A STUDY OF MULTISET ALGEBRAS

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AHMADU BELLO UNIVERSITY, ZARIA

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A STUDY OF MULTISET ALGEBRAS

Title page

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DEPARTMENT OF MATHEMATICS

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DECLARATION

I declare that the work in the dissertation entitled 'A study of Multiset Algebras' has been performed by me in the Department of Mathematics under the supervision of Professor D. Singh, Professor G. U. Garba and Dr. B. K. Jha.

The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at any university.

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CERTIFICATION

This dissertation entitled "A STUDY OF MULTISET ALGEBRAS" by IBRAHIM, Adeku Musa meets the regulation governing the award of the degree of Doctor of Philosophy of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This dissertation is dedicated to my supervisor, Professor Dasharath Singh, for his fatherly support, guidance and encouragement.

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I remain grateful and indebted to Allah for giving me the strength to carry out this research work to its logical conclusion (*Alhamdulillah*).

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ABSTRACT

This dissertation presents a concise history of the uses of multisets in various disguised forms which eventually led to a formalization of multiset theory. We present a comprehensive study of fundamentals of multisets and their applications which appeared in two of our publications (Singh *et al.*, 2007 and 2008). We study max–plus algebra and by exploiting the notion of compatibility relation (reflexive and symmetric), we develop a truncated symmetrized max–plus algebra called *minimal* $\mathbb{S}_{max}$ or $\approx \mathbb{S}_{max}$, and discuss its application which appeared in our publication (Singh *et al.*, 2010). We extend our study on application of compatibility relation to multisets and outline a theory of "*Tolerance multiset and its topological perspective*". We explicate the notion of multiset space, introduce two new operations on the multiset space and show that it shares properties of many algebraic structures. We also develop the concept of relations and functions on multiset spaces and show that the multiset space can be metrized thereby defining topology on a multiset. Finally, we delineate the problem related to difference and complementation in multiset theory. We show that none of the existing approaches succeeds in resolving the attendant difficulties without assuming some contrived stipulations.

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CHAPTER ONE

GENERAL INTRODUCTION

1.0 Introduction and Motivation

Diverse applications of multisets provided motivation for the study of various algebras of multiset. As semirings and monoids play a significant role in the development of formal language theory, the study of multiset algebras naturally became an area of research to aid the study of formal language. The essential peculiarity of multiset is the multiplicity of its elements which distinguishes it from a set and this portrays it as a new interesting mathematical concept. Wildberger (2003) considers a new way of looking at multisets which involve linear rather than multiplicative notation and operation which mirror those on natural numbers. The motivation is to create a framework for mathematics which more closely connects with scientific usage in which repetition and relative concepts of equality play an important role.

Kosters and Laros (2003) propose a new class of distance measures (metric), designed for multisets, which are a recurrent theme in many *data mining* applications. One important instance of this class originated from the necessity for a clustering of criminal behaviours. Here, the multisets are the crimes committed in a particular given time. These distance measures are parameterized by a function f which, given a few simple restrictions, will always produce a well defined metric. Also, Petrovsky (2003) considers techniques for group objects that are described with many quantitative and qualitative attributes and may exist in several copies. Such multi–attribute objects may be represented as multisets. Poplin (2000) develops some basic theory of multiset

algebras and show that multisets provide a connection between the eigenvalue/eigenvector equations for the max–plus and non–negative real numbers.

Brink (1987) defines multiset as a mapping from some universal set into some set of numbers. In particular, considering only integral multiples of elements, a multiset is a mapping $\alpha: U \to N$, where U is a universal set and N is a set of natural numbers including zero, unlike the mapping $\alpha: U \to \{0, 1\}$ which is a set or $\alpha: U \to [0, 1] = \{x \in R | \ 0 \leq x \leq 1\}$ and R is the set of real numbers is a fuzzy set (Zadeh, 1965). With this formal definition, (Brink, 1987) explicates various algebraic structures on U^N, where U^N is the set of all mappings, $\alpha: U \to N$.

1.1 Statement of Research Problem

Traditionally, classical set theory, for long, has been guided by the notions of uniqueness and equality. For example, traditional mathematics implicitly assumes that all mathematical objects occur without repetition. Thus there is only one number six, one field of real numbers, etc. Accordingly, the only possible relations between two objects are either they are equal or they are different.

Many contemporary Mathematicians expressed their concern over the Cantor's insistence on not accepting repeating elements in a set. In the history of mathematics and science in general, three possible relationships between any two objects arise naturally. Two physical objects;

(i) are completely different

(ii) are the same but separate

(iii) coinciding and identical.

For example, repeated roots of polynomials, repeated observations in statistical samples, repeated hydrogen atoms in a water molecule etc.

Knuth (1981) notes that "despite frequent occurrences of multisetlike structures in mathematics, there is currently no structured way to deal with multisets" and similar observations have been voiced by many others, like (Lake, 1976), (Rado, 1975) etc. The recent works of (Blizard, 1989; Blizard, 1991; Singh, 1994; Poplin, 2000; Singh, 2003) etc., on multiset theory and its application in mathematics and computer science have negated the above assertion.

1.2 Objectives

a. To study the comprehensive review of literature pertaining to development of fundamental of multisets and their applications especially in developing various algebraic structures.

b. To critically study various algebras already developed using multiset theory especially, (Bender, 1974; Rado, 1975; Meyer *et al.*, 1982; Bender *et al.*, 1984; Clement, 1988; Poplin, 2000 and Wildberger, 2003).

c. To develop some algebras of multiset either as extension of existing ones or as a new framework.

1.3 Justification

Many algebras such as Group, Monoid, Semigroup, Quasigroup, Loop, Ring, Semiring, Lattice, Semilattice, Bounded algebra, Heyting, Boolean algebra etc. were developed

using set as underlying structure. Since multiset is a generalization of set, various algebras based on multiset could be developed.

1.4 Methodology:

We outline preliminaries that help in making the dissertation maximally self–contained. We systematize the fundamentals of multisets and outline their applications. We present a survey of multiset algebras, max–plus, min–plus and Brink–algebra in particular. We develop some new multiset algebras with partitions of multisets as our basic tools and related results.

We adopt the method of numbering the *Subtitles*, *Theorems*, *Remarks* and *Definitions* chapterwise. For Subtitle, we write *a.b* followed by the subtitle where *a* is the chapter that contains the subtitle and *b* represents the sequential numbering of the subtitle. For the *Theorems*, *Remarks* and *Definitions*, we write Theorem *a.b*, *Remark a.b* and *Definition a.b* where *a* and *b* are as defined above.

1.5 Organization of the Dissertation

The Dissertation consists of six chapters. In chapter one, a succinct explication of the significance of multiset algebras, the aims, objectives, the methodology to be adopted in the dissertation and organization of the dissertation are presented.

In chapter two, an up–to–date literature survey is presented, while in chapter three, mathematical preliminaries required for the dissertation and fundamentals of multisets are presented. Our specific innovation is to maintain the distinction between an object

and its copies, called elements throughout. Also, the theory of compatibility relation (reflexive and symmetric) is augmented for application in chapter four.

In chapter four, basics of max–plus, max–multiplication and symmetrized max–plus algebras based on the research undertaken during the last three decades or so are presented. In the end we constructed a truncated symmetrized max algebra and indicate its application areas.

In chapter five, the major part of our work, is presented. Firstly, multiset algebras developed during the last two decades or so, specially based on Brink (1987) and Poplin (2000), is presented. Secondly, we describe the theory of tolerance multisets, largely based on Marcus (2001) as an application of reflexive symmetric relation. Finally, we develop our own systems of multiset algebras. In order to obtain a Boolean multiset algebra, we specially present an exhaustive research on complementation in multiset–like structures.

In chapter six, we present our concluding remarks and some future directions. In the end, a list of references cited in the dissertation is presented.

CHAPTER TWO

LITERATURE SURVEY

2.0 Introduction

The birth of set theory dates to 7th December, 1873 when, a German Mathematician, Georg Ferdinand Ludwig Philip Cantor (1845–1918) proved the uncountability of the real line and he wrote Dedekind informing him of his discovery. Ever since then set theory has occupied a central area in mathematics. Set theory is one of the greatest achievements of modern mathematics, since almost all mathematical concepts, methods and results could be represented within axiomatic set theory. To emphasize, in 1910, Hilbert wrote that;

> *Set theory is that mathematical discipline which today occupies an outstanding role in our science, and radiates its powerful influence into all branches of mathematics".*

In post–Sputnik era, besides numerous publications appearing in the core area of set theory, one can hardly come across a standard book in other areas of mathematics which does not begin with some discussion of set theory. In fact, set theory has eventually become the language of science.

The concept of a set appears deceivingly simple to a trained mathematician, to such an extent that it is difficult to appreciate adequately the efforts and contributions of the originators. The notion of a collection is as old as counting, thus one may arguably find

it difficult to sort out the exact origins of concept of set. However, Ernst Zermelo said that the theory of set had historically been created by Cantor and Dedekind.

Motivation for Cantor's discovery of set theory came from his work on Fourier series, which led him to introduce ordinal numbers and transcendental numbers. Real numbers that are solutions to polynomial equations with integer coefficients are called algebraic, and the search was on then for numbers that are not algebraic. Such numbers called transcendental numbers were discovered and Cantor shows that such numbers are infinite. Cantor's discovery of uncountable set led him to the subsequent development of ordinal and cardinal numbers.

In correspondence with Dedekind, Cantor asked the question whether the infinite sets N of the natural numbers and real numbers R can be placed in one–to–one correspondence. Dedekind offered a proof that the set A of all algebraic numbers is denumerable. That is, there is one–to–one correspondence with N. Cantor later showed that the set R is denumerable leads to contradiction. He later applied the Bolzano–Weierstrass principle of completeness and show that there are more elements in R than in N or in Q the set of rational numbers.

The Cantor's concept of a set is stated as follows:

> *"By a 'set', we are to understand any collection M of definite and distinct objects m of our intuition or thought (which will be called the 'elements' of M) into a whole."*

Accordingly, one of the underlying assumptions of Cantorian set theory dictates that no element shall be allowed to appear more than once. The collection $\{a, b, b, c\}$ consequently becomes a set viz: $\{a, b, c\}$ only after deleting the repeated element. Indeed, this aspect of Cantorian set theory did not go hand in hand with the requirements of various other sciences in seeking mathematical formulation of some of the challenging problems. For example, repeated roots of polynomial equations, repeated observations in statistical samples, repeated hydrogen atoms in a water molecule, H_2O, etc., need to be counted for attaining adequacy and exactness. The concept of multiset ensue repeated elements are admitted in a set.

A multiset (mset, for short) is an unordered collection of objects (called elements) in which, unlike a standard (Cantorian) set, objects are allowed to repeat. In other words, an mset is a collection in which objects may belong more than once and each individual occurrence of an object is called its element. It follows that the distinct elements of an mset are the objects of that mset. Despite the fact that the distinction made between the term *object* and *element* does enrich the multiset language, respecting the traditional sense of its use, the term element alone may be used if no confusion arises.

Knuth (1981) notes that "despite frequent occurrences of multisetlike structures in mathematics, there is currently no structured way to deal with multisets". It is gratifying to note that during the recent years a sizeable number of papers, specifically dealing with the development of the theory of multisets, have appeared and, not very surprisingly, some of the outcomes turned out to be exceedingly vexing. In part, it conforms to Cantor's resolute insistence on not admitting repeated elements in a set. One is tended to believe that time is getting ripe to have a clearer perspective of

Cantor's compassion for delimiting himself to the development of set theory in its simplest form, called the *standard* or *crisp* set theory. In our opinion, a convergent view point of multitude of researches that have gone into developing a comprehensive theory of multisets dictates that it should be Cantor's fascinating theory of infinity that could not have straightforwardly got through in the generalized cases, such as multisets. Over the years, besides sporadic evidences of applications of multisets in Philosophy, Logic, Linguistics and Physics, a good number of them witnessed in mathematics and computer science which have led to the formulation of a comprehensive theory of multisets.

In this dissertation, among other exertion, we endeavour to present some of the history of multisets, a systematization of various approaches towards formalizing the concept of multiset and operations under multisets, an overview of recent applications of multisets and finally, propose some further research problems.

2.2 Some early History of Multisets

The concept of multiple–membership collection is as old as the concept of number itself. For example, evidences of representing a number by a collection of tally marks or units are found in the work of the Babylonians ($200\ B.C.$), Egyptians (3500–1700 B.C), Greeks (Hallet, 1984) and (Ifrah, 1985). Knuth (1981) notes that enumeration of permutations of a set was known in ancient times and historically the first known document is the Hebrew Book of Creation ($C.100\ A.D.$), followed by the Indian Classic *Anuyogadvarā-sutra* ($C.500\ A.D.$); and the corresponding results for multiset seems to have appeared first in another Indian Classic *Lilávati of Bháscara Achárya* ($C.1150$).

He further notes that Kircher $(C.1650)$, correctly gave the number of permutations of multiset $\{m.C,\ n.D\}$ for several values of m and n, though without revealing his method of calculation except when $n = 1$. A generalization of the rule for enumerating the permutations of multiset appeared in *Prestet's Elémens de mathématiques* (Paris 1675, 351–352), and later in John Walli's Treatise of Algebra 2 $(Oxford\ 1685, pp.117-118)$. It is heartening to note that, by exploiting Dominique Floata's work done in 1965, Knuth (1981) presents a good number of significant results on multiset permutations.

An early reference to multiset is also found in the work of Marius Nizolius (1498–1576), cf. (Singh, 1994). "*It is not perhaps clear whether Nizolius' multitude comes closer to* '*class*' *or* '*heap*'*…. But Nizolius' multitudines, might still be heaps in the sense of Quine or Goodman*".

In Brink (1987), it is noted with '*emphasis*' (rightly, I think), which is missing in (Blizard, 1991): "*…attempting to* '*make sense*' *of Boole's algebra of Logic that multisets do have a history, and ...that honour should go to George Boole's (1854) Laws of Thought*".

Hailperin (1976) justified (by introducing signed heaps) that Boole's Laws of Thought may be interpreted as a treatise dealing with multisets. However, on the question of whether Boole himself had this interpretation in mind, Brink (1987) notes that "*Hailperin is silent on this point, but strong evidence comes from another quarter: at least one eminent 19th century Logician read Boole in precisely right way, and that was*

Charles Sanders Pierce". Brink (1978), notes that "*the axiom system for Boole's original system is an axiom system for signed heaps*".

Cantor himself, in his first (1895) definition of a set, defines the cardinality M^* of a set M as a collection of 'units' (one unit for each element of M), admitting repetitions (Cantor, 1955) and (Hallet, 1984). Dedekind, in his 1888 masterpiece "*was sind und was sollen die zahlen*?" ("The nature and meaning of numbers", English translation, cf. (Blizard, 1991) introduces the notion of msets. He considers a non–injective function ψ from the set Σ (with n elements) onto the set $\psi(\Sigma)$ (with m elements) to conclude that n is also the number of elements in $\psi(\Sigma)$ counted in this sense, while the number of its actual different elements is m. Dedekind remarks: "*In this way, we reach the notion; very useful in many cases, of systems (sets) in which every element is endowed with a certain frequency number which indicates how often it is to be reckoned as element of the system*". In fact, Dedekind's approach has been found to be the most fundamental.

Weierstrass defines real numbers as certain msets of rational numbers in which finitely many repetitions are allowed. For example, the quantity $\pi = 3.141\ldots$ can be identified with a multiset containing the number 1 with multiplicity 3, the element $\frac{1}{10}$ with multiplicity 1, the element $\frac{1}{100}$ with multiplicity 4, etc. (Hallet, 1984) and (Blizard, 1991).

Whitney (1933) indeed gave the first substantive mathematical treatment of msets, particularly in terms of algebra of characteristic functions of sets and subsequently, that of '*generalized*' sets (sets whose characteristic functions are integer valued). He cites "*chains in analysis situs*" in which "*each element is counted any number of times*".

The appendix B, entitled *Ars combinatoria*, of Weyl (1949) is another rich place where msets receive utmost vindication, in theoretical as well as applied directions. Weyl (1949) defines mset as a set with an equivalence relation defined on it. Equivalent elements are said to be "*in the same state*" ("*in the same sort*" in (Monro, 1987)). Weyl, (1949) notes: "… *electrons may be in this or that position; atoms in a molecule may be N, He, Li, ...atoms. ... no artificial differences between elements are introduced by their levels ... and merely the intrinsic differences of states are made use of …*". Weyl applies these concepts to a variety of problems in physics, chemistry and genetics: for example, in physics, "*Two individuals in the same 'complete state' (no further refinement is possible) are indiscernible by any intrinsic characters although they may not be the same thing*". He also notes: "*...one has to distinguish between equal* (*= of the same kind and identicals*)". Thus, an aggregate is identified by its distinct states which are precisely the concept of multiset.

Rado (1975) defines an mset to be any cardinal–valued function whose nontrivial domain (the collection of elements not mapped to zero) is a set. The class of msets is called the cardinal module, like "a module over the semi–group of all cardinals". An mset f represents a family of sets $\alpha = (x_i)$, $i \in I$ just in case $f(x_i)$ equals the number

of times x_i occurs in α (that is, $f(x) = |\{i \in I : x_i = x\}|$ for all x). Rado also makes use of signed msets (functions which may have negative cardinal values).

Parker–Rhodes (1981) observes that "… *there exists no branch of mathematics in which a third parity–relation, besides equality and inequality, is admitted*". Parker–Rhodes develops a theory of "*sorts*" (collections of indistinguishables), an elaborate mathematical system, and applies it to explicate some fundamental problems of physics. "*Copies of elements behave as identicals when they appear as elements of different classes, but as a plurality (each of them contributes to cardinality) when they are elements of the same class*". Elements of multisets conform to the Parker–Rhodes principle of indistinguishables. However, the system of Parker–Rhodes is a radical departure from classical mathematics because of its triparitous nature (objects may be identical, distinct or twins). The underlying idea lies in treating equality and identity as different relations.

In course of formalizing Category theory, Corcoran (1980) introduces msets, for example, $S_a\ [a, b]_{2,1} = [a, b]_{3,1}$ where $S_a =$ the successor function of a.

Meyer and McRobbie (1982) made use of multisets in the study of relevant implication by way of reckoning how often a premise is repeated in the course of derivation.

The principle of indistinguishability of (Parker–Rhodes, 1981) has been re-emphasized in (Wildberger, 2003). Wildberger argues that two physical objects are either different or they are the same (or equal) but separate or they are coinciding (and identical). He takes the example of a water molecule with two hydrogen atoms, say H^1 and H^2 and one oxygen atom say O and concludes that H^1 and O are obviously different (and, for that matter, H^2 and O are different), however H^1 and H^2 are the same but separate, while H^1 and H^1 are coinciding and identical. Wildberger (2003), by taking some examples from the real world, such as "*….to a shopkeeper, any two dollar coins are equal even if their years of minting are different, whereas a coin collector would regard them essentially different, etc*.", proposes some very innovative conclusions like "*the notion of equality is often a relative one*", "*traditional mathematics works with the implicit assumption that all objects exist uniquely in some ideal sense*" and "*relies heavily on equivalence relation*", etc., which, for the time being, appear to us only a contrived idea.

In order to gauze the pervasiveness of the use of msets, we endeavour to present an overview of results pertaining to application of msets, especially in mathematics (combinatorial and computational) and computer science, that have appeared in the recent years.

2.3 Theory of Multisets

As to the development of a formal (axiomatic) theory of multisets, despite having voiced by a number of authors (Fraenkel et al., 1973) and (Singh, 1985) for example, their disagreement with Cantor's insistence on disallowing repeated elements in a set, nothing substantial came through for a considerable period of time. The work of Lake (1976) inspired by Rado's remark that "*there is no axiomatization for msets*" during his 1974 London Mathematical Society lecture entitled "*Multisets and multicardinals*", is known to be the first axiomatization of multiset theory. In Lake (1976) function (like vonNeumann's axiomatization) is taken as primitive. Lake admits: '…. *It might be though desirable to have an axiomatization which does not go via functions, such an axiomatization …could be conveniently written out using* $x \in^z y$' (intended to stand for "x *belongs to* y *precisely* z *times*". This is the approach taken in Blizard (1989), the most recent and sustained work on this line. Unlike many other formalizations of non–Cantorian set theory, Blizard formulates a formal theory of msets as a conservative extension of standard set theory. Bunder (1987), using BCK–linear logic (weakened first–order logic), develops an elementary theory of msets (in fact, finite msets). Blizard, (1991) provides an exhaustive survey of the literature dealing with the development of the theory of msets.

2.4 Survey of some Applications of Multisets in Mathematics

In this section, a modest attempt is made to present a comprehensive survey of some various applications of multisets in Mathematics.

An early elaborate reference is Knuth (1981). Some of the Knuth's findings are as follows:

The prime factorization of an integer $n > 0$ is a multiset N whose elements are primes, where $\pi_{p \in N} = N$. Accordingly, as every positive integer can be uniquely factored into primes, one can obtain a bijection between the positive integers and the finite msets of prime numbers. For example, if $n = 2^2 \cdot 3^3 \cdot 17$, the corresponding mset is $N = [2, 2, 3, 3, 3, 17]$. A simple algebra of msets is also developed, for example, if M and N are msets corresponding to integers m and n, then $M \cap N = gcd(m, n)$, $M \cup N = lcm(m, n)$ and $M \uplus N = mn$.

The natural correspondence between a monic polynomial over the complex numbers and the unique mset containing its roots exists. That is, every monic polynomial $f(z)$ over the complex numbers corresponds in a natural way to the mset F of its roots. Hence, if $f(z)$ and $g(z)$ are the polynomials corresponding to the finite msets F and G of complex numbers, then

$$F \uplus G = f(z)\, g(z), \quad F \cup G = lcm\,(f(z), g(z)),$$

and $$F \cap G = gcd\,(f(z), g(z)).$$

Knuth (1981) also shows that zeros and poles of meromorphic functions, invariants of matrices in canonical form and invariants of finite abelian groups do correspond to multiset representations.

Generating functions and nonnegative integer coefficients correspond one–to–one with msets of nonnegative integers. Thus, if A and B are msets of nonnegative integers, and

if $G(z) = \sum_{n \in A} z^n$ and $H(z) = \sum_{n \in B} z^n$ be generating functions corresponding to A and B, then $G(z) + H(z)$ corresponds to $A \uplus B$, etc. Also, for msets A and B, the product of Dirichlet generating functions $g(z) = \sum_{n \in A} \frac{1}{n^z}$ and $H(z) = \sum_{n \in B} \frac{1}{n^z}$ corresponds to the mset product AB.

Not much is known about the history of permutations of an mset. Knuth (1973), in line with some early references cited by him, expounds the area of permutations of msets.

Goguen (1974), in course of developing a category–theoretic foundation of fuzzy set theory, investigates properties of semiring sets, as in Ei1enberg (1974), along with various applications to msets. He also discusses usefulness of msets in the study of combinatorics and formal languages.

Chapin (1974, 1975), while formalizing the theory of fuzzy sets and Boolean–lattice–valued sets, seems to have mset–model in mind. The interpretation of the atomic formula $\varepsilon\,(x, y, z)$ as *x is an element of y with degree of membership at least z* reflects seminality. Mathematics of multisets proves a potential tool to study computing of Gröbner Bases and straightening laws in polynomial rings (Buchberger, 1976) and (Baclawski, 1981).

Brink (1987), outlines an algebra of msets and shows that it could model relevant structures to a great extent, yet falls short of DeMorgan monoid.

Anderson (1987) can be considered as the first sustained development of combinatorics of msets. The area of concentration includes the study of sizes, numbers and properties of *chains* (collections of pair wise $\subseteq$ comparable subsets) and *antichains* (collections of pairwise $\subseteq$ incomparable subsets) in the subset and msubset lattices (Weyl, 1949; Reingold *et al.*, 1977; Bender *et al*, 1984; Clements, 1988_a and 1988_b; and Blizard, 1990).

Martin (1989) observes that many well–founded partial orderings of the set of finite msets on a given set have appeared that played significant roles in the study of invariant theory (Muirhead, 2003) and (Gerstenhaber, 1961) for early references, ring theory (Baclawski, 1981) and (Poplin, 2000) for some recent developments and the theory of partitions (Bender, 1974; Macdonald, 1979; Bender *et al*., 1984 and Blizard, 1993) for various recent developments. Msets can be gainfully exploited in the study of measure theory which, however, would require tools of functional analysis, especially when the msets considered are infinite (Blizard, 1993).

Singh and Singh (2003) pointed out a fundamental problem. It is observed that the best known efficient formula ${}^{n}H_r = {}^{r+n-1}H_r$, designed to compute the number of combinations of n distinct objects taking r at a time when objects may occur with repetitions, includes terms of the type $[a_i]_n$, for each i, that are also msubsets of an mset in which each distinct element occurs unlimited number of times (hence, help determining the cardinality of the power set of an mset), turns out to be unworkable and computationally inefficient even if adjusted to be applicable, if the mset in consideration is finite or infinite, but with finite multiplicities. For example, for a 6–element mset $\alpha = [x, y, z]_{1,2,3}$ the number of all combinations of size 6 (repetitions allowed and

consequently $[x]_6$ –like combinations included which is not an msubset of α) is ${}^3H_6 = {}^8C_6 = 28$ whereas the number of 6–msubsets of α is only one and that is α itself. Also, besides discussing computational inefficiency of inclusion–exclusion principle, the paper puts forward a reasonably efficient and workable formula.

Petrovsky (1994), outlines some very innovative mathematical applications of msets. An axiomatic foundation for metrization of multiset spaces is developed. Poplin (2000) develops a multiset algebra in his Ph. D. Dissertation work, titled *the semiring of multisets*. He explicates that msets provide a connection between eigenvalues/eigenvectors equations for the max–Plus and the nonnegative real number systems. It is shown that max–plus, max–times and the nonnegative real numbers can be viewed as a special case of msets. The guiding factors for undertaking research in the area of mathematics abound. For example, the (max, +) algebra has been extensively used in Discrete event systems, transportation networks, parallel computations, project management, machine scheduling, to name a few (Poplin, 2000).

The interest in multisets and subsets of commutative monoids has increased in the recent years (see (Claude *et al*., 2001) for various deliberations).

Hegarty and Larson (2006), study permutations π of the natural numbers for which the numbers $\pi(n)-n$ belongs to a given (multi)subset M of Z (the set of integers), for all $n \in S$ (a given subset of the set of natural numbers).

2.5 Survey of some Applications of Multisets in Computer Science

Knuth (1973) and (1981) are the early known references to applications of msets in computer science. Knuth notes:

> *The terminal string of a non-circular context free grammar form an mset that is a set if and only if the grammar is unambiguous* (Knuth, 1973).

He introduces multisets into algorithms that compute values of x^n where x is a real quantity and n is a large positive integer (Knuth, 1981). Multisets and permutations of multisets are applied in a variety of *search* and *sort* procedures (Knuth, 1973).

Eilenberg (1974) describes a general theory of msets and applies it to automata. Various semiring structures are exploited and also, extensions to the cases of msets and field structures are studied. Shoesmith and Smiley (1978), while studying multiple–conclusion logic, find application of msets quite useful to account for *arrays* of formulae.

Dersowitz and Manna (1979) introduce mset ordering and ingeniously exploit it for proving termination of certain programs, which has served as a basis for many alternative and equivalent orderings subsequently proposed to date. The major intent of this seminal work can be seen in the following: "*... the mset ordering ... permits the use of relatively simple and intuitive termination functions in otherwise difficult termination proofs*" (Dersowitz and Manna, 1979) and (Bachmair, 1986).

Huet and Oppen (1980), and Jouannuad and Lescanne (1982) introduced significant refinements of Dersowitz–Manna mset ordering. In (Jouannuad and Lescanne, 1982), it is shown that the standard mset ordering is a maximal extension function. It also provides an efficient implementation of Dersowitz–Manna mset ordering.

Peterson (1981) shows that the very foundation of Petri net theory, introduced by C. A. Petri in 1962, needs the use of msets. Mayer and McRobbie (1982) find the use of msets quite appropriate to account for how often a premise is repeated in characterizing relevance aspect of an argument. Thistlewaite *et al* (1988) make use of algebra of msets developed in (Mayer and McRobbie, 1982) in explicating automated theorem proving for relevance logics, especially in implementation while using the program KRIPKE. They also recognise that msets have been taken as *datatypes* in a number of programming languages.

Bundy (1983) exploits mset of numbers to illustrate the definition–and–conjecture–formation program of Lenat. Manna and Waldinger (1985) can be considered as a sustained exposition to substantiate the fact that mathematical logic plays a fundamental role in the realm of theoretical computer science. A theory of BAG is developed by way of introducing a novel binary primitive operation, $\aleph$: if an atom u has multiplicity $n \geq 0$ in a bag x, then u has multiplicity $n + 1$ in the bag $u \aleph x$.

Reisig (1986) uses msets to define *relation nets*. Interestingly, exploiting mathematical intent of the definition of msets as generalized (integer–valued) characteristic functions given in (Whitney, 1933), the novel idea of *negative multiplicity* in an mset is introduced. Reisig (1986) also outlines the concept of *multirelations* – an mset whose domain is the Cartesian product of a set of sorts.

Yager (1986), after developing an algebra of msets, introduces the notion of *fuzzy bag* as follows: …to each element x in a fuzzy bag A is associated a multiset containing elements α (real number in the interval [0, l]) with multiplicities n (non–negative integers). The number n indicates the number of times the element x appears with

membership grade α in the fuzzy bag A. Yager notes that Zadeh fuzzy sets are special cases of fuzzy bags.

Grzymala–Busse (1987) extends the notion of *rough* set (introduced by Pawlak) to rough mset. Martin (1987) constructs a number of extension functions for mset orderings. Also, Martin (1989) introduces various well–founded partial orderings of the set of all finite msets whose elements are taken from a given set. Martin exploits the notion of *cone* in R^n (n–dimensional real space) and provides a systematization of the construction and classification of various well founded partial orderings underlying in proofs of program termination.

Pratt (1987) besides giving several arguments for the use of *pomsets*, especially showing how pomsets can be used to represent *parallel processes*, he also describes how Petri nets can be modelled as pomsets.

Gischer (1988) exploits the notion of a partial string and a partial language introduced in (Grabowski, 1981) to show how pomsets can be used as a model of concurrency.

