$ in HD 98800 B

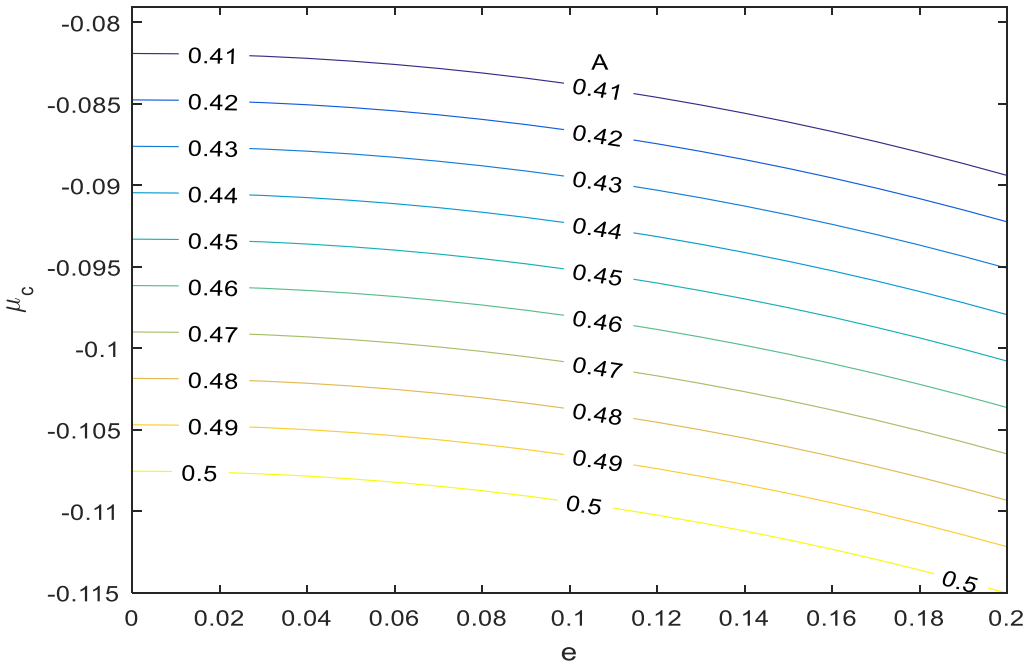


Figure 4. 12: Effect of oblateness on the stability $L_{4,5}$ in GJ 630.1A

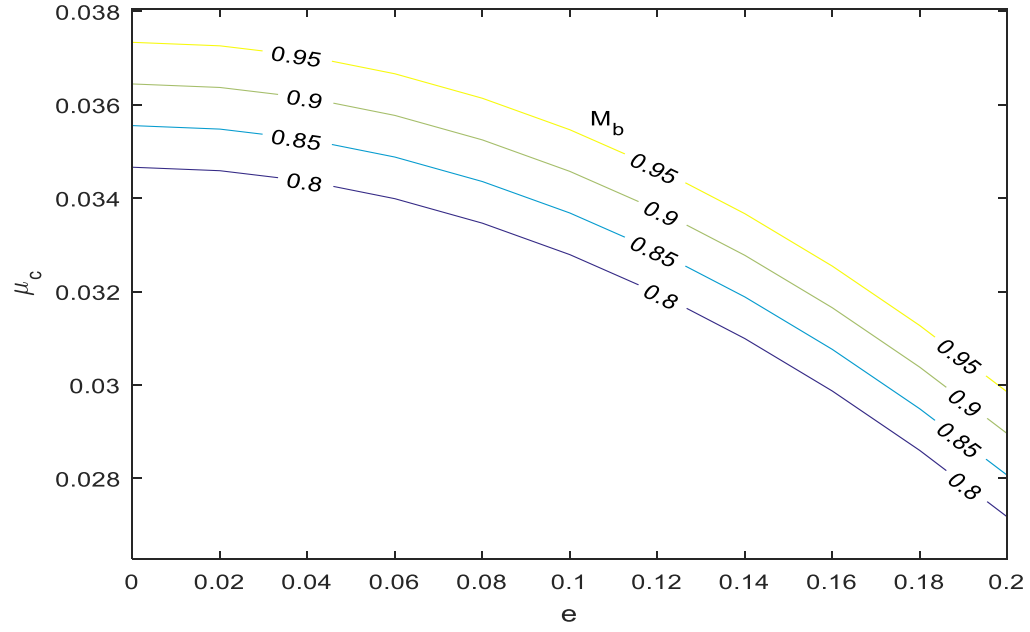


Figure 4. 13: Effect of potential from the disc on the stability $L_{4,5}$ in HD98800B

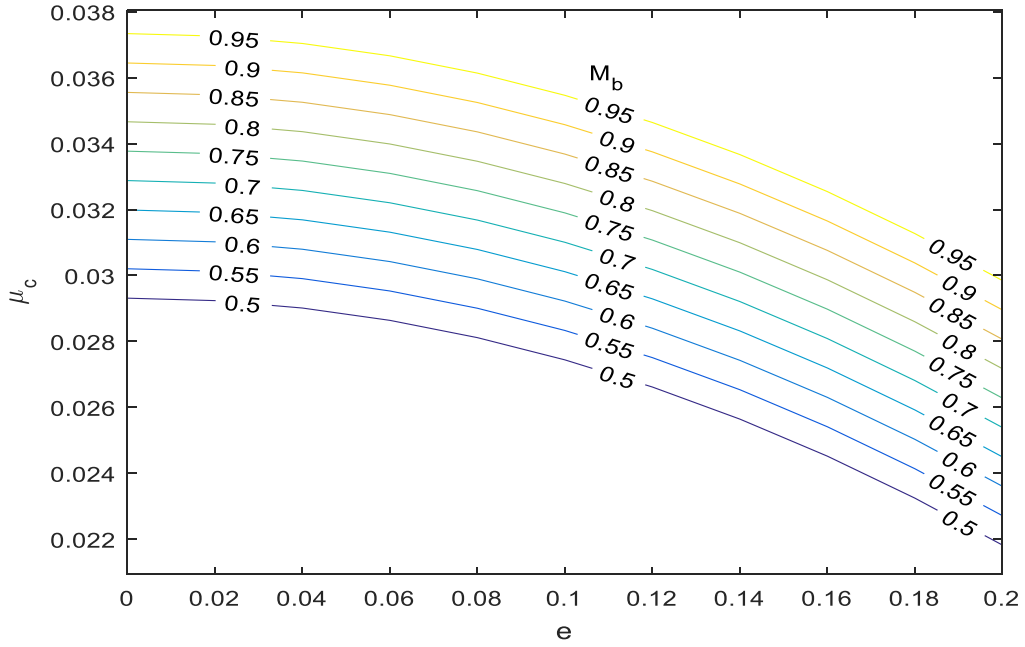


Figure 4. 14: Effect of potential from the disc on the stability $L_{4,5}$ in GJ 630.1A

4.3 Numerical Results of the Collinear Equilibrium Points

The coordinates of the collinear libration points for different cases as highlighted in table 4.2 will be obtained by the use of WOLFRAM MATHEMATICA 10.0 software package using equation (3.25a).

