CONSTRUCTIVE COMPACT LINEAR MAPPINGS

By

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DEDICATION

This work is dedicated
to
my mother
and
late father.

DECLARATION

I hereby declare that this work is a record of my own research work under supervision. It has not been submitted for a higher degree in any University. Information derived from published or unpublised work of others has been duly acknowledged by means of references.

Ngwa Matthias

Sign: .....

Date: 3/7/97

CERTIFICATION

This thesis titled "CONSTRUCTIVE COMPACT LINEAR MAPPINGS" by NGWA MATTHIAS meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria, and is approved for its contribution to knowledge and literary presentation.

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ABSTRACT

This dissertation opens with a discussion of the distinction between the classical and constructive notions of Mathematics. There then follows a description of the three main varieties of modern constructive mathematics: Bishop's constructive mathematics, the recursive constructive mathematics of the Russian School of Markov, and Brouwcr's intuitionistic mathematics. We investigate the relationship between compactness, finite rank and located kernel for a bounded linear mapping of a normed space into a finite-dimensional space in a constructive setting. One of the main results is that a bounded linear mapping *of* a normed space onto a finite-dimensional normed space is constructively compact if and only if its kernel is located. We proceed with the investigation of compact operators on a Hilbert space, and compact linear mappings of a normed space, within the framework of Bishop's constructive mathematics. We characterise the compactness of a bounded linear mapping of a Hilbert space H into C", and prove the theorems: Let A and B be compact operators on a Hilbert space H, let C be an operator on H and let *a £ C* Then aA is compact, A + B is compact, the Hilbert adjoint A* is compact; An operator on a Hilbert space has an adjoint if and only if it is weakly compact. We also look at constructive substitutes for the classically well-known theorems on compact linear mappings: T is compact if and only if T* is compact; if S is bounded linear operator and if T is compact, then TS is compact; if S and T are compact, then S+T is compact.

LIST OF ABBREVIATIONS

BISH	Bishop's Constructive Mathematics.
CLASS	Classical mathematics.
EM	Law (principle) of Excluded middle.
INT	Brouwer's intuitionistic mathematics.
LPO	Limited principle of omniscience
LLPO	Lesser Limited Principle of omniscience.
MP	Markov's principle.
RUSS	The constructive mathematics of the Russian School of Markov.

LIST OF DEFINITIONS AND FIRST APPEARANCE

Adjoint	5.1.4
Banach space	3.3.2
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LIST OF SYMBOLS

$A \subset B$	A is contained in B.
$B(x_o,\varepsilon)$	Open ball centred at x_o with radius ε.
$\mathbb{C}$	Complex plane or field of complex numbers.
$\mathbb{C}^n$	Space of all ordered n-tuples of complex numbers.
C(x,y)	Set of uniformly continuous functions from a totally bounded metric space X to a metric space Y.
d(X)	Diameter of X.
d(x,y)	Distance from x to y.
dom(f)	Domain of f.
E*	Set of bounded linear functionals on E.
E'	Set of normable linear functionals on E.
$\|\|f\|\|$	Norm of a bounded linear functional f.
$f:X \to Y$	f is a function (mapping) from X to Y.
$f \mid X_1$	Restriction of f to X_1.
Hom(E,F)	Set of bounded linear mappings of E into F.
$Hom_c(E,F)$	Set of compact linear mappings of E into F.
$Hom_n(E,F)$	Set of normable linear mappings of E into F.
Inf	Infimum (greatest lower bound).
$\exists$	Existential qualifier.
Ker f	Kernel of a function f.
Ker (T)	Kernel of an operator T.
ℓ^p	Sequence space such that $\Sigma\|x_i\|^p < \infty$
ℓ^∞	Space of bounded sequences.

$\wedge$	And
$\mathbb{N}$	Set of natural numbers.
$\mathbb{R}$	Set of real numbers.
ran (f)	Range of f.
S_E	Closed unit sphere of E.
$S_r(x)$	Open sphere centred at a point x with radius r.
Sup	Supremum (least upper bound).
$\|\|T\|\|$	Norm of a bounded operator T.
T^*	Hilbert-adjoint of T.
$T(S_E)$	Image of S_E under T.
Tx	Image of x under T.
$\vee$	Or
$\forall$	Universal quantifier.
$\|\|x\|\|$	Norm of x.
$\|\|\|x\|\|\|$	Double norm of x.
(x,y)	Inner product of x and y.
$x \perp y$	x is orthogonal to y.
X/Y	$\{x \in X: x \notin Y)\}$.
$X \cap Y$	The intersection of two sets X and Y.
$X \cup Y$	The union of two sets X and Y.
X x Y	Cartesian product of X and Y.
Z	Set of integers.
Z^+	Set of positive integers.
$\neg$	Not.

CHAPTER ONE

INTRODUCTION

The main aim of this research work is the constructive investigation of compact linear mappings. This differs in several respects from the classical theory of compact linear mappings. We shall base our analysis on constructivism or what has come to be known as constructive mathematics. Just as every interest group and profession has its own jargon - a specialized vocabulary understood by its members but seemingly incomprehensible to others, so is the case with constructive mathematics.

In chapter 2, we shall look into the meaning of constructive mathematics from its classical (or traditional) counterpart. The key feature of constructive mathematics is the identification *existence=computability*. Modern constructive mathematics originated from the work of several constructivists in the early part of the 20th century. The approaches of three constructivists (Bishop, Markov and Brouwer) based on logicism, formalism and intuitionism led to three major varieties of constructive mathematics that are currently the domain of serious research activity. It began in 1907 with the work of Brouwer (1881-1966) in his critique of classical mathematics, in which he demonstrated that it is deficient in numerical meaning. Subsequent work of Brouwer was devoted more to logical, philosophical, and foundational considerations than to positive constructive developments. A significant advance in the latter direction was made in

1967 by Errett Bishop (1928-1983) in California. He succeeded in developing a large portion of analysis in a realistic, constructive manner; see Bridges [2], [5], Bridges-Demuth [8], Bridges-Mines [10], Mandelkern [21], Richman [25].

Based on the underlying principles of which the notion of algorithm is admitted by all we shall look at these three varieties. With recent advances in constructivity, Bishop's constructive mathematics (BISH) is regarded as the constructive core of all the varieties and classical mathematics. Some classical theorems are found to have constructive versions as well as proofs while others are lacking in these. We shall look at some of such theorems and the various principles underlying them. This will lead us to the determination of a criterion that we can use to know that a classical result can never be proved constructively (omniscience principles). We shall define Brouwerian examples, which provide strong evidence for these omniscience principles.

In chapter 3, we shalll look at the various spaces on which compact linear mappings are defined. Hence, we shall review the constructive theory of metric and normed spaces.

Let E and F be normed linear spaces and let $S_E = \{x \in E: ||x|| \leq 1\}$ be the closed unit sphere of E. We say that a linear mapping T of E into F is *bounded* if $T(S_E)$ is bounded, *normable* if $||T| \equiv \sup\{||Tx||: x \in S_E\}$ exists, and compact if $T(S_E)$ is totally bounded, Ishihara [16]. If T is normable, then $||T||$ is called the norm of T.

We shall next look at some preliminary results which are required for the proofs of most theorems on compactness. Completeness and total boundedness are vital properties of compactness. Consequently complete normed (Banach) spaces and complete inner product (Hilbert) spaces with their defining properties are of importance here.

A mapping u between normed spaces is compact if given any $\varepsilon > 0$, we can construct a finite ε-net (or what is known as ε-approximation) of the image of the unit ball under u; see Bridges *et al* [6]. Total boundedness lies with the construction of this ε-approximation to the space in question.

Certain normed linear spaces are frequently used to construct Brouwerian examples here. The l^p ($1\leq p < \infty$) spaces, especially l^2, are very important examples of such spaces. Some classical normed spaces are found not to be normed spaces constructively. The space l^∞ of all bounded sequenes $x=\{x_n\}$ of numbers has the classical norm $||x||_{l\infty} \equiv \mathrm{Sup}|x_n|$. This norm cannot be computed everywhere. Constructively, this is not a normed space. However, by associating a sequence of seminorms $||x||_n \equiv |x_n|$, the space l^∞ becomes a quasinormed linear space. These quasinormed linear spaces enable us to deal with linear spaces on which the classical norm cannot be computed everywhere; Ishihara [13].

In chapter 4, we shall deal with compact operators on a Hilbert space. A bounded linear mapping T of a normed

linear space E into itself is called an operator on E.

The classical result that every bounded mapping into a finite-dimensional normed space is compact is not true constructively. A Brouwerian example will be used to establish this; see Bridges *et al* [6, example 5].

Classically, every bounded linear mapping of finite rank is compact. This is not constructively true, even when the domain of the mapping is a Hilbert space; Bridges *et al* [6, example 1]. A bounded linear mapping onto a one-dimensional normed space is constructively compact if and only if its norm can be computed.

One of the main results of this work is that a bounded linear mapping onto a finite-demensional normed space is compact if and only if its kernel is located i.e. the distance from any point to the kernel can be computed; Bridges *et al* [5, Theorem 1]. We shall look at three preliminary lemmas which are required for the proof of this result. In the case where the domain of the mapping is a Hilbert space, we can prove the following proposition:
Let u be a bounded linear mapping of a Hilbert space H into $\mathbb{C}^n$, and for i=1, ..., n, let p_i: $\mathbb{C}^n \to \mathbb{C}$ be a mapping defined by $p_i(x_1 \ldots, x_n) = x_i$. Then u is compact if and only if $p_i$ou is normable for each i = 1,..., n; Ishihara [16, prop. 1]. This result also tends to be valid when the set of all normable linear functionals on the domain of the mapping is a linear space.

By virtue of the above proposition, we shall prove the following theorem:

Let A and B be compact operators on a Hilbert space H, let C be an operator on H and let $\alpha \in \mathbb{C}$. Then

(1) αA is compact;

(2) A+B is compact;

(3) A* is compact;

(4) CA is compact;

(5) If C* exists, then AC is compact; Ishihara [16, Theorem. 1].

Classically, every operator on a Hilbert space has an adjoint, but as we shall see, this is not the case in the constructive setting, especially in the case where the Hilbert space is l^2.

In (5) of the above theorem, the existence of an adjoint of an operator plays an important role, and we shall prove the following theorem:

An operator on a separable Hilbert space has an adjoint if and only if it is weakly compact; Ishihara [16].

In chapter 5, we shall prove similar results in a more general setting. For this, we define quasinormed linear spaces, which enable us to deal with linear spaces, such as Hom(E,F) (the set of all bounded linear mappings of E into F, where E and F are normed linear spaces), on which the classical norm cannot be computed everywhere.

We shall also deal with compact linear mappings between normed linear spaces. Let E be a normed linear space. Then we consider E* as a quasinormed linear space with the quasinorms

$\{||.||_x : x \in S_E\}$ given by $||f||_x \equiv |f(x)|$ for all $f \in E^*$.

Let T be a bounded linear mapping of a normed linear space E into a normed linear space F. Then the mapping f → foT of F* into E* is called the adjoint of T and is written T^{x}; see Ishinhara [14]. The definition of the adjoint is slightly different from that of an operator on a Hilbert spac, in the sense that the adjoint in this definition always exist, but one on a Hilbert space exists when $T^{x}(F') \subseteq E'$, where E' and F' are normable linear mappings on E and F respectively.