Application of msets in Logic Programming languages is found to overcome *computational inefficiency* inherent in otherwise situation, especially in solving a sweep of real–life problems where multiple occurrences of an identical element are persistent. In fact, both the usual options viz, *attaching different levels* or *assigning numbers* to account for multiplicity turn out to be computationally inefficient. Representation of msets in a logic Programming language can be effected, for example, by introducing a binary associative and commutative function symbol ○, which also admits a unit

element. In a constraint logic programming language, constraints are built up from mset operations and relations and also an extended abstract on *Logic Programming with multisets*. Kiziltan (2004) is a recent work on multiset and strict multiset ordering constraints. Rule based multiset programming paradigm is extensively exploited to study synthetic biology (Bantre, 1993), (Claude *et al.*, 01) and (Krishinamwthy, 2005). Basically, multisets are interpreted to represent biological systems, such as molecules in a biochemical system. A host of simulators for biological systems has been developed and found useful to several fields of biology such as biochemistry, microbiology and evolutionary biology. For example, a system could be a multiset $[A, A, B, B, C]$, where A, B and C are elements that evolves by means of application of a set of rewriting rules (say, of the biological systems) viz;

$\{\{A, B\} \rightarrow \{C\}, \{C\} \rightarrow \{A, B\}\}$. If we choose the first rule to apply to the multiset in the example, the multiset $\{A, B, C, C\}$ is obtained. The interaction among elements in a multiset object space, which includes the environment, are like chemical reactions and the evolution of multiset can mimic the biological evolution leading to plausibility of DNA computing and programmable living machines (Krishinamwthy, 2005).

Multisets have been taken as datatypes in a number of programming languages. The specification of an abstract data type (ADT) for msets has been studied. Dovier *et al*., (1998), after developing mathematics of msets, substantiate that msets are the fundamental data structures for P systems. In fact, in computational sense, an mset is just a data structure, which differs from a set in permitting repetition of some of its elements and that from a *list* in being unordered, and hence it turns out to be a suitable modelling tool for a large class of real–life phenomena. Ross and Stoyanovich (2004),

study cardinality bounded msets in Database systems to overcome consistency and performance problems that conventional representations in relational database suffered from.

Petrovsky (1991) explores a new dimension. He provides an extensive treatment of mset operations and discusses *Cluster* analysis in mset spaces, and demonstrates their application in augmenting Decision Support System (DSS). It is shown that "*a multi-aspect analysis of problem and structuring alternatives allow us to gain an insight into the problem nature and find better decisions. ...suggested the tools for structuring a collection of objects represented by many qualitative attributes when a lot of copies of objects or values of attributes describing them exist. ...approach is based on a theory of multiset metric spaces and indexes of difference/similarity between multisets*". A "*rough*" set approach to multi–attribute decision analysis of (Pawlak and Slowinski, 1994) is a pioneering work in this direction.

CHAPTER THREE

FUNDAMENTALS OF MULTISETS

3.0 Introduction

In Sections 3.1 – 3.3 of this chapter, we outline some mathematical preliminaries that help in making the dissertation maximally self–contained in developing the theory of multiset algebras. In Section 3.4 we systematize the fundamentals of multisets. We derive some results regarding the algebraic properties of multiset operations that do not conform to standard set theoretic results.

3.1 Mathematical Preliminaries

Definition 3.1 (*Set*)

A set is a collection of well defined distinct objects called the elements or members, without repetition and order. In formal words, Cantor's concept of a set is defined as;

> *By a 'set', we are to understand any collection M of definite and distinct objects m of our intuition or thought (which will be called the 'elements' of M) into a whole.*

Sets are denoted by capital letters, $A, B, C, \ldots, X, Y$ and Z while elements of sets are denoted by lower case letters, $a, b, c, \ldots, x, y, z$. The symbols $\in$ and $\notin$ means "*belong to*" and "*does not belong to*" respectively. We shall also use the symbols $\cup$, for universal set, and $\emptyset$ for empty set.

Sets are represented as follows:

List notation of a set : $\{x_1, x_2, \ldots, x_n\}$ or $\{x_1, x_2, \ldots, x_n, \ldots\}$ according as it is finite or infinite respectively.

Set builder notation: $\{x \mid P(x)\}$ denotes a set of all elements x satisfying a definite predicate $P(x)$.

Operations on sets:

Intersection ($\cap$) and *union* ($\cup$) of sets are defined respectively as;

$$\forall A\, \forall B\, \forall z(z \in A \cap B \leftrightarrow (z \in A \wedge z \in B))$$

$$\forall A\, \forall B\, \forall z(z \in A \cup B \leftrightarrow (z \in A \vee z \in B))$$

Powerset: The family of all the subsets of a set S is called the powerset of S denoted by $\wp(S)$.

Difference: The difference of two sets A and B is the set of all elements that belong to set A which do not belong to set B. It is denoted by $A - B$ and it is also called the *relative complement* of B with respect to A. That is,

$$A - B = \{x | x \in A \text{ and } x \notin B\}.$$

The *complement* of a set A with respect to universal set ($\cup$) is the difference of the universal $\cup$ and A. It is denoted by A' or A^c.

$$A' = \{x | x \in \cup \text{ and } x \notin A\}$$

3.1.1 Relations and Functions

An ordered pairs otherwise known as ordered 2–tuples is an object denoted by (a, b) or $< a, b >$. It is the collection of objects a and b in which a is the first object and b is the second object. In set–theoretic terms, following Wiener–Kuratowski formulation, the ordered pair $(a, b) = \{\{a\}, \{a, b\}\}$, indicating that a is its first element. The set $\{a, b\}$ is an unordered pair. We note that $(a, b) \neq (b, a)$, unless $a = b$, whereas $\{a, b\} = \{b, a\}$ for all a and b. Also, $\{a, a\} = \{a\}$, whereas $(a, a) \neq \{a\}$.

In general, an ordered n–tuple, written as $(a_1, a_2, \ldots, a_n)$ or $< a_1, a_2, \ldots, a_n >$ is the collection in which a_1 is the first element, a_2 is the second element, . . ., a_n is its n^{th} element. It is instructive to note that the representation of an n–tuple as $(a_1, a_2, \ldots, a_n)$ does entail that $(a_1, a_2, \ldots, a_n) = ((a_1, a_2, \ldots, a_{n-1}), a_n)$, an ordered pair whose first element is an ordered $(n-1)$–tuple. Also, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_1 = b_1$, $a_2 = b_2$, . . ., $a_n = b_n$.

An ordered n–tuple is used to represent a number of notions in diverse contexts, such as, *strings* built from elements of a given set, *words* built from letters of the English alphabet, etc.

Definition 3.2 (*Cartesian Products*)

If A and B are any sets, then the set of all ordered pairs (a, b) such that $a \in A$ and b $\in$ B denoted by $A \times B$ is called the Cartesian product of A and B. Precisely, we write, $A \times B = \{(a, b) \mid a \in A, \ b \in B\}$, and $A \times A = A^2 = \{(a, b) \mid a \in A, \ b \in A\}$. In general, $A \times B \neq B \times A$ and if A and B are finite sets then $|A \times B| = |A||B|$.

We note, in general, that for any sets A_1, A_2, . . ., A_n, $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_k \in A_k$, for $k = 1, 2, \ldots, n\}$ and If $A_1 = A_2 = \ldots = A_n$, then we have $A \times A \times \ldots \times A = A^n$.

Definition 3.3 (*Relations*)

A set R is a binary relation or 2–ary relation if all elements of R are ordered pairs. In other words, a 2–ary relation from A to B is a subset of $A \times B$. By extension, an n–ary relation on sets A_1, A_2, . . ., A_n is a subset of $A_1 \times A_2 \times \ldots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the domains of the relation and n is called its degree.

Let R be a relation from A to B, we use the notation aRb to denote that $(a, b) \in R$, and $a \not R\, b$ or $\neg\, aRb$ to denote that $(a, b) \notin R$. The graph of R is denoted by $\{(a, b) \in A \times B \mid aRb\}$. The domain and the range of R are respectively defined by

$$\text{dom}\,(R) = \{(a \in A \mid (\exists\, (b \in B)\, (\, (a, b) \in R)\};$$

$$\text{ran}\,(R) = \{b \in B \mid (\exists\, a \in A)\, (\, (a, b) \in R)\}.$$

The inverse of R, denoted by R^{-1}, is defined by $R^{-1} = \{(b, a) \mid (a, b) \in R\} \subset B \times A$; and the complementary relation $\overline{R}$ is defined by $\overline{R} = \{(a, b) \mid (a, b) \notin R\}$.

Remark 3.1

Since relations are defined as sets; the union, intersection and difference operations on relations can be conveniently defined similar to the one for sets. For example, if R and S are relations, then $x\,(R \cup S)\,y \Leftrightarrow x\,R\,y \vee x\,S\,y$, etc.

Definition 3.4 (*Relations on a set*)

A relation on a set A is a relation from A to A. For example, if A is any set of real numbers, then $<$, $\leq$, $>$, $\geq$ and $=$ are all relations on A.

Definition 3.5 (*Empty or void relation*)

The empty relation on A is the subset $\emptyset$ of $A \times A$. That is, the relation on A is $\emptyset$ if for all $a, b \in A, \neg aRb$.

Definition 3.6 (*Identity relation*)

The identity relation, $=$, on a set A is the subset $\Delta_A = \{(a, a) \mid a \in A\}$, the diagonal in $A \times A$.

Definition 3.7 (*Universal relation*)

The universal relation on A is $A \times A$ itself. That is, the relation R on A is universal if and only if aRb for all $a, b \in A$.

Proposition 3.1

For all $n \in$N, if $|A| = n$, then the total number of relations on A is $2^{|A \times A|} = 2^{n^2} =$ the total number of subsets of $A \times A$. While the total number of subsets of $A = 2^n = |(\wp(A)|$, where $\wp(A)$ is the powerset of A, and $|A| = n$.

3.1.2 Some properties of relations

Let A be a set with n elements, $n = 1, 2, 3\ldots$. A relation R on A is called

(i). reflexive if, for all $a \in A$, aRa. This implies that R is reflexive if and only if $R = R^{-1}$. Also, if R is reflexive, R^n is reflexive.

(ii). irreflexive if, for all $a \in A$, $\neg aRa$ or equivalently, $\nexists\, a \in A$, aRa. This implies that R is irreflexive if and only if R^{-1} is reflexive. Also, if R is irreflexive, R^n is not necessarily irreflexive and so is the case with $\bar{R}$.

(iii). symmetric if, for all $a, b \in A$, $aRb \Rightarrow bRa$; also R^n is symmetric if R is symmetric.

(iv). asymmetric if, for all $a, b \in A$, $aRb \Rightarrow \neg\, bRa$

(v). antisymmetric if, for all $a, b \in A$, $aRb \wedge bRa \Rightarrow a = b$; in other world, $a \neq b \wedge aRb \Rightarrow \neg bRa$; in other words, a relation is antisymmetric if and only if there are no pairs of distinct elements a and b with aRb and bRa both. Also, R is antisymmetric if and only if $R \cap R^{-1} \subseteq \{(a, a) \mid a \in A\}$.

(vi). transitive if, for all $a, b, c \in A$, $aRb \wedge bRc \Rightarrow aRc$;

(vii). complete if, for all $a, b \in A$, $a \neq b \Rightarrow aRb$ or bRa.

Proposition 3.2

(i) If relations R and S are reflexive and symmetric, then $R \cup S$ and $R \cap S$ are also reflexive and symmetric.

(ii) If relations R and S are both transitive, then $R \cap S$ is also transitive, but $R \cup S$ is not necessarily transitive.

Remark 3.2

If a relation is not reflexive it is not necessarily irreflexive, and vice versa. Also, a relation can be both symmetric and antisymmetric. This is obviously the case when each element is either related to itself or not related to any element. Thus, a relation cannot be both symmetric and antisymmetric if it contains some pair of the form (a, b) where $a \neq b$. It can be shown that a relation is asymmetric if and only if it is irreflexive and antisymmetric. In general, any relation which is irreflexive and symmetric cannot be transitive.

3.1.3 Composite relations or Relational product

Let R_1 be a relation from A to B and R_2 a relation from B to C. Then the composite relation or relational product of R_1 and R_2 is the relation from A to C, consisting of ordered pairs (a, c) where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that aR_1b and bR_2c. The composite of R_1 and R_2 is denoted by $R_1 \circ R_2$. The domain of $R_1 \circ R_2$ is a subset of A and the range is a subset of C. Also, in general, $R_1 \circ R_2 \neq R_2 \circ R_1$. However, the composition of relations is associative. It follows that $R_1 \circ R_2 \neq \emptyset$ if and only if ran $(R_1) \cap$ dom $(R_2) \neq \emptyset$.

3.1.4 Powers of a relation

Let R be a relation on set A. The powers R^n, $n = 1, 2, 3$ ... are defined inductively by $R^1 = R$ and $R^{n+1} = R^n \circ R$. Hence, $R^3 = R^2 \circ R$ and $R^2 = R \circ R$. For example, let $A = \{a, b, c\}$ and $R = \{(a, b), (b, c), (c, a)\}$.

Then, $R^2 = R \circ R = \{(a, c), (b, a), (c, b)\}$,

$R^3 = \{(a, a), (b, b), (c, c)\}$,

$R^4 = R$.

Remark 3.3

If R is a relation on set A where $|A| = n$, then $R^{n+1} = R$

Proposition 3.3

The relation R on A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$. In other words, the powers of a transitive relation R are subsets of R. The proof follows by mathematical induction. In particular $R^2 \subseteq R$ implies R is transitive.

If R is reflexive and transitive, then $R^n = R$

3.1.5 Closure of relations

Let R be a relation on a set A which does not possess some property P. If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then S is called the closure of R with respect to P. In other

words, S, if it exists, is the smallest (or intersection) of all the relations that posses the property P and contain R.

In order to find the reflexive and the symmetric closures of a relation R on A, it is sufficient to add to it all those pairs of elements of A that are lacking in the respective cases. The reflexive closure of R is $R \vee \Delta$, where Δ is the diagonal relation on A, and the symmetric closure of R is $R \cup R^{-1}$. However, in general, this single–step approach does not work if one needs to find the transitive closure of a given relation R on A. Nevertheless, it may be eventually possible on repeating this process until no new ordered pairs are required. *Section 3.1.6* is a mathematical formulation of the aforesaid inductive approach.

3.1.6 Connectivity relation

Let R be a relation on set A. The connectivity relation symbolized by R^+ or $R^\star$ is defined as the set of pairs (a, b) such that there is a 'path' from a to b in R, where the latter is defined as a sequence of points $a, x_1, x_2, \ldots, x_{n-1}, b$ satisfying $aRx_1, x_1Rx_2, \ldots, x_{n-1}Rb$. Hence, R^+ or $R^\star = \bigcup_{n=1}^{\infty} R^n$, where R^n is the set of pairs (a, b) such that there is a path of length n from a to b.

For example, let A = {a, b, c}, and $R = \{(a, b), (b, c), (c, a)\}$ be a relation on A. Now,

$$R^\star = R \cup R^2 \cup R^3 \cup \ldots$$

$$= \{(a,b), (b,c), (c,a), (a,c), (b,a), (c,b), (a,a), (b,b), (c,c)\},$$

which is transitive and $R \subseteq R^\star$.

Proposition 3.4

The transitive closure of a relation R equals the connectivity relation $R^\star$ (Rossi, 1999) and also, $R^\star = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$, where n is the number of elements in the set A.

$R^\star$ is also used to symbolize the transitive closure of a relation. Details for constructing algorithms (Warshall's algorithm, in particular) for determining $R^\star$ can be found in (Rossi, 1999). We now present a simplified formulation of $R^\star$ as follows:

Let R be a relation on A. The transitive closure $R^\star$ of R is the relation on A such that

i. $R^\star$ is transitive

ii. $R \subseteq R^\star$

iii. $R^\star$ is the smallest set satisfying (i) and (ii) above

The algorithmic steps for the construction of $R^\star$ can be outlined as follows:

Step 1: Determine $R_1 = R \cup (R \circ R)$ and $R_{i+1} = R_i \cup (R_i \circ R_i)$

R_{i+1}, as defined above, is called the transitive extension of R_i, $i = 1, 2, \ldots$

Step 2: If $R_1 = R$, then $R^\star = R$ and stop. If $R_1 \neq R$, compute R_2.

Step 3: If $R_2 = R_1$, then $R^\star = R_1$ and stop. If $R_2 \neq R_1$, compute R_3, and so on.

In general, if $R_{i+1} = R_i$ for $i = 1, 2, 3, \ldots$, then $R^\star = R_i$. If $R_{i+1} \neq R_i$, compute the next R_i.

Remark 3.3

The closure of a relation with respect to a given property may not exist. We illustrate it as follows:

Let $A = \{0, 1, 2\}$, $R = \{(0, 0), (0, 1), (1, 1), (2, 2)\}$ and P is the property satisfying (a) "is not reflexive" and (b) "has an odd number of elements".

Then the closure of R does not exist with respect to both (a) and (b). Here,

(a) holds since R is reflexive, every relation containing it must also be reflexive; and

(b) holds since both {(0, 0), (0, 1), (1, 1), (2, 2)} and {(0, 0), (0, 1), (1, 0), (1, 1), (2, 2)} contain R and have an odd number of elements, but neither is a subset of the other. These emphasize that the closure of a relation with respect to a given property may not exist.

Remark 3.4

Transitive closures of relations have important applications in the areas of Networks, Syntactic analysis, Fault detection and diagnosis in switching circuits (Rossi, 1999).

3.1.7 Order relations

Quasi ordering: A relation R on a set S is called a quasi–ordering if it is reflexive and transitive.

Partial ordering: A relation R on a set S is called a partial ordering or partial order relation or simply an order if it is reflexive, antisymmetric, and transitive.

Conventionally, a partial ordering is denoted by the symbol ≤, which does not necessarily mean the usual "*less than or equal to*" as is used for real numbers. If ≤ is a partial ordering on S, then the ordered pair $(S, \leq)$ is called a (reflexive) partially ordered set (poset, for short). It follows by definition that if $(S, \leq)$ is a poset, then $(S, \geq)$ is also a poset called the dual of $(S, \leq)$. Moreover, if A and B are two posets, then $A \times B = \{(a, b) | \, a \in A, b \in B\}$ is a poset under the relation ≤ defined as $(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq a_2$ on A, and $b_2 \leq b_2$ on B. Note that we have denoted the three relations viz; on A, B, and $A \times B$ by the same symbol ≤ for convenience.

It can be seen that associated with every partial ordering ≤ on a set S, there is an ordering $<$, defined as follows:

For every $x, y \in S$, $x < y \iff x \leq y \wedge x \neq y$.

Clearly $<$ is irreflexive, asymmetric and transitive and sometimes called a strict ordering. That is, we have

(i) $x \not< x$ (irreflexibility),

(ii) $x < y \implies y \not< x$ (asymmetry),

(iii) $x < y \wedge y < z \implies x < z$ (transitivity).

We call x and y of S comparable if either $x \leq y$ or $y \leq x$, written as $x \sim y$ and non–comparable, written as $x \nsim y$, if neither $x \leq y$ nor $y \leq x$.

Total Ordering: Let $(S, \leq)$ be a poset. If for every $x, y \in S$, either $x \leq y$ or $y \leq x$, then $\leq$ is called a 'total' or 'simple' or 'linear' ordering, and $(S, \leq)$ is called a loset or a chain. In other words, every two elements of a chain are comparable. Note also that a poset $(S, \leq)$ is called an antichain if every two elements of S are incomparable.

Definition 3.8 (*Partition of a set*)

Let S be a non empty set and $A = \{A_1, A_2, \ldots, A_n\}$ be a collection of mutually disjoint subsets of S such that $\bigcup_{i=1}^{n} A_i = S$, then A is called a partition of S and the sets $A_1, A_2, \ldots, A_n$ are called the blocks of the partition. The blocks of a partition correspond to its R equivalence classes. Also, the partition of a given set containing all its singletons has the largest number of blocks and that containing only the set itself has the smallest. In fact, for any finite set S with n elements, no partition can have more than n blocks.

A partition P_1 is called a refinement of the partition P_2 denoted by $P_1 \leq P_2$, if every set in P_1 is a subset of one of the sets in P_2. More precisely, if P_1 is a refinement of P_2, then any two elements that are in the same block of P_1 must also be in the same block of P_2.

3.1.8 Equivalence relations

A relation on set A is called an equivalence relation if it is reflexive, symmetric, and transitive. Due to its diverse applications, the theory of equivalence relations has been extensively studied. In fact, to a large extent, it has been exploited as a tool for

construction of new mathematical objects; for instance, in the development of number systems. The fundamental result that has ushered into enormous applications of equivalence relation is the following:

Let R be an equivalence relation on a non empty set S. Then the equivalence classes of R form a partition of S and conversely, given a partition $\{A_i \mid i \in I\}$ of S, there is an equivalence relation R that has A_i, $i \in I$ as its equivalence classes. In other words, there is a one–to–one correspondence between the sets of "*equivalence relations*" on S and '*partitions*" of set S.

Let $P(n)$ denote the number of distinct equivalence relations on a set with n elements. Then, $P(n)$ satisfies the recurrence relation $P(n) = \sum_{j=0}^{n-1} C(n-1, j)P(n-j-1)$, and $P(0) = 1$.

The numbers $P(n)$ are called Bell numbers after the American Mathematician E. T. Bell.

Proposition 3.5

The transitive closure of the symmetric closure of the reflexive closure of a relation is an equivalence relation.

Definition 3.9 (*Covering of a Set*)

Let S be a given non empty set and $A = \{A_1, A_2, \ldots, A_m\}$ be a collection of nonempty subsets of S such that $\bigcup_{i=1}^{m} A_i = S$. Then, A is called a "*covering*" of S and the sets $A_1, A_2, \ldots, A_m$ are said to cover S. It follows that A becomes a partition of S

if $A_1, A_2, \ldots, A_m$ are mutually disjoint. Note that the equality of two coverings or two partitions or a covering and a partition is based on the definition of the equality of two sets.

3.1.9 Compatibility relations

A relation R on a set S is said to be a compatibility relation if it is reflexive and symmetric. It is sometimes denoted by $\approx$. Any two elements $x, y \in S$ are called compatible if xRy, where R is a compatibility relation defined on S.

By definition, a compatibility relation need not be necessarily transitive. Thus, all equivalence relations are compatibility relations but the converse need not be true. In fact, it is useful to exploit relations which are compatibility relations but not equivalence relations. Accordingly, a compatibility relation does not necessarily define a partition; however, it does define a covering of the set.

Compatibility relations are exploited in solving minimization problems, especially in switching theory.

3.2 Functions

A (total) function or mapping from set A to set B represents a relation where exactly one element of B is related to each element of A. Equivalently, if f be a function from A to B, and if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$. The set A is called the domain of f and B is called the codomain of f. The range of f is the set of all images of elements of A and as such it is a subset of the codomain of f. In case, f is defined on $D \subset A$, then it is called a partial function. A function defined by $g(a, b) = a - b$ in the set $\mathbb{N}$ of

natural numbers is a partial function, whereas $g(a, b) = a + b$ is total. The need for using partial function arises when the function is not effectively defined on its extended domain.

A function f is called one–to–one, or injective, if and only if $f(x) = f(y)$ implies $x = y$ for all x and y in the domain of f. That is, by contraposition, $x \neq y$ implies $f(x) \neq f(y)$. A function f is called onto, or surjective, if and only if $\text{ran}(f) = \text{codomain}(f)$. A function f is called bijective or one–to–one correspondence if and only if it is both injective and surjective. It follows by the definition of a function that its converse, unlike a relation, need not be a function. We note that the converse of a bijective function exists as a function called its inverse function which is also bijective. This is why one–to–one correspondence is called invertible.

If $f: A \to B$ and $S \subset A$, then $f \cap (S \times B)$ is a function from $S \to B$ called the restriction of f to A and is sometimes denoted by f/A. If g is a restriction of f, then f is called the extension of g.

Definition 3.10 (*Projection*)

The Projection $P_{i_1, i_2, \ldots, i_m}$ maps the n–tuple $(a_1, a_2, \ldots, a_n)$ to the m–tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$ where $m \leq n$. In other words, the projection $P_{i_1, i_2, \ldots, i_m}$ deletes $(n - m)$ of the components of an n–tuple, leaving the i_1^{th}, i_2^{th}, …, i_m^{th} components. The operation projection on a given functions enables us to obtain functions of smaller dimensions.

3.2.1 Composition of functions

Let f be a function from A to B and let g be a function from B to C. The composition or the relative product of f and g, denoted by $g \circ f$, is defined by $(g \circ f)\ (a) = g(f(a))$, where $a \in A$. More precisely, $g \circ f$ is called the left composition of g with f or, equivalently any function g for which $g \circ f$ can be obtained is called left–composible with f. Note that in order to define $g \circ f$, the range of f must be a subset of the domain of g. In general, $\circ f \neq f \circ g$. However, the composition of functions is associative. Also for any bijective function from A to B, $(f^{-1} \circ f) = I_A$ and $(f \circ f^{-1}) = I_B$, where I_A and I_B are identity functions on A and B, respectively. That is,$(f^{-1})^{-1} = f$.

3.2.2 Remainder function

Let m be any positive integer and k be any integer, we define $k(mod\ m) = r$, $0 \leq r < m$ when k is divisible by m. It follows that when k is positive, divide k by m to obtain r. If k is negative, divide $|k|$ by m to obtain r^1, and then $k\ (mod\ m) = r = (m - r^1)$. For example, $-21\ (mod\ 5) = 5 - 1 = 4$.

The function f_m defined as $f_m(k) = k(mod\ m)$, is known as the remainder function (or, the mod m function). Note that the remainder function is onto but not one–to–one.

3.2.3 Characteristic functions

Let S be a subset of a universal set $\sqcup$. The characteristic function of the set S is defined by

$$\chi_S\ (x) = \begin{cases} 1, & if\ \ x \in S \\ 0, & otherwise \end{cases}$$

In other words, $\chi_S(x)$ is the function from $\sqcup$ to $\{0, 1\}$ such that $\chi_S(x) = 1$ if $x \in S$ and $\chi_S = 0$ if $x \in S$.

Clearly, $\chi_{\sqcup} = 1$ and $\chi_{\emptyset} = 0$, where 1 and 0 stand for constant functions identically equal to 1 and 0, respectively, for each element in $\sqcup$. A typical example is Kronecker's delta, denoted by δ, and defined as $\delta = \chi_{\Delta}$, where $\Delta = \Delta_S = \{(a, a) \mid a \in S\}$ is the diagonal of $S \times S$.

That is, $\delta_{xy} = \delta(x, y) = \begin{cases} 1 & if \ x = y \\ 0, & if \ x \neq y \end{cases}$

Let $A, B, A_1, A_2, \ldots, A_m$ be subsets of a universal set $\sqcup$. Then for all $x \in \sqcup$,

(i). $A \subseteq B \Leftrightarrow \chi_A(x) \leq \chi_B(x)$; $A = B \Leftrightarrow \chi_A(x) = \chi_B(x)$

ii. $\chi_{A^C}(x) = 1 - \chi_A(x)$

iii. $\chi_{A \cap B}(x) = \chi_A(x)\chi_B(x) = \min(\chi_A(x), \chi_B(x))$

iv. $\chi_{A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_m} = \prod_{i=1}^{m} \chi_{A_i}$

v. $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$

$= 1 - (1 - \chi_A(x))(1 - \chi_B(x)) = \max(\chi_A(x), \chi_B(x))$

vi. $\chi_{A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_m} = 1 - \prod_{i=1}^{m} (1 - \chi_{A_i})$

vii. $\chi_{A \oplus B}(x) = |\chi_A(x) - \chi_B(x)| = \chi_A(x) + \chi_B(x) - 2\chi_A(x)\chi_B(x)$

viii. $\sum_{x \in \cup} \chi_A(x) = |A|$, where A is finite.

Remark 3.5

By the definition of characteristic function, it follows that $\leq$, +, ., −, $\oplus$ are the usual arithmetic operations and this is how the applications of characteristic functions abound. All equations in characteristics functions are identities which are true for each element of $\sqcup$. Whitney (1933) gave an early detailed presentation.

Also, as statements about sets and their operations can be represented on a computer in terms of binary numbers, characteristics functions are exploited to model it. For example, consider a set $S = \{a, b, c, d\}$, the value of the characteristic function of a subset $\{a, b\}$ of S will contain four binary digits {1 1 0 0}.

3.3 Operations and Algebraic Structures

3.3.1 Binary and n −ary operations

Let S be a set, then a mapping $f: S \times S \to S$ is called a binary operation (or a binary composition) on S. In general, a mapping $S^n \to S$ is called an n–ary operation and n is called the order of the operation. For $n = 1$, $f: S \to S$ is called a unary operation.

Conventionally, various symbols like +, −, ∗, ∪, ∩, ∨, ∧ etc., are used to denote binary operations, sometimes chosen pertaining to a specific subject area. For example, if S be the set of integers, and f be the usual addition +, then for any integers x and y, we write $x + y$ or $f(x, y)$ to represent the outcome.

3.3.2 Some properties of binary operations

(a) Closure Property:

A set S is said to be closed under an operation $*$ if $a * b$ is a unique element of S for any a and b in S. By the definition of a binary or an n–ary operation on a set, it is entailed that the set is closed under that operation. Accordingly, the closure property need not be mentioned explicitly.

(b) Commutativity:

A binary operation $*$ on S is said to commutative if $x * y = y * x$, for every $x, y \in S$.

(c) Associativity:

A binary operation $*$ on S is said to be associative if $x*(y * z) = (x * y) * z$, for every $x, y, z \in S$.

(d) Distributivity:

Let $*$ and o be two operations on S. Now, $*$ is said to be left distributive over o if $x * (y \text{ o } z) = (x * y) \text{ o } (x * z)$ and right distributive if $(y \text{ o } z)*x = (y * x) \text{ o } (z * x)$ for all $x, y, z \in S$.

3.3.3 Some distinguished elements of $(S, *)$

(a) Identity elements:

Let $*$ be a binary operation on S. If there exists an element $e_l \in S$ satisfying $e_l * a = a$ for every $a \in S$, then e_l is called a left identity element for $*$. Similarly, we can define a right identity elements e_r in S for $*$. It follows that $e_l = e_r = e$ (say) satisfying

$e \times a = a \times e = a$ for every $a \in S$, then e is unique and called the identity for $*$. Note that for a commutative binary operation, e_l and e_r coincide.