Table 4. 11: Locations of collinear points for

$\mu = 0.453125,\ \ q_1 = 0.99903,\ \ q_2 = 0.99941,\ \ A = M_b = T = 0.01$

Case	Roots ($L_{1,2,3}$)	L_1	L_2	L_3
0	L_1(-1, 0, -0.546875, 0.667334)	1.214209		
	L_2 (0, 0.480654, 0.546875)		0.066221	
	L_3 (0.271377, 0.999998, 1, 1.45313, 1.99999, 2.00001)			-1.181748
2	L_1 (-1, -0.546875, 0, 0.66729)	1.214165		
	L_2 (0, 0.48077, 0.546875)		0.066105	

	L_3 (0.271627, 1.453125)		-1.181498
3	L_1 (-0.546875, 0, 0.667194)	1.214096	
	L_2 (0, 0.480586, 0.546875, 1)	0.096289	
	L_3 (0.271399, 1.453125)		-1.181726
4	L_1 (-0.546875, 0, 0.582649)	1.129524	
	L_2 (0, 0.482086, 0.546875)	0.064789	
	L_3 (0.359901, 0.999997, 1, 1.453125, 1.99998, 2.00002)		-1.093224
5	L_1 (-0.546875, 0, 0.785561)	1.332436	
	L_2 (0, 0.479381, 0.546875)	0.067494	
	L_3 (0.148677, 0.999998, 1, 1.453125, 1.99999, 2.00001)		-1.304448
6	L_1 (-1, -0.593293, -0.546875, 0, 0.661491)	1.208367	
	L_2 (0, 0.434919, 0.546875, 0.610656, 1)	0.111956	
	L_3 (0.277327, 1.453125, 1.50283)		-1.175798
7	L_1 (-0.546875, 0.682789)	1.229664	
	L_2 (0.481426, 0.546875)	0.065449	
	L_3 (0.264124, 1.453125)		-1.189001
8	L_1 (-0.588686, -0.547349, -0.546402 0.697656)	1.244531	

L_2 (0.43811, 0.546749, 0.547002, 0.606963)			0.108765
L_3 (0.24835, 1.45357, 1.49986)			-1.204775

From the table above, we observe that there are several real roots to the solution of equation (3.25a) but only one physically reasonable root and only one in each case corresponds to the collinear points L_1, L_2 and L_3. The remaining real roots do not satisfy the slow-motion assumption as confirmed by (Yamada and Asada, 2011).

4.4 Diagrams of numerical Results of the Collinear Equilibrium Points of the models

To see the effects of eccentricity, semi-major axis, oblateness of the third body and potential from the disc in determining their stability or instability on the collinear points we use the values of $\Omega^0_{\xi\xi}$ and $\Omega^0_{\eta\eta}$. From these results we can also see the effects of eccentricity, semi-major axis, oblateness of the third body and potential from the disc on the collinear points L_1, L_2 and L_3 in this case we use the program in appendix 3.

Table 4. 12: Effect of eccentricity in HD 98800 B on the collinear point L_1

e	hL_1	L_1	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0.1	0.787143	1.334018	4.05489	-0.487323	-1.97604	Unstable
0.3	0.752985	1.29986	4.29765	-0.538467	-2.31414	Unstable
0.5	0.697656	1.244531	4.87254	-0.653676	-3.18506	Unstable
0.7	0.635939	1.182814	6.13299	-0.888847	-5.45130	Unstable
0.9	0.577068	1.123943	10.4751	-1.63513	-17.1281	Unstable

Table 4. 13: Effect of eccentricity in GJ 630.1A on the collinear point L_1

e	hL_1	L_1	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0.00001	0.919917	1.718734	3.82161	-0.392256	-1.49905	Unstable
0.01	0.919852	1.718669	3.82139	-0.392072	-1.49826	Unstable
0.1	0.913512	1.712329	3.80058	-0.374356	-1.42277	Unstable
0.3	0.866709	1.665526	3.71028	-0.26495	-0.983039	Unstable
0.5	0.791511	1.590328	3.93961	-0.216919	-0.854579	Unstable

Table 4. 14: Effect of semi major-axis in HD 98800 B on the collinear point L_1

a	hL_1	L_1	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0.95	0.596816	1.143691	14.7805	-5.21268	-77.0459	Unstable
0.85	0.663555	1.21043	14.8773	-5.25614	-78.1973	Unstable
0.75	0.703207	1.250082	17.0394	-6.26528	-106.757	Unstable
0.65	0.739893	1.286768	19.1574	-7.25463	-138.980	Unstable
0.55	0.774135	1.32101	21.3256	-8.26648	-176.287	Unstable

Table 4. 15: Effect of semi major-axis in GJ 630.1A on the collinear point L_1

a	hL_1	L_1	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0.9	0.359023	1.15784	6.55626	-1.52979	-10.0297	Unstable
0.7	0.591371	1.390188	4.69325	-0.783755	-3.67836	Unstable
0.5	0.888748	1.687565	3.86526	-0.4123	-1.59365	Unstable
0.3	1.00258	1.801397	3.70302	-0.336761	-1.24703	Unstable
0.1	1.103	1.901817	3.59495	-0.285959	-1.02801	Unstable

Table 4. 16: Effect of oblateness of third body in HD 98800 B on the collinear point L_1

A	hL_1	L_1	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0	0.682613	1.229488	4.89661	-0.716252	-3.5072	Unstable
0.01	0.697656	1.244531	5.1214	-0.772333	-3.95543	Unstable
0.02	0.711108	1.257983	5.07432	-0.704021	-3.57242	Unstable
0.1	0.788527	1.335402	6.32074	-0.968544	-6.12191	Unstable
0.2	0.851851	1.398726	5.83884	-0.6361	-3.71409	Unstable

Table 4. 17: Effect of oblateness of third body in GJ 630.1A on the collinear point L_1

A	hL_1	L_1	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0	0.910946	1.709763	7.45874	-2.22935	-16.6281	Unstable
0.01	0.920838	1.719655	3.89357	-0.42733	-1.66384	Unstable
0.02	0.930194	1.729011	3.88788	-0.406998	-1.58236	Unstable
0.1	0.991813	1.79063	4.37385	-0.510973	-2.23492	Unstable
0.2	1.04964	1.848457	4.31166	-0.396236	-1.70843	Unstable

Table 4. 18: Effect of potential from the disc in HD 98800 B on the collinear point L_1

M_b	hL_1	L_1	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0	0.701537	1.248412	4.86975	-0.6527	-3.17848	Unstable
0.01	0.697656	1.244531	4.81186	-0.624706	-0.624706	Unstable
0.02	0.693877	1.240752	4.8159	-0.626298	-3.01619	Unstable
0.1	0.666829	1.213704	4.4871	-0.467154	-2.09617	Unstable
0.2	0.639236	1.186111	4.48553	-0.464235	-2.08234	Unstable
0.4	0.597531	1.144406	4.29406	-0.370156	-1.58947	Unstable
0.5	0.581202	1.128077	4.65326	-0.538065	-2.50376	Unstable

Table 4. 19: Effect of potential from the disc in GJ 630 1.A on the collinear point L_1