We shall state the following constructive substitute for the classical theorem that a bounded linear mapping is compact if and only if the adjoint is compact. If T is a linear mapping of a normed linear space E into a separable normed linear space F, then T is compact if and only if ran(T^{x}) $\subseteq E'$ and T^{x} is compact; Ishihara [14].

As a consequence of this, we shall be able to prove the constructively nontrivial compactness of composition of compact and bounded linear mappings. In the constructive setting, this is stated as follows:

If S is a bounded linear mapping of a normed linear space E into a separable normed linear space F with $S^{x}(F') \subseteq E'$ and T is a compact linear mapping of F into a separable normed linear space G, then TS is compact; Ishihara [14, Theorem. 2].

From the last two theorems above, we note that in the constructive setting, the conditions ran (T^{x}) $\subseteq E'$ and $S^{x}(F') \subseteq E'$ cannot be dropped.

We have seen various aspects of constructive linear

mappings, and found that they behave as classically under suitable conditions. In the concluding part, we shall consider some of the reformulations which make a nonconstructive classical theorem to be constructive in detail.

CHAPTER TWO

PRELIMINARIES

In order to investigate compact linear mappings using constructive methods, one needs to explain the meaning of constructivism or what is known as constructive mathematics.

2.1 Meaning and varieties of Constructive Mathematics:

Classical mathematics refers to the sort of mathematics taught in virtually every school and college classroom in the world. Work in constructive mathematics, using only constructive methods, is presently carried on by only a small minority of mathematicians.

The distinction of constructive mathematics from its classical or traditional counterpart rests primarily upon the constructive mathematicians strict interpretation of the phrase "there exists": whereas in classical mathematics it is common to prove the existence of an object x with a property p by deducing a contradiction from the assumption that no such x exists, a constructive proof of the existence of such an x must embody both an algorithm for the construction of x (at least to any preassigned degree of accuracy) and an algorithm which verifies that x has the property p. Thus, in classical mathematics, "existence" means "nonexistence is contradictory", while in constructive mathematics, "existence" means "computability"; see Bridges [2], [5], Bridges-Mines [10], Mandelkern [21], Richman [25].

One can interpret an algorithm as, for example, a syntactically correct program written in a programming

language of one's own choice. Whatever interpretation one makes of the word "algorithm", it should only admit computations that can be performed, at least in principle, by a finite number of human beings or computers in a finite length of time. The algorithmic (routine) interpretation of existence do not require that one should know the efficiency or complexity of the algorithms we use. The search of efficient routines is a separate activity, a part of numerical analysis and computer science; Bridges [5], Bridges-Mines [10]. Errett Bishop (1928-1983) expressed the notion of a constructive procedure as follows:

'How do you know whether a proof is constructive? Try to write a computer program. If you can program a computer to do it, it should be constructive. Notice that I said write the program. Don't necessarily run it on the computer and wait around for the result.' See Mandelkern [21].

There are three varieties of constructive mathematics. These varietes differ in several respects, especially in the notion of algorithm that they all admit.

The first variety, Bishop's constructive mathematics (BISH), considers "algorithm" to be a primitive, undefined notion, and there is no commitment to any formal system (such as the system of natural numbers) or to special principles (such as the Church-Markov-Turing thesis); this allowed for future innovations. In consequence, it is consistent with each of the other two varieties and with classical mathematics (CLASS); in particular, every proof of a theorem within Bishop's constructive mathematics is also a proof of

that theorem in classical mathematics. In view of this consistency, BISH can be regarded as the constructive core of classical mathematics; see Bridges-Demuth [8].

The second variety, the constructive mathematics of the Russian School of Markov (RUSS), is based on the same underlying logical principles as BISH, but pins down the notion of algorithm by adopting the Church-Markov-Turing thesis that every sequence of natural numbers is recursive. It also studies only those mathematical objects which are effectively codable by natural numbers. Thus, in a highly consistent way, the practitioners of RUSS endorse Bishop's view that if algorithmic method is characteristic of constructive mathematics, so also is numerical content: according to Bishop,

> The primary concern of mathematics is number, and this means the positive integers. We feel about numbers the way Kant felt about space. The positive integers and their arithmetic are presupposed by the very nature of our intelligence and, we are tempted to believe, by the very nature of intelligence in general. The development of the theory of the positive integers from the primitive concept of the unit, the concept of adjoining a unit, and the process of mathematical induction carries complete conviction... . Building on the positive integers, weaving a web of ever more sets and more funtions, we get the basic structures of mathematics: the rational number system, the real number system, the euclidean spaces, the complex number system, the algebraic number fields, Hilbert space, the classical groups, and so forth. Within the frame work of these structures most mathematics is done. Everything attaches itself to number, and every mathematical statement ultimately expresses the fact that if we perform certain computations within the set of positive integers, we shall obtain certain results; see Bridges [5], Bridges-Demuth [8].

The third variety, and historically the oldest, is Brouwer's intuitionistic mathematics (INT). Based on Brouwer's intuitionistic philosophy, this variety uses a classically inexplicable notion of "free choice sequence" to develop principles that lead to results apparently inconsistent with both classical mathematics and RUSS. In this case, it is more appropriate to say that these results and the intuitionistic constructions used to obtain them, cannot be interpreted using the philosophy and methods of classical mathematics; Bishop-Mines [10].

If we work formally and forget the underlying philosophies, each of these two varieties can be viewed as BISH together with certain adjoined principles. In the case of RUSS, the essential principle adjoined to BISH is a form of the Church-Markov-Turing thesis that all sequences of natural numbers are recursive. On the other hand, Brouwer's intuitionistic mathematics (INT) can be viewed formally as BISH together with certain principles which lead to the *intuitionistic uniform continuity theorem:*

Every function from a compact metric space into a separable metric space is uniformly continuous; Bridges-Demuth [8].

This theorem, on first reading, appear to contradict classical mathematics (CLASS). It contradicts the uniform continuity theorem of classical mathematics:

There is a pointwise continuous function $f: [0,1] \to R$ *that is not uniformly continuous.*

The apparent contradiction disappears when we interpret our

statement more carefully:

Every intuitionistically defined function from the intuitionistic interval [0,1] to the intuitionistic real line is, intuitionistically, uniformly continuous.

There is a strong case for saying that, except at certain levels of formalism, INT and CLASS are so divergent that it is not possible to capture fully the spirit and meaning of intuitionistic statements, such as the intuitionistic uniform continuity theorem, within a classical framework.

2.2 Constructive parts of classical mathematics:

In practice, this involves our paying much closer attention to the meaning of mathematical statements, especially those dealing with existence.

The most extreme example of a classically acceptable, constructively unacceptable, principle is the *law of Excluded Middle* (EM). In contrast to the classical view that every meaningful mathematical statement is either true or false (even if we cannot say which alternative holds), the constructive mathematician does not consider a proposition P to be true or false unless he can produce either a proof of P or a proof that P is impossible. Thus, the law of excluded middle which asserts that for any statement P either P or its negation is true, cannot be accepted within constructive mathematics, see Bridges [2], [5], Bridges-Demuth [8], Bridges-Mines [10], Mandelkern [21].

In all the varieties of constructive mathematics, the algorthmic interpretation of existence forces one to

reconsider the meaning of each logical connective and quantifier. Constructive mathematicians interpret them according to inuitionistic logic; Ishihara [13]. The important differences from classical mathematics, as we shall see, centre upon the connective 'or', 'implies' and the quantifier 'there exists'.

2.2.1 **Definition**: A function P(x) is a propositional function on a set A if P(a) is a statement for each a∈A.

2.2.2 **Definition** (Quantifiers): Let A be a set and P(x) be a propositional function on A. ∀x P(x) means "for every element x in A, P(x) holds". The symbol ∀ which reads "for all", or "for every" or "for each" or "for any" is called the *universal quantifier*.

For a constructive proof of ∀x P(x), we must produce an algorithm which, applied to any object x in the universe of discourse, proves that P(x) holds; Bridges [5].

The symbol ∃ which reads "there exists" or "for some" or "for at least one" or "there is" is called the *existential quantifier*. If there is exactly one element x∈A for which P(x) holds, we write ∃!x P(x).

It is clear from the intuitive significance of quantification that the universal and existential quantifiers are not independent, and that either can be expressed in terms of the other with the aid of negation. The negation of "for every x, P(x) holds" is "there is some x for which P(x) does not hold", and thus ¬∀x P(x) reduces to ∃x (¬P(x)); see Kneebone [19].

2.2.3 **Definition** (Logical constants): We now introduce

suitable symbols for the basic logical operators that can be applied to propositions and propositional variables. These are: ¬ (not); ∨ (or); ∧ (and); ⇒ (implies); ⇔ (equivalent). These operators are sometimes referred to as *logical constants*. The last four ∨, ∧. ⇒, ⇔, which always stand between two propositional symbols are called *connectives*; Kneebone [19].

The constructive interpretation of ∧ (and) pose no problem: to prove $P_1 \wedge P_2$ we must produce both a proof of P_1 and a proof of P_2; to justify the assertion $P_1 \vee P_2$ for proposition P_1 and P_2, we must have either a proof of P_1 or a proof of P_2. In other words, $P_1 \vee P_2$ is equivalent to ∃i Pi; see Bridges [5].

The commonest constructive interpretation of the connective ⇒ (implies) can be paraphrased as follows: P⇒Q means that Q holds under the assumption that P holds, or that we can derive Q from the hypothesis P.
The statement ¬ P (not P) is interpreted as P⇒Q , where Q is a contradiction.

The standard classical interpretation of the connective ⇒ is "material implication", in which P⇒Q is, by definition, ¬P∨Q. Certainly, we can derive Q constructively from the two hypotheses ¬P∨Q and P, so that (P⇒Q) ⇒(¬P∨Q); but as P⇒P holds trivially, and we cannot prove the law of excluded middle, ¬P∨P, constructive implication and material implication are not equivalent constructively.

Another classical principle rejected by constructive mathematicians is ¬¬P⇒P: for taking P to be of the form ∃x Q(x), we see that if ¬¬P holds, then it is absurd to deny

that there exists x with the property Q(x); but such an absurdity does not provide us with an algorithm for constructing x such that Q(x) holds. However, we can prove $\neg\neg\neg P \Rightarrow \neg P$: for if P holds, then so does $\neg\neg P$; and in

constructive as in classical mathematics, if $A \Rightarrow B$, then $\neg B \Rightarrow \neg A$; See Bridges [5].

Bishop wrote that every classical theorem presents a challenge: to find a constructive version and to give it a constructive proof; Bridges-Mines (10].

According to the constructivist, a theorem proved by classical methods is merely incomplete; the degree of incompleteness varies to both extremes. A classical theorem may exhibit no numerical meaning, and there may be little chance of extracting any numerical meaning from it. At the other extreme is the classical theorem that is constructive as it stands, or becomes constructive after some quite minor reformulations. Situated between these extremes, most classical theorems are neither constructively valid nor completely devoid of numerical meaning. Such a theorem must be significantly modified or rephrased to show its true constructive content, and a considerable amount of work must be done to find a constructive proof. Often several new theorems are obtained, revealing different constructive aspects of the classical theorem which were hidden by its classical presentation; Mandelkern [21].