On the same line, the zero element of a set sometimes denoted by $\emptyset$ with respect to an operation can be defined. For example, the empty set $\emptyset$ is the zero for the operation of intersection and the universal set is the zero for the operation of union of subsets of a given universal set.

Any distinguished element of S such as an identity element or a zero element with respect to a binary operation is considered as a 0–ary operation, a particular case of an n–ary operation for $n = 0$.

(b) Idempotent element:

Let $*$ be a binary operation on S. An element $a \in S$ is called an idempotent element for $*$ if $a * a = a$. For example, every set is idempotent with respect to union and intersection.

(c) Invertible elements:

Let * be a binary operation on S with the identity element e. An element $a \in S$ is said to be left invertible if there exists an element $b \in S$ satisfying $b * a = e$. The element b is called a left inverse of a. Similarly we can define a right invertible element and its right inverse. If an element $a \in S$ is both left and right invertible, then a is called invertible.

(d) Cancellable element:

Let $*$ be a binary operation on S. An element $a \in S$ is called cancellable for $*$, if for every $x, y \in S$, $(a * x = a * y) \vee (x * a = y * a) \Rightarrow (x = y)$.

By definition, it follows that if $*$ is associative and the element $a \in S$ is invertible, then a is cancellable, but the converse need not be true. For example, in the set of integers any nonzero integer is cancellable for multiplication, but the only invertible integer is the identity element, the unity.

3.3.4 Algebraic systems

A system consisting of a set and one or more n–ary operations on the set will be called an algebraic system or simply Algebra, usually denoted by $(S, F_1, F_2, \ldots)$ where S is a nonempty set and $F_1, F_2, \ldots$ are operations on S. At times, as part of the system, relations which are not part of operations are also included. In view of the fact that relations and operations on the set S do define a structure on the element of S. An algebraic system is also called algebraic structure. It is also possible that more than one set is included in the system, along with operations on them. Two algebraic systems are said to be of the same parity or type if the degree or arity of operations on them are the same. For example, $(\mathbb{Z}, +, \times)$ and $(\mathbb{R}, +, \times)$ are of the same type, where $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers and $+$ and $\times$ are the operations of addition and multiplications respectively on $\mathbb{Z}$ and $\mathbb{R}$. Note that being of the same type does not entail that the two systems have the same properties.

In this thesis, we would be largely interested in the following algebraic systems:

(a) Group:

A group $(G, \cdot)$ is an algebra with a set G and binary operation $\cdot$ in which the following axioms are true:

$G1$. G is closed under the operation $\cdot$, that is $a \cdot b \in G$ for all $a, b \in G$

$G2$. The operation is associative, that is $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in G$

$G3$. There is an identity element $e \in G$ such that a$\cdot e = e \cdot a = a$ for all $a \in G$

$G4$. There exist $a^{-1} \in G$ such that for all $a \in G$, $a^{-1} \cdot a = a \cdot a^{-1} = e$

A group G is Abelian (or commutative) if in addition to the above axioms, $ab = ba$ for all $a, b \in G$. Let G be a group, a subset H of G is called a subgroup if under the product $(\cdot)$ H itself is a group.

Groups were one of the earliest concepts studied in algebra (*groups of substitutions appeared about two hundred years ago*). Groups are generalized to semigroups and monoids in one direction and to quasigroups and loops in another direction. Groups are extensively used in *error-correcting codes* and *design of fast adders*.

(b) Monoid:

A monoid $(M, *)$ consists of a set M together with a binary $*$ on M such that

$M1$. $a*(b * c) = (a * b) * c$ for all $a, b, c \in M$

$M2$. There exists an identity $e \in M$ such that $a*e = e * a = a$

A semigroup is also a monoid if it also satisfies identity property. Monoids are specially used in *syntactic analysis* and *formal languages*. A monoid is a group if it also possess the inverse property.

Let A be a set called alphabet whose elements are called letters. A word over A is a finite sequence of elements of A. Let A^n be the set of n–tupples (string of elements of A without any symbol between them). The elements of A^n are called the words of length n. A word of length 0 (zero) is an empty string (word) denoted by Λ.

Let $(A) = A^0 \cup A^1 \cup A^2 \cup \ldots = \cup_{=0}^{\infty} A^n$, then $(Fm(A), *)$ is called the *free monoid* generated by A usually denoted by A^* where the operation * is a concatenation and the identity element is the empty word Λ.

(c) Semigroup:

A semigroup $(G, \cdot)$ is an algebra with a set G and binary operation $\cdot$ satisfying the axioms $G1$ and $G2$ above. Semigroups are the simplest algebraic systems which satisfy closure and associativity. They play a very significant role in *sequential machines* and *formal language theory*. Free monoids without identity element is called *free semigroups*.

(d). Quasigroup:

A quasigroup is an algebra $(Q, /, \cdot, \backslash)$ with three binary operations $/, \cdot$ and $\backslash$ satisfying the following identities:

$Q1$ (i). $x \backslash (x \cdot y) \approx y$ (ii). $(x \cdot y) / y \approx x$

$Q2$ (i). $x \cdot (x \backslash y) \approx y$; (ii). $(x / y) \cdot y \approx x$

(e). Loop:

A loop is a quasigroup with identity.

(f). Lattice:

A lattice $(L, \vee, \wedge)$ is a set L together with two binary operations $\wedge$ and $\vee$ satisfying the following:

$L1$. (i). $x \vee y = y \vee x$, (ii). $x \wedge y = y \wedge x$ (*Commutative law*)

$L2$. (i). $x \vee (y \vee z) = (x \vee y) \vee z$, (ii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (*Associative law*)

$L3$. (i). $x \vee x = x$ (ii). $x \wedge x = x$ (*Idempotent law*)

$L4$. $x = x \vee (x \wedge y)$ (ii). $x = x \wedge (x \vee y)$ (*Absorption law*)

(g). Semilattice:

A Semilattice is a semigroup $(S, \cdot)$ which satisfies the commutative law, $x \cdot y = y \cdot x$ and idempotent law $x \cdot x = x \ \forall x, y \in S$.

(h). Bounded lattice:

A bounded lattice $(B, \vee, \wedge)$ is a set B with two binary operations, $\vee$ and $\wedge$ that satisfies the following:

$B1$. $(B, \vee, \wedge)$ is a lattice

$B2$. $x \wedge 0 = 0, \ x \vee 1 = 1$

(i). Boolean algebra:

A Boolean algebra $(B, \vee, \wedge)$ is a set together with two operations, $\vee$ and $\wedge$ that satisfies the following:

$B1$. $(B, \vee, \wedge)$ is a distributive lattice

$B2$. $x \wedge 0 = 0, \quad x \vee 1 = 1$

$B3$. $x \wedge x^1 = 0, \; x \vee x^1 = 1$

(j). Heyting algebra:

Heyting algebra $(H, \vee, \wedge, \rightarrow)$ is a set H together with three binary operations $\vee$, $\wedge$, and $\rightarrow$ satisfying the following:

$H1$. $(H, \vee, \wedge)$ is a distributive lattice

$H2$. $x \wedge 0 = 0, \; x \vee 1 = 1$

$H3$. $x \rightarrow x = 1$

$H4$. $(x \rightarrow y) \wedge y = y \quad x \wedge (x \rightarrow y) = x \wedge y$

$H5$. $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z), \quad (x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

(k). Ring:

A ring is an algebra $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ on R that satisfies the following axioms for all $x, y, z \in R$:

(i). $x + y = y + x$ (*Commutative law*)

(ii). $(x + y) + z = x + (y + z)$ (*associative law*)

(iii). $\exists\ 0 \in R$ such that $x + 0 = x$ (*identity law*)

(iv). $\exists\ -x \in R$ such that $x + (-x) = 0$ (*inverse law*)

(v). $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (*associative law*)

(vi). $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = yx + zx$

(*distributive law*)

This implies that

$R1$: $(R, +)$ is an abelian group

$R2$: $(R, \cdot)$ is a monoid

A ring without additive inverse is called a semiring**.** In addition to the above axioms of a ring if;

(i) $x \cdot y = y \cdot x$ for all $x, y \in R$, the ring R is said to be commutative

(ii) there exist an element $1 \in R$ such that $1 \cdot x = x \cdot 1 = x$ for all x, then 1 is called a *unit element* and the ring R is said to be a ring with unit element.

(iii) $R - \{0\}$ is a group under multiplication where R is a ring i.e., if each $x \in R - \{0\}$ has an inverse x^{-1} such that $xx^{-1} = x^{-1}x = 1$, where $1 \in R$ is the unit element, then R is called *division ring*.

Semiring with additional axiom,

(i) $a + a = a \ \forall a \in R$ is called *additively idempotent semiring*.

(ii) $a \cdot a = a \ \forall a \in R$ is called *multiplicatively idempotent semiring*.

(iii) if $a + b = a + c$ then $b = c \ \forall a, b, c \in R$ is *called cancellative semiring*.

(iv). if $a + b = 0$ then $a = b = 0 \ \forall a \in R$ is called *zerosumfree semiring*.

Let R be a ring and let $S \subset R$. We say that S is a subring of R, if with addition and multiplication operations (of R) restricted to S, the set S is itself a ring. In particular, $(S, +)$ is a subgroup of $(R, +)$.

We say that S is a (two–sided) ideal (or simply an ideal) if S is a subgroup of $(R, +)$ and if, in addition, $(\forall x \in \text{S})(\forall \text{y} \in R)\ (x \cdot y,\ y \cdot x \in S)$. Thus every ideal is a obviously a subring. The reverse is however false in general.

(l). Field:

A field is a commutative ring with unit element $1 \neq 0$ such that $(F\backslash\{0\}, \cdot)$ is a group. In other words, a field is simply a commutative division ring. A subset $K \subset F$ is said to be a *subfield* of F if K is a subring of F and $K\backslash\{0\}$ is a subgroup of $(F\backslash\{0\}, \cdot)$

(m). **Vector Space:**

Let F be a field. A nonempty set V whose elements are vectors, is called a vector space over F if V is an Abelian group under an operation $+ : V \times V \to V$, called vector addition and if there is an operation (.) : $F \times V \to V$ called scalar multiplication, written as $(a, v) = av$ for all $a \in$ F and $v \in$ V such that the following axioms hold for all $a,\ b \in F$ and $u, v \in V$

(i) $a(u + v) = au + av$

(ii) $(a + b)u = au + bu$

(iii) $a(bu) = (ab)u$

(iv) $1u = u$

A subset $U \subset V$ is said to be a (vector) *subspace* of V if under the addition and multiplication restricted to U, the set U itself is a vector space.

(n) Dioid:

Dioid is a set D together with two binary operations $\oplus$ and $\otimes$ such that

(i) (D, $\oplus$) is associative, commutative, idempotent with null element ε,

(ii) (D, $\otimes$) is associative with unit element e

(iii) $\otimes$ is distributive over $\oplus$,

(iv) the null element is absorbing i.e. $\forall a \in D\ a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$

A dioid is an idempotent semiring, also called *double monoid*.

Definition 3.11 (*Topology*)

Let X be any set and let τ be a family of subsets of X. Then τ is a topology on X if and only if:

(i) both the empty set and X are elements of τ,

(ii) arbitrary union of elements of τ is an element of τ,

and (iii) intersection of finitely many elements of τ is an element of τ

If τ is a topology on X, then the pair (X, τ) is called a topological space.

Definition 3.12 (*Topological group*)

A topological group is a group G endowed with a topology such that the multiplication and inverse operations of G are continuous. That is, the map;

$(TG1)$ $G \times G \to G$: $(x, y) \to xy$ and

$(TG2)$ $G \to G$: $x \to x^{-1}$

are continuous functions, and the topology on $G \times G$ is the product topology.

Definition 3.13 (*Topological ring*)

A topology τ on a ring R is a ring topology and R, furnished with τ, is a topological ring if the following conditions hold:

$(TR1)$ $(x, y) \to x + y$ is continuous from $R \times R$ to R

$(TR2)$ $x \to -x$ is continuous from $R \times R$ to R

$(TR3)$ $(x, y) \to xy$ is continuous from $R \times R$ to R

In a simple term, a topological ring R is a ring equipped with a topology so that its algebraic operations are continuous.

Remark 3.6

In *chapter five* we show that most of these algebraic structures could be defined under multisets setting.

3.4 Fundamentals of Multisets

The term multiset was first suggested by N. G. de Bruijn in a private communication to him (knuth, 1981). Due to its suitability, it has replaced a variety of terms viz. *list*, *heap*, *bunch* , *bag*, *sample*, *weighted set*, *occurrence set* and *fireset* (finitely repeated element set) used in different contexts but conveying synonimity with mset.

As mentioned earlier, elements are allowed to repeat in an mset, finitely in most of the known application areas, albeit in a theoretical development infinite multiplicities of elements are also dealt with (Eilenberg, 1974; Rado, 1975; Lake, 1976; Hickman, 1980; Monro, 1987 and Blizard, 1993) in particular.

The number of copies or duplicates ((Brink, 1987) prefers to call it '*multiples*') of an element appearing in an mset is called its multiplicity. Moreover, multiple occurrences of an element in an mset are treated without preference (perhaps to retain the force of classical concept of identity). We mention (Parker–Rhodes, 1981) for an earliest extensive treatment of indistinguishability of repeated elements without any preference. Monro (1983) and Wildberger (2003) gave alternative treatment.

The number of distinct elements in an mset A (which need not be finite) and their multiplicities jointly determine its cardinality, denoted by $C(A)$ or $|A|$. In other words the cardinality of an mset is the sum of multiplicities of all its objects. Relatedly, the cardinality of an object $x \in A$ is its multiplicity or characteristic value, denoted by $m_A(x)$ or $C_A(x)$, or simply by $C(A)$ and hence $C(A) = \sum C_A(x)$ for all distinct element $x \in A$.

An mset A is called finite if the number of distinct elements in A and their multiplicities are both finite, it is infinite otherwise. Thus, an mset A is infinite if either the number of elements in A is infinite or the multiplicity of one or more of its objects is infinite i.e., $C(A) \geq \aleph_o$.

The $root$ (or $support$ or $carrier$) of an mset A, denoted by A^* is the set containing all the distinct elements of A. The elements of the root set of an mset are also called the generators of that mset. Hence, by definition, it follows that every mset has a unique root set. The cardinality of the root set A is called the dimensionality of A, denoted by $|A|$. An mset is called *regular* or *constant* if all its elements occur with the same multiplicity. The common multiplicity of elements of a constant mset is called its *height*. Also an mset is called *simple* if all its elements are the same, for example, $[x]_3$ is simple mset containing x as its only object. The root set of a simple mset is a singleton.

A considerable amount of efforts have also gone into the study of msets with negative multiplicities (Whitney, 1933; Eilenberg, 1974; Rado, 1975; Reisig, 1986; Hailperin, 1976; Blizard, 1990 and Wildberger, 2003), in particular.

3.4.1 Representations of multisets

(a) Multiplicative form

Meyer and McRobbie (1982) used square brackets to represent an mset and ever since then it has become almost standard. An mset containing two occurrences of a, three occurrences of b, and one occurrence of c is represented as $[[a, a, b, b, b, c]]$ or $[a, a, b, b, b, c]$ or $[a, b, c]_{2,3,1}$ or $[a, b, c]_{2\,3\,1}$ or $[a^2, b^3, c^1]$ or $[a2, b3, c1]$, depending on one's taste and expediency. Nevertheless, for convenience, the curly

brackets can be used in place of square brackets. Note that, in the above example, a, b and c are the objects and each individual occurrence of these objects is element.

(b) Linear form

Wildberger (2003) puts forward a linear notation for multisets which seems quite innovative, especially when negative multiplicities (integral as well as rational) are to be dealt with, for example, the mset $M = [a, b, c]_{2,3,1}$ can be written as $M = 2[a] + 3[b] + 1[c]$.

Similarly, a rational mset can be represented. For example,

$N = \frac{1}{3}[5] - \frac{1}{2}[1_8]$.

In order to accommodate negative multiplicities, round brackets are used: the symbol (a), in an mset stands for negative of a; for example,

$[2, 4, (5), (5), 4] = [2] + 2[4] - 2[5]$.

(c) Multiset as a sequence

A multiset can also be represented as a sequence in which the multiplicity of an element equals the number of times the element occurs in the sequence, which is exactly Dedekind's '*frequency-number*'. The idea is to construct an mset as a sequence (a function with domain $\mathbb{N}$, the set of natural numbers) and ignore the ordering of its elements, which can be done by taking all permutations of the domain of the sequence.

(d) Multiset as a family of sets

A multiset can also be represented as a family of sets, which is altogether a generalization of the idea of a sequence described above. Thus, the family of sets $F = \{Fi\}$, $i \in I$, where $Fi = Fj$, if $i = j$, which identifies a repeated element, represents an mset. Clearly, the family F is a function: $I \to \{Fi \mid i \in I\}$, which in turn, is a sequence if $I = \mathbb{N}$.

(e) Multiset as a numeric-valued function

Representation of an mset as a numeric–valued or cardinal–valued function abounds, especially in the application areas. Formally, an mset is just a mapping from some ground or generic or universal set into some set of numbers. For example, an mset $\alpha = [x, y, z]_{1,2,3}$ is a mapping from a ground set S to $\mathbb{N}$, the set of natural numbers including zero, defined by

$$\alpha(t) = \begin{cases} 1, & \text{if } t = x \\ 2, & \text{if } t = y \\ 3, & \text{if } t = z \\ 0, & \text{for all the remaining } t \in S. \end{cases}$$

In general terms, for a given ground set D and a numeric set T, we call a mapping α: $D \to T$,

a set, if $T = \{0, 1\}$;

a multiset, if $T = \aleph$, the set of natural numbers including zero;

a signed multiset (or, hybrid/shadow set) if $T = Z$, the set of integers;

a fuzzy (or hazy) set if $T = [0,1] \subseteq R$, a two – valued Boolean algebra.

Accordingly, an mset A can also be represented by the set of pairs as follows:

$$A = \{\langle m_A(x_1), x_1\rangle, \ldots, \langle m_A(x_j), x_j\rangle, \ldots\}$$

or, $A = \{m_A(x_1) . x_1, \ldots, m_A(x_j). x_j, \ldots\}$

or $A = \{{}^{n_1}/_{x_1}, \ldots, {}^{n_j}/_{x_j}, \ldots\}$, where $m_A(x_j) = n_j$.

In other words, an mset A over a given ground set D can be represented by a '*function count A or multiplicity function A*', denoted by m_A: $D \longrightarrow \mathbb{N} = \{0, 1, 2, \ldots\}$, such that $m_A(x) = 0$ if $\notin A$, and $m_A(x) > 0$ if $x \in A$ for all $x \in D$. It follows that an mset A is a set if $m_A(x) = 0$ or 1 for all $\in D$. An mset is called empty, denoted by $\emptyset$ or [ ] if $m_A(x) = 0$ for all $\in D$.

(f) Multiset as a generalized characteristic function

Similar to the representation of a set by its characteristic function (function whose range is {0, 1}), a multiset or hybrid set is determined by its generalized characteristic function (whose range is the set of integers, positive, negative or zero) (Whitney 1933).

3.4.2 Operations under multisets

Despite the works of (Lake, 1976; Hailperin, 1976 and Hickman, 1980), one can considered (Knuth, 1981) as the earliest reference describing intuitively the properties of msets in a sufficient detail. But in recent years, a good number of papers (Blizard, 1989; Wi1dberger, 2003; Singh *et. al*, 2007 and 2008; Girish and John, 2009) and many others have appeared. We endeavour to present an overview.

Definition 3.14 (*Equal msets*)

Two msets A and B are equal or the same, written as $A = B$, iff for any object $x \in D$, $m_A(x) = m_B(x)$ or $A(x) = B(x)$. Equivalently, $A = B$ if every element of A is in B and conversely. Clearly, $A = B \Rightarrow A^* = B^*$, however the converse need not hold.

Definition 3.15 (*Dressed epsilon symbol*, $\in_+$)

For any object x occurring as an element of an mset A i.e., $m_A(x) > 0$, we write $x \in_+ A$, where $\in_+$, *dressed epsilon*, is a binary predicate intended to be '*belongs at least once*', as $\in$ is '*belongs to only once*' in the case of sets. Thus, $m_A(x) = 0$ implies $x \notin A$, and $x \in_+^k A$ implies 'x *belongs to A at least k times*', however $x \in^k A$ means 'x *belongs k times to A*'. Analogously, $\in_-$, standing for '*belongs at most once*' can be introduced. In this case, $x \in_-^k$ A implies 'x *belongs to A at most k times*'.

Definition 3.16 (*Multisubsets or submultisets* (*msubsets or submset for short*))

Let A and B be two msets, A is an msubset or a submultiset of B, written as $A \subseteq B$ or $B \supseteq A$, if $m_A(x) \leq m_B(x)$ for all $x \in D$. Also, if $A \subseteq B$ and $\neq B$, then A is called a proper submset of B, written as $A \subset B$. In other words, $A \subset B$ if $m_A(x)) \leq m_B(x)$ for all $x \in D$ and there exists at least one $x \in D$ such that $m_A(x) < m_B(x)$. An mset is called the parent multiset or overmultiset in relation to its msubsets.

It is easy to see that $\subseteq$ is antisymmetric i.e., $A \subseteq B$ and $\subseteq A \Rightarrow A = B$, and it is a partial ordering on the class of msets defined on a given generic domain. Clearly, $\emptyset$ is a submset of every mset.

We wish to emphasize that introduction of $\in_+$ greatly enhance the language of msets. For example, $A \subseteq B$ stands for $\forall_Z \forall k\ (z \in^k A \rightarrow z \in_+^k B)$. Also, the use of $\in_-$ can be exploited.

A *whole* msubset of a given mset contains all multiplicities of common elements; while a *full* msubset contains all objects of the parent mset and accordingly, every mset contains a unique full msubset, called its root set. Clearly, for any two msets A and B, if $A \subseteq B$ and Dom (A) = Dom (B), then A is a full msubset of B.

Definition 3.17 (*Similar msets*)

Two msets A and B are said to be *cognate* or *similar*, written as $A \simeq B$ if $\forall x(x \in A \Leftrightarrow x \in B)$, where x is an object. Thus, similar msets have equal root sets but need not be equal themselves. In (Thistlewaite *et al*., 1988), the notion of *strong* multiset containment is introduced as A is strongly contained in B iff $A \subseteq B$ and $\simeq B$.

Definition 3.18 (*Ordered pair of two mset terms*)

Let u and v be two mset terms, the ordered pair of u and v, denoted by $[u, v]$, is defined as follows:

$$[u, v] = \{u, v\} \text{ if } u \neq v, \text{ and } [u, v] = \{[u]_2\} \text{ if } u = v.$$

Here, $[u]$ is written as $\{u\}$, and $\langle u, v\rangle$ is actually the ordered pair set, where set (u) stands for $u = \emptyset \vee \forall x\, \forall n (x \in^n u \Longrightarrow n = 1)$, though x itself may be an mset term (Blizard, 1989).

Definition 3.19 (*Power multiset*)

In Cantorian spirit, the power multiset of a given mset A, denoted by $\widetilde{\wp}(A)$ to distinguish it from the symbol $\wp(A)$ used for power set of A, is the multiset of all submultisets of A. For example, let $A = [x, y]_{2,1} = [x, x, y]$. Then,

$$\widetilde{\wp}(A) = [\emptyset, \{x\}, \{x\}, [x]_2, \{y\}, \{x, y\}, \{x, y\}, [x, y]_{2,1}].$$

In this case, $C(\widetilde{\wp}(A)) = 2^{C(A)}$, for any mset A.

The following combinatorial formula for determining the number of times an element occurs in the power multiset of a given mset is as follows: Let X be an mset. An msubset Y of X appears k times in $\wp(X)$ where $k = \prod_{z \in Y^*} \binom{|X_z|}{|Y_z|}$.

However, as has been voiced by many researchers, (Hickman, 1980) and (Blizard, 1989) in particular, in the area of msets and their applications, that there is no *good* reason for admitting repeated elements into a power multiset. Accordingly, in multiset theory, the axiom of exact multiplicity is admitted that is, multiplicity with which an element belongs to an mset is unique. Hence, a power multiset may be called a

power set only and denoted by $\wp(A)$. For example if $A = [x, y]_{2,1}$ then $\wp(A) = \{\emptyset, \{x\}, [x]_2, \{y\}, \{x, y\}, [x, y]_{2,1}\}$ and hence $C(\wp(A)) < 2^{C(A)}$ which implies that Cantor's power set theorem: $C(A) < C(\wp(A))$ fails. However, for finite msets, Cantor's theorem holds for power mset.

Definition 3.20 (*Union*)

Let A and B be two msets over a given domain set D. The union of A and B denoted by, $A \cup B$, is defined as $A \cup B = max\{m_A(x), m_B(x)\}$, for all $x \in D$. That is, an object z occurring n_1 times in A and n_2 times in B, occurs maximum $\{n_1, n_2\}$ times in $A \cup B$, if such a maximum exists otherwise the minimum of (n_1, n_2) is taken which always exists. In other words, $A \cup B$ is the smallest mset which contains both A and B. It follows that for any given mset X there exists an mset Y which contains elements of elements of X, where the multiplicity of an element z in Y is the maximum multiplicity of z as an element of elements of X along with the above stipulation on the existence of such a maximum. We denote this fact by $Y = \cup X$

Clearly, $Dom(\cup X) = \cup\{Dom(A), A \in X\}$ and that the multipliticity of z in Y is the maximum of its multiplicities as an element of elements of X, if it exists, otherwise, the minimum is taken.

For example, if $A = [2\ 3\ 4\ 4]$, $B = [1\ 4\ 3\ 3]$ then $A \cup B = [1\ 2\ 3\ 3\ 4\ 4]$.

Also, it follows that for a finite mset X, the maximum multiplicity of elements of elements of X always exists. However, for certain infinite sets like $X = \{\{y\}, [y]_2, [y]_3 \ldots\}$, the maximum multiplicity of elements of elements of X does not exist, and hence $\cup X = \{y\}$. It is obvious by definition of the union that multiplicity of any $y \in X \neq \emptyset$ is irrelevant to $\cup X$, and hence $\cup X = \cup X^*$ (Blizard, 1989).

Note that if $A \cup C = B \cup C$ or $A \cap C = B \cap C$, we cannot conclude that $A = B$. For example, we consider the multisets $A = [1,1, 2, 2, 3]$, $B = [1, 1, 3]$ and $C = [1,1, 2, 2]$ $A \cup B = B \cup C = [1,1, 2, 2, 3]$ but $A \neq B$.

Definition 3.21 (*Intersection*)

Let A and B be two msets over a given domain set D. The intersection of two msets A and B denoted by $A \cap B$, is the mset defined by $A \cap B = \min\{m_A(x), m_B(x)$ for all $x \in D$. In other words, $A \cap B$ is the smallest mset which is contained in both A and B. That is, an object x occurring n_1 times in A and n_2 times in B, occurs minimum (n_1, n_2) times in $A \cap B$, which always exists.

In general, for a given mset X, $\mathrm{Dom}(\cap X) = \cap\{\mathrm{Dom}(A): A \in X\}$ and $z \in \cap X$ implies that the multiplicity of z is the minimum of its multiplicities as element of elements of X.

For example, if $A = [3\ 3\ 3\ 4\ 4]$, $B = [1\ 4\ 3\ 3]$, then $A \cap B = [3\ 3\ 4]$.

Note that for any mset X, we have $\cap X \subseteq \cup X$.

Definition 3.22 (*addition or sum or merge*)

Let A and B be two msets over a given domain set D. The direct sum or arithmetic addition of A and B denoted by $A + B$ or $A \uplus B$ is the mset defined by $A \uplus B = m_{A+B}(x) = m_A(x) + m_B(x)$, for all $x \in D$.

That is, an object x occurring n_1 times in A and n_2 times in B, occurs $n_1 + n_2$ times in $A \uplus B$.

For example if A = [1 1 2 2 3 4 4], B = [2 3 3] then A⊎ B = [1 1 2 2 2 3 3 34 4].

Clearly, $|A \uplus B| = |A \cup B| + |A \cap B|$.

It follows that the insertion of an element x into an mset A gives rise to a new mset $A' = A + x$ such that $m_{A'}(x) = m_A(x) + 1$ and $m_{A'}(y) = m_A(y)$ for all $\neq x$. Also, if x is an infinite mset, then the multiplicity of some mset $z \in \uplus x$ may not be finite. In that case, the multiplicity of z in $\cup x$ is used.

For example, if $x = \{\{z\}, [z]_2, [z]_3 \ldots\}$ then $\uplus x = \cup x = \{z\}$

Definition 3.23 (*Difference*)

Let A and B be two msets over D and $B \subseteq A$, then the difference of B from A, otherwise known as the relative complement of B in A, denoted by $A - B$, is the mset such that $m_{A-B}(x) = m_A(x) - m_B(x)$ for all $x \in D$.

In general, $m_{A-B}(x) = m_A(x) - m_{A\cap B}(x)$, for all $x \in D$.

$$= max\{m_A(x) - m_B(x), 0\}$$

It is also sometimes called the arithmetic difference of B from A. If $B \nsubseteq A$, this definition still hold good. It follows that the deletion of an element x from an mset A gives rise to a new mset $A' = A - x$ such that $m_{A'}(x) = max\{m_A(x) - 1, 0\}$.

Similarly, the symmetric difference, $A \Delta B$, is defined by

$$m_{A\Delta B}(x) = m_{A-B}(x) \cup m_{B-A}(x) = |m_A(x) - m_B(x)|$$

Remark 3.7

The properties of difference in sets fail in multisets. For example, given multisets $A = [a, b]_{4,5}$ and $B = [a, b]_{2,3}$ then $A - B = [a, b]_{2,2} \subseteq B$, contradicting the classical law: $(A - B) \cap B = \emptyset$. Also, in classical set theory, given any sets A, B and C, we have $A - (B \cup C) = (A - B) - C$. But in general, $A - (B \cup C) \neq (A - B) - C$, if A, B and C are multisets. For example, let $A = [y]_6$, $B = [y]_4$ and $C = [y]_2$.
$A - (B \cup C) = [y]_2$ and $(A - B) - C = \emptyset$. Hence, $A - (B \cup C) \neq (A - B) - C$.