M_b	hL_1	L_1	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0	0.928054	1.726871	3.36876	-0.171068	-0.576287	Unstable
0.01	0.920838	1.719655	3.75841	-0.361395	-1.35827	Unstable
0.02	0.913866	1.712683	3.76476	-0.36432	-1.37158	Unstable
0.1	0.865473	1.66429	3.49292	-0.230615	-0.80552	Unstable
0.2	0.818599	1.617416	3.52191	-0.243727	-0.858385	Unstable
0.4	0.751935	1.550752	3.40939	-0.187683	-0.639883	Unstable
0.5	0.727063	1.52588	3.71349	-0.333904	-1.23995	Unstable

Table 4. 20: Effect of eccentricity in HD 98800 B on the collinear point L_2

e	hL_2	L_2	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0.1	0.437348	0.109527	49.3914	-22.511	-1111.85	Unstable
0.3	0.437604	0.109271	46.4206	-21.0072	-975.167	Unstable
0.5	0.43811	0.108765	42.7908	-19.0925	-816.982	Unstable
0.7	0.438856	0.108019	41.9364	-18.3209	-768.312	Unstable

Table 4. 21: Effect of eccentricity in GJ 630.1A on the collinear point L_2

e	hL_2	L_2	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0.00001	0.347727	0.45109	68.4374	-30.1609	-2064.13	Unstable
0.01	0.347728	0.451089	68.4312	-30.158	-2063.75	Unstable
0.1	0.347824	0.450993	67.5307	-29.7486	-2008.94	Unstable
0.3	0.348596	0.450221	62.811	-27.5265	-1728.97	Unstable
0.5	0.350141	0.448676	57.5218	-24.924	-1433.67	Unstable

Table 4. 22: Effect of semi major-axis in HD 98800 B on the collinear point L_2

a	hL_2	L_2	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0.95	0.439466	0.107409	29.8696	-12.8236	-383.035	Unstable
0.85	0.438494	0.108381	38.0985	-16.8159	-640.661	Unstable
0.75	0.438054	0.108821	43.587	-19.4788	-849.022	Unstable
0.65	0.437713	0.109162	49.05	-22.1305	-1085.50	Unstable
0.55	0.437441	0.109434	54.5661	-24.8054	-1353.53	Unstable

Table 4. 23: Effect of semi major-axis in GJ 630.1A on the collinear point L_2

a	hL_2	L_2	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0.9	0.389847	0.40897	8.82479	-2.67958	-23.6468	Unstable
0.7	0.357687	0.44113	26.7166	-10.9427	-292.352	Unstable
0.5	0.348232	0.450585	63.4923	-27.8836	-1770.40	Unstable
0.3	0.346655	0.452162	81.9444	-36.3806	-2981.19	Unstable
0.1	0.345652	0.453165	100.409	-44.8826	-4506.60	Unstable

Table 4. 24: Effect of oblateness of third body in HD 98800 B on the collinear point L_2

A	hL_2	L_2	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0	0.434964	0.111911	19.4163	-8.07783	-156.842	Unstable
0.01	0.43811	0.108765	43.0075	-19.187	-825.182	Unstable
0.02	0.440795	0.10608	46.9469	-20.4981	-962.322	Unstable
0.1	0.453845	0.09303	77.6969	-30.6021	-2377.69	Unstable
0.2	0.461783	0.085092	109.366	-40.697	-4450.85	Unstable

Table 4. 25: Effect of oblateness of third body in GJ 630 1.A on the collinear point L_2

A	hL_2	L_2	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0	0.339001	0.459816	283.93	-140.465	-39882.3	Unstable
0.01	0.347727	0.45109	71.4859	-31.4132	-2245.6	Unstable
0.02	0.354403	0.444414	78.7045	-32.7365	-2576.51	Unstable
0.1	0.380239	0.418578	152.3	-50.8544	-7745.12	Unstable
0.2	0.392017	0.4068	204.084	-62.9523	-12847.6	Unstable

Table 4. 26: Effect of potential from the disc in HD98800B on the collinear points L_2

M_b	hL_2	L_2	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0	0.481242	0.065633	746.027	-266.056	-198485.0	Unstable
0.01	0.43811	0.108765	41.191	-18.3966	-757.776	Unstable
0.02	0.419538	0.127337	46.6098	-21.0028	-978.936	Unstable
0.1	0.357876	0.188999	52.7265	-24.0249	-1266.75	Unstable
0.2	0.323608	0.223267	58.6544	-26.5447	-1556.96	Unstable
0.4	0.286165	0.26071	59.0404	-26.4482	-1561.51	Unstable
0.5	0.273758	0.273117	67.0366	-29.5647	-1981.92	Unstable

Table 4. 27: Effect of potential from the disc in GJ 630 1.A on the collinear point L_2

M_b	hL_2	L_2	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0	0.35026	0.448557	76.5063	-34.589	-2646.28	Unstable
0.01	0.347727	0.45109	67.6439	-29.8347	-2018.14	Unstable
0.02	0.345283	0.453534	68.4441	-30.1529	-2063.79	Unstable
0.1	0.32831	0.470507	62.595	-27.6253	-1729.20	Unstable
0.2	0.311677	0.48714	61.257	-26.8886	-1647.12	Unstable
0.4	0.287253	0.511564	56.1216	-24.435	-1371.33	Unstable
0.5	0.277802	0.521015	60.9814	-26.2288	-1599.47	Unstable

Table 4. 28: Effect of eccentricity in HD 98800 B on the collinear point L_3

e	hL_3	L_3	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0.1	0.154406	-1.298719	3.89717	-0.413644	-1.61204	Unstable
0.3	0.190171	-1.262954	3.75429	-0.279106	-1.04785	Unstable
0.5	0.24835	-1.204775	4.65878	-0.555408	-2.58753	Unstable
0.7	0.313657	-1.139468	4.78364	-0.25094	-1.20041	Unstable

Table 4. 29: Effect of eccentricity in GJ 630.1A on the collinear point L_3

e	hL_3	L_3	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0.00001	-0.41637	-1.617553	3.22307	-0.103662	-0.334109	Unstable
0.01	-0.416297	-1.61748	3.22324	-0.103673	-0.334164	Unstable
0.1	-0.409089	-1.610272	3.24097	-0.104928	-0.340069	Unstable
0.3	-0.35552	-1.556703	3.39385	-0.115433	-0.391764	Unstable
0.5	-0.267902	-1.469085	3.76562	-0.139231	-0.524292	Unstable

Table 4. 30: Effect of semi major-axis in HD 98800 B on the collinear point L_3

a	hL_3	L_3	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0.95	0.355317	-1.097808	4.93925	-0.676421	-3.34101	Unstable
0.85	0.284377	-1.168748	4.7427	-0.591999	-2.80767	Unstable
0.75	0.242498	-1.210627	4.646	-0.549804	-2.55439	Unstable
0.65	0.203909	-1.249216	4.567	-0.515012	-2.35206	Unstable
0.55	0.168013	-1.285112	4.96714	-0.710794	-3.53062	Unstable