In most cases, it is not at all easy to formulate a conjecture about the constructive content, if any, of a

classical theorem. As such many classical theorems are unlikely to have any constructive version: an example is the Bolzano-Weierstrass Theorem (Any bounded sequence of real numbers has a convergent subsequence); a more extreme example is Zorn's Lemma (Let $M \neq \emptyset$ be a partially ordered set. Suppose that every chain $C \subseteq M$ has an upper bound. Then M has at least one maximal element).

The fundamental theorem of Algebra (Any nonconstant polynomial f with complex coefficients has at least one complex root) is an example where the classical statement remains valid when interpreted constructively, but the usual classical proofs are not constructive. At the risk of being repetitive we emphasize the importance of appreciating that, although the statement of the Fundamental theorem of Algebra is the same in both classical and constructive mathematics, its constructive interpretation is quite different from the classical one. When correctly interpreted as an assertion about the computability of zeros of polynomials, the constructive version is seen to be considerably more informative than the classical. We bring out this distinction by rewriting Brouwer's version in the form:
There is a finite routine which, applied to any nonconstant polynomial f with complex coefficients, enables us to compute a root of f; Bridges-Mines [10].

In situations where a classical concept has a constructive version, the single classical concept may split into several interesting constructive distinct ones, that are classically, but not constructively, equivalent. This

phenomenon of branching of concepts is excellently illustrated by the reinterpretations of the Big Picard's Theorem in complex analysis. It states that

If f has an essential singularity at z_o, then the range of f is either $\mathbb{C}$ or else $\mathbb{C}$ with one point omitted; Bridges [2], Bridges-Mines [10].

In above theorem, f is a differentiable complex function on the punctured disc

$$U \equiv \{z \in \mathbb{C}: 0 < |z - z_o| < r\}.$$

Bridges *et al* (1982) showed that the classical form of the theorem is essentially nonconstructive, and then proved the following constructive version of the theorem:

If f has an essential singularity at z_o, and if α, α' are distinct complex numbers, then there exists z in U such that either $f(z) = \alpha$ or else $f(z) = \alpha'$; Bridges [2].

Now, there is another classically equivalent version that turns out also to be provable within BISH.

If the range of f omits two complex values, then f has a pole of finite (determinate) order at z_o; Bridges [2].

The two constructive versions are not equivalent within BISH, as they embody totally different algorithms.

Some classical theorems have constructive versions with these as well as the classical statements having similar proofs. The *Uniform Boundedness Principle* is an example of such a theorem in which the constructive version has a proof that is similar to a standard classical one; Bridges-Ishihara [9]. The classical statement is:

If $\{f_n\}$ is a sequence of continuous linear functionals on a

Banach space X and, for every x∈X, the sequence $\{|f_n(x)|\}$ *is bounded, then the sequence* $\{||f_n||\}$ *of norms is bounded;* Goffman-Pedrick [12].

This has the following constructive version:

Let $\{T_n\}$ *be a sequence of bounded linear mappings from a* Banach *space X into a normed space Y, and let* $\{x_n\}$ *be a sequence of unit vectors in X such that* $||T_n x_n|| \to \infty$ *as* $n \to \infty$. *Then there exists a vector x in X such that the sequence* $\{||T_n x||\}$ *is unbounded;* Bishop[1], Bridges-Ishihara [9].

In order to understand what is true and what is not true in constructive mathematics, let us consider a comparatively simple mathematical notion: that of a binary (decision) sequence. Classically, this is simply a sequence in which each term belongs to $\{0,1\}$; but from our constructive point of view, a binary sequence $\{a_n\}$ is an algorithm which, applied to any positive integer n, produces an output a_n belonging to $\{0,1\}$. This choice of only two values represents the situation where we use some finite process to look for something and either we find it or we don't; the 0 or 1 simply record our results.

So, in order to justify constructively the assertion that a_n belongs to $\{0,1\}$, we must be able to tell whether $a_n = 0$ or $a_n = 1$; it does not suffice to prove that a_n cannot fail to belong to $\{0,1\}$; Bridges [5], Bridges-Demuth [8], Mandelkern [22].

2.2.4 **Definition:** Let $\{a_n\}$ be a binary sequence. A classically true statement of the form "$P(a_n)$ for all binary sequences $\{a_n\}$" which is considered not to have a

constructive proof is an *omniscience principle*; Bridges [2], Ishihara [13], [15], Mandelkern [22].

In a systematic manner, Bishop formulated several general omniscience principles, each of which implies the solution to a vast number of unsolved problems or at least leads to certain information about such problems which in fact is not available. One advantage of this is that it shows more clearly the nonconstructivities in classical mathematics, since there will always be unsolved problems which the omniscience principles would solve.

Now we consider the following statement, which Bishop called the *limited principle of omniscience* (LPO):

If $\{a_n\}$ is a binary sequence, then either there exists n such that $a_n=1$, or else $a_n=0$ for all n.

This is an acceptable classical definition of a binary sequence. In the classical setting, since according to EM every statement is either true or false, pure logic dictates that LPO is trivially true. But, interpreted constructively, LPO is a much stronger statement than at first appears: if LPO holds constructively, then there exists an algorithm α whose input set consists of all binary sequences, whose outputs belong to $\mathbb{N}\cup\{-1\}$, and which, on being applied to a binary sequence $\{a_n\}$, behaves as follows:

If there exist n with $a_n=1$, then α outputs such a natural number n;

if $a_n=0$ for all n, then α outputs -1.

To anyone familiar with computer programming, a moment's reflection should suggest that such an algorithm is unlikely

ever to materialize, and therefore LPO should not be accepted as a principle of constructive mathematics. To turn this suggestion into complete conviction, we need to observe that within the context of recursion theory, in which binary sequences are recursive functions with values in $\{0,1\}$, there is no recursive algorithm which, applied to any binary sequence $\{a_n\}$, will output 1 if there exists an index n with $a_n=1$, and output 0 if $a_n=0$ for all n: indeed such an algorithm would provide us with a procedure for deciding, for any program P with integer inputs and any integer n, whether or not P will halt when given the input n. It is a basic and famous result of elementary recursion theory that no such decision procedure exists, even classically; Bridges [2], Bridges-Demuth [8], Bridges-Mine [10], Mandelkern [22].

More simply, if we let $P(a_n) \equiv \exists n\ (a_n=1)$, LPO is of the form $P(a_n) \vee \neg P(a_n)$, that is an instance of the law of excluded middle (EM), which is not accepted constructively.

One other result which is constructively unacceptable is the Goldbach conjecture (every even integer greater than 2 is the sum of two primes). Let us consider a function $f:\mathbb{N}\to\mathbb{N}$ defined by

$$f(n) = \begin{cases} 0 \text{ if the Goldbach conjecture is false.} \\ 1 \text{ if the Goldbach conjecture is true.} \end{cases}$$

We observe that a constructive proof of LPO would provide an algorithm for deciding the Goldbach conjecture.

Similar arguments shows that a constructive proof of LPO would lead to decision procedures for many unsolved problems; See Bridges [2], Mandelkern [22].

Another classically trivial principle which is

constructively rejected for similar reasons is the *lesser limited principle of omniscience* (LLPO).

If $\{a_n\}$ is a binary sequence containing at most one term equal to 1, then either $a_n=0$ for all even n, or else $a_n=0$ for all odd n.

LPO and LLPO are really weak forms of the law of excluded middle. Any classical theorem which, if proved constructively, would entail LPO, LLPO or some other non-algorithmic principle is considered to be essentially non-constructive, and is therefore constructively unacceptable. Among the more elementary of such results are the following:

(i) The law of trichotomy for real numbers: for each real number x, $x > 0$ or $x=0$ or $x<0$. This is equivalent to LPO: that is, $\forall x \in \mathbb{R}$ ($x=0$ or $x \neq 0$) implies LPO.

(ii) The least-upper bound principle for sequences: to each sequence $\{a_n\}$ of real numbers that is bounded above, there corresponds an upper bound s such that for each $\varepsilon > 0$ there exists n with $a_n > s-\varepsilon$; it implies LPO.

(iii) The sequential compactness property of the closed interval [0,1]: each sequence in [0,1] contains a convergent subsequence.

(iv) The weak form of trichotomy for $\mathbb{R}$: $\forall x \in \mathbb{R}$ ($x \geq 0$ or $x \leq 0$) $\Rightarrow$ LLPO.

See Bridges [2], [5], Bridges-Demuth [8], Mandelkern [22].

2.2.5 Definition (Real Numbers): A real number is a

sequence of approximating rational numbers expressed as finite decimals that is, a real number is a sequence $x \equiv \{x_n\}$ of rational numbers that is *regular*, in the sense that $|x_m - x_n| \leq 1/m + 1/n$ for all positive integers m and n; then the term x_n is called the n^{th} *rational approximation* to the real number x. Each rational number is a ratio of integers; See Bridges [5], Mandelkern [21], [22].

Fortunately, there are good constructive substitutes that enable us to get round the inadmissible trichotomy laws within BISH. Two constructive principles, that reflect the essence of a real number, given only by approximations, are commonly used.

(1) If $x > 0$ is contradictory, then $x \leq 0$.

(2) (Constructive Dichotomy Lemma): If a and b are real numbers with a<b, then for any real number x, either x<b or x>a. See Bridges [2], [5], Mandelkern [22].

This second result is commonly needed to split a constructive argument into cases, where a split of the form "either x>a or x≤a" would be unacceptable.

There is another classically trivial proposition whose role in constructive mathematics is more controversial than that of LPO or LLPO. The proposition in question is *Markov's Principle* (MP):

If for a binary sequence $\{a_n\}$, it is impossible that all the terms equal 0, then there exists n such that $a_n=1$.

This principle is accepted, with some feelings of doubt, by constructive mathematicians of the Russian School of Markov, but not by proponents of BISH and INT. Markov's principle,

although not a consequence of the law of excluded middle, is regarded as an omniscience principle

Markov's principle states that we can construct the number n with $a_n=1$ by successively computing a_1, a_2, ... until we get a 1. Our argument against MP is that we have no prior bound to the number of terms that we must look at before we can be sure that we have found one equal to 1. For this reason, most constructivists either reject outright, or refrain from using any propositions that are constructive equivalent to Markov's principle.

As in the case of LPO, given a real number x, let $\{a_n\}$ be a binary sequence with the property that

$x > 0$ if and only if $\exists n\ (a_n=1)$

$x \leq 0$ if and only if $\forall\ n\ (a_n=0)$.

Then constructively, we have

Markov's principle $\Leftrightarrow \forall\ x \in \mathbb{R}\ (\neg\ (x\leq 0) \Rightarrow x > 0)$; Bridges[2], [5].

2.2.6 Definition (Function): A function from a set A to a set B is an algorithm f which produces an element f(x) of B when applied to an element x of A, and is extensional: has the property that $f(x)=f(x')$ whenever $x = x'$; Bridges [5].