Definition 3.24 (*Complement*)

In order to define complement in multiset we follow (Petrovsky, 1997).
Let $F = \{A_1, A_2, \ldots\}$ be a family of multisets composed of the elements of the generic set D. Then, the maximum multiset Z is defined by $m_Z(x) = max_{A \in F}\, m_A(x)$ for all $x \in D$ and for all $A \in F$.

Now, the complement of an mset A, denoted by $\overline{A}$, is defined as fallows:

$$\bar{A} = Z - A = \{m_{\bar{A}}(x)\,.\,x | m_{\bar{A}}(x) = m_Z(x) - m_A(x), \text{ for all } x \in D\}.$$

3.4.3 Some algebraic properties of mset operations

In this section, we present some algebraic properties of multiset operations following (knuth, 1981; Blizard, 1989 and Singh *et al.*, 2007, 2008). Let A, B and C be any multisets then we have the following:

(a) *Commutativity*: (i) $A \uplus B = B \uplus A$ (ii) $A \cup B = B \cup A$

(iii) $A \cap B = B \cap A$

(b) *Associativity*: (i) $A \uplus (B \uplus C) = (A \uplus B) \uplus C$

(ii) $A \cup (B \cup C) = (A \cup B) \cup C$

(iii) $A \cap (B \cap C) = (A \cap B) \cap C$

(c) *Idempotency*: (i) $A \cup A = A$ (ii) $A \cap A = A$ (iii) $A \uplus A = 2A \neq A$

In fact, Wildberger (2003) and Petrovsky (1997) suggested that in order to obtain a linear combination of msets, kA may be interpreted to designate the sum of k number of A's, where k is a natural number.

(d) *Identity laws*: (i) $A \cup \emptyset = A$ (ii) $A \cap \emptyset = \emptyset$ (iii) $A \uplus \emptyset = A$

(e) *Distributivity*: (i) $A \uplus (B \cup C) = (A \uplus B) \cup (A \uplus C)$

(ii) $A \uplus (B \cap C) = (A \uplus B) \cap (A \uplus C)$

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Remark 3.8

The proof of all these identities follows from the interpretation of ∪, ∩ and ⊎ of two natural numbers as maximum, minimum and (direct) sum respectively. It is observed that the addition or sum operation (⊎) is stronger than both union operation (∪) and intersection operation (∩) in the sense that neither ∪ nor ∩ distributes over ⊎, whereas ⊎ distributes over both ∪ and ∩. Also, $\cap x \subseteq \cup x \subseteq \uplus x$. These points are justified in *theorems 3.3* and *3.4*.

3.5 Functions on Multisets

The underlying assumption in defining a function between msets has been invariably not to allow mapping of identical elements to non–identical elements and hence, it amounts to defining the function between their root sets, which is just the classical definition of a function.

The function $f: A \longrightarrow B$ is an injection iff

(i) $f: A^* \longrightarrow B^*$ is an injection, and

(ii) $\forall z\,(z \in A^* \Rightarrow m_A(z) \leq m_B(f(z)))$

The function $f: A \longrightarrow B$ is a surjection iff

(i) $f: A^* \longrightarrow B^*$ is a surjection, and

(ii) $\forall z\,(z \in A^* \Rightarrow m_A(z) \geq m_B(f(z)))$

The function $f: A \longrightarrow B$ is a bijection iff

(i) $f: A^* \longrightarrow B^*$ is a bijection and

(ii) $\forall z\,(z \in A^* \Rightarrow m_A(z) = m_B(f(z)))$

For example, $f: [x]_3 \longrightarrow [x]_{10}$ is an injection, $f: [x]_5 \rightarrow [y]_4$ is a surjection and $f: [x, y]_{6,2} \longrightarrow [x, y]_{3,4}$ is neither an injection nor a surjection.

One can find some other various details in (Blizard, 1989) and (Hickman, 1980).

Remark 3.9

The consequences of the definitions of functions in multisets give rise to some conflicting results with some fundamental theorems of the classical set theory. For example, the Cantor's power set theorem does not hold, there is no injection from multiset A into power multiset of A, $(\wp(A))$. That is, there is no function f such that $f: A \rightarrow \wp(A)$, is injection (Hickman, 1980). More so, msets of equal cardinality need not have a bijection between them. For example, $[a, b]_{2,2}$ and $[a, b, c, c]$ both contain four elements, but there can be no bijection between them because the objects and their multiplicities are different in the domain and the range of any such function. In other words, there is no bijection between their root sets viz; $:\{a, b\} \rightarrow \{a, b, c\}$ cannot be a bijection. Schröder–Bernstein theorem also fails in multisets (Hickman, 1980) and (Blizard, 1989). For example, let $A = [x_1, x_2, \ldots]_{2,4,6, \ldots}$ and $B = [y_0, y_1, y_2, \ldots]_{1,3,5, \ldots}$ be two multisets, then the function $f: A^* \rightarrow B^*$ defined as $f(x_n) = y_n$ makes $f: A \rightarrow B$ an injection so that $A \leq B$. The function $g: B^* \rightarrow A^*$ defined by $g(y_n) = x_{n+1}$ makes $g: B \rightarrow A$ an injection so that $B \leq A$. But there cannot be a bijection $h: A \rightarrow B$ since all multiplicities in A are even and that in B are odd.

3.6 Some General Algebraic Results on Multisets

Given any multisets A and B the following identities hold:

(i) $A = (A - B) \uplus (A \cap B), B = (B - A) \uplus (A \cap B), (A - B) \cap (B \cap A) = \emptyset$

(ii) $A \cup B = (A \uplus B) - (A \cap B) = (A \cap B) \uplus (A \triangle B)$

$= A \uplus (B - A) = B \uplus (A - B)$

(iii) $A \cap B = (A \uplus B) - (A \cup B) = (A \cup B) - (A \triangle B)$

$= A - (A - B) = B - (B - A)$

(iv) $A - B = A - (A \cap B) = A \Delta (A \cap B) = (A \cap B) - B = (A \cup B) \triangle B$

(v) $A \triangle B = (A \cup B) - (A \cap B) = (A \uplus B) - 2(A \cap B) = (A - B) \uplus (B - A)$

(vii) $A \uplus B = (A \cup B) \uplus (A \cap B) = (A \triangle B) \uplus 2(A \cap B)$

Theorem 3.1 (*partial order relation of* $\subset$)

Given any set M of multisets then $\subset$ is a partial order relation on M.

Proof

The theorem requires to show that the strict containment ($\subset$) is irreflexive and transitive. For the irreflexive, it is obvious since for any $A, B \in M$, $A \subset B$ does not implies that $B \subset A$. For transitivity, suppose $A \subset B$ and $B \subset C$, then we have for every object $x \in A$, $m_A(x) \leq m_B(x)$ and for every $x \in B$, $m_B(x) \leq m_C(x)$. Since $\subset$ is a strict containment, there exist $x \in A$, such that $m_A(x) < m_B(x)$. Also, there exist

$x \in B$, such that $m_B(x) < m_C(x)$ which implies that $m_A(x) < m_C(x)$. That is, $A \subset C$ which shows that $\subset$ is transitive.

Theorem 3.2 (properties of $\subseteq$)

Let A, B and C be multisets.

(i) If $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$ and $A \cap B \subseteq C$.

(ii) If $C \subseteq A$ and $C \subseteq B$ then $C \subseteq A \cap B$ and $C \subseteq A \cup B$.

(iii) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Proof

(i) If $A \subseteq C$ and $B \subseteq C$, then $m_A(x) \leq m_C(x)$ and $m_B(x) \leq m_C(x)$ for all x. This implies that $max\ (m_A(x), m_B(x)) \leq m_C(x)$. That is $A \cup B \subseteq C$.

Since $m_{A\cap B}(x) \leq m_{A\cup B}(x)$ for all x, it follows that $A \cap B \subseteq C$.

(ii) If $C \subseteq A$ and $C \subseteq B$, then $m_C(x) \leq m_A(x)$ and $m_C(x) \leq m_B(x)$. This implies that $m_C(x) \leq min\ (m_A(x), m_B(x))$ which implies that $C \subseteq A \cap B$. Also, Since $m_{A\cap B}(x) \leq m_{A\cup B}(x)$ for all x, it follows that $C \subseteq A \cup B$.

(iii) The proof follows from *theorem 3.1*.

Theorem 3.3 (*adequate set of operations*)

The set $\{\uplus, -\}$ of operations are adequate to define the operations $\cup$ and $\cap$ in multisets.

Proof

Given any multisets A and B we have:

$$A \cup B = (A - B) \uplus B$$
$$= (A \uplus B) - (A \cap B)$$
$$= (A \uplus B) - (A - (A - B))$$

Also, $$A \cap B = (A \uplus B) - (A \cup B)$$
$$= (A \uplus B) - ((A - B) \uplus B)$$

Therefore, $\cup$ and $\cap$ are completely defined by the set of operations $\uplus$ and $-$.

Theorem 3.4 (*independence of the operation* $\uplus$)

Given any multisets A and B, the set of operations $\{\cup, \cap, -\}$ is not adequate to define $A \uplus B$ or $B \uplus A$.

Proof

Let E be any well–formed multiset expression constructed from the set $\{A, B, \cup, \cap, -\}$ where A and B are multisets, we first of all claim that $E \subseteq A \cup B$. We claim this by induction on the number of $\cup$, $\cap$ and $-$ occurring in E. Suppose E has none of the operations, $\cup$, $\cap$ and $-$. Then either $E = A$ or $E = B$ which implies that $E \subseteq A \cup B$ from the *theorem 3.2* (ii).

Now let E be an expression having exactly $n > 0$ operators, and suppose that the claim is true for an expression having fewer than n. Then E has one of the following forms:

$$E = E_1 \cup E_2$$

$$E = E_1 \cap E_2$$

$$E = E_1 - E_2$$

$$E = E_2 - E_1$$

Since the number of operators in either E_1 or E_2 is less than n, so, by the induction hypothesis, $E_1 \subseteq A \cup B$ and $E_2 \subseteq A \cup B$. Again, it follows from the *theorem 3.2* (i), $E_1 \cup E_2 \subseteq A \cup B$, $E_1 \cap E_2 \subseteq A \cup B$ and $E_1 - E_2 \subseteq A \cup B$ or $E_2 \cup E_1 \subseteq A \cup B$.

Finally, suppose there exist an expression, E, constructed from the set $\{A, B, \cup, \cap, -\}$ where A and B are multisets that is equivalent to $A \uplus B$ or $B \uplus A$. Then $E \subseteq A \cup B$ and by the claim $A \uplus B \subseteq A \cup B$ which is a contradiction since for any multisets A and B where $A \cap B \neq \emptyset$, $A \uplus B \not\subseteq A \cup B$. Hence, the theorem follows.

Theorem 3.5 (*independence of the operation* $-$)

Given any multisets A and B, the set of operations $\{\cup, \cap, \uplus\}$ is not adequate to define $A - B$ or $B - A$.

Proof

The proof is similar to the proof of the theorem 3.4. Let E be any well–formed multiset expression constructed from the set $\{A, B, \cup, \cap, \uplus\}$ where A and B are multisets, we first of all claim that $A \cap B \subseteq E$. We claim this by induction on the number of $\cup$, $\cap$ and $-$ occurring in E. Suppose E has none of the operations, $\cup$, $\cap$ and $\uplus$. Then either $E = A$ or $E = B$ which implies that $A \cap B \subseteq E$ from the *theorem 3.2* (i).

Now let E be an expression having exactly $n > 0$ operators, and suppose that the claim is true for an expression having fewer than n. Then E has one of the following forms:

$$E = E_1 \cup E_2$$

$$E = E_1 \cap E_2$$

$$E = E_1 \uplus E_2$$

Since the number of operators in either E_1 or E_2 is less than n, so, by the induction hypothesis, $A \cap B \subseteq E_1$ and $A \cap B \subseteq E_2$. Again, it follows from the *theorem 3.2* (i), $A \cap B \subseteq E_1 \cup E_2$, $A \cap B \subseteq E_1 \cap E_2$ and $A \cap B \subseteq E_1 \uplus E_2$.

Finally, suppose there exist an expression, E, constructed from the set $\{A, B, \cup, \cap, \uplus\}$ where A and B are multisets that is equivalent to $A - B$ or $B - A$. Then $E = A - B$ or $E = B - A$ and by the claim $A \cap B \subseteq A - B$ or $A \cap B \subseteq B - A$. Now if $A = [x]$ and $B = [x]$ then $[x] = A \cap B \subseteq A - B = \emptyset$ and $[x] = A \cap B \subseteq B - A = \emptyset$ which is a contradiction hence the theorem.

To emphasize the inadequacy of difference operator in generating a proper complement operator, we give an example of a property of difference in set theoretic environment and show that such may not hold in multiset environment.

Given any set A, B and C then we have;

$$\begin{aligned} A - (B \cup C) &= A \cap \overline{(B \cup C)} \\ &= A \cap (\bar{B} \cap \bar{C}) \\ &= (A \cap \bar{B}) \cap \bar{C} = (A - B) - C \end{aligned}$$

Now, let $A = [x, x, y, y, z]$, $B = [y, y, z]$ and $C = [y, z]$ be multisets, then $A - (B \cup C) = [x, x]$ and $(A - B) - C = \emptyset$.

Remark 3.10

Multiset difference operator is similar to set difference operator. However, given any multiset as a universe, this difference operator does not induce a complement operator relative to this universe. Hence we cannot define Boolean algebra on multiset since the properties enjoyed by complement are a subset of the axioms of Boolean algebra. However, we can define distributive lattice on multiset. We present, in *chapter five*, *theorems 5.5* and *5.6* to ascertain these points.

Theorem 3.6 (*Inclusion and exclusion principles*)

Let $A_1, A_2, \ldots, A_n$ be multisets.

Then $A_1 \cup A_2 \cup \ldots A_n = \sum_{1 \leq i \leq n} A_i - A_{i_1} \cap A_{i_2} + \ldots + (-1)^{n+1} A_1 \cap A_2 \cap \ldots \cap A_n$.

Proof

See (Wildberger, 2003)

3.7 Alternative way of proving Multisets Identities

As a forerunner to the study of max–plus algebra in *chapter four*, we present an alternative way of proving multisets identities.

Let x, y, z, be any multisets. We have the following *tropical* equations:

Let $x + y = z\,(T) \quad \Leftrightarrow \quad x \cup y = z$ 3.1

In general, $\sum_i^n x_i = z\,(T) \quad \Leftrightarrow \quad \cup_i^n x_i = z$

Let $xy = z\,(T) \quad \Leftrightarrow \quad x + y = z$ 3.2

In general, $\prod_i^n x_i = z(T) \quad \Leftrightarrow \quad \sum_i^n x_i = z$

Let $y = x^n\ (T) \quad \Leftrightarrow \quad y = nx$ 3.3

The transformation 3.3 gives room for what we called multiplicative inverse of any multiset x, denoted by $x^{-1}\,(T)$, hence the idea of division is indirectly introduced in multiset theory by the rule that

$$x/y = xy^{-1}(T) = x(T)y^{-1}\,(T) = x - y \qquad 3.4$$

In particular, $x/y = z\,(T) \quad \Leftrightarrow \quad x - y = z$

The tropical addition enjoys the idempotent property. That is, $x + x = x\,(T)$ since $x \cup x = x$. Also, identity property is feasible, that is, $\emptyset + x = x\,(T)$ if and only if the multiplicity of x is positive since $\emptyset \cup x = x$ if and only if x is maximum. If the multiplicity of x is negative then $\emptyset + x = \emptyset\,(T)$.

Finally, Addition and multiplication are connected by the distributive law $x(y + z) = xy + xz\,(T)$. 3.5

This concept is similar to max–plus algebra hence the essence of *chapter four*. We now present a theorem that will enable us to use these innovative ideas to prove some properties of multisets operations.

Theorem 3.7

For any multisets x, y and z, $\quad z = \frac{xy}{x+y}(T) \quad \Leftrightarrow \quad z = x \cap y$

Proof

$z = \frac{xy}{x+y}(T) = \frac{xy(T)}{(x+y)(T)} = xy(T)(x+y)^{-1}(T) = (x+y) - (x \cup y) = (x \cap y)$,

Hence, $z = \frac{xy}{x+y}(T) \quad \Leftrightarrow \quad z = (x \cap y)$, for any multisets x, y and z.

Example 3.1

Show that for any multisets x, y and z

$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$$

Solution

We proceed as follows:

The corresponding tropical equation is $x + \frac{yz}{y+z} = \frac{(x+y)(x+z)}{(x+y)+(x+z)}(T)$

$$\frac{x(y+z)+yz}{y+z} = \frac{x^2+xz+yx+yz}{x+y+x+z}(T)$$

$$\frac{xy+xz+yz}{y+z} = \frac{x^2+xz+yx+yz}{x+y+z}(T)$$

$$(xy + xz + yz)(x + y + z) = (y + z)(x^2 + xz + yx + yz)(T)$$

$$x^2y + x^2z + xyz + xy^2 + xyz + y^2z + xyz + xz^2 + yz^2$$

$$= x^2y + x^2z + xyz + xy^2 + xyz + y^2z + xz^2 + yz^2(T)$$

$$x^2y + x^2z + xyz + xy^2 + y^2z + xz^2 + yz^2$$

$$= x^2y + x^2z + xy^2 + xyz + y^2z + xz^2 + yz^2(T)$$

(Idempotent law)

LHS = RHS, hence the proof.

We can similarly prove many other properties of multisets operations.

Theorem 3.8

Let a, b and c be any multisets, then $a \cap b + c \cap (b - a \cap b) \leq (a + c) \cap b$

Proof

Let $t = a \cap b + c \cap (b - a \cap b) - (a + c) \cap b$, using our tropical equations concept, we have

$$t = \frac{ab}{a+b} \cdot \frac{cb(\frac{ab}{a+b})^{-1}}{c+b(\frac{ab}{a+b})^{-1}} \cdot \left(\frac{acb}{ac+b}\right)^{-1}(T)$$

$$= \frac{ab}{a+b} \cdot \frac{cb.\frac{a+b}{ab}}{c+b(\frac{a+b}{ab})} \cdot \frac{ac+b}{acb}(T)$$

$$= \frac{ac+b}{ac+a+b}(T)$$

$$t + 0 = \frac{ac+b}{ac+a+b} + 0 = \frac{ac+b}{ac+a+b} + \frac{ac+a+b}{ac+a+b}(T)$$

$$= \frac{(ac+b)+(ac+a+b)}{ac+a+b}(T)$$

$$= \frac{ac+a+b}{ac+a+b}(T) = 0, \text{ using the idempotent law}$$

That is $t \leq 0$, hence the proof.

Example 3.2

Let $A = [1\ 1\ 2\ 2\ 2\ 3\ 4]$, $B = [1\ 1\ 2\ 2\ 3\ 4\ 5]$ and $C = [2\ 2\ 2\ 3\ 3\ 5\ 5]$

$A \cap B = [1\ 1\ 2\ 2\ 3\ 4]$, $B - A \cap B = [5]$, $C \cap (B - A \cap B) = [5]$

$$A \cap B + C \cap (B - A \cap B) = [1\ 1\ 2\ 2\ 3\ 4\ 5] \quad 3.6$$

$$A + C = [1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 4\ 5\ 5], \quad 3.7$$

$$(A + C) \cap B = [1\ 1\ 2\ 2\ 3\ 4\ 5] \quad 3.8$$

From 3.6, 3.7 and 3.8, for any given multisets A, B and C

$$A \cap B + C \cap (B - A \cap B) = (A + C) \cap B$$

Theorem 3.9 (*The Pigeonhole principle*)

Pigeonhole principle for multisets state that suppose that A_i $(i = 1, 2, 3, \ldots k)$ are submultisets of a multiset X satisfying $A_i \cap A_j = \emptyset$ where $i \neq j$ for all i and j. Then $\sum_{i=1}^{k} A_i \subseteq X$.

Proof

Let $x \in X$, if $x \in A_i$ then $x \notin A_j$ for $i \neq j$ since $A_i \cap A_j = \emptyset$. We conclude that $A_1 \uplus A_2 \uplus \ldots \uplus A_k \subseteq X$. That is $\sum_{i=1}^{k} A_i \subseteq X$.

Theorem 3.10 (*Hall theorem*)

Let A be set and $A_1, A_2, \ldots, A_r$ be any r subsets of A. Then there exist a set of distinct representatives $a_1, a_2, \ldots, a_r$ of $A_1, A_2, \ldots, A_r$ such that $a_1 \in A_i$ and $a_i \neq a_j$ for all $i \neq j$, if and only if for all k, $1 \leq k \leq r$, the union of any k of the sets $A_1, A_2, \ldots, A_r$ has at least k elements.

Proof

See (Hall, 1935).

Hall theorem, also known as the marriage theorem, is usually credited to mathematician Philip Hall. It gives the condition allowing the selection of a distinct element from each of a collection of subsets.

Example 3.3

A typical internet order, O, involves the total number n of items ordered and the multiset A of items requested for with $n = |A|$. This order can be filled from the multiset X of inventory of items in the store if and only if $A \subseteq X$, that is, $|A| \leq |X|$.

Given a catalogue of orders $B = [O_1, O_2, \ldots, O_k]$ with $O_i = [n_i, A_i]$, then $L = [F_1, F_2, \ldots, F_k]$ of submultisets of X must be found such that $F_i = A_i$ for $i = 1, 2, \ldots, k$ and $F_i \cap F_j = \emptyset$, for all $i \neq j$. We emphasized that L can be found if and only if $|\sum_{i=1}^{k} A_i| \leq |X|$ and this is an ideal situation.

A large catalogue of orders at a particular time may not satisfy this condition. To avoid this problem, suppose discount is given to orders that are flexible. That is, $O = [n, A]$ such that $n \leq |A|$ instead of strict condition $n = |A|$. The question is how can these orders be filled? This is an inventory version of Hall theorem.

Theorem 3.11

A catalogue $B = [o_1, o_2, \ldots, o_k]$ of orders $o_i = [n_i, A_i]$ from a multiset X can be filled if and only if for any submultisets $B' = [O_{i_1}, O_{i_2}, \ldots, O_{i_m}]$ of B, $|B'| \leq |A_{B'} \cap X|$.

Proof

Suppose we define a filling of a flexible order $= [n, A]$ to be a submultisets $F \subseteq X$ with properties that $F \leq A$ and $|F| = n$. Suppose $B = [o_1, o_2, \ldots, o_k]$ is a catalogue of flexible orders, we define its size to be $|B| = n_1 + n_2 + \ldots + n_k$ and its range to be $A_B = A_1 + A_2 + \ldots + A_k$.

A filling of B is the submultisets $L = [F_1, F_2, \ldots, F_k]$ of X such that F_i is a filling of order $O_{i,}$ for $i = 1, 2, \ldots, k$ and $i \neq j, F_i \cap F_j = \emptyset$. The Pigeonhole principle implies that $F_1 + F_2 + \ldots + F_k \subseteq X$, hence the theorem.

Conversely, we use induction method.

Let $n = |B|$, for $n = 1$, it is clear that B can be filled. Suppose $B = [o_1, o_2, \ldots, o_k]$ is k orders with $O_i = [n_i, A_i]$ for $i = 1, 2, \ldots, k$, from a multiset X, that satisfies the

condition that for any proper submultisets $B' = [O_{i_1}, O_{i_2}, \ldots, O_{i_m}]$ of B, $|B'| \leq |A_{B'} \cap X|$.

Now either $|B'| = |A_{B'} \cap X|$ or $|B'| < |A_{B'} \cap X|$.

Case one.

Suppose $|B'| = |A_{B'} \cap X|$.

Let $B_o = [o_1, o_2, \ldots, o_m]$ such that $A_o = A_1 + A_2 + \ldots + A_m$ and $n_o = n_1 + n_2 + \ldots + n_m$ for some $m < k$.

Since $|B'| = n_o < n$, by induction there is a filling of B_o namely $F_o = [F_1 + F_2 + \ldots + F_m]$ of submultisets of X such that F_i is a filling of order O_i for $i = 1, 2, \ldots, m$ and such that $F_i \cap F_j = \emptyset$ for all i, j from 1 to m and $i \neq j$.

It follows that $F_o \subseteq X$ and $F_o \leq A_o \cap X$

Since we are considering case one $F_o = A_o \cap X$, i.e. $|F_o| = |A_o \cap X|$ where $|F_o| = n_o$ (3.9)

Let $X' = X - F_o = X - A_o \cap c$, we now consider the remaining $k-m$ orders $B' = [o_{m+1}, o_{m+2}, \ldots, o_k]$ and show that there is a fill for orders B' from X'. Let $B'' = [o_{j_1}, o_{j_2}, \ldots, o_{j_s}]$ be any submultisets of B' and let

$A'' = A_{j_1} + A_{j_2} + \cdots \ldots + A_{j_s}$

$n'' = n_{j_1} + n_{j_2} + \cdots \ldots + n_{j_s}$

Applying *theorem 3.8* for any multiset A_o, A'' and X we have

$A_o \cap X + A'' \cap (X - A_o \cap X) = (A_o + A'') \cap X$

$A_o \cap X + A'' \cap X' = (A_o + A'') \cap X$ (3.10)

By assumption that $B_o + B'' = [o_1, o_2, \ldots, o_m, o_{j_1}, o_{j_2}, \ldots, o_{j_s}]$ satisfies the condition

$| B_o + B'' | = n_o + n'' \leq | (A_o + A'') \cap X | = |A_o \cap X | + | A'' \cap X' | = n_o + | A'' \cap X' |$ from (3.9) and (3.10)

Thus, $n'_o \leq | A'' \cap X'|$

This shows that the order B' from X' satisfies the condition of the theorem so by induction it can be filled, say by submultisets $F' = [F_{m+1}, F_{m+2}, \ldots, F_k]$ then $[F_1, F_2, \ldots, F_m, F_{m+1}, F_{m+2}, \ldots, F_k]$ is a filling of order B.

Case two,

$| B'| < |A_{B'} \cap X|$

Suppose we pick a submultiset $[x_1]$ of X with $x_1 \in A_1$.

Let $A'_1 = A_1 - x_1$

$n'_1 = n_1 - 1$ and $o'_1 = [n'_1, A'_1]$. If $n'_1 = 0$, then o'_1 is an empty order. We then claim that the modified order $B' = [o'_1, o_2, \ldots, o_k]$ from $X' = X - [x_1]$ satisfies the condition of the theorem. If a submultiset B'' from B' does not contain o'_1 then

$|B_o + B''| < |A_{B''} \cap X |$. Since $| X'| = |X| - 1$, $|B_o + B''| \leq |A_{B''} \cap X'|$

If a submultiset B'' from B' does contain o'_1 then it is obtained from a submultiset $B_o{}''$ from B by replacing o_1 by o'_1 so that

$|B''| = | B_o{}''| - 1 < |A_{B''} \cap X | - 1 \leq |A_{B''} \cap X'| + 2 - 1.$

Thus $| B''| \leq |A_{B''} \cap X'|$.

This shows the claim, therefore by induction there is a filling $F = [F_1', F_2, \ldots, F_k]$ of orders $B' = [o_1', o_2, \ldots, o_k]$ from X'.

To fill the original order it is suffices to set $F_1 = F_1' + [x_1]$, then $F = [F_1, F_2, \ldots, F_k]$ is a filling of B.

3.8.3 Conclusions

Multiset theory, has occupied a central place among a number of non–classical (nonstandard) set theories developed during the last few decades due to its multitudinal applications. Some examples of non–classical set theories are:

> *Boolean– and Heyting-valued set theory, Zadeh's Fuzzy set theory, Vopenka's Alternative (or Semi-) set theory, Intuitionistic and constructive set theory, Quantum set theory,. daCosta's paraconsistent set theory, Pawlak's Rough set theory, Tarski's set theory without variables, Aczel's Non-well-founded (or Hyper) set theory, Extremal set theory, etc.*

None of these theories endeavours to replace classical set theory rather intend to supplement or generalize it. Zermelo–Fraenkel (ZF) set theory is still the best candidate for a secure foundation for mathematics.

CHAPTER FOUR

MAX – PLUS AND MAX – TIMES ALGEBRAS

4.0 Introduction

During the last twenty five years or so a lot of research work has been carried out to systematize various Path Algebras. Path algebras have evolved over the past few decades initially in response to resolving certain issues that arise in weighted graph theory, for example, computing the shortest or the longest path in a graph. During the first phase of its development, several path algebras, following desultory approaches, were developed. However, as research evolved, it was found that a unified theory was underlying all path algebras and idempotent semiring (also called *dioid*) proved to be the unifying thread. Path algebra is a specific subclass of idempotent semirings or dioids. (See *appendix 1* for a concise list of different types of path algebras).

In this chapter, we outline a comprehensive study of max – plus and max – times, the two major classes of path algebras. Finally, a truncated symmetrized max – plus algebra, along with its innovative application is presented. We have studied max – plus, max – times, symmetrized max – plus algebras as a stepping stone to develop multiset algebras which is the main goal in our thesis.

The *max-plus* (denoted by (max, +) or R_{max} or M^{+}) algebra is also called *tropical calculus or minima algebra*. The basic ingredient in this algebra is that addition is defined as $max(a, b)$ and *multiplication* as $a + b$, the latter being distributive over the former. The basic advantage of using R_{max} lies in getting linearlized a problem which is

nonlinear in a conventional algebra system. R_{max} has been found useful in diverse fields, for example, graph theory, discrete event systems, transportation network, parallel computations, project management, machine scheduling and language theory (automata with multiplicities), see (Cuninghame–Green, 1979; Baccelli *et al*., 1992; Gunawardena, 1996 and Butkovic, 2008), in particular.

Besides its initial application, essentially found in the study of discrete event systems, the theory of dioids, involving max (or min) and + as the basic operations, is found appropriate for a large class of systems dealing with *input-output* relation, suitably interpreted as an inf. (or a sup.) convolution. Yet another situation where dioid algebra shows up is the asymptotic behaviour of exponential functions. The conventional operations + and × over positive numbers, say, are transferred into max and + respectively by the mapping $x \to lim_{s\to+\infty} \exp(sx)$, which is found relevant, for example, in the study of large deviations. Note that certain classes of dioids have been extensively studied under other names (Gunawardena, 1996), for example,

To illustrate the usefulness of max–plus algebra, we begin with an example to show how max – plus algebra system is used to linearize a problem which is nonlinear in the conventional algebra. We consider a simple train network. The train network is between three cities, also with intercity routes. Suppose that there are two trains at each station; one for the intracity track and one for the intercity track. *Figure 4.1* shows a diagram which represents the train network. To simplify the problem we will assume that the times given include the time needed to board the train, the time required for exiting, and the time required for change–overs. Similar examples can be found in (Heidergot *et al*., 2006) and (Farlow, 2009).