Table 4. 31: Effect of semi major-axis in GJ 630.1A on the collinear point L_3

a	hL_3	L_3	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0.9	0.311484	-0.889699	6.55626	-1.52979	-10.0297	Unstable
0.7	-0.020639	-1.221822	4.69325	-0.783755	-3.67836	Unstable
0.5	-0.379789	-1.580972	3.86526	-0.4123	-1.59365	Unstable
0.3	-0.508233	-1.709416	3.70302	-0.336761	-1.24703	Unstable
0.1	-0.619094	-1.820277	3.59495	-0.285959	-1.02801	Unstable

Table 4. 32: Effect of oblateness of third body in HD 98800 B on the collinear point L_3

A	hL_3	L_3	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0	0.255407	-1.197718	63597.9	-31797.2	-0.02*10^9	Unstable
0.01	0.24835	-1.204775	4.75882	-0.603416	-2.87155	Unstable
0.02	0.241646	-1.211479	4.82629	-0.597228	-2.8824	Unstable
0.1	0.197043	-1.256082	6.01014	-0.871764	-5.23942	Unstable
0.2	0.154822	-1.298303	6.32825	-0.82775	-5.23821	Unstable
0.4	0.092254	-1.360871	7.2823	-0.971629	-7.07569	Unstable
0.5	0.0672203	-1.3859047	6.38336	-0.636858	-4.06529	Unstable

Table 4. 33: Effect of oblateness of third body in GJ630 1.A on the collinear point L_3

A	hL_3	L_3	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0	-0.412822	-1.614005	4.68532	-0.842639	-3.94804	Unstable
0.01	-0.416349	-1.617532	3.2395	-0.111761	-0.362051	Unstable
0.02	-0.419825	-1.621008	3.25449	-0.11148	-0.362812	Unstable
0.1	-0.445998	-1.647181	3.48615	-0.16529	-0.576224	Unstable
0.2	-0.475342	-1.676525	3.62958	-0.172721	-0.626904	Unstable

Table 4. 34: Effect of potential from the disc in HD 98800 B on the collinear point L_3

M_b	hL_3	L_3	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0	0.244425	-1.2087	4.65737	-0.55498	-2.58474	Unstable
0.01	0.24835	-1.204775	4.60479	-0.529482	-2.43815	Unstable
0.02	0.25217	-1.200955	4.60742	-0.530507	-2.44427	Unstable
0.1	0.279475	-1.17365	4.3099	-0.386167	-1.66434	Unstable
0.2	0.307252	-1.145873	4.30433	-0.382096	-1.64467	Unstable
0.4	0.349038	-1.104087	4.13297	-0.298301	-1.23287	Unstable
0.5	0.365316	-1.087809	4.43484	-0.440969	-1.95562	Unstable

Table 4. 35: Effect of potential from the disc in GJ 630 1.A on the collinear points L_3

M_b	hL_3	L_3	$\Omega^0_{\xi\xi}$	$\Omega^0_{\eta\eta}$	$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}$	Remark
0	-0.423797	-1.62498	10.1937	-3.51394	-35.8199	Unstable
0.01	-0.416349	-1.617532	3.18904	-0.086828	-0.276898	Unstable
0.02	-0.409163	-1.610346	3.19001	-0.0872923	-0.278463	Unstable
0.1	-0.359576	-1.560759	3.21952	-0.101768	-0.327645	Unstable
0.2	-0.312162	-1.513345	3.02164	-0.0041197	-0.012448	Unstable
0.4	-0.246203	-1.447386	3.20118	-0.0927914	-0.297042	Unstable
0.5	-0.222179	-1.423362	3.10445	-0.04528	-0.140569	Unstable

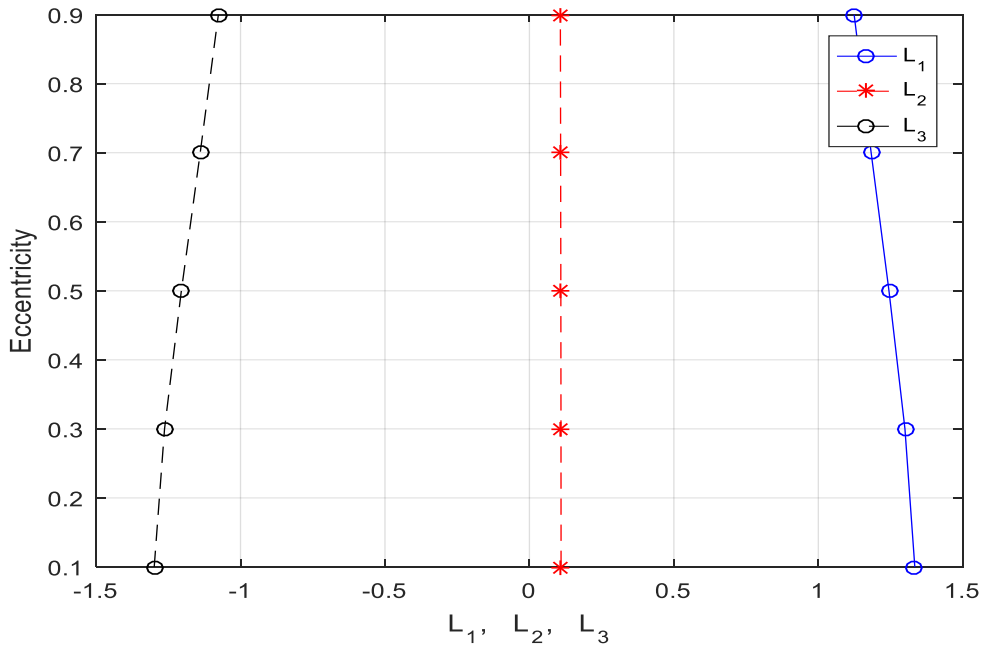


Figure 4. 15: Effect of eccentricity on the collinear points L_i *(i=1,2,3)* of HD98800 B