The equality relation on the set $\mathbb{N}$ of natural numbers (known as standard equality relation) is the relation of identity: two natural numbers are equal if and only if they are one and the same.

2.2.7 The Axiom of choice: There is widespread belief that a constructive mathematician can be identified by his rejection of the axiom of choice (AC):

AC: *Let A and B be sets. If S is a subset of A x B, and for each x in A there exists y in B such that (x,y) ∈ S, then there is a function f: A → B, called a choice function for S, such that (x, f(x))∈S for each x in A;* Bridges [5].

The axiom of choice is unacceptable within BISH: it entails the law of excluded middle.

To see this, let P be any constructively meaningful statement. Let A = {0, 1}, with equality defined by 0 = 1 if and only if P holds.

Let B = {0,1} with the standard equality, and let S ≡ {(0,0), (1,1)} ⊂ A x B, where the equality on S is derived in the usual way from those on A and B: (x,y) = (x',y') if and only if x = x' in A and y = y' in B.

If f: A → B is a choice function for S, then we have two possibilities:

(i) either f(0) = 1 or f(1) = 0;

(ii) f(0) = 0 and f(1) = 1

In case (i), if, for example, f(0) = 1, then (0,1) ∈ S. So either (0,1) = (0,0), which forces 0 and 1 to be equal elements of ℕ and is therefore impossible; or, as must be the case, (0,1)=(1,1). Thus 0 and 1 are equal elements of A, and so P holds.

On the other hand, if P holds, then as 0 and 1 are equal elements of A, and as f is a function, we have f(0) = f(1), which rules out case (ii). Hence in case (ii) P is false. Thus we have derived P∨¬P; see Bridges [2], [5].

Although the axiom of choice is not part of BISH, a

restricted form of choice the principle of *dependent choice* (DC) - is generally accepted and widely used by constructive mathematicians:

DC: *Let A be a set. If $a \in A$ and $S \subset A \times A$, and for each x in A there exists y in A such that $(x,y) \in S$. Then there exists a sequence of elements $a_1, a_2, \ldots$ of A such that $a_1 = a$ and $(a_n, a_{n+1}) \in S$ for each positive integer n.*

A consequence of the principle of dependent choice (DC) is the principle of *countable choice* (CC), which is the case A = N of the axiom of choice; Bridges [5], Ishihara [18].

CC: *Let B be a set, $S \subset N \times B$ such that for each n in N there is an element y in B such that $(n,y) \in S$. Then there is a function $f: N \to B$, called a choice function for S, such that $(n, f(n)) \in S$ for each n in N.*

Another choice principle which is generally accepted and widely used by constructive mathematicians is the principle of *unique choice*:

Let A and B be sets, $S \subset A \times B$ such that for each x in A there is an element y in B such that $(x,y) \in S$, and if (x,y_1), (x, y_2) are elements of S, then $y_1 = y_2$. Then there is a function $f: A \to B$ such that $(x,f(x)) \in S$ for each x in A; Ishihara [13].

2.3 Criterion for unprovability of statements in constructive mathematics:

We use omniscience principles as a criterion for unprovability of statements in constructive mathematics. We consider this to be a criterion because we cannot prove that

neither the statements nor their denials are valid.

Brouwer's critique of many classical theorems, his claim that they lacked numerical meaning, consisted of demonstrations showing that their truth would imply solutions to problems for which no solutions were known. He concluded that certain classical theorems could not be true, for if they were true, many people would try to use them to solve the unsolved problems. Such a demonstration is called a Brouwerian counter-example. With the formulation of omniscience principles by Errett Bishop, a Brouwerian counter-example then becomes a demonstration that a certain statement implies one of these omniscience principles; Mandelkern [22].

2.3.1 **Definition** (Brouwerian example): For a classical statement P(x), if P(a) implies some omniscience principle, we say that 'a' is a *Brouwerian example* for P(x). If the omniscience principle is Markov's principle, we speak of a Markovian example, rather than a Brouwerian one; Bridges [5], Ishihara [13].

Brouwerian examples are not examples in the usual sense, but provide strong evidence that certain classical propositions will never be proved constructively; Bridges [5].

As an example, we look at a Brouwerian example for the axiom of choice which is widely used in classical mathematics.

2.3.2 **Brouwerian Example** (Ishihara [13]): A set A and a subset S of A x {0,1} for which a choice function does not exist.

Let $\{a_n\}$ be a binary sequence and let A = {x,y}, where x = y if and only if a_n=1 for some n. Consider the subset S={(x,0),(y,1)} of A x {0,1}. If there exists a choice function f for S, then either f(x) = f(y), in which case a_n = 1 for some n; or else f(x) ≠ f(y), and a_n = 0 for all n. Thus a constructive proof of the axiom of choice, even from sets with at most two elements, would lead to one of LPO.

From this, we can find a rule (ie algorithm) ψ such that (a, ψ(a) ∈ S for each a in A by the constructive interprettion of existence; but it is not a function. A function f should be a rule which is extensional in the sense that if x=y then f(x) = f(y). Moreso, when a constructivist says that a number is either this or that, he is prepared to say which; or at least to give a finite procedure by which it may be determined.

Considering the fact that the main objective of this research work is to investigate constructive compact linear mappings, it is appropriate to end this brief disussion of the differences between classical and constructive mathematics here.

CHAPTER THREE

CONSTRUCTIVE METRIC AND NORMED LINEAR SPACES

In this section of the thesis, we may assume the reader is familiar with the elementary classical theory of metric and normed linear spaces; see Goffman-Pedrick [12], Kreyszig [20], Rudin [26], Taylor-Lay [28], Yosida [29].

We are now going to review the constructive theory of metric and normed spaces. Since these are defined on a set, we start with the constructive notion of a set.

3.1 Definition: A set X is defined when we describe how to construct its members from objects that have been, or could have been, constructed prior to X, and describe what it means for two members of X to be equal.

The equality relation, =, on a set is an essential part of its description, and must satisfy the defining properties of an equivalence relation:

reflexivity: $x = x$;

symmetry: if $x = y$, then $y = x$;

transitivity: if $x = y$ and $y = z$, then $x = z$;

see Bridges [5].

The equality relation on the set Z^+ of positive integers, and that on the set $\mathbb{N}$ of natural numbers, is the relation of identity: two natural numbers are equal if and only if they are one and the same.

A *unary relation* on a set X is a property P which is applicable to elements of X, and is *extensional*, in the sense that if x=y, then P(x) if and only if P(y). We are only

concerned with extensional properties; we are not concerned with *intensional* properties - those that depend on the manner in which the objects are presented to us.

If Y is a subset of a set X, and $x \in Y$, we write $x \notin Y$ to mean $x \in Y$ is impossible; and we denote by $X \setminus Y$ the set $\{x \in X : x \notin Y\}$. A subset Y of a set X is *detachable* from X if for each x in X either $x \in Y$ or $x \notin Y$.

If X and Y are subsets of a set Z, and if every element of X is an element of Y, then we say that X is *contained* in Y, and write $X \subseteq Y$. two subsets X and Y of Z are equal if $X \subseteq Y$ and $Y \subseteq X$;this is clearly an equivalence relation on subsets of Z.

A set is *non-empty*, or *nonvoid*, if we can construct an element of it; an empty set ϕ is a set that cannot be non-empty.

The *union*, $X \cup Y$, of two subsets X and Y of a set Z is the subset of Z defined by

$X \cup Y \equiv \{x \in Z : x \in X \vee x \in Y\}$. The intersection $X \cap Y$ of two subsets X and Y of Z is the subset of Z defined by $X \cap Y \equiv \{x \in Z : x \in X \wedge x \in Y\}$.

The *Cartesian product* X x Y of two sets X and Y consists of the ordered pairs (x,y) with $x \in X$ and $y \in Y$; that is,

$X \times Y \equiv \{(x,y) : x \in X \wedge y \in Y\}$.

Two ordered pairs (x,y) and (x',y') are equal if $x = x'$ and $y = y'$.

A subset X x X, or equivalently, a property applicable to elements of X x X, is called a binary relation on X.

A *function* from a set X to a set Y is an algorithm

(rule) f which produces an element f(x) of Y when applied to an element x of X, and is extensional in the sense that f(x) = f(y) whenever

x = y. The notation f:X→Y indicate that f is a function from X to Y. A function is also called a *mapping*, or *map*. The set X is called the *domain* of f written dom(f), and the set {f(x): x∈X} is called the *range* of f, written ran(f). If the range of f equals Y, we say that f maps X onto Y; if x = y whenever f(x) = f(y), then f is one-one.

A set X is *finitely enumerable* if there exists a non-negative integer n and a mapping f from {1, ..., n} onto X. If a map f is also one-one, we say that X is *finite*, and that it has n elements. A set X is *countable* if there is a function from a detachable subset of N onto X. We say that X is countably infinite if there is a one-one mapping from N onto X. As in classical mathematics, the cartesian product NxN is countable: infact, there exists a one-one map of N onto N x N.

The following result characterises countable sets:

3.2 Theorem (Bridges [5, 2.3]: An empty set is countable; a non-empty set is countable if and only if it is the range of a function with domain N.

3.3 Definition: A *metric space* is a set X together with a function d, called a *metric*, from XxX to the set of non negative real numbers such that

(1) d(x,y) = 0 if and only if x = y;

(2) d(x,y) = d(y,x)

(3) d(x,z) ≤ d(x,y) + d(y,z).

The open sphere centered at a point $x \in X$ with radius $r>0$ is the subset $S_r(x) \equiv \{y \in X: d(x,y)<r\}$ of X. A subset Y of X is *open* if for each $y \in Y$ there exists $r > 0$ such that $S_r(y) \subseteq Y$. The *interior* of a subset Y of X consists of all $y \in Y$ such that there exists $r > 0$ such that $S_r(y) \subseteq Y$.

A subset Y of X is *closed* if for each $r>0$ there exists $y \in Y$ such that if $y \in S_r(x)$, then $x \in Y$. The closure of a subset Y of X consists of all $x \in Y$ such that $y \in S_r(x)$.

A subset Y of a metric space X is *dense* in X if its closure equals X, that is,

$\overline{Y}=X$

. A metric space is separable if it contains a countable dense subset; see Ishihara [13].

3.4 Definition: A *Cauchy sequence* in a metric space X is a sequence $\{x_n\}$ in X such that for each $\varepsilon > 0$, there exists $N \geq 1$ such that $d(x_n, x_m) \leq \varepsilon$ whenever $m, n \geq N$. A sequence $\{x_n\}$ in X *converges* to $y \in X$ if for each $\varepsilon > 0$, there exists $N \geq 1$ such that $d(x_n, y) \leq \varepsilon$ whenever $n \geq N$. If $\{x_n\}$ converges to y, we say that y is the limit of $\{x_n\}$.

The proof of a constructive convergence test embodies algorithms which, if applied to an infinite series of the appropriate type, calculate the sum of the series and provide the rate of convergence to that sum. Thus convergence tests for infinite series play a role in constructive analysis; Bridges [5].

If each cauchy sequence converges to some element of X,

we say that X is *complete*.

The set of real numbers $\mathbb{R}$ forms a metric space under the metric $d(x,y) \equiv |x-y|$.