Example 4.1

Suppose we let x_i represent the time that a train leaves station i, for $i = 1, 2, 3$. We let $x_i\ (k)$ represent the k^{th} time a train leaves station i. Let us suppose, first of all, that each of the trains leaves at time = 0. Therefore we have $x_1(1) = x_2(1) = x_3(1) = 0$.

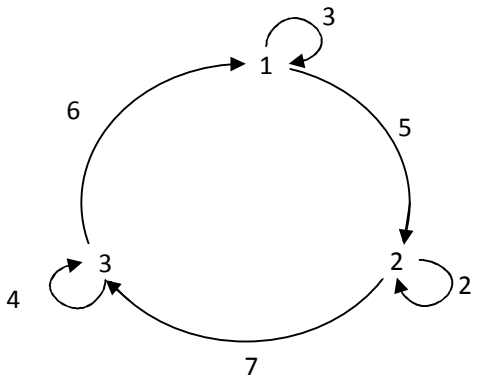


Figure 4.1: Graph for Transportation Network

The time at which the trains leave the station the second time is:

$$\begin{cases} x_1(2) = max\ \{x_1(1) + 3,\ x_3\ (1) + 6\} \\ x_2(2) = max\ \{x_2(1) + 2,\ x_1(1) + 5\} \\ x_3(2) = max\ \{x_3(1) + 4,\ x_2(1) + 7\} \end{cases} \quad 4.1$$

This set of equations is certainly nonlinear. These equations can be changed into a linear system if we change to a different algebra system.

Let $max(a, b) = a \oplus b$ and $a + b = a \otimes b$. Also, $\theta \oplus a = a$ and $\theta \otimes a = -\infty$. Then the set of equations (1) above can be written as:

$$\begin{cases} x_1(2) = x_1(1) \otimes 3 \oplus x_3(1) \otimes 6 \\ x_2(2) = x_2(1) \otimes 2 \oplus x_1(1) \otimes 5 \\ x_3(2) = x_3(1) \otimes 4 \oplus x_2(1) \otimes 7 \end{cases} \quad 4.2$$

This looks very similar to a linear algebra problem. We assume that the usual laws of arithmetic (and matrix arithmetic) apply in this example.

If we let $\underline{x}(k) = [x_1(k), x_2(k), x_3(k)]^T$ then $x(k)$ represents the k^{th} time that each train leaves the station.

Equations (2) can be written in matrix form as $\underline{x}(2)^T = \underline{x}(1)^T \otimes A$, where $A = \begin{bmatrix} 3 & 5 & -\infty \\ -\infty & 2 & 7 \\ 6 & -\infty & 4 \end{bmatrix}$ and $a_{i,j}$ is the travel time from station i to station j. $-\infty$ means there is no path (route).

We also include a result which is quite useful in the theory of algebra of asymptotic growth rates of functions. The max – plus algebra operations $\oplus$ and $\otimes$ are operations on exponents induced by conventional algebra that can be applied to exponential functions. For example

$$e^{sa} e^{sb} \asymp e^{s(a \otimes b)}$$

$$e^{sa} + e^{sb} \asymp e^{s(a \oplus b)}$$

Definition 4.1

If $p: (0, \infty) \to (0, \infty)$ and $u \in [-\infty, \infty)$, we define $p \asymp e^{su}$ to mean

$\lim_{s \to \infty} s^{-1} \ln(p) = u$

We take note of the fact that $\ln(0) = -\infty$ and $e^{-\infty} = 0$.

Theorem 4.1

If $f \asymp e^{sa}$ and $g \asymp e^{sb}$, then $f + g \asymp e^{s(a \oplus b)}$ and $fg \asymp e^{s(a \otimes b)}$

Proof

$\lim_{s\to\infty} s^{-1} \ln(fg) = \lim_{s\to\infty} s^{-1} \ln(f) + \lim_{s\to\infty} s^{-1} \ln(g) = a + b = a \oplus b$

Hence, $fg \asymp e^{s(a \otimes b)}$.

Observe that $max(f, g) \leq f + g \leq 2\, max(f, g)$ and applying *definition 4.1* we have

$$\lim_{s\to\infty} s^{-1} \ln(max(e^{sa}, e^{sb})) \leq \lim_{s\to\infty} s^{-1} \ln(e^{sa} + e^{sb})$$

$$\leq \lim_{s\to\infty} s^{-1} \ln\Big(max(e^{sa}, e^{sb})\Big) + \ln(2)$$

But $s^{-1} \ln(max(e^{sa}, e^{sb}) = max(a, b)$ and by applying squeeze theorem we have

$$\lim_{s\to\infty} s^{-1} \ln(f + g) = max(a, b) = a \oplus b$$

Many properties in max – plus algebra have been proved using this connection with conventional algebra. In particular, it is used to generalize Cramer's rule and the Cayley–Hamilton theorem (Heidergot *et al*., 2006).

We shall now investigate two particular path algebras in our thesis, the max–plus algebra and the max–times algebra. These two algebras are quite similar. In fact, they are isomorphic.

4.1 Basics of Max-Plus Algebra

In this section we present the basic definitions and algebraic properties of max–plus algebra. Most of the literature on developing the concept of max–plus can be found in (Olsder and Roos, 1988; Baccelli *et al*., 1992; Gunawardena, 1996; DeSchutter and DeMoor, 1997; Heidergot *et al*., 2006; McEneaney, 2006 and Butkovic, 2008), in particular.

Let $\varepsilon = -\infty$ and $e = 0$, called the additive identity and multiplicative identity respectively. We denote the set $R \cup \{\varepsilon\}$ by R_{max}, where R is the set of real numbers. For all elements $a, b \in R_{max}$, we define operations $\oplus$ and $\otimes$ by

(i) $a \oplus b = max(a, b)$

(ii) $a \otimes b = a + b$.

Clearly, $max(a, -\infty) = max(-\infty, a) = a$ and $a + (-\infty) = -\infty + a = -\infty$, for any $a \in R_{max}$. We have,

(iii) $a \oplus \varepsilon = \varepsilon \oplus a = a$

(iv) $a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$, for any $\in R_{max}$.

These definitions are illustrated with some numerical examples as follows:

$8 \oplus 4 = max(8, 4) = 8$

$9 \oplus \varepsilon = max(9, -\infty) = 9$

$5 \otimes \varepsilon = 5 + (-\infty) = -\infty = \varepsilon$

$e \oplus 7 = max(0, 7) = 7$

$9 \otimes 5 = 9 + 5 = 14$

The set R_{max} together with the operations $\oplus$ and $\otimes$ is called max – plus algebra and is denoted by $M^+ = (R_{max}, \oplus, \otimes, \varepsilon, e)$.

Clearly, $\oplus$ and $\otimes$ are commutative, associative and obey many properties similar to + and × in conventional algebra. For instance, it is easy to see that $\otimes$ distributes over $\oplus$ as follows: $\forall x, y, z \in R_{max} : x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$.

$[\, x \otimes (y \oplus z) = x + max(y, z) = max(x + y, x + z) = (x \otimes y) \oplus (x \otimes z)]$

But $x \oplus (y \otimes z) \neq (x \oplus y) \otimes (x \oplus z)$,

since $max(x, y + z) \neq max(x, y) + max(x, z)$, in general.

In particular, $max(9, 3 + 4) \neq max(9, 3) + max(9, 4)$.

These algebraic properties of $\oplus$ and $\otimes$ and many others are listed below. The proofs are elementary and are not included.

For all *x, y, z* $\in R_{max}$,

a. *Associativity*: (i) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

(ii) $x \otimes (y \otimes z) = (x \otimes y) \otimes z$.

b. *Commutativity*: (i) $x \oplus y = y \oplus x$

(ii) $x \otimes y = y \otimes x$

c. *Distributivity*: $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$.

d. *Existence of a zero element*: $x \oplus \varepsilon = \varepsilon \oplus x = x$

e. *Existence of a unit element*: $x \otimes e = e \otimes x = x$.

f. *The zero is absorbing for* $\otimes$: $x \otimes \varepsilon = \varepsilon \otimes x = \varepsilon$.

g. *Idempotency of* $\oplus$**:** $x \oplus x = x$.

h. *Existence of order*: $a \leq b$ if $a \oplus b = b$

i. *Multiplicative inverse*: if $x \neq \varepsilon$ then there exists a unique y such that $x \otimes y = e$

j. *Exponential property*: let N be the set of natural numbers including zero then for $x \in R_{max}$, $x^{\otimes n} = x \otimes x \otimes \cdots \otimes x$ (n times)

$$= x + x + \cdots + x = nx$$

Hence, the following hold:

a) if $x \neq \varepsilon$, then $x^{\otimes 0} = e\ (= 0)$

b) if $\alpha \in R$, then $x^{\otimes \alpha} = \alpha x$

c) if $k > 0$, then $\varepsilon^{\otimes k} = \varepsilon$, ($k \leq 0$ is not defined)

d) for $m, n \in N$ and $x \in R_{max}$,

(i) $x^{\otimes m} \otimes x^{\otimes n} = mx + nx = (m + n)x = x^{\otimes(m \otimes n)}$

(ii) $x^{\otimes m} \otimes y^{\otimes m} = mx + my) = m(x + y) = (x \otimes y)^{\otimes m}$

Remark 4.1

Max–plus algebra is an example of an algebraic structure, called idempotent semiring, since $\oplus$ is associative and commutative with zero element ε; $\otimes$ is associative, distributes over $\oplus$, and has unit element e and ε is absorbing for $\otimes$. It is important to note that max–plus algebra is idempotent and it does not have an additive inverse. This point is captured in *theorem 4.2*.

Theorem 4.2

The idempotency of $\oplus$ in the max–plus semiring implies that for every $\in R_{max} \setminus \{\varepsilon\}$, a does not have an additive inverse.

Proof.

Suppose $a \in R_{max}$ such that $a \neq \varepsilon$ has an inverse with respect to $\oplus$, Let b be the inverse of a. Then we have

$a \oplus b = \varepsilon$ 4.3

By adding a to both sides of (1) we have

$a \oplus (a \oplus b) = a \oplus \varepsilon$

$a \oplus b = a \oplus \varepsilon$ by idempotency law

Hence $a \oplus b = a$ which contradicts (4.3) that $a \oplus b = \varepsilon$

Remark 4.2

Theorem 4.2 exposes the limitation of max–plus algebra. For example, there exists no number that makes equation, $7 \oplus x = 4$ true. Similarly, *min* $(5, x) = 13$ has no solution in min–plus algebra (see *Table 4.1* below). The existence of a solution in a given equation depends on the algebra. This shows that there is no idempotent semiring for which negative numbers can be defined. The fact that the subtraction operation cannot be defined in an idempotent semiring makes significant difference in the study of max-plus algebra versus the conventional algebra.

Table 4.1 Family of (max, +)

Notation	Semiring	Name	Description
$\mathbb{R}_{max}$	$(\mathbb{R} \cup \{-\infty\}, max, +)$	(Max, +)-semiring max algebra	idempotent semifield
$\overline{\mathbb{R}}_{max}$	$(\mathbb{R} \cup \{\pm \infty\}, max, +)$	Completed (Max, +)-semiring	$-\infty + (+\infty) = -\infty$
$\mathbb{R}_{max, \times}$	$(\mathbb{R}^+, max, \times)$	(max, ×) semiring	for $\theta \otimes a = \theta$
$\mathbb{R}_{min}$	$(\mathbb{R} \cup \{+\infty\}, min, +)$	(min, +) semiring	
$\mathbb{N}_{min}$	$(\mathbb{N} \cup \{+\infty\}, min, +)$	Tropical semiring	Famous in Language Theory
$\mathbb{R}_{max, min}$	$(\mathbb{R} \cup \{\pm \infty\}, min, min)$	Bottleneck algebra	
$\mathbb{B}$	({false, true}, or, and)	Boolean semiring	isomorphic to $(\{\theta, e\}, +, \times)$, for any of the above semirings
$\mathbb{B}_h$	$(\mathbb{R} \cup \{-\infty\}, \oplus_h, +)$ $a \oplus_h b = hlog(e^{a/h} + e^{b/h})$	Maslov semirings	isomorphic to $(\mathbb{R}^+, +, \times)$ $lim_{h \to 0^+} \mathbb{R}_h = \mathbb{R}_0 = \mathbb{R}_{max}$

Note that a statement that is proved for a semiring will immediately hold in any of the above algebras.

4.2 Matrices and Vectors in Max–Plus Algebra

In this Section, we present the concepts of matrices over R_{max} to enable us develop max – plus versions of the standard results in linear algebra in Section 4.3. Much of the material discussed in this section can be found in (Bapat *et al*., 1985; Baccelli *et al.,* 1992; Gaubert, 1997 and Heidergot *et al*., 2006).

Let $R_{max}^{n\times m}$ be the set of $n\times m$ matrices for $n,m \in N$ over R_{max} where n and m are the number of rows and columns respectively. Following conventional algebra, a matrix $A \in R_{max}^{n\times m}$ is written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \ddots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{21} & a_{22} & \cdots & a_{mm} \end{pmatrix}$$

The entry in the i^{th} row and j^{th} column of A is denoted by a_{ij} or $[A]_{ij}$ depending on handiness.

The elements of $R_{max}^{n\times 1}$ are called vectors. The jth element of a vector $x \in R_{max}^{n\times 1}$ is denoted by x_j, which can also be written as $[x]_j$. The vector in $R_{max}^{n\times 1}$ with all elements equal to e is called the *unit vector* and is denoted by $\mathbf{u}$ i.e. $[\mathbf{u}]_j = e$. Then $\alpha \otimes \mathbf{u}$ denotes a vector with all elements equal to α, for any $\alpha \in R_{max}^{n\times 1}$.

4.2.1 Some Definitions on matrix of max – plus algebra

1. The *sum of matrices A, B* $\in R_{max}^{n\times m}$, denoted by $A \oplus B$, is defined by

   $[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = max\,(a_{ij}, b_{ij})$, for $i \in n$ and $j \in m$.

2. For matrices $A \in R_{max}^{n\times l}$ and $B \in R_{max}^{l\times m}$ the *matrix product* $A \otimes B$ is defined as

   $[A \otimes B]_{ik} = \oplus_{j=1}^{l} (a_{ij} \otimes b_{jk}) = max\,(a_{ij} + b_{jk})$

3. The *transpose* of an element $A \in \mathrm{R}_{max}^{n \times m}$, denoted by A^T, is defined in the usual way as in conventional matrix algebra by $[A^T]_{ij} = [A]_{ji}$, for $i \in n$ and $j \in m$.

4. Let $\mathcal{E}(n,m)$ denote the $n \times m$ matrix with all elements equal to ε, and denote by $E(n,m)$ the $n \times m$ matrix defined by $[E(n,m)]_{ij} = \begin{cases} e & for\ i = j \\ \varepsilon & else \end{cases}$

   If $n = m$, then $E(n,n)$is called the $n \times n$ *identity matrix* denoted by E_n

5. If $A \in R_{max}^{n \times n}$ (square matrices) and k is a natural number, the k^{th} *power* of A denoted by $A^{\otimes k}$ is defined as $A^{\otimes k} = A \otimes A \otimes \ldots \otimes A$ (k times).

   If $k = 0$, $A^{\otimes k} = E_n$.

6. For any matrix $A \in R_{max}^{n \times m}$ and any scalar $\alpha \in R_{max}$ then $\alpha \otimes A$ is defined by $[\alpha \otimes A]_{ij} = \alpha \otimes [A]_{ij}$ for $i \in n$ and $j \in m$.

7. For any matrix $A \in R_{max}^{n \times m}$,

$$A \otimes E(m,m) = A = E(n,n) \otimes A,$$

$$A \oplus \mathcal{E}(n,m) = A = \mathcal{E}(n,m) \oplus A.$$

Example 4.2

Let $A = \begin{pmatrix} 10 & e \\ 4 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 5 \\ 6 & \varepsilon \end{pmatrix}$, we have $A \oplus B = B \oplus A = \begin{pmatrix} 10 & 5 \\ 6 & 7 \end{pmatrix}$

Also, $A \otimes B = \begin{pmatrix} 7 & 15 \\ 13 & 9 \end{pmatrix}$ and $B \otimes A = \begin{pmatrix} 9 & 12 \\ 16 & 6 \end{pmatrix}$

Remark 4.3

For $R_{max}^{n \times m}$, the *matrix addition* $\oplus$ is *commutative*, *associative*, and has *zero element* $\mathcal{E}(n,m)$, while the *matrix product* $\otimes$ is *noncommutative*, *associative*, *distributive* with respect to $\oplus$, has *unit element E(n , n),* and $\mathcal{E}(n, n)$ *is absorbing* for $\otimes$. It is also

interesting to note that $\oplus$ is idempotent in $R_{max}^{n\times m}$. Therefore, the set $R_{max}^{n\times m}$ with respect to $\oplus$ and $\otimes$ is an idempotent semiring in which $\otimes$ is noncommutative.

4.3 Max–Plus Linear Algebra

In this Section we develop max–plus version of some concepts of conventional linear algebra, in particular, matrix inverses, generalization of the determinant of a matrix.

Definition 4.2 (*Inverse Matrices*)

A matrix $A \in R_{max}^{n\times m}$ is said to be invertible if there exists a matrix $B \in R_{max}^{n\times m}$ such that $A \otimes B = E$ and we write $A^{\otimes -1} = B$.

Definition 4.3 (*Permutation matrix*)

A permutation matrix is a matrix in which each row and each column contains exactly one entry e and all other entries are ε. If σ: $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a permutation, the max – plus permutation matrix is defined as $P_\sigma = [P_{ij}]$ where

$$P_{ij} = \begin{cases} e & i = \sigma(j) \\ \varepsilon & i \neq \sigma(j) \end{cases}$$

So that the j^{th} column of P_σ has e in the $\sigma(j^{th})$ row.

Definition 4.4 (*Diagonal matrix*)

If $\lambda_1, \lambda_2, \ldots, \lambda_n \in R_{max}$ and $\lambda_i \neq \varepsilon$, the diagonal matrix, $D(\lambda_i)$, is defined as

$$D(\lambda_i) = \begin{pmatrix} \lambda_1 & \varepsilon & \cdot & \cdot & \cdot & \varepsilon \\ \varepsilon & \lambda_2 & \varepsilon & \cdot & \cdot & \varepsilon \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \varepsilon & \cdot & \cdot & \cdot & \varepsilon & \lambda_n \end{pmatrix}$$

Theorem 4.3

A matrix $A \in R_{max}^{n \times n}$ has a right inverse if and only if there is a permutation σ and values $\lambda_i > \varepsilon$, $i \in \{1, 2, \ldots, n\}$ such that $A = P_\sigma \otimes D(\lambda_i)$.

Proof

Suppose there exists B such that $A \otimes B = E$, then

(i) $max_k(a_{ik} + b_{ik}) = e = 0$ for each i

(ii) $max_k(a_{ik} + b_{ik}) = \varepsilon = -\infty$ for all $i \neq j$

By (i) for each i there exists a k so that $(a_{ik} + b_{ik}) = e$. Therefore we have a function $k = \theta(i)$ with $a_{i\theta(i)} > \varepsilon$ and $b_{\theta(i)i} > \varepsilon$. From (ii) we find that $a_{i\theta(j)} = \varepsilon$ for all $i \neq j$. Since $a_{i\theta(i)} > \varepsilon = a_{i\theta(j)}$ for all $i \neq j$, it follows that θ is an injection and therefore a permutation. The expression $a_{i\theta(j)} = \varepsilon$ for all $i \neq j$ also show that $a_{i\theta(i)}$ is the only entry of the $\theta(i)^{th}$ column of A that is not ε.

Now let $\bar{A} = P_\theta \otimes A$. The $\theta(i)^{th}$ row of $\bar{A}$ is the i^{th} row of A, which has an entry greater than ε in the $\theta(i)^{th}$ column. This implies that all the diagonal entries of $\bar{A}$ are greater than ε. Also, A has only one non – ε entry in each column. This is also true for $\bar{A}$.

Thus $P_\theta \otimes A = \bar{A} = D(\lambda_i)$ with $\lambda_i = a_{\theta^{-1}(i)i} > \varepsilon$.

Let $\sigma = \theta^{-1}$. Since $P_\sigma \otimes P_\theta = E$, it follows that $A = P_\sigma \otimes D(\lambda_i)$.

Conversely, suppose $A = P_\sigma \otimes D(\lambda_i)$ with $\lambda_i \in R_{max}$ and $\lambda_i > \varepsilon$. Let $B = D(-\lambda_i) \otimes P_{\theta^{-1}}$ where $-\lambda_i = \lambda_i^{\otimes -1}$.

Then $A \otimes B = P_\sigma \otimes D(\lambda_i) \otimes D(-\lambda_i) \otimes P_{\theta^{-1}} = P_\sigma \otimes P_{\theta^{-1}} = E$. This shows that B is the right inverse of A.

Theorem 4.4

Given matrices $A, B \in R_{max}^{n \times n}$ if $A \otimes B = E$ then $B \otimes A = E$ and B is uniquely determined by A.

Proof

We know that $A = P_\sigma \otimes D(\lambda_i)$ for some values $\lambda_i > \varepsilon$ and permutation σ from the theorem above. Also, $\bar{B} = D(-\lambda_i) \otimes P_{\theta^{-1}}$ is a left inverse of A. Now if $A \otimes B = E$, then $\bar{B} = \bar{B} \otimes A \otimes B = (\bar{B} \otimes A) \otimes B = E \otimes B = B$ which shows that B is the left inverse and it is uniquely determined.

Theorem 4.5

If $A \in R_{max}^{n \times n}$ and $B \in R_{max}^{n \times n}$ are invertible then $A \otimes B$ is also invertible.

Proof.

If A and B are invertible in $R_{max}^{n \times n}$, then $A = P_{\sigma a} \otimes D(\lambda_i^a)$ and $B = P_{\sigma b} \otimes D(\lambda_i^b)$. Consequently,

$A \otimes B = P_{\sigma a} \otimes D(\nwarrow_i^a) \otimes P_{\sigma b} \otimes D\left(\nwarrow_i^b\right) = P_{\sigma a} \otimes D(\nwarrow_i^a) \otimes D\left(\nwarrow_i^b\right) \otimes P_{\sigma b}$.

Since product of diagonal matrices is also a diagonal matrix we have $A \otimes B = P_{\sigma a} \otimes D(\nwarrow_i^a \otimes \nwarrow_i^b) \otimes P_{\sigma b}$.

This shows that $A \otimes B$is a permuted matrix hence it is invertible.

More theorems concerning left and right inverses of max – plus matrices can be found in (Heidergot *et al*., 2006).

Definition 4.5 (*Determinants*)

The determinant of a matrix $A \in R^{n \times n}$ in a conventional algebra is defined as $|A| = \sum_{\sigma \in P_n} Sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$, where P_n is the set of all permutations of $\{1, 2, \ldots, n\}$ and $Sgn(\sigma)$ is the sign of the permutation σ. Determinant in Max – plus algebra could not be immediately captured in this mode since it lack additive inverses hence the idea of permanent of A and dominant of A. See *Section 4.6* for symmetrized max–plus algebra version of determinant.

Definition 4.6 (*Permanent of a matrix*)

Let $A \in R_{max}^{n \times n}$ be a matrix, the permanent of A is defined as $\text{Perm}(A) = \oplus_{\sigma \in P_n} \otimes_{i=1}^{n} (a_{i\sigma(i)})$, where P_n is the set of all permutations of $\{1, 2, \ldots, n\}$.

Theorem 4.6

If matrix A in $R_{max}^{n\times n}$ is invertible then perm(A)$\neq \varepsilon$

Proof

By the definition of an invertible matrix in max – plus algebra, the matrix A is a diagonal matrix times a permutation matrix. Therefore, perm(A) is the max – plus product of the diagonal entries of the diagonal matrix and since none of the diagonal entries of the diagonal matrix in max – plus algebra is ε, perm(A) $\neq \varepsilon$.

Remark 4.4

The converse of the above lemma is not true. That is perm(A) $\neq \varepsilon$ is not sufficient for matrix A to be invertible. For example, the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix}$ has perm(A)= $max\{2 + 0,\ 3 + 4\} = 7 \neq \varepsilon$ and it is not an invertible matrix since it is not a permuted diagonal matrix.

The permanent of a matrix A is defined similar to the determinant in a conventional algebra only that it has no $sign(\sigma)$.

Definition 4.7 (*Dominant of a matrix*)

Let $A \in R_{max}^{n\times n}$ z^A is the $n \times n$ matrix with entries $z^{a_{ij}}$, where z is a variable. The dominant of A is defined as follows:

$$\text{Dom}(A) = \begin{cases} highest\ exponent\ in \det(z^A) & if \quad \det(z^A) \not\equiv 0 \\ \varepsilon & if \quad \det(z^A) \equiv 0 \end{cases}$$

Following *Theorem 4.1*, if we replace z by e^s, we can reframe the *definition 4.7* as follows:

$$\text{Dom}(A) = \begin{cases} lim_{s\to\infty} \frac{1}{s} lin|\det(e^{sA})| & if \ \det(e^{sA}) \not\equiv 0 \\ \varepsilon & if \ \det(e^{sA}) \equiv 0 \end{cases}$$

Given a matrix $A \in R_{max}^{n\times n}$, the matrix e^{sA} has entries $e^{sa_{ij}}$ where $a_{ij} \in R_{max}$ are the entries in A.

Remark 4.5

With reference to *definition 4.1*, we have $|\det(e^{sA})| \asymp e^{s\ dom(A)}$ which gives asymptotic connection with the conventional determinants that provides the basic approach to generalizing Cramer's Rule and Cayley – Hamilton theorem (Olsder and Roos, 1988).

We list some theorems connecting the permanent and the dominant of a matrix in max – plus algebra whose proof can be found in (Olsder and Roos, 1988), (Heidergot *et al*., 2006) and (Farlow, 2009).

Theorem 4.7

(i) Given a matrix $A \in R_{max}^{n\times n}$, dom($A$) ≤ perm(A).

(ii) If $A \in R_{max}^{n\times n}$ is invertible then dom(A)≠ ε.

(iii) If $A \in R_{max}^{n\times n}$ is invertible then dom(A)= perm(A).

(iv) If $A, B \in R_{max}^{n\times n}$ are invertible then dom($A \otimes B$) = dom(A) $\otimes$ dom(B).

Proof

See (Farlow, 2009)

Gondran and Minoux, (1984) also define what is called the bideterminant of a matrix A in max – plus algebra which is another approach in handling the absence of subtraction.

Definition 4.8 (*Bideterminant*)

Given $A \in R_{max}^{n \times n}$, let $w_A(\sigma) = a_{1,\sigma(1)} \otimes a_{2,\sigma(2)} \otimes \ldots \otimes a_{n,\sigma(n)},$ the bideterminant of A is $\begin{pmatrix}\Delta_1(A)\\ \Delta_2(A)\end{pmatrix}$ where $\Delta_1(A) = \oplus_{\sigma \in p_n^e} w_A(\sigma)$, $\Delta_2(A) = \oplus_{\sigma \in p_n^o} w_A(\sigma)$ p_n^e is the set of even permutations and p_n^o is the set of odd permutations of $\{1, 2, \ldots, n\}$.

Using the *definition 4.8*, permanent of a matrix can be rewritten as perm(A)= $\Delta_1(A) \oplus \Delta_2(A)$.

Definition 4.9 (*Eigenvalue and Eigenvector*)

Let $A \in R_{max}^{n \times n}$ be a square matrix. If $\lambda \in R_{max}$ is a scalar and there exists a vector $v \in R_{max}^{n}$ that contains at least one element not equal to ε such that $A \otimes v = \lambda \otimes v$ then λ is called an eigenvalue of A and v is the corresponding eigenvector of A.

In general there can be more than one eigenvalue and $A \otimes v = \lambda \otimes v$ implies that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \lambda \otimes \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

$$\Rightarrow \begin{cases} max\{a_{11} + x_1, \ a_{12} + x_2, & \ldots, \ a_{1n} + x_n = \lambda + x_1 \\ max\{a_{21} + x_1, \ a_{22} + x_2, & \ldots, \ a_{2n} + x_n = \lambda + x_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ max\{a_{n1} + x_n, a_{n2} + x_2, & \ldots, \ a_{nn} + x_n = \lambda + x_n \end{cases}$$

4.4 Max–Times Algebra

Max–Times Algebra is similar to max–plus algebra. We present here briefly the concept of max–times algebra. All the results in max–plus algebra system can be extended to max–times algebra. We give some definitions and results in max–times algebra.

Let R^+ represent a non-negative number and define for all $a, b \in R^+$

(a) $a \oplus b = max(a, b)$

(b) $a \odot b = a \cdot b$, the usual multiplication.

With these definitions, the algebraic system $(R^+, \oplus, \odot)$ is called max – times algebra denoted by R^*_{max}. As in max–plus algebra, the operations, $\oplus$ and $\odot$ are extended to matrices as follows.

Let $A, B \in R^{*(n\times n)}_{max}$, then

(i) $[A \oplus B]_{ij} = max(a_{ij}, b_{ij})$

(ii) $[A \odot B]_{ij} = max(a_{ik}, b_{kj}), 1 \leq k \leq n$

We show that there is an isomorphism between the max–plus and the max–times algebras.

Theorem 4.8

Max–plus algebra is isomorphic to max–times algebra.

Proof

Define a map φ: $R_{max} \rightarrow R^*_{max}$ by

(i) $\varphi(x) = e^x$ for all $x \in R$, with $\varphi(-\infty) = 0$

(ii) $\varphi^{-1}(y) = \ln(y)$ for all $y \in \mathrm{R} \geq 0$ with $\varphi^{-1}(0) = -\infty$

For matrices, the isomorphism is defined elementwise as $[\ \varphi(A)]_{ij} = [e^{a_{ij}}]$

Now, (i) $\varphi(a \oplus b) = e^{a \oplus b} = e^{max\,(a,\ b)} = max(e^a, e^b) = e^a \oplus e^b = \varphi(a) \oplus \varphi(b)$

(ii) $\varphi(a \otimes b) = e^{a \otimes b} = e^{(a+\,b)} = e^a \cdot e^b = \varphi(a) \odot \varphi(b)$

Hence max – plus is isomorphic to max – times algebra and with this isomorphism it follows that $A \otimes B = \varphi^{-1}(\varphi(A) \odot \varphi(B))$.