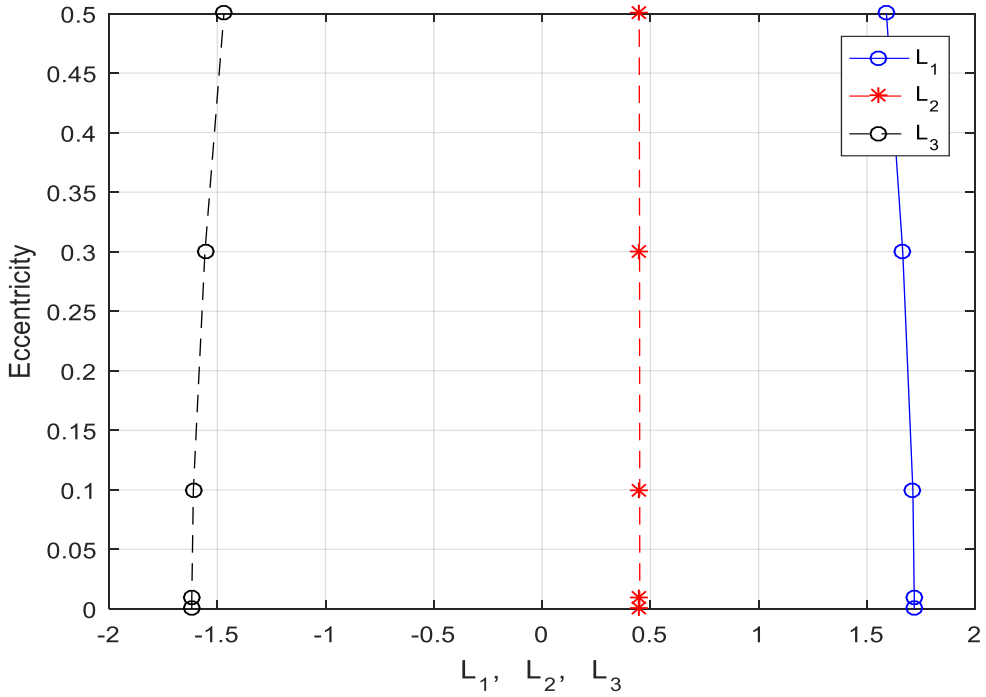


Figure 4. 16: Effect of eccentricity on the collinear points L_i *(i=1,2,3)* of GJ 630.1A

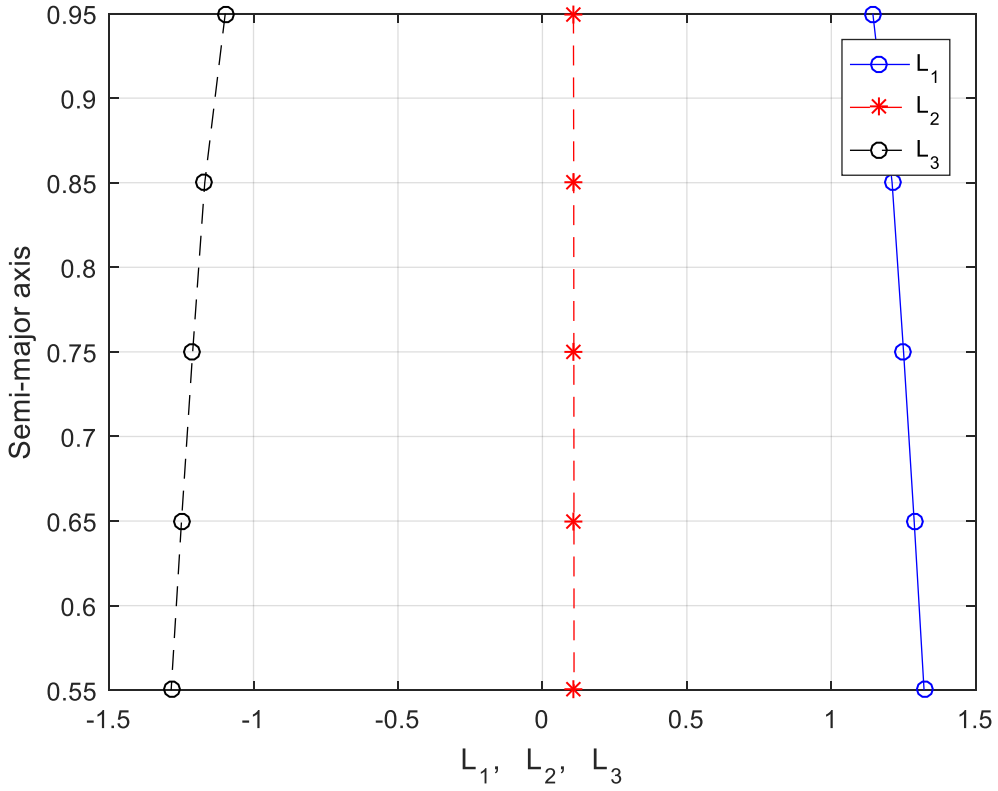


Figure 4. 17:Effect of semi-major axis on the collinear points L_i *(i=1,2,3)* of HD98800 B

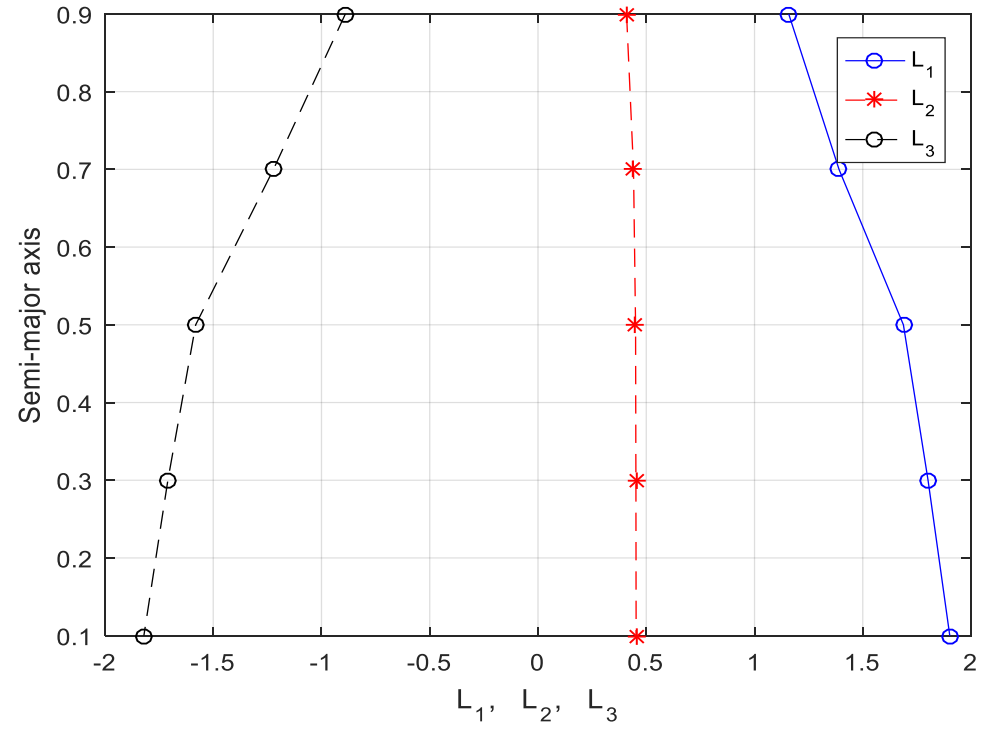


Figure 4. 18: Effect of semi-major axis on the collinear points L_i *(i=1,2,3)* of GJ 630.1A