3.5 Theorem (Bridges [5]): $\mathbb{R}$ is a complete metric space.

3.6 Definition (Completion): The property of completeness of the field of real numbers plays an important role in mathematical analysis. The property of completeness of metric spaces occupies a similar crucial role in constructive analysis. We shall now examine a process according to which every non-complete metric space can be extended to a complete space, analogous to the extension of a set of rationals to a set of all real numbers.

Let (X,d) and (Y,ρ) be metric spaces. (X,d) and (Y,ρ) are said to be *isometric* if there is a transformation T of (X,d) into (Y,ρ) such that, for every $x, y \in X$,

$\rho(Tx,Ty) = d(x,y)$.

Let X_o be a given metric space. If this space is non-complete, that is, in this space there is a cauchy sequence but no element which will be its limit. Then there exists another complete metric space X, such that it has a subset W everywhere dense in X and isometric to X_o. This space X is called the *completion* of the space X_o.

3.7 Definition: (Suprema and Infima): Let X be a non-void subset of $\mathbb{R}$. A real numbers b is

an *upper bound* of X if $x \leq b$ for all $x \in X$;

the *supremum*, or *least upper bound*, of X if it is an upper bound of X, and if, for each $\varepsilon > 0$, there exists x in X such that $x > b-\varepsilon$.

If X has an upper bound, then we say that X is *bounded above*. Also, if the supremum of X exists, it is unique, and we denote it by sup X.

On the other hand, b is

a *lower bound* of X if $b \leq x$ for all x in X;

the *infimum*, or *greatest lower bound*, of X if it is a lower bound of X, and if, for each $\varepsilon > 0$, there exists x in X such that $x < b + \varepsilon$.

If X has a lower bound, then we say that X is *bounded below*. The infimum of X, if it exists, is unique, and is written infX. We say that X is *bounded* if it is both bounded above and bounded below; Bridges [5].

One result of classical analysis that is constructively unaceptable is the expression of the completeness of $\mathbb{R}$ in the least upper bound principle: every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound. In fact, the construction of the least upper bound of an increasing binary sequence is not certain. To see this, let $\{a_n\}$ be any increasing binary sequence, and suppose that $s \equiv \sup a_n$ exists. Then either $s > 0$, in which case there exists n with $a_n = 1$, or else $s < 1$, so that $a_n = 0$ for all n. Thus the classical least upper bound principle implies LPO; Bridges [2].

As an application of the completeness of $\mathbb{R}$, we have the following useful constructive version of the least upper bound principle:

3.8 Theorem: (Bridges [2], [5]): Let S be a nonvoid subset of $\mathbb{R}$ that is bounded above. Then supS exists if and only if

for all α,β in $\mathbb{R}$ with $\alpha < \beta$, either β is an upper bound of S or there exists x in S with $x > \alpha$.

3.9 **Definition:** A non empty subset Y of X is a *bounded* set if its *diameter*

$$d(X) \equiv \sup \{d(x,y) : x, y \in Y\}$$

exists and is finite.

3.10 **Definition:** A *subset* Y *of* a metric space (X,d) is *located* in X, or is a located subset of X if the distance

$$d(x,Y) \equiv \inf \{d(x,y) : y \in Y\}.$$

from x to Y exists for each $x \in X$.

The completeness of a metric space assures us of the following very useful lemma.

3.11 **Lemma** (Ishihara [13]): Let A be a complete, located subset of a metric space X, and x a point of X. Then there exists a point a in A such that d(x,a) > 0 entails d(x,A) > 0.

3.12 **Definition**: A function f from a metric space (X,d) to a metric space (Y,ρ) is *continuous* if for each $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $\rho(f(x), f(y) < \varepsilon$ for all $y \in X$.

3.13 **Definition**: A function f from a metric space (X,d) to a metric space (Y,ρ) is *uniformly continuous* if for each $\varepsilon > 0$ there exists $\omega > 0$, called a *modulus of continuity*, such that for each

$x, y \in X$

$\rho(f\,x), f(y)) \leq \varepsilon$ whenever $d(x,y) \leq \omega$.

3.14 **Definition:** We say that a mapping f between metric spaces X and Y is *sequentially continuous* if $x_n \to x$ as $n \to \infty$

implies

$f(x_n) \to f(x)$ as $n \to \infty$.

3.15 Definition: We say that a mapping f between metric spaces is *sequentially nondiscontinuous* if $x_n \to x$ as $n \to \infty$ and

$d(f(x_n), f(x)) \geq \delta$ for all n imply $\delta \leq 0$.

See Ishihara [17], [18].

Classically, for a linear mapping boundedness and sequential continuity are equivalent. But we know of no constructive proof that these notions are equivalent; Bridges [4], Ishihara [17], [18].

3.16 Definition: An *ε-approximation* to a metric space (X,d) is a subset Y of X such that for each $x \in X$ there exists $y \in Y$ with

$d(x,y) < \varepsilon$. The metric space X is said to be *totally bounded* if for each $\varepsilon > 0$ there exists a finite ε-approximation to X; Bridges [5].

Some authors allow ε-approximations, and therefore totally bounded sets, to be empty. To avoid complicating the relation between total boundedness and locatedness, in this dissertation, we shall only allow nonvoid ε-approximations.

3.17 Definition: A metric space (X,ρ) is said to be *compact* if it is totally bounded and complete; *locally compact* if each bounded set in X is contained in a compact set; Bridges-Demuth [8].

3.18 Proposition (Ishihara [13]): If X is a totally bounded space, and f is a uniformly continuous map of X into a metric space Y, then $f(X) \equiv \{f(x) : x \in X\}$ is totally bounded.

We shall give an application of proposition 3.18. For that, we need the following lemma:

3.19 Lemma (Bridges [5, 9.2]): If S is a nonvoid totally bounded subset of $\mathbb{R}$, then infS and supS exist.

Applying proposition 3.18 and lemma 3.19, we have the following theorem:

3.20 Theorem (Ishihara [13]): If f is a uniformly continuous map of a totally bounded space X into $\mathbb{R}$, then inf f $\equiv$ inf f(X) and sup f $\equiv$ sup f(X) exist.

If x is a point of a metric space X, then the mapping $y \to d(x,y)$ is uniformly continuous; so we have the following corollary to theorem 3.20.

3.21 Corollary (Ishihara [13]: A totally bounded subset Y of a metric space X is located.

If X is a totally bounded space, then since the mapping $x \to \sup \{d(x,y) : y \in X\}$ is uniformly continuous, the diameter

$d(X) \equiv \text{Sup} \{\text{Sup} \{d(x,y) : y \in X\} : x \in X\} = \text{Sup} \{d(x,y) : x, y \in X\}$ of X exists. That is, X is bounded.

From corollary 3.21 and the last result, we have the following lemma.

3.22 Lemma (Bridges [5]): A totally bounded subset of $\mathbb{R}$ is bounded and located.

3.23 Lemma (Bridges [5]): If S is a totally bounded subset of $\mathbb{R}$, and A is a located subset of S, then A is totally bounded.

We denote by C(X,Y) the set of uniformly continuous functions from a totally bounded metric space X to a metric

space Y. The metric ρ on C(X,Y) is defined by

ρ (f,g) $\equiv$ sup $\{$ d(f(x), g(x)) : x $\in$ X$\}$, (f,g $\in$ C(X,Y)).

3.24 Definition: A subset A of C(X,Y) is *equicontinuous* if the functions in A have a common modulus of continuity.

Classically, if X is a compact metric space, a subset k $\subset$ C(X) is relatively compact if and only if it is uniformly bounded and equicontinuous (Arzela-Ascoli Theorem).

The Arzela-Ascoli theorem has the following constructive version:

3.25 Lemma (Ishihara [13]): Let X be a totally bounded metric space, let Y be a metric space, and let S be an equicontinuous subset of C(X,Y) such that for each $\varepsilon > 0$ there exists an

ε-approximation $\{x_1, \ldots, x_n\}$ to X for which the subset

$$A \equiv \{(f(x_1), \ldots, f(x_n)) : f \in S\}$$

of Y^n is totally bounded. Then S is totally bounded.

3.26 Definition: A normed linear space is a linear (or vector) space E together with a function $||.||$, called a norm, from E to the set of non negative real numbers such that

(1) $||x|| = 0$ if and only if $x = 0$;

(2) $||\lambda x|| = |\lambda|\ ||x||$;

(3) $||x+y|| \leq ||x|| + ||y||$.

The linear space $\mathbb{C}^n$ admits a number of natural norms, one of which is given by

$||(\lambda_1, \ldots, \lambda_n)|| \equiv |\lambda_1| + \ldots |\lambda_n|$.

3.27 Definition: A normed linear space E is finite-dimensional if it contains a finitely enumerable set $\{e_1, \ldots$

$e_n\}$, called a *metric basis* of E, such that the map f: $\mathbb{C}^n \to$ E, defined by setting

$f(\lambda_1, \ldots, \lambda_n) \equiv \Sigma\, \lambda_i e_i$ has continuous inverse in the sense that there exists a uniformly continuous map

g : E → $\mathbb{C}^n$ such that

$f(g(x)) = x$ for each $x \in E$ and

$g(f(\lambda_1, \ldots, \lambda_n)) = (\lambda_1, \ldots, \lambda_n)$ for each $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$.

3.28 **Theorem** (Ishihara [13]): The following are equivalent conditions on a normed linear space E.

(1) E is finite-dimensional;

(2) The closed unit sphere of E is compact;

(3) E is locally compact.

3.29 **Definition:** Let E and F be normed linear spaces and let $S_E \equiv \{x \in E: ||x|| \leq 1\}$ be a closed unit sphere of E. Then we say that a linear mapping T of E into F is

bounded if $T(S_E)$ is bounded;

normable if $||T|| \equiv \sup \{||Tx|| : x \in S_E\}$ exists, and

compact if $T(S_E)$ is totally bounded.

Every compact map is normable, every normable map is bounded, and every bounded map is uniformly continuous. If T is normable, then ||T|| is called the *norm* of T.

We denote the set of all bounded linear mappings of E into F by Hom(E,F), the set of all normable linear mappings by $\mathrm{Hom}_n(E,F)$, and the set of all compact linear mappings by $\mathrm{Hom}_c(E,F)$. Then, since every compact map is normable, every

normable map is bounded, and every bounded map is uniformly continuous, we have

$$Hom_c(E,F) \subseteq Hom_v(E,F) \subseteq Hom(E,F).$$

3.30 Definition: A mapping from a normed space X into itself is called an *operator*. A mapping from a normed space X into the scalar field $\mathbb{R}$ or $\mathbb{C}$ is called a *functional*.

We recall from section 2.3 (definition 2.3.1) that: For a classical statement P(x), if P(a) implies some omniscience principle, we say that 'a' is a Brouwerian example for P(x). Here, we give examples of normed linear spaces which will be frequently used to construct Brouwerian examples in the rest of this disseration.

For $1 \leq p < \infty$, let l^p be the space of sequences $x \equiv \{x_n\}$ of numbers such that $\Sigma\ |x_n|^p$ converges. With each $x \equiv \{x_n\}$ in l^p, associating a non negative real number $||x||$ by $||x|| \equiv (\Sigma\ |x_n|^p)^{1/p}$, l^p becomes a normed linear (Banach) space. We know that l^p has the standard basis; for each n, we denote by e_n the n^{th} basis whose n^{th} term is 1 and all other terms are 0.