We now give some basic results in the max – times algebra.

Theorem 4.9

Let $A, B \in R_{max}^{*(n \times n)}$ and $\alpha \in R_{max}^{*}$ then we have the following results:

(i) $\alpha \otimes A = \alpha \odot A$, where α is a scalar.

(ii) $A \otimes B = A \odot B$ if A has only one positive element in each row.

(iii) $A \otimes B = A \odot B$ if $\boldsymbol{A}$ has only one positive element in each column.

(iv) $P \otimes A \otimes P^T = P \odot A \odot P^T$

(v) If D is a diagonal matrix, $D = diag(a_{11},\ a_{12},\ \ldots, a_{nn})$ where $a_{ii} \neq 0$ for each i then D^{-1} exists in max – times algebra and

$D^{-1} = diag(\frac{1}{a_{11}},\ \frac{1}{a_{22}}, \ldots, \frac{1}{a_{nn}})$

More results and proof of these theorems can be found in (Poplin, 2000).

4.5 Symmetrization of the Max–Plus Algebra

Symmetrized max – plus, denoted by $\mathbb{S}_{max}$ or $(\mathbb{S}, \oplus, \otimes)$ was developed [Gau 92] in response to removing the deficiency of R_{max} which was found lacking additive inverses. However, $\mathbb{S}_{max}$ was found lacking one of the characteristics property of the conventional algebra viz; $a - a = 0$. Gaubert (1992) introduced the notion of *balances* to overcome the aforesaid deficiency. But, the balance relation turns out to be non-transitive. In other to achieve transitivity of the balance relation, a new relation, closely related to the balance relation, was introduced.

It is straight forward to observe that R_{max} is a *zerosum free idempotent semiring*. Moreover, it is a semifield since all nonzero elements do have a multiplicative inverse. However, it is a bit disappointing that R_{max} lacks an additive inverse and as a result, even some simple linear equations do not have a solution. For example, $x \oplus 2 = 1$, has no solution.

In order to remove this deficiency, a standard technique is exploited i.e. embedding R_{max} into a larger system which will have an additive inverse, akin to embedding the system of natural numbers $\aleph$ into that of integers $\mathbb{Z}$. This technique is known as "Symmetrization". That is, "$a \oplus x = \theta$ (additive identity), $a \in R_{max}$, has a solution" is to say that "a is symmetrized" (Gaubert, 1992) and (Poplin, 2000).

We outline here the *symmetrized max –plus algebra*, denoted by $S_{max,}$ developed in (Gaubert, 1992) as an extension of R_{max}.

4.5.1 Algebra of Pairs

Definition 4.10 (*Algebra of pairs*)

Let β = $R_{max} \times R_{max}$ or R_{max}^2, and $x = (x_1, x_2)$, $y = (y_1, y_2)$ be elements of β. We define:

1. $x \oplus y = (x_1, x_2) \oplus (y_1, y_2) = (x_1 \oplus x_2, y_1 \oplus y_2)$.
2. $x \otimes y = (x_1, x_2) \otimes (y_1, y_2) = ((x_1 \otimes y_1) \oplus (x_2 \otimes y_2), (x_1 \otimes y_2) \oplus (x_2 \otimes y_1))$.

Thus, (θ, θ) is the additive identity and (e, θ) is the unit element (multiplicative identity) of the set β. Note that the use of the same symbols ⊕ and ⊗, both for R_{max} and S_{max} (where they operate on the elements of β) will be contextually understood.

Let $x \in (x_1, x_2)$, we define:

1. $\ominus x = (x_2, x_1)$
2. $|x| = x_1 \oplus x_2 = max(x_1, x_2)$, called the absolute value of x
3. $x^* = x \ominus x = (|x|, |x|)$, called the *balance* operator

Note that

$x \ominus x = x \oplus (\ominus x) = (x_1, x_2) \oplus (x_2, x_1) \equiv (x_1 \oplus x_2, x_2 \oplus x_1) = (|x|, |x|)$.

Theorem 4.10

(i). (β, ⊕) is commutative, associative and idempotent with additive identity (θ, θ)

(ii). (β, ⊗) is commutative, associative and distributive over ⊕ with multiplicative identity (e, θ).

(iii). The additive identity is absorbing under ⊗.

(iv). The algebraic structure (β, ⊕, ⊗) is a commutative dioid.

Descriptively, this algebra is also called the "algebra of pairs".

Proof

Let $x, y \in \beta$ then

$$x \oplus y = (x_1, x_2) \oplus (y_1, y_2) = max((x_1, x_2), (y_1, y_2)) = max((y_1, y_2), (x_1, x_2))$$

$$= (y_1, y_2) \oplus (x_1, x_2)) = y \oplus x$$

Hence $(\beta, \oplus)$ is commutative.

Also, let $x, y, z \in \beta$.

$$(x \oplus y) \oplus z = [(x_1, x_2) \oplus (y_1, y_2)] \oplus (z_1, z_2)$$

$$= max[(x_1, x_2), (y_1, y_2)] \oplus (z_1, z_2)$$

$$= max[max[(x_1, x_2), (y_1, y_2)], (z_1, z_2)]$$

$$= max[(x_1, x_2), (y_1, y_2), (z_1, z_2)]$$

$$= max[(x_1, x_2), [(y_1, y_2) \oplus (z_1, z_2)]]$$

$$= (x_1, x_2) \oplus ((y_1, y_2) \oplus (z_1, z_2)) = x \oplus (y \oplus z)$$

Hence $(\beta, \oplus)$ is associative. Similarly other properties can be verified.

Theorem 4.11

For any $x, y \in \beta$, the following properties hold:

(i)	$x^* = (\ominus x)^* = (x^*)^*$	idempotence
(ii)	$x \otimes y^* = (x \otimes y)^*$	absorption
(iii)	$\ominus(\ominus x) = x$	involution
(iv)	$\ominus(x \oplus y) = (\ominus x) \oplus (\ominus y)$	additive morphism
(v)	$\ominus(x \otimes y) = (\ominus x) \otimes y = x \otimes (\ominus y)$	Sign rule

Proof

(i) $x^* = x \ominus x = x \oplus (\ominus x) = (x_1, x_2) \oplus (x_2, x_1)$

$= (x_1 \oplus x_2, x_2 \oplus x_1) = (|x|, |x|)$ 4.4

Also, $\ominus x^* = \ominus x \ominus (\ominus x) = (x_2, x_1) \ominus (x_2, x_1)$

$= (x_2, x_1) \oplus \ominus (x_2, x_1)$

$= (x_2, x_1) \oplus (x_1, x_2)$

$= (x_2 c\, x_1),\ (x_1 \oplus x_2)$

$= max(x_2, x_1),\ max(x_1, x_2) = (|x|, |x|)$ 4.5

From (4.4) and (4.5) $x^* = \ominus x^*$

Finally, $(x^*)^* = x^* \ominus x^* = x^* \oplus \ominus x^* = x^* \oplus x^* = x^*$. Hence $x^*=(\ominus x)^*= (x^*)^*$

Similarly the remaining properties can be verified.

Remark 4.7

In the conventional algebra, we have $a - a = 0$, but in the *algebra of pairs*, $a \ominus a = a^* \neq (\theta, \theta)$ for all $a \in \beta$ unless $a = (\theta, \theta)$. This suggests for introducing a new relation on β.

4.5.2 Quotient Structure

Definition 4.11 (*Balance relation*)

Let $x = (x_1, x_2),\ y = (y_1, y_2)$. We say that x balances y, denoted by $x \Delta y$, if and only if $x_1 \oplus y_2 = x_2 \oplus y_1$.

It follows that $a \ominus a = a^* = (|a|, |a|) \Delta (\theta, \theta)$ for all $a \in \beta$.

It is fundamental to notice that Δ is not transitive. For instance, $(0,1)\,\Delta\,(1,1)$, $(1,1)\,\Delta\,(1,0)$ but $(0,1)\not\Delta\,(1,0)$. Since Δ is not an equivalence relation we cannot use it to form a partition and, by implication, a quotient set. However, we can introduce a new relation $\mathcal{R}$, closely related to the balance operator Δ, on $\beta = R_{max} \times R_{max}$ or R_{max}^2 which is transitive.

$$x\,\mathcal{R}\,y = \begin{cases} x\Delta y, & if\ x_1 \neq x_2\ and\ y_1 \neq y_2 \\ x = y, & otherwise \end{cases}$$

Here = is conventional: $x = y \Leftrightarrow (x_1, x_2) = (y_1, y_2) \Leftrightarrow x_1 = x_2, y_1 = y_2$.

Hence, we can define the quotient $\mathbb{S} = \beta/R$

Definition 4.12 (*symmetrized max-plus algebra*)

The algebraic structure $\mathbb{S}_{max} = (\mathbb{S}, \oplus, \otimes)$ is called the symmetrized max – plus algebra. The quotient set $\mathbb{S} = \beta/R$ has three types of equivalence classes, the max – positive, max – negative and the balanced classes. They are denoted as follows:

$\overline{(t,-\infty)} = \{(t,x) \in \beta: x < t\}$, called max – positive

$\overline{(-\infty,t)} = \{(x,t) \in \beta: x < t\}$, called max – negative

$\overline{(t,t)} = \{(t,t) \in \beta$, called balanced.

By associating $\overline{(t,-\infty)}$ with $t \in R_{max}$, we let $R_{max}^{\oplus}$ = the set of max – positive elements, $R_{max}^{\ominus}$ = the set of max – negative (of the form $\ominus\, x$), and R_{max}^{*} = the set of balanced classes (of the form x^*). We have

$$\mathbb{S}_{max} = R_{max}^{\oplus} \cup R_{max}^{\ominus} \cup R_{max}^{*}$$

Note that $R_{max}^{\oplus} \cap R_{max}^{\ominus} \cap R_{max}^{*} = \overline{(\theta, \theta)}$ and $\theta = \ominus\, \theta = \theta^{*}$. This is similar to $\mathbb{Z} = \mathbb{N}^{+} \cup \mathbb{N}^{-}$, where $\mathbb{N}^{+} \cap \mathbb{N}^{-} = 0$.

Also, $R_{max}^{v} = R_{max}^{\oplus} \cup R_{max}^{\ominus}$ is called the set of signed elements of $\mathbb{S}_{max}$ and $R_{max}^{v} \backslash \{\theta\}$ is the set of all invertible elements in $\mathbb{S}_{max}$.

Example 4.4

(i) $3 \ominus 2 = 3 \oplus (\ominus 2) = \overline{(3, -\infty)} \oplus \overline{(-\infty, 2)} = \overline{(3,2)} = \overline{(3, -\infty)} = 3$

(ii) $2 \ominus 3 = 2 \oplus (\ominus 3) = \overline{(2, -\infty)} \oplus \overline{(-\infty, 3)} = \overline{(2,3)} = \overline{(-\infty, 3)} = \ominus 3$

Generally, calculations in $\mathbb{S}_{max}$ can be summarized as follows:

$$a \oplus (\ominus b) = \begin{cases} a & if\ a > b \\ \ominus b & if\ b > a \end{cases}$$

$$a \ominus a = a \oplus (\ominus a) = a^{*}$$

Example 4.5

$$\begin{aligned} 4 \ominus 3^{*} \oplus 5 \ominus 6 &= 4 \ominus (3 \ominus 3) \oplus 5 \ominus 6 \\ &= 4 \ominus 3 \oplus 3 \oplus 5 \ominus 6 \\ &= 4 \oplus 3 \oplus 5 \ominus 3 \ominus 6 \\ &= (4 \oplus 3) \oplus 5 \ominus (3 \oplus 6) \\ &= 4 \oplus 5 \ominus 6 \\ &= 5 \ominus 6 = \ominus 6 \end{aligned}$$

4.5.3 Linear balances

In this section we show how the introduction of balances in $\mathbb{S}_{max}$ does help resolving the difficulties in solving equations in R_{max}. First of all, we state the general properties of balances and some propositions and provide an example to demonstrate the applications of linear balances. The proof of some of these properties and propositions can be found in Akian *et al*. (1990).

Properties of balances

Some immediate algebraic features of balances are as follows:

(i) $a \vartriangle a$ (reflexivity)

(ii) $a \vartriangle b \Leftrightarrow b \vartriangle a$ (symmetry)

(iii) $a \vartriangle b \Leftrightarrow a \ominus b \vartriangle \theta$

(iv) $a \vartriangle b, c \vartriangle b \Rightarrow a \oplus c \vartriangle b \oplus d$

(v) $a \vartriangle b \Rightarrow ac \vartriangle bc$

Despite the fact that balance operator $\vartriangle$ is not transitive, we can manipulate balances in the same way as we manipulate equations if the variables involved are signed. We list some propositions that make these manipulations possible.

(i) *Weak substitution*:

$\begin{cases} x \vartriangle a \\ cx \vartriangle b \end{cases}$ and $x \in R_{max}^{v} \Longrightarrow ca \vartriangle b$

(ii) *Weak transitivity*:

If we let $c = e = 0$ in (i), we have $a \vartriangle x, x \vartriangle b$ and $x \in R_{max}^{v} \Longrightarrow a \vartriangle b$

(iii) *Reduction of balances*:

$x \vartriangle y$ where x and $y \in R_{max}^{v} \Longrightarrow x = y$ (translating balances into equations)

(iv) *Equations to balances*:

Equation $Ax \oplus b = Cx \oplus d$ in R_{max} can be rewritten in $\mathbb{S}_{max}$ as $Ax \oplus b \triangle Cx \oplus d$ and by reflexivity and property of balance (iii) we have $(A \ominus C)x \oplus (b \ominus d) \triangle \theta$. Where A, B and C are matrices.

Note that it is immediate to extend balances to vectors and matrices by applying the balance operator elementwise with appropriate dimensions, provided we replace "$\in R_{max}^{v}$" by "every entry $\in R_{max}^{v}$". The above general properties of balances and the associated propositions also hold in matrix form of balances.

Example 4.6

Find the solutions of:

$$\begin{cases} \max(x, y-4, 1) = \max(x-1, y+1, 2) \\ \max(x+3, y+2, -5 = \max(y+2, 7) \end{cases} \quad 4.6$$

Solution

Matrix form of (4.6) is

$$\begin{bmatrix} 0 & -4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} \oplus \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -\infty & 2 \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 7 \end{bmatrix} \quad 4.7$$

Recall that if $a > b$, the equation $a = x \oplus b$ is equivalent to $a = x$ and this suggests the naive rule that: $a \ominus b = a$ if $a > b$ 4.8

Equations 4.7 $\Leftrightarrow$ (i) $x \oplus (-4)y \oplus 1 = (-1)x \oplus 1y \oplus 2$

(ii) $3x \oplus 2y \oplus (-5) = 2y \oplus 7$

$(i) \Longrightarrow [0 \ominus (-1)]x = [1 \ominus (-4)]y \oplus 2 \ominus 1)$

$$\Longrightarrow x = 1y \oplus 2 \qquad 4.9$$

From (ii) and (4.9) $\Longrightarrow 3(1y \oplus 2) \oplus 2y \oplus (-5) = 2y \oplus 7$

$\Longrightarrow (4 \oplus 2 \ominus 2)y = 7 \ominus (-5) \ominus 5$

$\Longrightarrow 4y = 7 \qquad \Longrightarrow y = 3$

From equation (4.9) $x = 1 \otimes 3 \oplus 2 = 4$

It is easy to see that $(x, y) = (4, 3)$ is a solution of system (4.6).

4.6 Determinants in Symmetrized Max–Algebra

We need to extend the notion of determinant to the $\mathbb{S}_{max}$ before dealing with general systems. Given a matrix $A \in \mathbb{S}_{max}$, the determinant of A denoted by $det(A)$ or $|A|$ is defined as follows:

$|A| = \oplus_{\sigma \in P_n} Sgn(\sigma) \otimes_{i=1} a_{i\sigma(i)}$

P_n is the set of all permutations of $\{1, 2, \ldots, n\}$ and $Sgn(\sigma)$ is the sign of the permutation σ defined as follows:

$$Sgn(\sigma) = \begin{cases} e & if\ \sigma\ is\ even \\ \ominus e & if\ \sigma\ is\ odd \end{cases}$$

Remark 4.8

Besides developing some extra properties of balances, Gaubert (1992) demonstrates that the balance operation can be extended to vectors and matrices by applying it component wise. With the definition of determinant Akian *et al*. (1990) shows that a non–degenerate systems of linear balances always have a unique solution. Since our interest

lies in translating some main results of R_{max} algebras into multiset environment, we will not go into many other finer details. Nevertheless, we propose to develop at this juncture a new algebra and call it a "minimal $\mathbb{S}_{max}$" algebra, abbreviated as $\approx \mathbb{S}_{max}$.

4.7 An outline of $\approx \mathbb{S}_{max}$:

We confine our attention to those properties of balances which are only reflexive and symmetric but not transitive. Such a relation is called a *compatibility* relation. We construct an algebra defined by compatibility relation defined on $\beta = R_{max} \times R_{max}$ or R_{max}^2 (Singh, et al., 2009).

Definition 4.13 (*Covering of a set*)

Let X be a given set and let $A = \{A_1, A_2, \ldots, A_m\}$ where each A_k, $k = 1, 2, \ldots, m$ is a nonempty subset of X and $\bigcup_{k=1}^{m} A_k = X$, then A is called a cover of X. Note that A_k's are not necessarily disjoint, and hence it may not define a partition.

Definition 4.14 (*Compatibility relation*)

A relation R in set X is said to be a "compatibility" relation, sometimes denoted by $\approx$, if it is reflexive and symmetric. Obviously, all equivalence relations are compatibility relations. We shall, however, be concerned herewith those compatibility relations which are not equivalence relations.

For an illustration, let us consider

X = {{2, 1, 6, 6}, {2, 4, 3,}, {3, 7, 5}, {6, 4, 8}, {4, 5, 5}}

= $\{x_1, x_2, x_3, x_4, x_5\}$, where x_1 = {2, 1, 6, 6}, x_2 = {2, 4, 3,}, x_3 = {3, 7, 5}, etc. respectively; and R be given by

$R = \{(x, y) \ / \ x, y \ \in X \wedge xRy$ if x and y contain some common elements$\}$

Clearly, $\{2, 1, 6, 6\}\ R\ \{2, 4, 3\} \wedge \{2, 4, 3\}\ R\ \{4, 5, 5\} \nRightarrow \{2, 1, 6, 6\}\ R\ \{4, 5, 5\}$

Then R is a compatibility relation. We say that x, y are compatible if xRy. Note that the elements of X could be multisets as well.

Definition 4.15 (*Compatibility relation on* $\beta = R_{max} \times R_{max}$ or R_{max}^2)

Let $x = (x_1,\ x_2),\ y = (y_1, y_2)$. We say that $x \approx$ y iff $x \Delta y$.

Hence, it follows that all the properties of $\mathbb{S}max$ algebra, except only those which require transitivity, hold in $\approx \mathbb{S}max$, called *minimal* $\mathbb{S}max$.

4.8 Conclusions

The prime motivation for introducing $\approx \mathbb{S}max$ lies in the fact that a compatibility relation defines a covering, which has been found useful in solving certain *minimization* problems of switching circuit algebra, particularly for the class of minimization problems that are incompletely specified (Tremblay and Manohar, 1997).

CHAPTER FIVE

MULTISET ALGEBRAS

5.0 Introduction

In this final chapter, we present techniques for constructing multiset algebra, study their basic properties. We emphasize that the field of real number $\mathbb{R}$ is the underlying structure which motivates the development of multiset algebras (mset algebras, for short). In *Section 5.1* we present an outline of an mset algebra constructed in Poplin (2000) and Brink (1987) and point out some vexing problems. We briefly explicate the notion of tolerance multiset in *Section 5.2*. In *Section* 5.3 and 5.4, we present multiset space algebra, which provides a new direction in the study of multiset algebras. In *Section 5.5* and *5.6*, we develop the concept of functions and metrics on multiset spaces. Finally, in *Section 5.7*, we present an exhaustive research on complementation in multiset–like structures, including multiset algebras.

5.1 Constructing Multiset Algebras

The technique for constructing multiset algebra is to begin with a set S along with two binary operations $\oplus$ (addition) and $\odot$ (multiplication) such that both the operations are commutative and associative, there exist additive and multiplicative identities (denoted by θ and e respectively) and $\odot$ is distributive over $\oplus$. We let $\mathcal{M}(\mathrm{S})$ be the set containing all finite msets built from S. In other words, $\mathcal{M}(\mathrm{S})$ is the power set of all finite msubsets of S, which is sometimes denoted by 3^{S} (see (poplin, 2000)) in order to

distinguish it from 2^S which denotes the powerset of S. We choose to use $\alpha, \beta, \gamma, \ldots$ to denote objects of 3^S.

Let $\uplus$, called *fusion*, an addition and $\otimes$, a *multiplication* be defined on 3^S. Various properties of $\uplus$ were mentioned in *Section 3.4*. However, to emphasize our findings in regards to 3^S, we state some properties of $\uplus$ and $\otimes$ under 3^S.

5.1 Properties of $\uplus$ and $\otimes$ under 3^S

(i) 3^S is closed under $\uplus$

(ii) $\uplus$ is commutative, associative with $\emptyset$, the empty mset, as the additive identity, which is the additive identity of S as well.

(iii) 3^S is cancellative, that is, if α, β and $\gamma \in 3^S$ such that $\alpha \uplus \beta = \alpha \uplus \gamma$, then $\beta = \gamma$.

(iv) $\uplus$ is not idempotent, that is, if $\alpha \in 3^S$ then $\alpha \uplus \alpha \neq \alpha$

(v) There are no additive inverses in 3^S, that is, if $\alpha \in 3^S$ and $\alpha \neq \emptyset$, then $\nexists\, \beta \in 3^S$, such that $\alpha \uplus \beta = \emptyset$.

(vi) 3^S is zerosumfree, that is, $\alpha \uplus \beta = \emptyset$ implies that $\alpha = \beta = \emptyset$.

(vii) $|\alpha \uplus \beta| = |\alpha| + |\beta|$.

(viii) 3^S is closed under $\odot$.

(ix) $\odot$ is commutative, associative with $\{1\}$ as the multiplicative identity, which is the multiplicative identity of S as well.

(xi) $\odot$ is distributive over $\uplus$.

(x) $|\alpha \odot \beta| = |\alpha| \cdot |\beta|$.

(xi) $\emptyset$ is absorbing under $\odot$, that is, $\emptyset \odot \alpha = \emptyset$.

(xii) If $\alpha \odot \beta = \gamma \odot \beta$, then it does not follows that $\alpha = \gamma$, for example

$\{4\} \odot \{3, -3\} = \{-4\} \odot \{3, -3\}$

(xiii) Multiplicative inverses in 3^S: If $a \in S$ is invertible in S, then $\{a\} \odot \{a^{-1}\} = \{1\}$

Note that a multiset $\alpha \in 3^S$ can be splited into three parts such as $\alpha = (\alpha^+, \theta_\alpha, \alpha^{-1})$, where $\alpha^+$is the positive elements of α, $\alpha^{-1}$is the negative elements of α and θ_α is the mset consisting of all the zeros of α.

With these tools (Poplin, 2000) show that multisets provide a connection between the eigenvalue and eigenvector equations for the max–plus and nonnegative real number systems and concludes that multisets provide a more general setting.

Brink, (1987) formally define a multiset as a mapping from universal set U into some set of numbers (integral values only). That is, $\alpha: U \to N$ where U is a universal set and N the set of natural numbers including zero. The set of all mappings $\alpha: U \to N$ is denoted by U^N. The multisets have a natural lattice structure, already noted in (Knuth, 1981). Namely, the definitions

$(\alpha \vee \beta)(x)$ for every $x \in U$

$(\alpha \wedge \beta)(x)$	for every $x \in U$
$0(x) = 0$	for every $x \in U$
$\alpha = \beta$ iff $\alpha(x) = \beta(x)$	for every $x \in U$
$\alpha \leq \beta$ iff $\alpha(x) = \beta(x)$	for every $x \in U$
$(\alpha + \beta)(x) = \alpha(x) + \beta(x)$	for every $x \in U$
$(\alpha\beta)(x) = \alpha(x)\beta(x)$	for every $x \in U$
$1(x) = x$	for every $x \in U$

With these definitions, (Brink, 1987) concludes that the structure (U^N, $\vee$, $\wedge$, 0) is a distributive lattice in which $\alpha \leq \beta$ iff $\alpha \vee \beta = \beta$.

Also, for every α and β there is a multiset $\beta \rightarrow \alpha$ which is the largest multiset p such that $p\beta \leq \alpha$. Actually, $\beta \rightarrow \alpha$ is that mapping such that for every $x \in U$, $\beta \rightarrow \alpha$ is the quotient upon division in N of $\alpha(x)$ by $\beta(x)$, without remainder. These properties are precisely those used in (Anderson and Belnap, 1975) to define the concept of a *Dunn monoid*, that is, the algebra correspond to the positive part of the relevant logic $\mathcal{R}$. Then, the structure (U^N, $\vee$, $\wedge$, 0, $\rightarrow$, 1) is a Dunn monoid.

Remark 5.1

With reference to *Section 5.1*, 3^S is a semiring with a cancellative property. Cancellative is defined as dropping elements of a multiset from each side of an equation. R^*_{max} is an

idempotent semiring. In 3^S idempotency is traded for cancellation property. In comparing the two semirings, (Poplin, 2000) dropped the addition operation on S ($\oplus$) and used only multiplication in the development of multiset algebras.

It would be nice if we could obtain from multiset in a natural way the algebraic structure corresponding to the relevant logic $\mathcal{R}$ itsef. That is, if we could obtain a De Morgan monoid (Anderson and Belnap, 1975), instead of just its positive version, a Dunn monoid. To do so requires a suitable notion of complementation. See *Section 5.7* in search for this complementation.

5.2 Tolerance Multiset and Applications

5.2.1 Concept of tolerance multiset

In this section, we develop the concept of tolerance multisets, largely based on (Marcus, 2001) as an application of our work on reflexive symmetric relation in chapter three. The theory of multisets presented in chapter three is based on the fundamental assumption that a multiset involves an equivalence relation between the copies of the same object. Nevertheless, in dealing with some real–world problems for example, Pogonowski (1981), in his work titled "*Tolerance spaces with applications in Linguistics*", showed that a binary relation relating an object to its copies is not an equivalence relation, but a weaker relation such as tolerance or compatibility (reflexive and symmetric) relation.

Exploiting this aspect of relationship between an object and its copies in an mset leads in a natural way to the notion of *tolerance multisets* as an extension of multisets. For

example, let us consider a multiset of coins of various values where for every value of a coin there are a large number of copies. Clearly, the relation between two coins of the same value is generally speaking an equivalence relation. However, there is no sharp border between an accepted and non–accepted coin by a machine. In this case, an equivalence relation is only a tolerance one. Another significant example is the relation of synonym in linguists. A more sophisticated situation appears in phonology, in which a basic building block is a class of equivalent sounds. It is shown in (Marcus, 2001) that the collection of *abstract* sounds is not a multiset, but a tolerance multiset. There exists an instinctive relation between tolerance relations and information systems.

Let us consider a class of objects with multiattributes and let f be a correspondence associating with each object attribute values. Clearly, f determines in the class of objects a tolerance relation t defined as follows: For any objects x and y, $x\ t\ y$ if and only if they have at least one attribute in common.

5.2.2 Topological perspective of the tolerance relations

Let t be a tolerance relation and (X, t) be a tolerance space. Define a closure operator cl which associates to any subset S of X a set $cl(S)$ contained in X, such that $cl(\emptyset) = \emptyset$, $S \subseteq cl(S)$, and if $A \subseteq B$, then $cl(A) = cl(B)$. Then the ordered pair (X, cl) is called a *Céch topological space* (Céch, 1966). In case $cl(S) = cl(cl(S))$ for any subset S of X, it becomes the usual closure operator associated with the usual topology. Now, it is only one step to see that any tolerance relation t in X, by defining $cl(S) = \{x|$ there exists y in S such that $xty\}$, generates a Céch topological space. However, the converse is not true. In fact, the tolerance closure operator cl_t is additive. That is,

$cl_t(A \cup B) = cl_t(A) \cup cl_t(B)$, while the Céch closure operator does not always possess this feature. The startling conclusion is that the tolerance multiset can be presented in the language of Céch topology (Pogonowski, 1981). We understand that the aforesaid established condition between tolerance space and Céch topological space is a tip of iceberg; in fact, it may lead to many other deep researches in mathematics.

5.3 Multiset Space Algebra

Let X be a finite set with cardinality m. Let $X^n(m)$ denote the set of all multisets built from elements in X such that no object in any of these multisets occurs more than n times. We call $X^n(m)$, the *multiset space* for X. It is easy to see that there exist $(n+1)^m$ number of such elements of $X^n(m)$. That is $|X^n(m)| = (n+1)^m$, where n and m are positive integers.

Definition 5.1

Let $X = \{a_1, a_2, a_3, \ldots, a_m\}$ be a set. Consider the multiset space, $X^n(m)$. In general, any multiset in $X^n(m)$ will have $m-$object with multiplicities less or equal to n, we can represent an arbitrary element of $X^n(m)$ by X_{p_i} where $p_i \in \{0, 1, 2, \ldots, n\}$. For simplification, multiset $[a_1, a_2, a_3, \ldots, a_m]_{7,\ 2,\ 5,\ldots,9}$ in $X^n(m)$ is written as $X_{7\ 2\ 5\ldots.9}$. Also, X_p means all the elements of X have the same multiplicity p and X_0 is referred to as the origin of $X^n(m)$ which is also called empty multiset.

Example 5.1

We compute $X^n(2)$ for various values of n.

Let $X = \{a, b\}$,

$X^0(2) = \{[a,b]_{0\,0}\}$

$X^1(2) = \{[a,b]_{0\,0}, [a,b]_{0\,1}, [a,b]_{1\,0}, [a,b]_{1\,1}\}$

$X^2(2) = \{[a,b]_{0\,0}, [a,b]_{0\,1}, [a,b]_{1\,0}, [a,b]_{1\,1}, [a,b]_{2\,0}, [a,b]_{0\,2}, [a,b]_{2\,1}, [a,b]_{1\,2}, [a,b]_{2\,2}\}$.