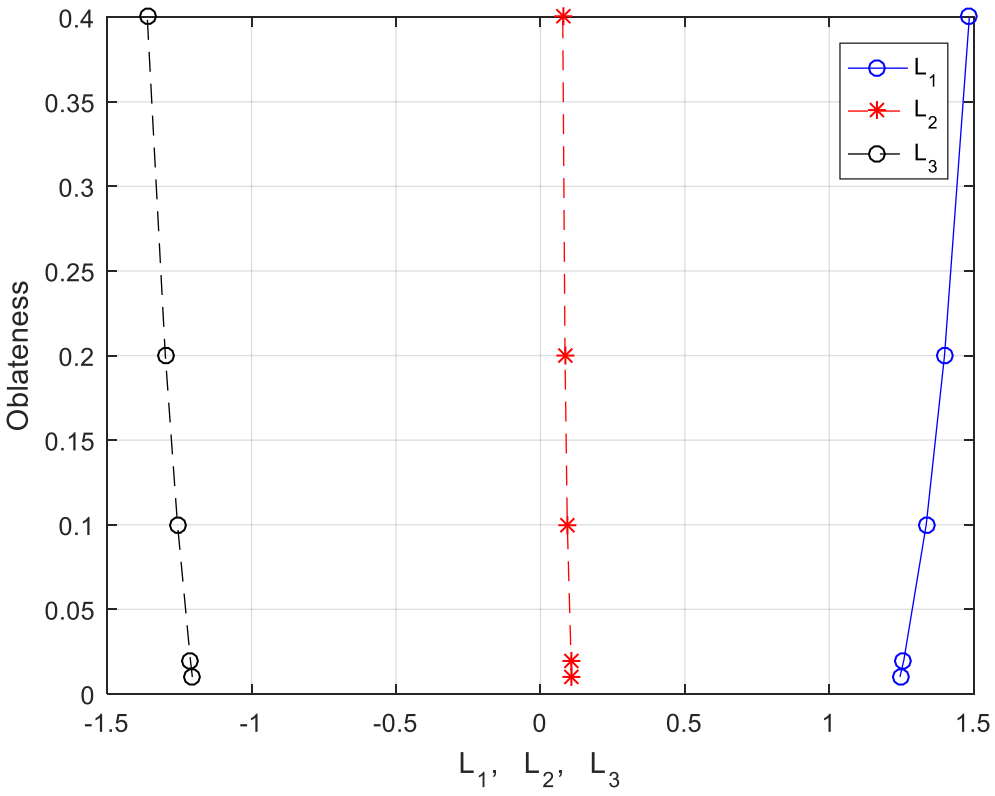


Figure 4. 19: Effect of third body on the collinear points L_i *(i=1,2,3)* of HD 98800 B

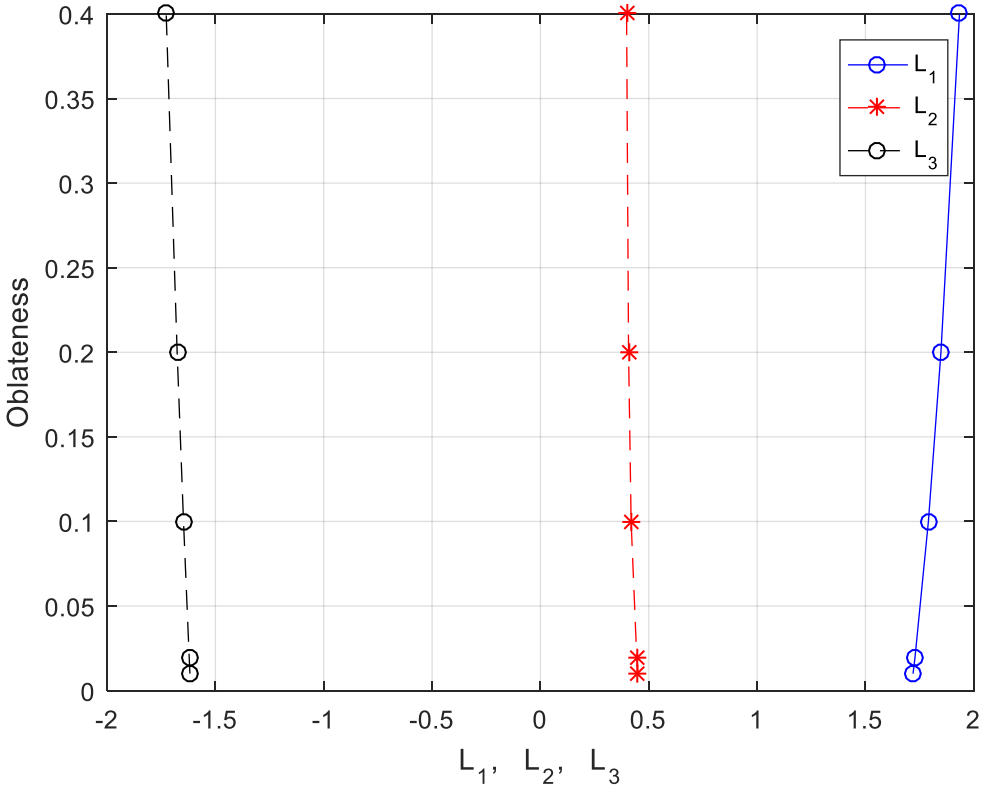


Figure 4. 20: Effect of third body on the collinear points L_i *(i=1,2,3)* of GJ 630.1A

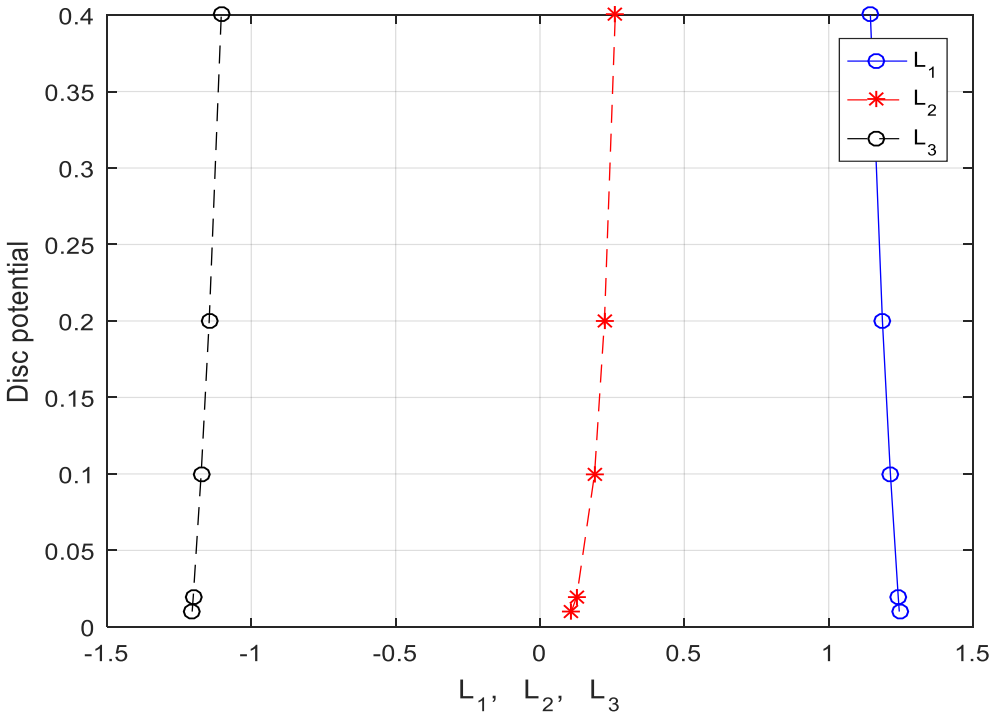


Figure 4. 21: Effect of potential from the disc on the collinear points L_i *(i=1,2,3)* of HD98800 B