The space l^∞ of all bounded sequences $x \equiv \{x_n\}$ of numbers is not a normed space as classically.

The following Bouwerian example shows that $Hom_v(E,F) \neq Hom(E,F)$; whence Hom(E,F) is not a normed linear space as classically.

3.31 Brouwerian Example (Bridges *et al* [61]): A bounded linear functional on l^2 which is not normable.

Let l^2 be the Hilbert space of square-summable sequences of real numbers and for each $x \in l^2$, let x_n be the n^{th}

component of x. Let $\{a_n\}$ be a binary sequence such that $x_n=1$ for at most one value of n. Define a bounded linear functional u: $l^2 \to \mathbb{R}$ by

$$u(x) \equiv \sum_{n=1}^{\infty} a_n x_n.$$

If u is normable, then a constructive proof for this would enable us to compute the norm of u.

Using the familiar dot product with one factor kept fixed, the above function can be written as

$$u(x) \equiv \sum_{n=1}^{\infty} a_n x_n = a.x,$$

where $a \equiv \{a_n\} \in l^2$ is fixed, $x \equiv \{x_n\} \in l^2$. Now since u is bounded, we have

$$|u(x)| = |\sum_{n=1}^{\infty} a_n x_n| \leq \sum_{n=1}^{\infty} |a_n x_n| \leq \sqrt{\sum_{n=1}^{\infty} |a_n|^2} \cdot \sqrt{\sum_{n=1}^{\infty} |x_n|^2}$$

(Cauchy-Schwarz inequality).

i.e

$$|u(x)| \leq \sqrt{\sum_{n=1}^{\infty} |a_n|^2} \cdot \sqrt{\sum_{n=1}^{\infty} |x_n|^2} = \|a\| \|x\|.$$

$$\|u\| = \sup_{\substack{x \in l^2 \\ x \neq 0}} \frac{|u(x)|}{\|x\|} = \sup_{\substack{x \in l^2 \\ \|x\|=1}} |u(x)| \leq \sup_{\|x\|=1} \|a\| \|x\| = \|a\|$$

Let S denote the set of values of the binary sequence $a \equiv \{a_n\}$. The set S has at least one element, and at most two, but in general we donot know exactly how many; in any event

it is a nonvoid bounded set of real numbers. Using the Constructive Dichotomy Lemma (See (2) of 2.2.5), since $0 < 1$, then for $||u||=y\in\mathbb{R}$, either $y > 0$, or $y < 1$.

If $||u|| > 0$, then there exists n such that $a_n=1$; if $||u|| < 1$, then $a_n=0$ for all n. It follows that we would have an algorithm for deciding whether or not an arbitrary sequence in $\{0,1\}$ has a nonzero term; but no such algorithm will ever materialize.

3.32 Definition: A *Banach space* is a complete normed linear space.

In the constructive development of the theory of bounded linear mappings it may be important to locate the kernel (null space).

$$\ker (T) = \{x \in E : Tx = 0\},$$

of such a mapping T.

We recall that a subset Y of a metric space (X,d) is *located* in X if the distance from x to Y,

$$d(x,Y) \equiv \inf \{d(x,y) : y \in Y\}$$

exists for all $x \in X$.

With the aid of the above notion, we can give a necessary and sufficient condition for normability of linear mappings. Of special interest is the case $F \equiv \mathbb{C}$; in which case, we have the following:

3.33 Proposition (Ishihara [13]): A nonzero bounded linear functional on a normed linear space E is normable if and only if its kernel is located.

Proposition 3.33 can be viewed as a special case of the following theorem:

3.34 Theorem (Bridges *et al* [6]): A bounded linear mapping of a normed linear space E onto a finite-dimensional normed linear space is compact if and only if its kernel is located in E.

3.35 Definition: An *inner product* space is a linear space E together with a function (.,.), called an inner product, from E x E to $\mathbb{C}$ such that

(1) $(x,x) \geq 0$; and $(x,x) = 0$ iff $x = 0$;

(2). $(x,y)=\overline{(y,x)}$

(3) $(\lambda x + \mu y, z) = \lambda(x,z) + \mu(y,z)$.

An inner product space E is a normed linear space with norm given by $||x||=(x,x)^{\frac{1}{2}}$ in which the Cauchy-Schwarz inequality $|(x,y)| \leq ||x||\,||y||$ and Minkowski's inequality $||x+y|| \leq ||x|| + ||y||$ are valid; Ishihara [13].

3.36 Definition: A *Hilbert space* is a complete inner product space.

3.37 Definition: We say that an operator A on a Hilbert space H has an adjoint (operator) A* on H if

$$(Ax,y) = (x, A^*y)$$

for all $x,y \in H$. An operator is selfadjoint if $A = A^*$.

As we shall see later on, the proofs of results on Hilbert spaces depend on the geometric properties of Hilbert spaces, hence we cannot expect to generalize certain results (theorems) to arbitrary normed linear spaces in the same manner.

Now, we define quasinormed linear spaces, which enable us to deal with linear spaces, such as Hom (E,F), on which

the classical norm cannot be computed everywhere.

3.38 Definition: A linear space E is said to be a *quasinormed linear space* if there is associated a family of seminorms $\{||.||_\alpha : \alpha\in A\}$, called the *quasinorm* on E, in such a way that for each $x\in E$, the set $\{||x||_\alpha : \alpha\in A\}$ of $\mathbb{R}$ is bounded.

The equality and inequality relations on the quasinormed linear space E are defined by

$$x = y \Leftrightarrow ||x-y||_\alpha = 0 \text{ for all } \alpha\in A.$$

and

$$x\neq y \Leftrightarrow ||x-y||_\alpha > 0 \text{ for some } \alpha\in A;$$ see Ishihara [13].

An element $x\in E$ is said to be normable if $||x|| \equiv \sup\{||x||_\alpha : \alpha\in A\}$, called the norm of x, exists.

The set l^∞ of all bounded sequences $x \equiv \{x_n\}$ of numbers which is not a normed linear space constructively, is a quasinormed linear space with a sequence of seminorms defined by $||x||_n \equiv |x_n|$ $(n=1,2,\ldots,\ x\in l^\infty)$.

Also, the set Hom(E,F) of all bounded linear mappings between normed linear spaces E and F is a quasinormed linear space with a family of seminorms defined by

$$||T||_x \equiv ||Tx|| \quad (x\in S_E,\ T \in \mathrm{Hom}\ (E,F)).$$

From now onwards, we shall write the set Hom(E,$\mathbb{C}$) of all bounded linear functionals on E by E* and the set $\mathrm{Hom}_\nu(E,\mathbb{C})$ of all normable linear functionals by E′.

CHAPTER FOUR

CONSTRUCTIVE COMPACT OPERATORS ON A HILBERT SPACE

By definition, a Hilbert space is a complete inner product space.

Here, we deal with compact operators on a Hilbert space, within the framework of Bishop's constructive mathematics (BISH). Most of the results (theorems) on Hilbert spaces depend on the geometric properties of Hilbert spaces (such as orthogonality properties, Riesz representation of bounded linear functionals by inner products, Hilbert-adjoint operator T* of a bounded linear operator T, etc). So we cannot expect to generalize certain results to arbitrary normed linear spaces in the same manner.

Let E and F be normed linear spaces and let $S_E \equiv \{x \in E: ||x|| \leq 1\}$ be the closed unit sphere of E. We say that a linear mapping T of E into F is *bounded* if $T(S_E)$ is bounded, *normable* if $||T|| \equiv \text{Sup}\{||Tx||: x \in S_E\}$ exists, and *compact* if $T(S_E)$ is totally bounded, that is, given any $\varepsilon > 0$, we can construct a finite ε-net of the image of S_E under T. If T is normable, then ||T|| is called the norm of T.

A bounded linear mapping T of E into itself is called an operator on E.

Classically, we know that every bounded linear mapping into a finite-dimensional normed linear space is compact, because every closed and bounded subset of a finite-dimensional normed space is compact. But, as the following example shows, we cannot expect to prove constructively that

a bounded linear mapping into a finite-dimensional normed linear space is compact.

4.1 Brouwerian Example (Bridges *et al* [6]): A bounded linear functional on l^2 which is not compact.

Let l^2 be the Hilbert space of square-summable sequences of numbers and for each $x \in l^2$, let x_n be the n^{th} component of x. Let $\{a_n\}$ be a binary sequence containing at most one 1 and define a bounded linear functional f on l^2 by

$$f(x) = \sum_{n=1}^{\infty} a_n x_n$$

If f is compact, and so normable, then it follows from Brouwerian example 3.31 that, either $||f|| > 0$ in which case there is a finite procedure which results in the construction of an integer n such that $a_n = 1$; or else $||f|| < 1$, and $a_n = 0$ for all n. As can easily be seen, there is no computational procedure for this.

4.2 Theorem (Bridges *et al* [6, Theorem 1]): A bounded linear mapping u of a normed linear space E onto a finite-dimensional normed linear space F is compact if and only if its kernel is located in E.

For the proof of this theorem, we require the following observations and some preliminary lemmas.

From Brouwerian example 4.1, we observe that our inability to prove the compactness of u is due solely to our lack of knowledge of the dimension of the domain of f. For any bounded linear mapping into $\mathbb{R}$ that appears in practice, we will know the dimension (finite or infinite) of the

domain. If the domain has finite dimension, then the mapping is certainly compact. If the domain has infinite dimension, then, as we have seen in Brouwerian example 4.1, we may be unable to prove the compactness of the mapping even when its kernel is located (that is, the distance from any point to the kernel can be computed).

We observe that in order to obtain the conclusion of theorem 4.2, we need the mapping u to be onto, rather than just into, a finite-dimensional space. This is because for the range of a compact linear mapping u: E→F, finite-dimensionality and completeness are equivalent conditions (Bridges *et al* [6]); in other words, we can construct a basis for u(E) if and only if given a Cauchy sequence $\{u(x_n)\}_{n \geq 1}$, we can construct x in E such that $u(x) = \lim u(x_n)$.

Next we look at the dimension of the range. If u: E→ F and dim ran (u) < ∞, we say that u is of finite rank. The linear mapping u is of *finite rank n* if the dimension of its range is n.

Classically, every bounded linear mapping u of finite rank is compact, because the image u(S) of a unit ball S in E is a bounded set in the finite-dimensional ran(u) and hence is relatively compact; see Taylor-Lay [28].

But, as the following example shows, we cannot expect to prove constructively that a bounded linear mapping of finite rank is compact, even when the domain of the mapping is a Hilbert space.

4.3 Example (Bridges *et al* [6]): Let a∈ℝ, and let H be the completion of ℝ x ℝa in the real Hilbert space $\mathbb{R}^2$. Then

$u(x,y) = x+y$ defines a bounded linear mapping of H into $\mathbb{R}$. A constructive proof of the compactness of u would enable us to compute its norm.

It follows that, when $a \in \mathbb{R}$, $\mathbb{R}a$ is complete if and only if we can decide whether $a=0$ or $a \neq 0$.