$X^3(2) = \{[a,b]_{0\,0}, [a,b]_{0\,1}, [a,b]_{1\,0}, [a,b]_{1\,1}, [a,b]_{1\,2}, [a,b]_{2\,1}, [a,b]_{0\,2}, [a,b]_{2\,0}, [a,b]_{2\,2}, [a,b]_{3\,0}, [a,b]_{0\,3}, [a,b]_{1\,3}, [a,b]_{3\,1}, [a,b]_{3\,2}, [a,b]_{2\,3}, [a,b]_{3\,3}\}$

$X^4(2) = \{[a,b]_{0\,0}, [a,b]_{0\,1}, [a,b]_{1\,0}, [a,b]_{1\,1}, [a,b]_{1\,2}, [a,b]_{2\,1}, [a,b]_{0\,2}, [a,b]_{2\,0}, [a,b]_{2\,2}, [a,b]_{2\,3}, [a,b]_{3\,2}, [a,b]_{3\,0}, [a,b]_{0\,3}, [a,b]_{3\,1}, [a,b]_{1\,3}, [a,b]_{3\,3}, [a,b]_{4\,0}, [a,b]_{0\,4}, [a,b]_{4\,1}, [a,b]_{1\,4}, [a,b]_{2\,4}, [a,b]_{4\,2}, [a,b]_{4\,3}, [a,b]_{3\,4}, [a,b]_{4\,4}\}$.

$X^5(2) = \{[a,b]_{0\,0}, [a,b]_{0\,1}, [a,b]_{0\,2}, [a,b]_{0\,3}, [a,b]_{0\,4}, [a,b]_{1\,0}, [a,b]_{1\,1}, [a,b]_{1\,2}, [a,b]_{1\,3}, [a,b]_{1\,4}, [a,b]_{2\,0}, [a,b]_{2\,1}, [a,b]_{2\,2}, [a,b]_{2\,3}, [a,b]_{2\,4}, [a,b]_{3\,0}, [a,b]_{3\,1}, [a,b]_{3\,2}, [a,b]_{3\,3}, [a,b]_{3\,4}, [a,b]_{4\,0}, [a,b]_{4\,1}, [a,b]_{4\,2}, [a,b]_{4\,3}, [a,b]_{4\,4}, [a,b]_{5\,0}, [a,b]_{0\,5}, [a,b]_{5\,1}, [a,b]_{1\,5}, [a,b]_{2\,5}, [a,b]_{5\,2}, [a,b]_{3\,5}, [a,b]_{5\,3}, [a,b]_{5\,4}, [a,b]_{4\,5}, [a,b]_{5\,5}\}$.

Similarly, $X^n(2)$ are obtained for $n = 6, 7, 8, \ldots$

We consider the cardinality of the element of $X^n(m)$ to generate a pattern with reference to frequency of its occurrence. See the tables 5.1 and 5.2, of $X^n(2)$ and $X^n(3)$ for various values of *n*.

Table 5.1 Frequency values of the cardinality of element of $X^n(2)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$X^0(2)$	1																
$X^1(2)$	1	2	1														
$X^2(2)$	1	2	3	2	1												
$X^3(2)$	1	2	3	4	3	2	1										
$X^4(2)$	1	2	3	4	5	4	3	2	1								
$X^5(2)$	1	2	3	4	5	6	5	4	3	2	1						
$X^6(2)$	1	2	3	4	5	6	7	6	5	4	3	2	1				
$X^7(2)$	1	2	3	4	5	6	7	8	7	6	5	4	3	2	1		
$X^8(2)$	1	2	3	4	5	6	7	8	9	8	7	6	5	4	3	2	1

In the *table 5.1*, the first row 0, 1, 2, 3, . . ., 16, represent the cardinality of the elements in $X^n(2)$ while the remaining rows represent the corresponding frequency. The set $X^0(2)$ contains only one element with cardinality zero, while the set $X^1(2)$ contains four elements; one element with cardinality zero, two elements with cardinality one and one element with cardinality two and so on for $X^n(2)$ for the remaining $n = 2, 3, 4, \ldots$

The number of elements in $X^n(m)$ is given by $(n + 1)^m$ which is the same as the summation of the frequency of the element with the same cardinality in $X^n(m)$ as

easily seen on the table above. The highest cardinality of an element ∈) is

which is unique while the smallest is zero. The highest frequency is (and the corresponding cardinality is .

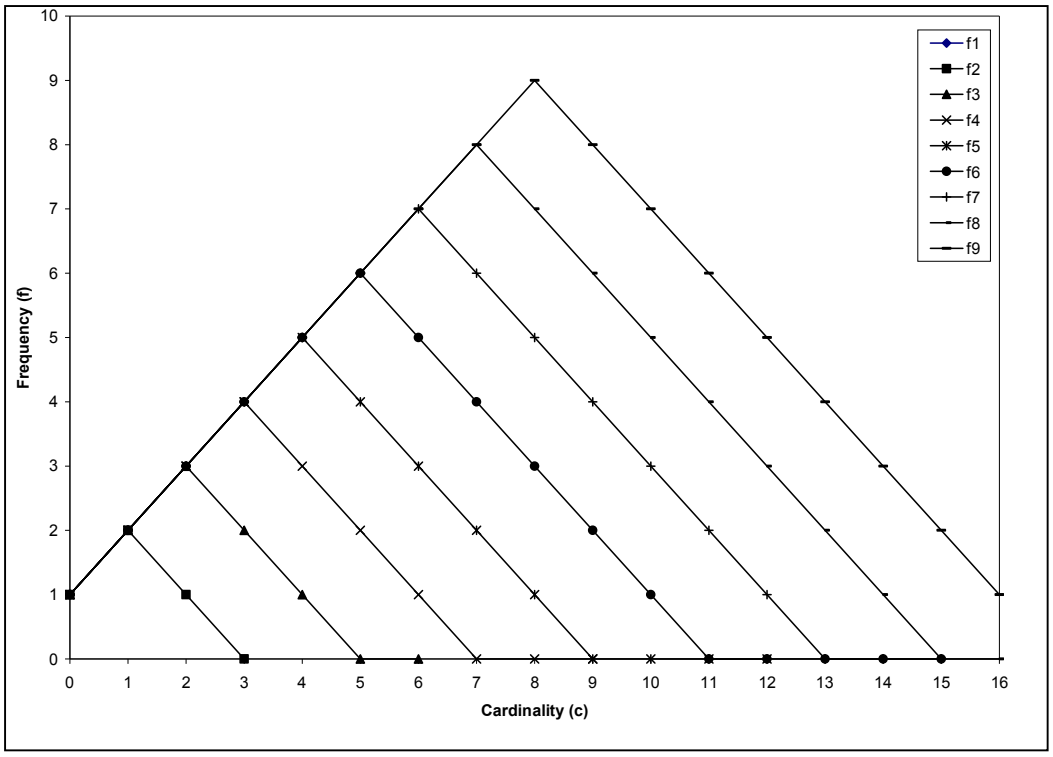


Fig. 5.1 Graphical representation of)

Remark 5.2

The graph of) is symmetric about .

Example 5.2

Let , we compute) for various values of .

Let

)

$X^1(3) = \{[a,b,c]_{0\,0\,0}, [a,b,c]_{0\,0\,1}, [a,b,c]_{0\,1\,0}, [a,b,c]_{0\,1\,1}, [a,b,c]_{1\,0\,0}, [a,b,c]_{1\,0\,1},$

$[a,b,c]_{1\,1\,0}, [a,b,c]_{1\,1\,1}\}.$

$X^2(3) = \{[a,b,c]_{0\,0\,0}, [a,b,c]_{0\,0\,1}, [a,b,c]_{0\,1\,0}, [a,b,c]_{0\,1\,1}, [a,b,c]_{1\,0\,0}, [a,b,c]_{1\,0\,1},$

$[a,b,c]_{1\,1\,0}, [a,b,c]_{1\,1\,1}, [a,b,c]_{0\,0\,2}, [a,b,c]_{0\,1\,2}, [a,b,c]_{0\,2\,0}, [a,b,c]_{0\,2\,1},$

$[a,b,c]_{0\,2\,2}, [a,b,c]_{1\,2\,0}, [a,b,c]_{1\,0\,2}, [a,b,c]_{1\,2\,1}, [a,b,c]_{1\,1\,2}, [a,b,c]_{1\,2\,2},$

$[a,b,c]_{2\,0\,0}, [a,b,c]_{2\,1\,0}, [a,b,c]_{2\,0\,1}, [a,b,c]_{2\,1\,1}\ [a,b,c]_{2\,0\,2}, [a,b,c]_{2\,2\,0},$

$[a,b,c]_{2\,1\,2}, [a,b,c]_{2\,2\,1}, [a,b,c]_{2\,2\,2}\}.$

Similarly, $X^n(3)$ are obtained for $n = 3, 4, \ldots$.

Table 5.2 Frequency values of $X^n(3)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$X^0(3)$	1																								
$X^1(3)$	1	3	3	1																					
$X^2(3)$	1	3	6	7	6	3	1																		
$X^3(3)$	1	3	6	10	12	12	10	6	3	1															
$X^4(3)$	1	3	6	10	15	18	19	18	15	10	6	3	1												
$X^5(3)$	1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1									
$X^6(3)$	1	3	6	10	15	21	28	33	36	37	36	33	28	21	15	10	6	3	1						
$X^7(3)$	1	3	6	10	15	21	28	36	42	46	48	48	46	42	36	28	21	15	10	6	3	1			
$X^8(3)$	1	3	6	10	15	21	28	36	45	52	57	60	61	60	57	52	45	36	28	21	15	10	6	3	1

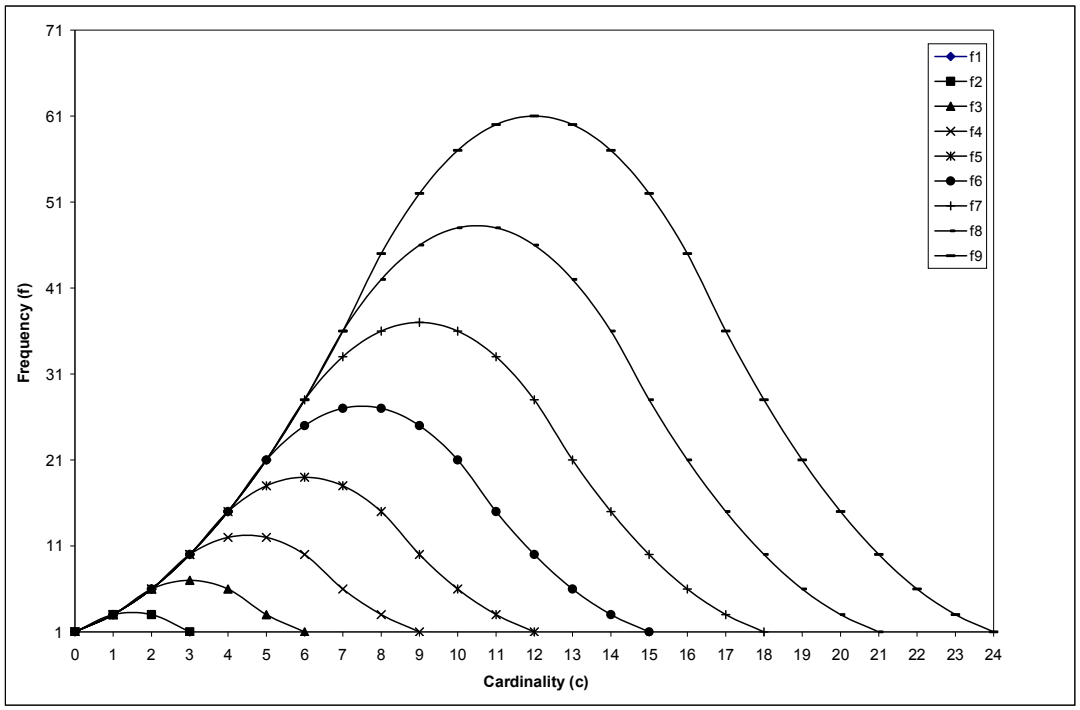


Fig. 5.2 Graphical representation of $X^n(3)$ for various values of n

Remark 5.3

$X^n(m)$ can be obtained for various values of n and m where n and m are positive integers. For m greater or equal to three, the graph of $X^n(m)$ is a normal graph.

5.3.1 Frequency number of multiset space

Let us call the number of elements of $X^n(m)$ with the same cardinality as the frequency number of that class of elements and denote it by $FX^n(m)$. This number can be generated using the recurrence formula as follows:

The frequency number of $X^n(2)$ is given by $FX^n(2)$ where

$$FX^n(2) = \begin{cases} \{k+1\}_n & 0 \le k \le n \\ 2n-k+1 & n+1 \le k \le 2n \end{cases}$$

The frequency number of $X^n(3)$ is given by

$$FX^n(3) = \begin{cases} \sum_{i=1}^{k}(i+1) & 0 \le k \le n \\ \sum_{i=1}^{n}(i+1) + \sum_{i=n+1}^{k}(3n-2i+1) & n+1 \le k \le 2n \\ \sum_{i=1}^{n}(i+1) + \sum_{i=n+1}^{2n}(3n-2i) + \sum_{i=2n+1}^{k}(i-3n-2) & 2n+1 \le k \le 3n \end{cases}$$

For $X^n(3)$, it is observed that for n < 2 the first two summations would generate the frequency numbers but for $n \geq 2$, all the three summations are necessary to generate this number.

Remark 5.4

No general rule is known for obtaining the frequency numbers for $X^n(m)$ for arbitrary set of values of m and n.

5.4 Algebraic Structures on Multiset Spaces

We define two new operations called union modulo (⊍) and product modulo (⊎), on the multiset spaces and explicate some of their properties.

Definition 5.2 (*Union Modulo* $\uplus$)

Let $X^n(m)$ be an mset space. Consider the set $X = \{a_1, a_2, a_3, \ldots, a_m\}$. Let $X_{p_i} = [a_1, a_2, a_3, \ldots, a_m]_{p_1\, p_2 \ldots p_m}$ and $X_{q_i} = [a_1, a_2, a_3, \ldots, a_m]_{q_1\, q_2 \ldots q_m}$ be any two elements of $X^n(m)$. The union modulo on $X^n(m)$ is defined as follows:

$X_{p_i} \uplus X_{q_i} = X_{(p_i + q_i)\ mod\ (n+1)}$ for $i = 1, 2, \ldots, m$

This operation is well defined for if $X_{p_i} = [a_1, a_2, a_3, \ldots, a_m]_{p_1\, p_2 \ldots p_m} \in X^n(m)$ and $X_{q_i} = [a_1, a_2, a_3, \ldots, a_m]_{q_1\, q_2 \ldots q_m} \in X^n(m)$ then

$X_{p_i} \uplus X_{q_i} = X_{k_i} \in X^n(m)$.

Since

$X_{p_i} \uplus X_{q_i} = [a_1, a_2, a_3, \ldots, a_m]_{p_1\, p_2 \ldots p_m} \uplus [a_1, a_2, a_3, \ldots, a_m]_{q_1\, q_2 \ldots q_m}$

$= [a_1, a_2, a_3, \ldots, a_m]_{(p_1 + q_1)\ mod\ (n+1),\ (p_2 + q_2)\ mod\ (n+1),\ \ldots, (p_m + q_m)\ mod\ (n+1)}$

$= X_{k_i}$

But $(p + q) mod\ (n + 1) = k$

$\Rightarrow p + q = (n + 1)z + k, \quad k < (n + 1),\ z \in \mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$

Then $(p_i + q_i)\ mod\ (n + 1) = k_i$

$\Rightarrow (p_i + q_i) = (n+1)z_i + k_i$, and $k_i < (n + 1)$ for $i = 1, 2, \ldots m$

It follows that $X_{p_i} \uplus X_{q_i} = X_{(p_i + q_i)\ mod\ (n+1)} = X_{(p_i + q_i) < (n+1)} \in X^n(m)$

For example, let $X = \{a, b, c\}$ such that $X_{3,2,4}$ and $X_{4,1,1} \in X^5(3)$ then

$X_{3,2,4} \uplus X_{4,1,1} = X_{1,3,5}$

5.4.1 Properties of union modulo $\uplus$ on $X^n(m)$

In this section, we list the basic properties of the operation $\uplus$ on $X^n(m)$. Let $X_{p_i} = X_{p_1 p_2 \ldots p_m}$, $X_{q_i} = X_{q_1 q_2 \ldots q_m}$, $X_{r_i} = X_{r_1 r_2 \ldots r_m} \in X^n(m)$ then we have the following:

a. *Closure*:

$X^n(m)$ is closed under $\uplus$

b. *Commutativity*:

$X_{p_i} \uplus X_{q_i} = X_{(p_i + q_i) \ mod \ (n+1)} = X_{(q_i + p_i) \ mod \ (n+1)} = X_{q_i} \uplus X_{p_i}$

Hence, the operation $\uplus$ is commutative.

c. *Associativity*:

$$X_{p_i} \uplus (X_{q_i} \uplus X_{r_i}) = X_{p_i} \uplus \left(X_{(q_i + r_i) \ mod \ (n+1)}\right) = X_{(p_i + (q_i + r_i)) \ mod \ (n+1)}$$

$$= X_{((p_i + q_i) + r_i) \ mod \ (n+1)} = (X_{p_i} \uplus X_{q_i}) \uplus X_{r_i})$$

Hence, $\uplus$ is associative.

d. *Identity law*:

$\emptyset = X_0$, the empty set, is the union modulo identity element of $X^n(m)$

Let $X_e \cup X_p = X_p$ then X_e is the identity element in $X^n(m)$

But $X_e \cup X_p = X_p \implies (e + p)\ mod\ (n + 1) = p$

$\implies \exists\ z$ such that $e + p = (n + 1)z + k$ where z and $k \in \mathbb{N}$

But $k = p$, then $e + p = (n + 1)z + p$

$\implies e = (n + 1)z = 0$

Therefore, X_0 is the identity element of $X^n(m)$.

e. Existence of inverses:

Let $X_p \in X^n(m)$, if $\exists\ X_q \in X^n(m)$ such that $X_p \cup X_q = X_0$ then X_p and X_q are said to be inverses of each other.

$X_p \cup X_q = X_0 \implies (p + q)\ mod\ (n + 1) = 0$

i.e. $p + q = (n + 1)z \qquad z \in \mathbb{N} \cup \{0\}$

$p = (n + 1)z - q$ or $q = (n + 1)z - p$

Consequently, $p_i = (n + 1)z_i - q_i$ since $q_i < (n + 1)$, $p_i > 0$ for $i = 1, 2, \ldots, m$

or $q_i = (n + 1)z_i - p_i$ since $p_i < (n + 1)$, $q_i > 0$ for $i = 1, 2, \ldots m$

Hence, given $X_p \in X^n(m)$, $X_{(n+1)z-p} \in X^n(m)$ is its inverse.

Remark 5.5

If $n = 1$, every element of $X^n(m)$ is its own inverse.

f. Idempotency:

The operation, $\uplus$, is idempotent if $X_p \uplus X_p = X_p$,

i.e., $2p = (n+1)z + p$

i.e., $p = (n+1)z$

$= 0$

$\Longrightarrow X_0$ is the only trivial idempotent element in $X^n(m)$ for the operation $\uplus$

g. $|X_p \uplus X_q| \leq |X_p| + |X_q|$

h. $X^n(m)$ is not cancellative under $\uplus$.

If $X_p \uplus X_q = X_p \uplus X_r$, it is not necessarily implies that $X_q = X_r$. For instance if $|X_p \uplus X_q| = |X_p \uplus X_r|$ it is not necessary means that $X_q = X_r$. For example, $|X_{o\,o} \uplus X_{2\,1}| = |X_{0\,0} \uplus X_{1\,2}| = 3$, but $X_{2\,1} \neq X_{1\,2}$.

i. $X^n(m)$ is not zerosumfree since there is existence of inverse for each element in $X^n(m)$, i.e. $X_p \uplus X_q = X_o$, it is not necessary means $p = q = 0$

j. Distributive properties:

For any X_p, X_q, and $X_r \in X^n(m)$

(i) $X_p \uplus (X_q \cup X_r) \neq (X_p \uplus X_q) \cup (X_p \uplus X_r)$

(ii) $X_p \cup (X_q \uplus X_r) \neq (X_p \cup X_q) \uplus (X_p \cup X_r)$

(iii) $X_p \uplus (X_q \cap X_r) \neq (X_p \uplus X_q) \cap (X_p \uplus X_r)$

(iv) $X_p \cap (X_q \uplus X_r) \neq (X_p \cap X_q) \uplus (X_p \cap X_r)$

Remark 5.6

The operation $\uplus$ is at par with the operations $\cup$ and $\cap$ since none is distributed over each other.

Definition 5.2.1(*Propositions*)

(i) $(\uplus X_p)^c = X_p \uplus X_p \uplus \ldots \uplus X_p$ (c times)

(ii) If $(\uplus X_p)^c = X_e$, then X_p is said to be a circle of length c

(iii) Every element of $X^n(m) \setminus X_e$ is a circle of length $(n+1)$.

Definition 5.3 (*Product modulo* $\uplus$)

Let $X^n(m)$ be an multiset space. Consider the set $X = \{a_1, a_2, a_3, \ldots, a_m\}$. Let $X_{p_i} = [a_1, a_2, a_3, \ldots, a_m]_{p_1 p_2 \ldots p_m}$ and $X_{q_i} = [a_1, a_2, a_3, \ldots, a_m]_{q_1 q_2 \ldots q_m}$ be any two elements of $X^n(m)$. The product modulo, $\uplus$, on $X^n(m)$ is defined as

$$X_{p_i} \uplus X_{q_i} = X_{(p_i q_i) \bmod n} \text{ for } i = 1, 2, \ldots, m$$

This operation is well defined for if $X_{p_i} = [a_1, a_2, a_3, \ldots, a_m]_{p_1 p_2 \ldots p_m} \in X^n(m)$ and $X_{q_i} = [a_1, a_2, a_3, \ldots, a_m]_{q_1 q_2 \ldots q_m} \in X^n(m)$, then

$$X_{p_i} \uplus X_{q_i} = X_{k_i} \in X^n(m).$$

Since $X_{p_i} \uplus X_{q_i} = [a_1, a_2, a_3, \ldots, a_m]_{p_1 p_2 \ldots p_m} \uplus [a_1, a_2, a_3, \ldots, a_m]_{q_1 q_2 \ldots q_m}$

$= [a_1, a_2, a_3, \ldots, a_m]_{(p_1 q_1) \bmod (n+1),\ (p_2 q_2) \bmod (n+1),\ \ldots,\ (p_m q_m) \bmod (n+1)}$

$= X_{k_i}$.

But $(pq) mod\ (n+1) = k$

$\Longrightarrow pq = (n+1)z + k, \quad k < (n+1),\ z \in \mathbb{N}$

Then $(p_i\, q_i)\ mod\ (n+1) = k_i$

$\Longrightarrow (p_i\, q_i) = (n+1)z_i + k_i$, and $k_i < (n+1)$ for $i = 1, 2, \ldots m$

It follows that $X_{p_i} \uplus X_{q_i} = X_{(p_i\, q_i)\ mod\ (n+1)} = X_{(p_i\, q_i)\ <(n+1)} \in X^n(m)$

For example, let $X = \{a, b, c\}$ such that $X_{3,2,4}$ and $X_{4,1,1} \in X^5(3)$ then

$X_{3,2,4} \uplus X_{4,1,1} = X_{0,2,4}$

5.4.2 Properties of product modulo $\uplus$ on $X^n(m)$

In this section we list the basic properties of the operation $\uplus$ on $X^n(m)$. Let $X_{p_i} = X_{p_1 p_2 \ldots p_m}$, $X_{q_i} = X_{q_1 q_2 \ldots q_m}$, $X_{r_i} = X_{r_1 r_2 \ldots r_m} \in X^n(m)$ then we have the following:

a. *Closure*:

   $X^n(m)$ is closed under $\uplus$

b. *Commutativity:*

   $X_{p_i} \uplus X_{q_i} = X_{(p_i\, q_i)\ mod\ (n+1)} = X_{(q_i\, p_i)\ mod\ (n+1)} = X_{q_i} \uplus X_{p_i}$

   Hence, the operation $\uplus$ is commutative.

c. Associativity:

$$X_{p_i} \uplus (X_{q_i} \uplus X_{r_i}) = X_{p_i} \uplus \left(X_{(q_i r_i)\ mod\ (n+1)}\right) = X_{(p_i(q_i r_i))\ mod\ (n+1)}$$

$$= X_{((p_i\, q_i) r_i)\ mod\ (n+1)} = (X_{p_i} \uplus X_{q_i}) \uplus X_{r_i}.$$

Hence, $\uplus$ is associative.

d. Identity law:

Let $X_e \uplus X_p = X_p$, then X_e is the identity element in $X^n(m)$.

$X_e \uplus X_p = X_p \implies (ep)\ mod\ (n+1) = p$

$\implies \exists\ z$ such that $ep = (n+1)z + k$ where z and $k \in \mathbb{N} = 0, 1, 2 \ldots$

But $k = p$, then $ep = (n+1)z + p$

and $(n+1)z = 0$

$\implies ep = p$

i.e., $e = 1$

Therefore, X_1 is the identity of the product modulo operation on $X^n(m)$

f. X_0 is absorbing under $\uplus$. i.e. $X_0 \uplus X_p = X_0$

g. $|X_p \uplus X_q| \leq |X_p||X_q|$

h. If $X_p \uplus X_r = X_q \uplus X_r$, it does not necessary follows that $X_p = X_q$. That is,

$X^n(m)$ is not cancellative under $\uplus$.

For example, let $X_{1\,0\,1}, X_{1\,1\,0}, X_{1\,1\,1} \in X^2(3)$.

$X_{101} \uplus X_{111} = X_{101}$

$X_{101} \uplus X_{101} = X_{101}$. But $X_{111} \neq X_{101}$

i. Existence of inverses:

Let $X_p \in X^n(m)$, if $\exists\ X_q \in X^n(m)$ such that $X_p \uplus X_q = X_1$ then X_p and X_q are said to be inverses of each other.

$X_p \uplus X_q = X_1 \Longrightarrow (pq)\ mod\ (n+1) = 1$

i.e. $pq = (n+1)z + 1$

$p = {}^{q}/_{(n+1)z+1)}$ or $q = {}^{p}/_{(n+1)z+1)}$ for minimum value of z

Consequently, $p_i = {}^{q_i}/_{(n+1)z_i+1)}$ or $q = {}^{p_i}/_{(n+1)z_i+1)}$ for minimum value of z_i

Hence, given $X_{p_i} \in X^n(m)$, $X_{p_i}/_{(n+1)z_i+1)} \in X^n(m)$ is its inverse.

j. Idempotency:

The operation, $\uplus$, is idempotent if $X_p \uplus X_p = X_p$,

i.e. $p^2 = (n+1)z + p$

i.e. $p^2 - p - (n+1)z = 0$

$$p = \frac{1 \pm \sqrt{1+4(n+1)z}}{2} \qquad 5.1$$

$p = 0, 1$ or X_0 and X_1 are the trivial idempotent elements of $X^n(m)$ under product modulo. We shall generate the non–trivial idempotent elements later.

Remark 5.7

(i) Intersection modulo coincide with the usual intersection in standard set theory.

(ii) Arbitrary union or product modulo of two or more elements of $X^n(m)$ degenerates into singleton.

Remark 5.8

In general, $X^n(m)$, with respect to the operations union modulo and product modulo, is not idempotent, but there are some special elements in $X^n(m)$ that are nontrivial idempotent. Below is a *table 5.3* for some nontrivial idempotent elements in $X^n(m)$ for various values of *n* and *m*.

Table 5.3 Non trivial idempotent elements in $X^n(m)$

n	5	11	19	29	41	55	71	……
p	3	4	5	6	7	8	9	…….

These values of *p* and *n* are obtained from equation 5.1 for $z = 1$.

For example, consider the multiset spaces $X^5(3)$, $X^{11}(2)$.

Let $X_3 \in X^5(3)$, $X_4 \in X^{11}(2)$.

$X_3 \in X^5(3) \Longrightarrow [a, b, c]_{3\,3\,3} \in X^5(3)$.

Now, $[a, b, c]_{3\,3\,3} \uplus [a, b, c]_{3\,3\,3} = [a, b, c]_{3\,3\,3}$

$X_4 \in X^{11}(2) \Longrightarrow [a, b]_{4\,4} \in X^{11}(2)$.

$[a, b]_{4\,4} \uplus [a, b]_{4\,4} = [a, b]_{4\,4}$.

Remark 5.9

No other element in $X^5(3)$ and $X^{11}(2)$ that has this property under $\uplus$ except the trivial elements X_0 and X_1.

The relation between n and p is $n = p^2 - p - 1$, while from $eq.1$ and using Lagrange's theorem, the relation between n and z is obtained as $n = z^2 + 3z + 1$

k. Distributive properties:

For any X_p, X_q, and $X_r \in X^n(m)$, in general

(i) $X_p \uplus (X_q \biguplus X_r) = (X_p \uplus X_q) \biguplus (X_p \uplus X_r)$

(ii) $X_p \biguplus (X_q \uplus X_r) \neq (X_p \biguplus X_q) \uplus (X_p \biguplus X_r)$

(iii) $X_p \uplus (X_q \cup X_r) \neq (X_p \uplus X_q) \cup (X_p \uplus X_r)$

(iv) $X_p \cup (X_q \uplus X_r) \neq (X_p \cup X_q) \uplus (X_p \cup X_r)$

(v) $X_p \uplus (X_q \cap X_r) \neq (X_p \uplus X_q) \cap (X_p \uplus X_r)$

(vi) $X_p \cap (X_q \uplus X_r) \neq (X_p \cap X_q) \uplus (X_p \cap X_r)$

Remark 5.10

The operation $\uplus$ is distributed over $\biguplus$ while the reverse is not true, similar to multiplication and addition operations on the set of real numbers.

Theorem 5.1

A nonempty multiset space $X^n(m)$, equipped with the operations of union modulo ($\uplus$) and product modulo ($\biguplus$), resembles with many algebraic structures as listed below.