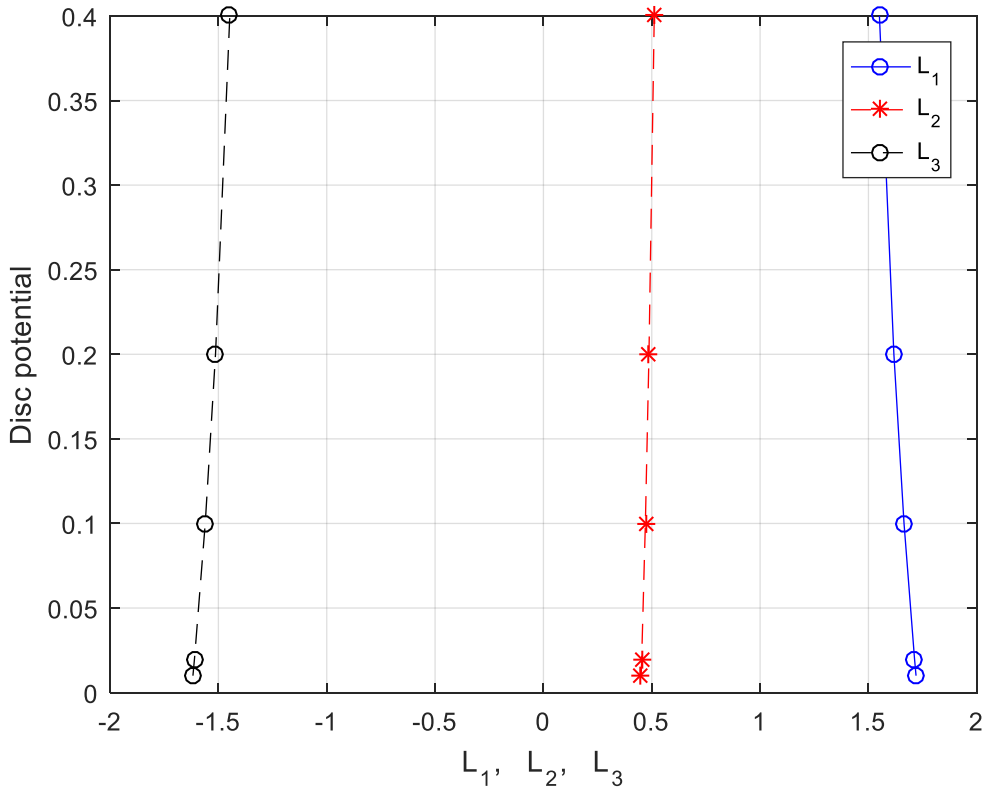


Figure 4. 22: Effect of potential from the disc on the collinear points L_i *(i=1,2,3)* of GJ 630.1A

4.5 Discussion

The critical mass value μ_c represents the total effect of eccentricity, semi-major axis, radiation of the primaries and potential from the disc, on the critical mass value. However, in the absence of these perturbations, the critical mass value μ_c becomes μ_0 in equation (3.43), which corresponds to the classical restricted three body problem (Szebehely, 1967). The effects of these perturbations are shown in Table (4.2); it indicates that the perturbations reduce the range of stability while the potential from the disc increase the range of stability.

When $A = M_b = 0,\ \ q_1 = q_2 = 1$equation (3.43) fully coincide with the classical case of Szebehely (1967); when both primaries are sources of radiation in the elliptic framework without potential from the disc, this agrees with Singh and Umar (2012) and Singh and Umar (2013c), and it coincides with the potential from the disc in the work of Singh and Taura (2014) under the influence of radiation and oblateness. In each case there is one acceptable root which corresponds to one of the collinear libration points $L_i (i = 1,2,3)$ as also confirmed by Yamada and Asada (2011).

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

This dissertation has examined for the first time the motion of an oblate infinitesimal body in the generalized photogravitational ER3BP with circumbinary disc and has revealed that the locations of equilibrium points exists, though they are affected by the eccentricity, semi-major axis, luminosity of the primaries, oblateness of the third body and the potential from the disc. In the case of collinear points we see that the equation determining the distance ratio among the three masses is an eleventh-order polynomial having at most six roots. Even for such cases, there exists one physically reasonable root and only one. The collinear points however remain unstable even under the effects of the aforementioned parameters.

The analytical results obtained have been applied to the binaries (HD98800B, and GJ630.1A). Numerical results are obtained with the help of Mathematica, Matlab software's, and excel spreadsheet.

5.2 Conclusion

This dissertation has focused on the use of analytical and numerical techniques to obtain the locations and linear stability of the equilibrium points of the stated equations of motion around HD 98800 B and GJ 630.1A. The locations and nature of stability of the triangular equilibrium points reveals the effects of the various parameters on the dynamics of the system. The locations of these triangular equilibrium points are significantly dependent on the parameters. The stability of the triangular equilibrium points coincides with Szebehely (1967) when eccentricity, semi-major axis, radiation,

oblateness and potential from the disc are relaxed, and is conditionally stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$.

When only the classical ER3BP is considered the triangular points are stable for $0 < \mu < \mu_c$ while the collinear points are unstable, which holds true in the case of motion of an oblate infinitesimal body in the generalized photogravitational ER3BP with circumbinary disc.

For increase in eccentricity and semi-major axis there is a shift of $L_{4,5}$ towards the line joining the primaries and away from the bigger primary for HD98800B (see tables 4.3,4.5 and figures 4.1,4.3), while the increase in eccentricity and semi-major axis makes $L_{4,5}$ to shift towards the line joining the primaries only for GJ630.1A (see tables 4.4,4.6 and figures 4.2,4.4).

For increase in oblateness of the third body this cause $L_{4,5}$ to shift away from the line joining the primaries and away from the bigger primary for the model Hd98800B (see table 4.7 and figure 4.5), while the increase in oblateness of the third body on GJ630.1A cause $L_{4,5}$ to move away from the line joining the primaries only (see table 4.8 and figure 4.6).

The increase in potential from the disc cause $L_{4,5}$ to move towards the line joining the primaries and there is a slight shift of $L_{4,5}$ towards the smaller primary on HD9800B (see table 4.9 and figure 4.7), while in the case of GJ630.1A an increase in potential from the disc causes a shift on $L_{4,5}$ away from the line joining the primaries only (see table 4.10 and figure 4.8).

In the case of stability the effects of eccentricity, semi-major axis, radiation, oblateness of the third body reduces the size of the region of stability, while the potential from the disc increases the size of the region of stability (see figures 4.9,4.10).

For collinear points, with increase in eccentricity and semi-major axis for both the models under consideration we observe that L_1 moves towards the smaller primary, L_2 moves away from the smaller primary and L_3 moves towards the bigger primary (see figures 4.11-4.14). With increase in oblateness of the third body both the models HD98800B and GJ630.1A we observe that L_1 move away from the smaller primary, L_2 towards the bigger primary and L_3 moves away from the bigger primary (see figures 4.15 and 4.16). With increase in potential from the disc this causes L_1 to move towards the smaller primary, L_2 to move away from the bigger primary and L_3 to move towards the bigger primary for both the models HD98800B and GJ630.1A (see figures 4.17 and 4.18).