Now, if $a=0$, then $\mathbb{R}a$ is the set whose only member is 0. Then

$$\mathbb{R} \times \mathbb{R}a = \{(x,0) : x \in \mathbb{R}\}.$$

By definition, $u(x,0) = x + 0 = x$

$$||u|| = \sup_{||x||=1} |u(x,0)| = \sup_{||x||=1} |x| \leq \sup_{||x||=1} ||x|| = 1.$$

On the other hand, if $a \neq 0$, then $\mathbb{R}a = \mathbb{R}$.

$$u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$$

$$u(x,y) = x + y$$

$$\|u\| = \sup_{\substack{x\neq 0 \\ y\neq 0}} \frac{|u(x,y)|}{|(x,y)|} \geq \sup_{\substack{x\neq 0 \\ y\neq 0}} \frac{\|x+y\|}{\|x\|\|y\|}$$

Since $x \perp y$, we have

$||x+y||^2 = ||x||^2 + ||y||^2$ (Pythagorean theorem).

$$\therefore \quad \|u\| \geq \sup_{\substack{x\neq 0 \\ y\neq 0}} \frac{\sqrt{\|x\|^2+\|y\|^2}}{\|x\|\,\|y\|} = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \sqrt{\|x\|^2+\|y\|^2} = \sqrt{1+1} = \sqrt{2}$$

It can be observed that if $||u|| > 1$, then $a \neq 0$; while if $||u|| < \sqrt{2}$, then $a=0$. Thus we would have an algorithm for deciding whether or not an arbitrary real number a is zero; nobody believes that such an algorithm willl be found.

It is worth noting that, from a constructive point of view, the space $\mathbb{R} \times \mathbb{R}a$ is complete if and only if either $a=0$

or $a\neq 0$; but there is no known computational procedure for making this decision; Bridges *et al* [6], [7].

With example 4.3 in mind, we now attempt o determine the necessary and sufficient conditions for a bounded linear mapping of finite rank to be compact. We can only do this if the rank is 1.

A bounded linear mapping u onto a one-dimensional normed space is constructively compact if and only if its norm can be computed. Bishop showed that this occurs precisely when the kernel is *located*, in the sense that the distance from any point to the kernel can be computed; Bridges *et al* [6].

4.4 Definition: Let X be a set. An open ball $B(x_0, \varepsilon)\equiv\{x\in X: d(x,x_0)<\varepsilon\}$ of radius $\varepsilon > 0$ is called an ε-neighbourhood of $x_0\in X$. By a neighbourhood of x_0 we mean any subset of X which contains an ε-neighbourhood of x_0.

4.5 Definition: A set X is said to be *convex* if for any two points in the set, the line segment joining these two points lies in the set, that is, X is convex if x_1, $x_2\in X$ imply $\lambda x_1+(1-\lambda)x_2\in X$, where $0\leq\lambda\leq 1$.

The set of all convex combinations of sets of points from X is called the *convex hull* of X.

4.6 Definition: If $f: X \to Y$ and $X_1 \subseteq X$, the function $f|X_1: X_1 \to Y$ is called the restriction of f to X_1 if $f \mid X_1(x) = f(x)$ for every $x \in X_1$.

4.7 Definition: A set X is symmetric if $X = -X$, that is, if

$-a\in X$ whenever $a\in X$.

Three preliminary lemmas are required for the proof of

theorem 4.2

4.8 **Lemma** (Bridges *et al* [6]): Let E be a normed space over $\mathbb{R}$, B a convex neighbourhood of 0 in E, and f a continuous linear mapping of E into $\mathbb{R}^n$. Then there exists a continuous retraction g of E onto kerf such that g(B) = B∩ker f.

Proof:

If n=0, we need only take g to be the identity mapping.

If n=1, and we choose a∈B so that f(a) > 0 and -a∈B, the required retraction g is obtained by setting.

$$g(x) = \frac{f(a)x - f(x)a}{f(a)+|f(x)|}$$

We note that

$$-f(x)a = \mathrm{Sup}\,(-f(x), 0)a + \mathrm{Sup}\,(f(x), 0)(-a) \qquad (x\in E);$$

taken with the convexity of B, this ensures that $g(x) \in B$ whenever $x\in B$.

Now let $k \geq 2$, assume we have proved the lemma for n = k-1, and consider the case n=k.

Define $\pi: \mathbb{R}^k \to \mathbb{R}$ by $\pi(x_1, \ldots, x_k) = x_k$. Then πof is a continuous linear mapping of E onto $\mathbb{R}$; so that, by the case n = 1, there exists a continuous retraction g_1 of E onto ker (πof) such that

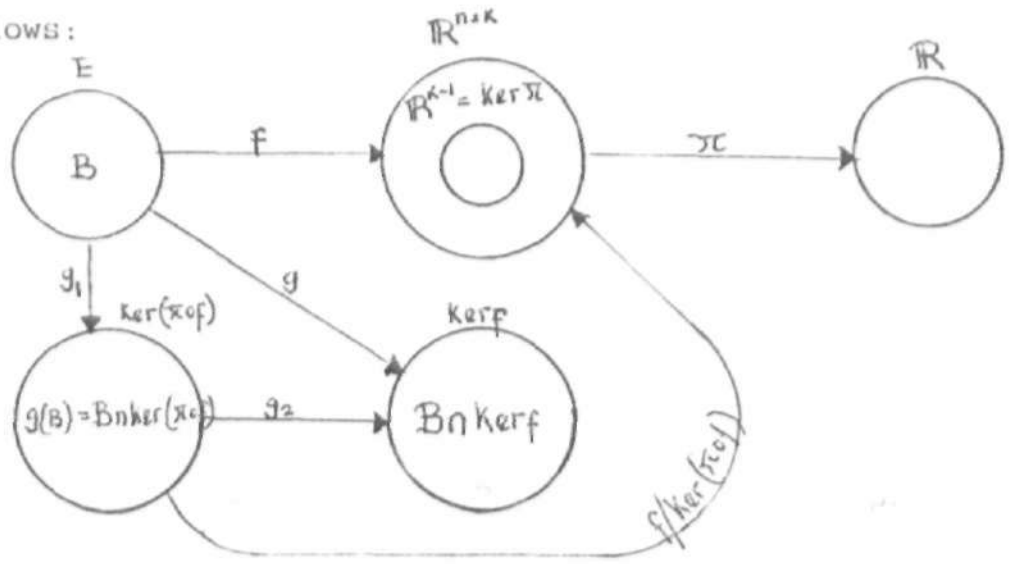
$g_1(B)=B \cap \ker(\pi of)$.

On the other hand, the restriction of f to ker (πof) is a continuous linear mapping of ker(πof) onto $\mathbb{R}^{k-1} = \ker \pi$; and $B \cap \ker(\pi of)$ is a convex neighbourhood of 0 in ker (πof). It follows from our induction hypothesis that there exists a continuous retraction g_2 of ker (πof) onto kerf such

g_2 (B ∩ ker (πof)) = B ∩ ker (πof) ∩ ker f = B ∩ kerf.

We now see that $g = g_2 o g_1$, is a continuous retraction of E onto kerf such that g(B) = B ∩ kerf. This completes the proof.

Diagrammatically, we can illustrate the above result as follows:



4.9 <u>Lemma</u> (Bridges *et al* [6]): Let B be a totally bounded, convex, symmetric neighbourhood of 0 in $\mathbb{R}^n$, and let a be a point of $\mathbb{R}^n$. Then $S(a)=\{\lambda\in\mathbb{R}:\lambda a \in B\}$ is a located neighbourhood of 0 in $\mathbb{R}$.

<u>Proof</u>

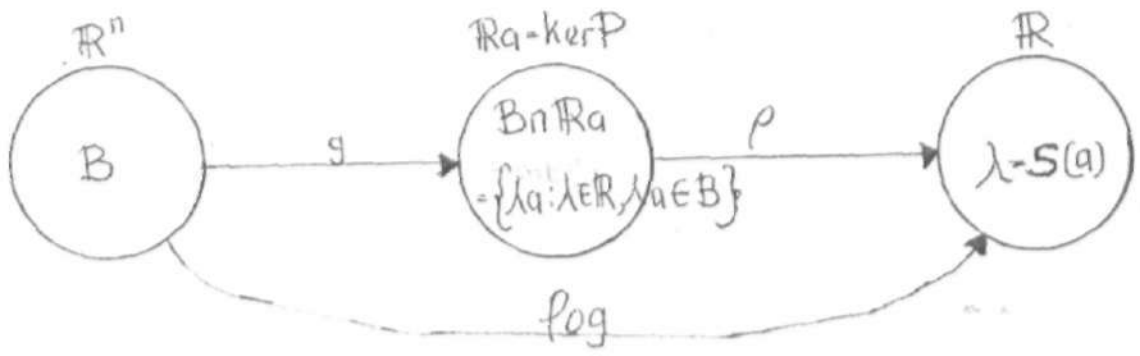
Choose $\delta > 0$ so that $B(0, \delta) \subset B$. Given ξ in $\mathbb{R}$, we have either $|\xi a| < \delta$ or $0 < |\xi a|$. In the former case, $\xi a \in B$; so that $\xi\in S(a)$, and therefore $d(\xi, S(a)) = 0$. In the case $0 < |\xi a|$, $\mathbb{R}a$ is a one-dimensional linear subspace of $\mathbb{R}^n$. Let P be the projection of $\mathbb{R}^n$ on the orthogonal complement of $\mathbb{R}a$. By lemma 4.8, there exists a continuous retraction g of $\mathbb{R}^n$ onto $\mathbb{R}a$ = ker P such that

$$g(B) = B\cap \mathbb{R}a = \{\lambda a: \lambda\in \mathbb{R}, \lambda a \in B\}.$$

As $a \neq 0$, $\rho(\lambda a) = \lambda$ defines a continuous mapping of $B\cap \mathbb{R}a$

onto S(a). Thus $\rho \circ g$ is a continuous mapping of B onto S(a). As B is totally bounded, so is S(a); whence $d(\xi, S(a))$ is computable. That S(a) is a neighbourhood of 0 in $\mathbb{R}$ follows from the fact that B is a neighbourhood of 0 in $\mathbb{R}^n$.

We illustriate this result as follows:



4.10 Lemma (Bridges *et al* [6]): Let E be a normed space over $\mathbb{R}$, and u a continuous linear mapping of E onto $\mathbb{R}^n$. Then u is an open mapping (i.e. the image of an open set in E under u is open in $\mathbb{R}^n$).

Proof

If n = 0, this is trivial.

If $n \geq 1$, and $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$, choose $r > 0$ so that $e_k \in u(B(0,r))$ for $k = 1, \ldots, n$. As $u(B(0,r))$ is convex and symmetric, it contains the convex hull of $\{-e_1, \ldots, -e_n, e_1, \ldots, e_n\}$ and so contains an open ball in $\mathbb{R}^n$ with centre 0 and positive radius.

With the above preliminary results, we can now prove

With the above preliminary results, we can now prove theorem 4.2: Let E be a normed linear space, F a finite-dimensional normed space, and u a bounded linear mapping of E onto F. Then u is compact if and only if its kernel is located in E.