(i) $(X^n(m), \uplus)$ is a monoid, a semigroup and an abelian group

(ii) $(X^n(m), \biguplus)$ is a monoid, a semigroup and an abelian group

(iii) $(X^n(m), \uplus, \biguplus)$ commutative ring

(iv) $(X^n(m), \bigcup, \biguplus)$ is a semidioids since $X^n(m)$ is not idempotent with respect to union or product modulo.

Proof

See *Sections 5.4.1* and *5.4.2* for properties of union modulo and product modulo under $X^n(m)$ and *Section 3.3.4* for algebraic systems.

5.5 Metrization of Multiset Spaces

For simplification, multiset $[a_1, a_2, a_3, \ldots, a_m]_{p_1, p_2, p_3, \ldots, p_m}$ in $X^n(m)$ is written as $X_{p_1, p_2, p_3, \ldots, p_m}$. Also, X_p means all the elements of X have the same multiplicity p and X_0 is referred to as the origin of $X^n(m)$. For example,

$X^2(2) = \{X_{0\,0}, X_{0\,1}, X_{1\,0}, X_{1\,1}, X_{1\,2}, X_{2\,1}, X_{0\,2}, X_{2\,0}, X_{2\,2}\}$.

$X^3(2) = \{X_{0\,0}, X_{0\,1}, X_{1\,0}, X_{1\,1}, X_{1\,2}, X_{2\,1}, X_{0\,2}, X_{2\,0}, X_{2\,2}, X_{3\,0}, X_{0\,3}, X_{1\,3}, X_{3\,1}, X_{3\,2}, X_{2\,3}, X_{3\,3}\}$.

$X^4(2) = \{X_{0\,0},\ X_{0\,1},\ X_{1\,0},\ X_{1\,1},\ X_{1\,2},\ X_{2\,1},\ X_{0\,2},\ X_{2\,0},\ X_{2\,2}, X_{3\,0}, X_{0\,3}, X_{1\,3}, X_{3\,1},$

$X_{3\,2},\ X_{2\,3}, X_{3\,3}, X_{4\,0}, X_{0\,4}, X_{4\,1}, X_{1\,4}, X_{2\,4}, X_{4\,2}, X_{4\,3}, X_{3\,4}, X_{4\,4}\}.$

$X^1(3) = \{X_{0\,0\,0}, X_{0\,0\,1}, X_{0\,1\,0}, X_{0\,1\,1},\ X_{1\,0\,0}, X_{1\,0\,1}, X_{1\,1\,0}, X_{1\,1\,1}\}.$

$X^2(3) = \{X_{0\,0\,0}, X_{0\,0\,1}, X_{0\,1\,0}, X_{0\,1\,1},\ X_{1\,0\,0}, X_{1\,0\,1}, X_{1\,1\,0}, X_{1\,1\,1}, X_{0\,0\,2}, X_{0\,2\,0}, X_{2\,0\,0},$

$X_{0\,1\,2}, X_{0\,2\,1}, X_{0\,2\,2}, X_{1\,2\,0},\ X_{1\,0\,2}, X_{1\,2\,1}, X_{1\,1\,2}, X_{1\,2\,2}, X_{2\,1\,0}, X_{2\,0\,1},\ X_{2\,1\,1},$

$X_{2\,0\,2}, X_{2\,2\,0},\ X_{2\,1\,2}, X_{2\,2\,1}, X_{2\,2\,2}\}.$

5.5.1 Distance function on multiset spaces

Let $X_{p_i} = [a_1, a_2, a_3, \ . \ . \ ., a_m]_{p_1,\ p_2,\ .\ .\ .\ p_m}$ and

$X_{q_i} = [a_1, a_2, a_3, \ . \ . \ ., a_m]_{q_1,\ q_2,\ .\ .\ .\ ,q_m}$ be any two elements of $X^n(m)$. We define distance on the elements of $X^n(m)$ as follows:

$$d(X_{p_i}, X_{q_i}) = \sqrt{\sum_{i=1}^{m}(p_i - q_i)^2}.$$

We denote it by $k \in R^+$.

For example, given $X^2(2) = \{X_{0\,0},\ X_{0\,1},\ X_{1\,0},\ X_{1\,1},\ X_{1\,2},\ X_{2\,1},\ X_{0\,2},\ X_{2\,0},\ X_{2\,2}\}$

The possible k's are 0, 1, $\sqrt{2}$, 2, $\sqrt{5}$ and $\sqrt{8}$

This distance is well defined since for all $X_{p_i},\ X_{q_i} \in \ X^n(m),\ d(X_{p_i},\ X_{q_i}) \geq 0,$ $d(X_{p_i}, X_{q_i}) = 0$ iff $X_{p_i} = X_{q_i}$

Remark 5.11

For any $X^n(m)$ the maximum distance between the elements is $n\sqrt{m}$. That is, $d(X_{p_i}, X_{q_i}) \leq n\sqrt{m}$, for any $X_{p_i}, X_{q_i} \in X^n(m)$.

5.5.2 Relations on $X^n(m)$

Let $d(X_{p_i}, X_{q_i}) = k$ where $X_{p_i}, X_{q_i} \in X^n(m)$ and $k \in \boldsymbol{R}^+$. We say that X_{p_i} is R_k related to X_{q_i} written as $X_{p_i} R_k X_{q_i}$ or $(X_{p_i}, X_{q_i}) \in R_k$.

For example, consider the set $X^2(2)$

$X^2(2) = \{X_{0\,0}, X_{0\,1}, X_{1\,0}, X_{1\,1}, X_{1\,2}, X_{2\,1}, X_{0\,2}, X_{2\,0}, X_{2\,2}\}$.

$R_0 = \{(X_{0\,0}, X_{0\,0}), (X_{0\,1}, X_{0\,1}), (X_{1\,0}, X_{1\,0}), (X_{1\,1}, X_{1\,1}), (X_{1\,2}, X_{1\,2}), (X_{2\,1}, X_{2\,1}),$
$(X_{0\,2}, X_{0\,2}), (X_{2\,0}, X_{2\,0}), (X_{2\,2}, X_{2\,2})\}$.

$R_1 = \{(X_{0\,0}, X_{0\,1}), (X_{0\,0}, X_{1\,0}), (X_{0\,1}, X_{1\,1}), (X_{0\,1}, X_{0\,2}), (X_{1\,0}, X_{1\,1}), (X_{1\,0}, X_{2\,0}),$
$(X_{1\,1}, X_{1\,2}), (X_{1\,1}, X_{2\,1}), (X_{1\,2}, X_{0\,2}), (X_{1\,2}, X_{2\,2}), (X_{2\,1}, X_{2\,0}), (X_{2\,1}, X_{2\,2})\}$.

$R_{\sqrt{2}} = \{(X_{0\,0}, X_{1\,1}), (X_{0\,1}, X_{1\,2}), (X_{0\,1}, X_{1\,0}), (X_{1\,0}, X_{2\,1}), (X_{1\,1}, X_{2\,2}), (X_{1\,1}, X_{2\,0}),$
$(X_{1\,1}, X_{0\,2}), (X_{1\,2}, X_{2\,1})\}$.

$R_2 = \{(X_{0\,0}, X_{0\,2}), (X_{0\,0}, X_{2\,0}), (X_{0\,1}, X_{2\,1}), (X_{1\,0}, X_{1\,2}), X_{0\,2}, X_{2\,2}), (X_{2\,0}, X_{2\,2})\}$.

$R_{\sqrt{5}} = \{(X_{0\,0}, X_{1\,2}), (X_{0\,0}, X_{2\,1}), (X_{0\,1}, X_{2\,0}), (X_{0\,1}, X_{2\,2}), (X_{1\,0}, X_{0\,2}), (X_{1\,0}, X_{2\,2}),$
$(X_{1\,2}, X_{1\,2}), (X_{2\,1}, X_{0\,2})\}$.

$R_{\sqrt{8}} = \{(X_{0\,0}, X_{2\,2}), (X_{2\,0}, X_{0\,2})\}$.

Theorem 5.2

If R_i are the possible relations defined on $X^n(m)$ then $R = \cup_i R_i$ is an equivalence relation on $X^n(m)$.

Proof

(i) $R = \cup_i R_i \Rightarrow R_0 \in R$, hence, R is a reflexive relation.

(ii) Let $(X_{p_i}, X_{q_i}) \in R$ for all $X_{p_i}, X_{q_i} \in X^n(m)$

$\Rightarrow X_{p_i} R_k X_{q_i}$ for some $R_k \in R$ and $k \in \boldsymbol{R}^+$

$\Rightarrow d(X_{p_i}, X_{q_i}) = k$

$\Rightarrow d(X_{q_i}, X_{p_i}) = k$ from the definition of distance

Hence, $X_{q_i} R_k X_{pi}$ where $R_k \in R$ .i.e. $(X_{q_i}, X_{p_i}) \in R$. Therefore R is a symmetric relation.

(iii) Let $(X_{p_i}, X_{ri}), (X_{r_i}, X_{q_i}) \in R$ for all $X_{p_i}, X_{r_i}, X_{q_i} \in X^n(m)$

$\Rightarrow \exists\ k_1$ and k_2 such that $X_{p_i} R_{k_1} X_{ri}$ and $X_{r_i} R_{k_2} X_{qi}$

$\Rightarrow d(X_{p_i}, X_{ri}) = k_1$

and $d(X_{ri}, X_{qi}) = k_2$

since d is a metric, by triangle inequality $d(X_{p_i}, X_{q_i}) = k \leq k_1 + k_2$, where $k \in R^+$ which shows that R is a transitive relation.

Since R is reflexive, symmetric and transitive, it is an equivalence relation.

5.5.3 Functions on multiset spaces

Let $f: X^n(m) \to S \subseteq R^+$ such that $f(X_{p_i}) = d(X_{p_i}, X_0)$.

Where S is the set of all possible distances between the elements of $X^n(m)$.

This function is surjective, since $f(X_{p_i}) = d(X_{p_i}, X_0) \in S, \forall i$. The function is not injective since there exist $f(X_{p_i}) = f(X_{qi})$ and $X_{p_i} \neq X_{qi}$ for some $X_{p_i}, X_{qi} \in X^n(m)$.

For instance, consider the set,

$X^2(2) = \{X_{0\,0}, X_{0\,1}, X_{1\,0}, X_{1\,1}, X_{1\,2}, X_{2\,1}, X_{0\,2}, X_{2\,0}, X_{2\,2}\}$.

$f: X^2(2) \to S \subseteq R^+$ such that

$f(X_{p\,q}) = d(X_{p\,q}, X_{0\,0})$

From this definition, $f(X_{0\,0}) = d(X_{0\,0}, X_{0\,0}) = 0$

$$f(X_{0\,1}) = d(X_{0\,1}, X_{0\,0}) = 1$$

$$f(X_{1\,0}) = d(X_{1\,0}, X_{0\,0}) = 1$$

$$f(X_{1\,1}) = d(X_{1\,1}, X_{0\,0}) = \sqrt{2}$$

$$f(X_{2\,0}) = d(X_{2\,0}, X_{0\,0}) = 2$$

$$f(X_{0\,2}) = d(X_{0\,2}, X_{0\,0}) = 2$$

$$f(X_{1\,2}) = d(X_{1\,2}, X_{0\,0}) = \sqrt{5}$$

$$f(X_{2\,1}) = d(X_{2\,1}, X_{0\,0}) = \sqrt{5}$$

$$f(X_{2\,2}) = d(X_{2\,2}, X_{0\,0}) = \sqrt{8}$$

$S = \{0,\ 1,\ 2,\ \sqrt{2},\ \sqrt{5},\ \sqrt{8}\} \subseteq R^+$.

Since $f(X_{0\,p}) = f(X_{p\,0})$ and $X_{0\,p} \neq X_{p\,0}$, the function is not an injective function.

5.5.4 Partitions on multiset spaces

We define partition on $X^n(m)$ as follows:

Let $\succ_k = \{X_{pi} : f(X_{pi}) = k\}$, 5.2

Where $f: X^n(m) \to S \subseteq R^+$ such that

$f(X_{p_i}) = d(X_{p_i}, X_0) = k \in S$ and $X_{p_i} \in X^n(m)$

Also, S is the set of all possible distances between the elements of $X^n(m)$.

Consider the set $X^2(2) = \{X_{0\,0},\ X_{0\,1},\ X_{1\,0},\ X_{1\,1},\ X_{1\,2},\ X_{2\,1},\ X_{0\,2}, X_{2\,0},\ X_{2\,2}\}$.

The mset space is partitioned using (5.2) above as follows:

$$\succ_k = \{X_{pq} : f(X_{pq}) = k\}$$

$$\succ_0 = \{X_{0\,0}\}$$

$$\succ_1 = \{X_{0\,1},\ X_{1\,0}\}$$

$$\succ_{\sqrt{2}} = \{X_{1\,1}\}$$

$$\succ_2 = \{X_{0\,2},\ X_{2\,0}\}$$

$$\succ_{\sqrt{5}} = \{X_{1\,2},\ X_{2\,1}\}$$

$$\succ_{\sqrt{8}} = \{X_{2\,2}\}$$

We now define a function on mset space $X^n(m)$, using the partition as follows:

$g: X^n(m) \to S \subseteq R^+$, such that $g(\succ_k) = k$.

Theorem 5.3

The function $g: X^n(m) \to S \subseteq R^+$ defined as $g(\succ_k) = k$, where $\succ_k$ is a partition defined on $X^n(m)$ is a bijective function.

Proof.

From the definition of partition the result follows.

5.6 Metric on Multiset Spaces

Abstracting from the definition of distance function (a metric) $d: X^n(m) \to R^+$, $(X^n(m), d)$ is a metric space since the real valued function d satisfies the following conditions:

(i) $d(X_{p_i}, X_{q_i}) \geq 0$, $d(X_{p_i}, X_{q_i}) = 0$ iff $X_{p_i} = X_{q_i}$

(ii) $d(X_{p_i}, X_{q_i}) = d(X_{q_i}, X_{p_i})$ (symmetric property)

(iii) $d(X_{p_i}, X_{q_i}) \leq d(X_{p_i}, X_{z_i}) + d(X_{z_i}, X_{q_i})$ (triangle inequality property)

Definition 5.3 (*Continuity in multiset space*)

Let $(X^n(m),\ d)$ be a metric space. The function $g: X^n(m) \to S \subseteq R^+$ defined by $g(\succ_k) = k$ where

$\succ_k = \{X_{pi} : f(X_{pi}) = k\}$ is a partition defined on $X^n(m)$ and $f(X_{p_i}) = d(X_{p_i}, X_0) = k \in S$, is a continuous function if given $\in > 0$, $\exists\ \delta(\in) > 0$ such that $d(\succ_x, \succ_k) < \in$ whenever $|x - k| < \delta(\in)$.

Definition 5.4 (*Submetric spaces of multiset space*)

Let $M = (X^n(m), d)$ be a metric space. If there exists $X^p(m) \subseteq X^n(m)$, $(p \leq n)$, then the function $d_{X^p(m)}$ the restriction of d to $X^p(m)$ is called the metric on $X^p(m)$ induced by d, and $(X^p(m), d_{X^p(m)})$ is a submetric space of M. That is, the distance between any two points in the space $(X^p(m), d_{X^p(m)})$ is the same as their distance in the space $(X^n(m), d)$.

Definition 5.5 (*Boundedness of Submetric space*)

Let $M = (X^n(m), d)$ be a metric space and $X^p(m) \subseteq X^n(m)$, $(p \leq n)$, then $X^p(m)$ is bounded if there exits $k \in R$ such that for all $X_{p_i},\ X_{q_i} \in X^p(m)$, $d(X_{p_i}, X_{q_i}) \leq k$.

Remark 5.12

If n is finite then the metric space, $(X^n(m), d)$, is bounded by $n\sqrt{m}$

Definition 5.6 (*Radius*)

Let $X^n(m)$ be a non empty bounded multiset then the radius of $(X^n(m)$ is defined as $rad(X^n(m)) = sup\ [d(X_{p_i}, X_{q_i})] = d(X_n, X_0)$, for all $X_{p_i},\ X_{q_i} \in X^n(m)$.

Proposition 5.1

(i) If $X^{n_1}(m)$ and $X^{n_2}(m)$ are bounded submetric spaces of a metric space $(X^n(m), d)$, then $X^{n_1}(m) \uplus X^{n_2}(m)$ and $X^{n_1}(m) \biguplus X^{n_2}(m)$ are also bounded metric spaces

(ii) If $X^{n_1}(m)$, $X^{n_2}(m)$, . . ., $X^n(m)$ are bounded submetric spaces of a metric space $(X^n(m), d)$, then $\biguplus_{i=1} X^{n_i}(m)$ and $\biguplus_{i=1} X^{n_i}(m)$ are also bounded metric spaces.

(iii) In any metric space, $(X^n(m), d)$, $B_{\in}(X_{p_i})$ is bounded and $rad\left(B_{\in}(X_{p_i})\right) \leq \in$.

Definition 5.7 (*Open balls in terms of Multiset Space*)

Let $(X^n(m), d)$ be a metric space and let $X_{p_i} \in X^n(m)$. Now for $\in > 0$, the open ball of radius $\in$ around X_{p_i} is defined as $B_{\in}(X_{p_i}) = \{X_{q_i} : d(X_{p_i}, X_{q_i}) < \in\}$.

Theorem 5.4 (Topology)

Let $X^n(m)$ be any multiset space and let τ be a family of subsets of $X^n(m)$, then τ is a topology on $X^n(m)$.

Proof

τ is a family of subsets of $X^n(m)$ then X_0, $X^n(m) \in \tau$. Arbitrary union modulo of elements of τ is an element of τ (see remark 5.7 (i)). Finally, by Remark 5.7 (ii), intersection of finitely many elements of τ is an element of τ. Hence, the result follows.

Definition 5.8 (*Topological space*)

$(X^n(m), \tau)$ is a topological space.

5.7 Complementation in Multiset Theory

In this section of the dissertation, we endeavour to survey the problem related to difference and complementation in multiset theory. We show that the attendant difficulties do remain unresolved unless some contrived stipulations are assumed.

Let A and B be two msets, and $B \subseteq A$. The (arithmetic) difference of B from A, denoted by $A \backslash B$ or $A - B$, is the mset $C \subseteq A$ such that $m_C(x) = m_A(x) - m_B(x)$, for all objects x.

In general, $m_C(x) = m_A(x) - m_{A \cap B}(x) = max\ \{m_A(x) - m_B(x), 0\}$, for all objects x.

It is also called the relative complement of B in A.

It follows that the deletion of an element x from an mset A gives rise to a new mset $A' = A - x$ such that $m_{A'}(x) = max\ \{m_A(x) - 1,0\}$.

Remark 5.13

It is easy to see that in multiset theory, an element may occur in an msubset and in its relative complement. Accordingly, some of the consequences of the aforesaid definition turn out to be *disturbing* (Hickman, 1980) and (Blizard, 1989).

For example, if $A = [a, b]_{4,5}$ and $B = [a, b]_{2,3}$ then $A - B = [a, b]_{2,2} \subset B$ contradicting the classical laws that $(A - B) \cap B = \emptyset$ and $(A - B) \cup B = A$. Therefore, $<\mathbb{p}(Y), \cup, \cap, -, \emptyset, Y>$ is only a lattice and not a boolean algebra.

Remark 5.14

We note that if the complementation operator is restricted to sets, then the classical laws hold. That is,

$[\text{Set}(Y) \wedge X \subseteq Y] \rightarrow [\text{Set}(X) \wedge \text{Set}(Y - X) \wedge X \cap (Y - X) = \emptyset \wedge X \cup (Y - X) = Y]$.

Recall that whole msubsets are *set-like* i.e., elements do not spill over into the relative complement. Thus, if X is a whole in Y, then

$X \cap (Y - X) = \emptyset$ and $X \cup (Y - X) = Y$ hold.

In other words, complementation operator behaves classically when restricted to sets and whole msubsets (Blizard, 1989).

Apparently, in order to define complementation such that multisets may have a boolean algebraic structure, one needs to introduce a restricted multiset universe, like the one above in which there underlie some contrived stipulations.

In this direction, a seminal work (Brink, 1987) appeared, though with a specific motivation to capture relevant logic structure. The power multiset $\widetilde{\wp}(A)$ of a given mset A is taken as the working universe. Hence, for any mset $\alpha \in \widetilde{\wp}(A)$, $m_\alpha(x) \leq m_A(x)$ for every object x. The complement $\bar{\alpha}$ of α is defined as the mset such that $m_{\bar{\alpha}}(x) = m_A(x) - m_\alpha(x)$ for every object x. It is easy to observe that

(i) $\bar{\bar{\alpha}} = \alpha$ (involution holds)

(ii) $\overline{(\alpha \vee \beta)} = \bar{\alpha} \wedge \bar{\beta}$ and $\overline{(\alpha \wedge \beta)} = \bar{\alpha} \vee \bar{\beta}$ (DeMorgan properties hold).

In general, neither $\vee \bar{\alpha} = A$, nor $\alpha \wedge \bar{\alpha} = \emptyset$ holds.

In course of delineating algebraic properties of bag data types, a very fascinating approach was adopted in (Albert, 1991) in order to provide a boolean algebraic structure on a truncated part of $\widetilde{\wp}(U)$, where U is a bag (or a multiset). The paper introduces the following two specific operations:

Let A, B and C be bags.

$\delta(B) =_{def} \{x | x \in B\}$, called the duplicate elimination function.

$\delta_{\varphi}(A) =_{def}$ the bag C such that, for all objects x,

$$x \in\in C = x \in\in A \text{ if } \varphi(x);$$

$$x \in\in C = 0, \text{ otherwise.}$$

This is called the boolean selection function.

Further, in (*theorem 3, p. 214*), besides other results, a key result appears in item (5) viz:

$x \in\in \delta_{\varphi}(A) = (x \in\in A).(x \in\in \delta_{\varphi}(\delta(A))$, where $\delta(\delta_{\varphi}(A)) = \delta_{\varphi}(\delta(A))$, which formally entails that the bag selection operator has *all-or-nothing* semantics.

Finally, in (*theorem 8, PP. 216–217*), it is shown that with respect to selection operations, bags behave like sets and consequently the operations of union, intersection, and difference for bags give rise to a boolean algebra when the operations arise from boolean selection. The conclusion is the following:

$(\mathrm{S}(U), \cup, \cap, -)$ is a boolean algebra where U is a bag, $\mathrm{S}(U)$ is a structure analogous to $\mathbb{p}(X)$ but containing subbags (or, submsets) obtained by applying a selection operator to U. That is, $\mathrm{S}(U) =_{def} \{A \subseteq U | (\exists \text{ a predicate } \varphi)\ (A = \delta_{\varphi}(U)\}$.

Remark 5.15

It can be observed that the subbags obtained by applying selection operator to a given bag U are exactly the whole msubsets of the given mset U and hence, the conclusion is a rephrasing of observations made in *Remark 5.13*.

In order to define complementation, Petrovsky (1997) adopts a somewhat new approach as follows:

Let $\mathfrak{J} = \{A_1, A_2, \ldots\}$ be a family of msets. The maximum mset $\mathbb{Z}$ is defined such that $m_{\mathbb{Z}}(x) = max\, m_A(x)$, $A \in \mathfrak{J}$, for all objects x and all $A \in \mathfrak{J}$. Then, the complement of an mset A, denoted by $\bar{A}$, is defined as $\bar{A} = \mathbb{Z} - A$ such that $m_{\bar{A}}(x) = m_{\mathbb{Z}}(x) - m_A(x)$, for all objects x.

However, the attendant difficulties remain unresolved; for example, $\bar{A} \cup A \neq \mathbb{Z}$, and $\bar{A} \cap A \neq \emptyset$.

In fact, the case of bounded multiplicities has already been addressed in (Brink, 1987) and it was not found of much help in resolving the issues at hand.

A similar approach, recently explicated in Girish and John (2009), is as follows:

Let X be a set. The multiset space X^n is defined as the set of all msets built from the elements of X such that an object in any mset of X^n occurs at most n times. That is, the multiplicities of objects in any mset of the space X^n are bounded. Then $\bar{A}$, the complement of A in X^n, is defined as an mset in X^n such that $m_{\bar{A}}(x) = n - m_A(x)$. Here, the regular mset of height n can be viewed as the working universe.

Note that, fort X^n to be closed with respect to $(+)$, the definition of the arithmetic sum $(+)$ needs to be modified as follows:

$$m_{A+B}(x) = min\{n, m_A(x) + m_B(x)\}.$$

It follows that X^n is closed with respect to union, intersection, sum and complementation. However, similar to other explications discussed above, not all the properties of complementation hold, for example, $\alpha \wedge \bar{\alpha} \neq \emptyset$ in general.

Theorem 5.5 (*non existence of complement in multisets*)

If M is a multiset, then there exist no unary operator, $-$, that can be defined on $\wp(M)$ such that $(\wp(M), \cup, \cap, -)$ is a Boolean algebra.

Proof

Suppose there exist operator, $-$, which defined a complement on M, then for every $A \in \wp(M)$, $A \cup (-A) = M$ and $A \cap (-A) = \emptyset$.

Now, given that M is a multiset then there exist $x \in M$ such that $m_M(x) > 1$. Pick an arbitrary multiset $A \subset M$ such that $0 < m_A(x) < m_M(x)$. Then $A = [x] \in \wp(M)$ which always exist.

Since $-$ is a complement operator, $A \cup (-A) = M$ which implies that $m_{(-A)}(x) = m_M(x)$. Similarly, $A \cap (-A) = \emptyset$ which implies that $m_{(-A)}(x)$, that is $m_M(x) = 0$ contradicting the fact that $m_M(x) > 1$. Therefore, there is no unary operator, $-$, that can be defined on $\wp(M)$ such that $(\wp(M), \cup, \cap, -)$ is a Boolean algebra.

Theorem 5.6 (*Distributive lattice*)

The structure $(\wp(M), \cup, \cap, -)$ is a distributive lattice induced by the partial order $\subset$.

Proof

By definition, $\cup$ and $\cap$ are the least upper bound and greatest lower bound of the partial order $\subset$ respectively and $\wp(M)$ is closed with respect to $\cup$ and $\cap$. We then show that $\cup$ distributes over $\cap$ and $\cap$ distributes over $\cup$.

Given any $x \in M$, let $x \in A \cup (B \cap C)$, where $A, B\ C \in \wp(M)$. Then

$$m_{A\cup(B\cap C)}(x) = max\ [m_A(x), min(m_B(x), m_C(x))]$$

$$= min[max\ (m_A(x), m_B(x)), max\ (m_A(x), m_C(x))]$$

$$= m_{(A\cup B)\cap(A\cup C)}(x)$$

$$\Rightarrow x \in (A \cup B) \cap (A \cup C).$$

Similarly,

Let $x \in A \cap (B \cup C)$. Then

$$m_{A\cap(B\cup C)}(x) = \min\ [m_A(x), max(m_B(x), m_C(x))]$$

$$= max\ [min\ (m_A(x), m_B(x)), min\ ((m_A(x), m_C(x))]$$

$$= m_{(A\cap B)\cup(A\cap C)}(x)$$

$$\Rightarrow x \in (A \cap B) \cup (A \cap C).$$

Therefore, the structure $(\wp(M), \cup, \cap, -)$ is a distributive lattice induced by the partial order $\subset$.

5.8 Concluding Remarks

It has been demonstrated that the technique of introducing a cardinality–bounded multiset space proves advantageous in many ways especially in Database systems (Ross and Stoyanovich, 2004) and (Albert, 1991). However, as investigated above none of the existing approaches succeeds if the goal is to augment the theory of multiset endowed with a boolean algebraic structure without assuming some contrived situations.

In fact, the obvious alternatives seem to be exhausted. However, some further adept explications cannot be ruled out.

CHAPTER SIX

SUMMARY AND CONCLUSIONS

6.0 Concluding Remarks

This dissertation presents a concise account of the uses of multisets in various disguised forms which eventually led to a formalization of multiset theory. It includes the study of a systematization of representations of multisets, basic operations under multisets, and overview of applications of multisets in mathematics and computer science and related areas.

It is shown that there exists a connection between multisets and max–plus algebras. By exploiting the notion of compatibility relation, a truncated symmetrized max–plus algebra called *minimal* $\mathbb{S}_{max}$ or $\approx \mathbb{S}_{max}$ is developed and some of its applications are indicated. The application of this technique is extended to multisets as well and subsequently the notion of tolerance multiset along with some of it applications is introduced.

Finally, we explicate the notion of multiset space. We define algebraic structures on multiset spaces and show that they share properties many algebraic properties. We develop the concept of relations and functions on multiset spaces with partitions of multisets as our basic tools and show that the multiset space can be metrized thereby defining topology on a multiset.

We also delineate the problem related to difference and complementation in multiset theory. We show that none of the existing approaches succeeds in resolving the attendant difficulties without assuming some contrived stipulations.

6.1 Future Directions

Constructing multiset algebras to obtain Boolean algebra by basically defining complementation on multisets that satisfies DeMorgan's laws is a future area of interest. Fuzzy multiset theory still needs a proper vindication. Application of compatibility relations in order to characterize formal language theory in multiset environment remains a rich area of research.

Having the aforesaid tools developed on multiset space, it seems plausible to pursue a good deal of topological analysis on multisets. In particular, the concept of topological group and topological ring in multiset space could be fully exploited. Also, a number of theorems on algebraic structures on multiset spaces could be developed.

Finally, we wish to note at this end that multiset space algebras would be relevant if the goal is to keep an optimal amount of information throughout the computations.

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Appendix 1

Set	$x \oplus y$	$x \otimes y$	θ	e	Applications
$\{0, 1\}$	Max	min	0	1	Existence of Paths
$\mathbb{R} \cup \{\infty\}$	Min	+	∞	0	Shortest Path
$\mathbb{R} \cup \{-\infty\}$	Max	+	$-\infty$	0	Longest path (critical path)
$\{x \in \mathbb{R} \mid x \geq 0\}$	Max	×	0	1	Critical path
$\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$	Max	×	0	1	Most reliable path network transmissions
$\mathbb{R}^+ \cup \{\infty\}$	Max	min	0	∞	Path of greatest capacity
$\mathfrak{B}(\Sigma^*)$	$x \cup y$	concat	Φ	Λ	Listing all paths Σ = alphabet, Λ = empty
$\mathfrak{B}(S)$	$x \cup y$	concat	Φ	Λ	Listing simple paths
B	b($x \cup y$)	concat	Φ	Λ	Listing elementary paths Computer logic circuits

Examples of path algebras