The locations of the collinear equilibrium points are obtained within their respective intervals.

5.3 Recommendations

This research has considered radiation pressure within the range $0 < q < 1$, it is recommended that it be extended to $-\infty < q < +\infty$.

5.4 Future Research

This dissertation can be extended for future research in the following ways:

1. The "Out - of - plane" points of motion of an oblate infinitesimal body in the generalized photogravitational ER3BP with circumbinary disc
2. Investigation of these problems with variable mass
3. Periodic orbits around triangular equilibrium points of the motion of an oblate infinitesimal body in the generalized photogravitational ER3BP with circumbinary disc

5.5 Contribution to Knowledge

This dissertation obtained the existence, location and stability of equilibrium points of an oblate body in a generalized photogravitational ER3BP with circumbinary disc when the finite bodies are luminous.

- This research will contribute to the study of dynamical systems in space, especially in the elliptic restricted three body problem when the bodies are radiating with a circumbinary disc.

- These equilibrium points are important in space science as they mark places where particles / satellites can be trapped (L_4 and L_5).
- Introduce new perspectives on our individual and collective place in the universe
- The research highlights the existence of binary stars beyond our solar system
- The research establish the existence of the locations and stability of equilibrium points in the ER3BP.

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APPENDICES

Appendix 1: Triangular Points

```
x1=[-0.4500 0.0500 0.5500 0.0500 -0.4500];
y1=[0 0.422372 0 -0.422372 0];
plot(x1,y1,'g'),grid;
holdon
x2=[-0.4500 0.0500 0.5500 0.0500 -0.4500];
y2=[0 0.678884 0 -0.678884 0];
plot(x2,y2,'b');
xlabel('x'),ylabel('y');
holdon
x3=[-0.4500 0.0500 0.5500 0.0500 -0.4500];
y3=[0 0.750737 0 -0.750737 0];
plot(x3,y3,'r');
xlabel('x'),ylabel('y');
holdon
g='e=0.70';
r='e=0.40';
b='e=0.20';
xlabel('\xi'),ylabel('\eta');
legend(g,r,b);
holdoff
```

Appendix 2: Stability Region

```
% critical mass
    [e,a]=meshgrid(0.0:0.001:0.03,0.0:0.001:0.08);
     w=(e-e+1).*(a-a+1);
     q1=0.999;
     q2=0.999;
%a=0.35;
     p=0.45;
%e=0.0;
     T=0.001;
mb=0.001;
rc=sqrt(1.-p+p.^2);
x1=1.-a;
     b1=1.-q1;
     b2=1.-q2;
x2=sqrt(69);
x3=x2./9;
    xx1=0.5.*(1.-x3);
x4=27.*x2;
x5=4.*x1;
    xx2=x5./x4;
x6=22.*A;
x7=9.*x2;
    xx3=x6./x7;
x8=14.*e.^2;
    xx4=x8./x7;
x9=2.*(b1+b2);
    xx5=x9./x4;
x10=76.-8.*rc;
x11=((rc).^2)+T.^2;
x12=(1.+6.*(rc).^2);
    x13=x11.^5./2;
    x14=x11.^3./2;
```

```
x15=x11.^4;
x16=(x10.*x13)-(9.*x12.*x14);
x17=x4.*x15;
x18=x16./x17;
    xx6=x18.*mb;
    u=xx1-xx2-xx3-xx4-xx5+xx6;
     [C,h]=contour(e,u,a);
set(h,'showText','on','textStep',get(h,'levelStep'));
colormap
xlabel('e')
ylabel('\mu_{c}')
```

Appendix 3: Collinear Points

```
x1=[0.9 0.8 0.7 0.6 0.5];
y1=[4.4208 4.5019 4.5992 4.7191 4.8721];
plot(y1,x1,'o-b'),grid;
holdon
y2=[-0.6374 -0.6779 -0.7264 -0.7862 -0.8625];
plot(y2,x1,'*--r');
holdon
y3=[-2.8179 -3.0517 -3.3410 -3.7103 -4.2022];
plot(y3,x1,'o--k');
g='\Omega_\xi_\xi^0';
r='\Omega_\eta_\eta^0';
k='\Omega_\xi_\xi^0\Omega_\eta_\eta^0';
xlabel('\Omega_\xi_\xi^0,   \Omega_\eta_\eta^0,
\Omega_\xi_\xi^0\Omega_\eta_\eta^0'),ylabel('semi-major axis (a)');
legend(g,r,k);
holdoff

x1=[0.5 0.4 0.3 0.2 0.1];
y1=[4.9089 4.5700 4.3360 4.1822 4.0944];
plot(y1,x1,'o-b'),grid;
holdon
y2=[-0.7217 -0.6475 -0.5950 -0.5597 -0.5392];
plot(y2,x1,'*--r');
holdon
y3=[-3.5428 -2.9289 -2.5800 -2.3407 -2.2075];
plot(y3,x1,'o--k');
g='\Omega_\xi_\xi^0';
r='\Omega_\eta_\eta^0';
k='\Omega_\xi_\xi^0\Omega_\eta_\eta^0';
xlabel('\Omega_\xi_\xi^0,   \Omega_\eta_\eta^0,
\Omega_\xi_\xi^0\Omega_\eta_\eta^0'),ylabel('eccentricity (e)');
legend(g,r,k);
```

```
holdoff

x1=[0.1 0.01 0.001 0.0001 0.00001];
y1=[4.9303 4.4254 4.3446 4.3360 4.3351];
plot(y1,x1,'o-b'),grid;
holdon
y2=[-0.5498 -0.5891 -0.5945 -0.5950 -0.5951];
plot(y2,x1,'*--r');
holdon
y3=[-2.7109 -2.6068 -2.5827 -2.5800 -2.5798];
plot(y3,x1,'o--k');
g='\Omega_\xi_\xi^0';
r='\Omega_\eta_\eta^0';
k='\Omega_\xi_\xi^0\Omega_\eta_\eta^0';
xlabel('\Omega_\xi_\xi^0,   \Omega_\eta_\eta^0,
\Omega_\xi_\xi^0\Omega_\eta_\eta^0'),ylabel('oblateness (A)');
legend(g,r,k);
holdoff
x1=[0.1 0.01 0.001 0.0001 0.00001];
y1=[4.3727 4.3360 4.3324 4.3321 4.3320];
plot(y1,x1,'o-b'),grid;
holdon
y2=[-0.6133 -0.5950 -0.5932 -0.5931 -0.5930];
plot(y2,x1,'*--r');
holdon
y3=[-2.6819 -2.5800 -2.5702 -2.5692 -2.5691];
plot(y3,x1,'o--k');
g='\Omega_\xi_\xi^0';
r='\Omega_\eta_\eta^0';
k='\Omega_\xi_\xi^0\Omega_\eta_\eta^0';
xlabel('\Omega_\xi_\xi^0,   \Omega_\eta_\eta^0,
\Omega_\xi_\xi^0\Omega_\eta_\eta^0'),ylabel('potential from disc (M_b)');
legend(g,r,k);
holdoff
```