Proof

Without loss of generality, we assume that E and F are real linear spaces, and that $F \equiv \mathbb{R}^n$ for some integer $n \geq 0$.' Let u be compact, and let $x \in E$. Then noting lemma 4.9 and applying lemma 4.8 with $B = u(B(0,1))$, we see that $S(x) = \{\lambda \in \mathbb{R}: \lambda u(x) \in B\}$ is a located neighbourhood of 0 in $\mathbb{R}$. Lex x_o be a positive element of $S(x)$. The mapping $\lambda \to 1/\lambda$ is uniformly continuous on the located subset $S(x) \cap [\lambda_o, \infty)$ of $\mathbb{R}$; whence

$\inf\{1/\lambda: \lambda \in S(x), \lambda_o \leq \lambda\}$ is computable. By the symmetry of B, this infimum equals $\inf \{|\lambda|: \lambda \in \mathbb{R}, u(x) \in \lambda B\}$. But

$\inf \{|\lambda|: \lambda \in \mathbb{R}, u(x) \in \lambda B\}$

$= \inf \{|\lambda|: \lambda \in \mathbb{R}$ and $u(x-\lambda z) = 0$ for some $z \in E$, $||z|| < 1\}$

$= \inf \{|\lambda|: \lambda \in \mathbb{R}$ and $||x-y|| < \lambda$ for some $y \in$ ker u$\}$

$= d(x,$ ker u$\}$.

Hence ker u is located in E.

Conversely, suppose that ker u is located in E. Then $||u(x)||_1 = d(x,$ ker u$)$ defines a norm on $\mathbb{R}^n$. As $\mathbb{R}^n$ is finite-dimensional, the open unit ball in $(\mathbb{R}^n, ||\ ||_1)$ is totally bounded. As this ball is just the set $u(B(0,1))$, u is compact.

When the domain of the mapping is a Hilbert space, we can prove the following:

4.11 Proposition (Ishihara [16]): Let u be a bounded linear mapping of a Hilbert space H into $\mathbb{C}^n$, and for i=1, ..., n, let $p_i : \mathbb{C}^n \to \mathbb{C}$ be a mapping defined by $p_i(x_1, \ldots, x_n) = x_i$. Then u is compact if and only if $p_i \circ u$ is normable for each i=1, ..., n.

We need the following results for the proof of proposiion 4.11.

4.12 Lemma (Riesz Representation Theorem): Let f be a bounded linear functional on a Hilbert space H. Then f is normable if and only if

$$f(x) = (x,y) \qquad (x \in H)$$

for some vector y in H; in which case y is unique, and $||f|| = ||y||$; Ishihara [13].

4.13 Lemma (Ishihara [13]): Lex $x_1, \ldots, x_n$ be elements of a normed linear space E, M the set of all linear combinations of $x_1, \ldots, x_n$, and $\varepsilon > 0$. Then there is a finite-dimensional subspace V of E with $V \subseteq M$, such that $d(x_i, V) < \varepsilon$ for each i.

Proof of Proposition 4.11:

Let u be compact. Then $p_i \circ u$ is compact, and therefore is normable, for each i=1, ..., n.

Conversely, suppose that $p_i \circ u$ is normable for each i=1, ..., n. Then, by lemma 4.12, for each i=1, ..., n, there exists $f_i \in H$ such that $(p_i \circ u)(x) = (x, f_i)$. Let $\varepsilon > 0$. Then by lemma 4.13, there exists a finite-dimensional subspace Y of H such that $d(f_i, Y) < \varepsilon/2n$ for all i=1, ..., n.

Since $u(S_Y) \equiv \{u(x) : x \in Y, ||x|| \leq 1\}$ is totally bounded, there

exists an $\varepsilon/2$-approximation

$\{u(x_1), \ldots, u(x_m)\}$ to $u(S_Y)$. Thus for each $x \in S$, there exists k such that $||u(Px) - u(x_k)|| < \varepsilon/2$, where P is the projection onto Y, and since $d(f_i, Y) \equiv \inf\{d(f_i, Pf_i): Pf_i \in Y\} = ||fi-Pf_i|| < \varepsilon/2n$ $(i=1, \ldots, n)$ and P is self-adjoint, we have

$$||u(x) - u(x_k)|| = ||\{u(x) - u(Px)\} + \{u(Px) - u(x_k)\}||$$

$$\leq ||u(x) - u(Px)|| + ||u(Px) - u(x_k)\}||$$

$$< \|u(x)-u(Px)\| + \frac{\varepsilon}{2}$$

$$= \sum_{i=1}^{n} |(p_i o u)(x - Px)| + \frac{\varepsilon}{2}$$

$$= \sum_{i=1}^{n} |(x - Px, fi)| + \frac{\varepsilon}{2}$$

$$= \sum_{i=1}^{n} |(x, fi - Pfi)| + \frac{\varepsilon}{2}$$

$$\leq \sum_{i=1}^{n} \|x\| \|f_i - Pf_i\| + \frac{\varepsilon}{2} < \sum_{i=1}^{n} \frac{\varepsilon}{2n} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $\{u(x_1), \ldots u(x_m)\}$ is also an ε-approximation to $u(S)$. Proposition 4.11 is still valid when the set of all normable linear functionals on the domain of the mapping is a linear space (see Corollary 5.1.7); but as we shall see in the next example, we cannot expect to generalise the result to

arbitrary normed linear spaces.

4.14 **Brouwerian Example** (Ishihara [16]): A bounded linear mapping u of l^1 into $\mathbb{C}^2$ such that $p_i$ou is normable for i=1,2, but u is not compact.

Let l^1 be the Banach space of absolutely summable sequences of real numbers and for each $x \in l^1$, let x_n be the n^{th} component of x. Let $\{a_n\}$ be a binary sequence and define bounded linear functionals f and g by

$$f(x) \equiv x_1 + \sum_{n=1}^{\infty} a_n x_{n+1}$$

$$g(x) \equiv -x_1$$

Then, by definition, it follows that $p_1 ou = f$ and $p_2 ou = g$ are normable, with

$$\|f\| = Sup_{\substack{x \in l^1 \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = Sup_{\substack{x \in l^1 \\ \|x\| = 1}} |f(x)| = 1$$

$$\|g\| = Sup_{\substack{x \in l^1 \\ \|x\| = 1}} |g(x)| = 1$$

If u is compact, then

$$\|f + g\| = \|\sum_{n=1}^{\infty} a_n x_{n+1}\|$$

Let $S_1 1$ be a closed unit sphere of l^1. Then

$\|f+g\| = Sup\{|z_1 + z_2| : (z_1, z_2) \in u(S_1 1)\}$.

From Brouwerian example 3.31, we have either $||f+g|| > 0$, in which case there exists n such that $a_n=1$; or else $||f+g|| < 1$, and $a_n=0$ for all n. As we can easily see, there is no computational procedure for this.

4.15 Lemma (Ishihara [16]): Let A be a linear mapping of a Hilbert space H into itself, such that for each $y\in H$, $\{(Ax,y): ||x|| \leq 1\}$ is totally bounded. Then A* exists.

Proof

For each $y \in H$, the hypotheses ensure that $x \to (Ax,y)$ is a normable linear functional on H. Now we know that a bounded linear functional f on a Hilbert space H is normable if and only if there exists $z\in H$ such that $f(x) = (x,z)$ (by lemma 4.12). Therefore, $(Ax,y) = (x,z)$ for all $x \in H$. Thus $z = A^*y$.

Constructively, it seems difficult to prove the compactness of the sum of compact linear operators, which is classically proved by means of the constructively unacceptable property of sequential compactness. From proposition 4.11 and lemma 4.15, we can prove the following:

4.16 Theorem (Ishihara [16]): Let A and B be a compact operators on a Hilbert space H, let C be an operator on H and $\alpha\in\mathbb{C}$. Then

(1) αA is compact;

(2) A+B is compact;

(3) A* exists and is compact;

(4) CA is compact;

(5) If C* exists, then AC is compact.

Proof

(1) Let S be a closed unit sphere of H. Since A is a compact operator, A(S) is totally bounded. Let $\alpha \in \mathbb{C}$. Then the set

$A(\alpha S) = \alpha A(S)$ is totally bounded. Hence αA is compact.

(2) Let $\varepsilon > 0$. Let Y be a finite-dimensional subspace of H. Then $||(A+B)x-(PA+PB)x|| < \varepsilon \quad (x \in S)$,

where P is the projection onto Y. Let $\{e_1, \ldots, e_n\}$ be a metric basis of Y. Then we can write PA and PB in the forms

$$PA(x) = f_1(x)e_1 + \ldots + f_n(x)e_n$$

$$PB(x) = g_1(x)e_1 + \ldots + g_n(x)e_n.$$

Since PA and PB are compact, proposition 4.11 shows that for each i, f_i and g_i are normable linear functionals on H; whence by proposition 4.11, $f_i + g_i$ are normable. Thus by proposition 4.11, the linear mapping

$$(PA+PB)(x) = (f_1+g_1)(x)\, e_1 + \ldots + (f_n+ g_n)(x)e_n$$

of H into Y is compact. Therefore, A + B is compact.

(3) Since A is compact, $\{(Ax,y) : x \in S\}$ is totally bounded for all $y \in H$; whence, by lemma 4.15, A* exists.

Let $\{Ax_1, \ldots, Ax_n\}$ be an $\varepsilon/3$-approximation to A(S) and let $u: H \to \mathbb{C}^n$ be the bounded linear mapping defined by

$$u(x) \equiv ((Ax_1,y), \ldots, (Ax_n , y)) \qquad (y \in H).$$

Then u is compact by proposition 4.11.

Let $\{u(y_1), \ldots, u(y_m)\}$ be an $\varepsilon/3$-approximation to u(S). Then for each $y \in S$, there exists k such that

$$|(Ax_j, y) - (Ax_j, y_k)| < \varepsilon \text{ for all } j=1, \ldots, n.$$

Let $x \in S$ and let j be such that $||(Ax - Ax_j|| < \varepsilon/4$. Then we have

$|(x, A^*y - A^* y_k)| \le |(Ax_j, y) - (Ax_j, y_k)| + |(Ax - Ax_j, y)| + |(Ax - Ax_j, y_k)|$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$

and hence

$||(A^*y - A^*y_k)|| = \text{Sup}\{|(x, A^*y - A^*y_k)| : ||x|| \le 1\}$

Therefore, $\{A^*y_1, \ldots, A^*y_m\}$ is an ε-approximation to $A^*(S)$.

(4) Let S be a closed unit sphere of H. Since C is an operator on H, the set $C(A(S)) = CA(S)$ is totally bounded because A is compact. Hence CA is compact.

(5) By (3), A^* is compact, so C^*A^* is compact. Hence, by (3),

$AC = (C^*A^*)^*$ is compact.

Classically, every operator on a Hilbert space has an adjoint. Thus, the hypothesis "if C^* exists" in (5) of theorem 4.16 may seem more than needed. But the following example shows that there is an operator on a Hilbert space which does not have an adjoint, and so we cannot drop the existence of C^* from the hypothesis of theorem 4.16 (5).

4.17 **Brouwerian example** (Ishihara [16]): An operator on l^2 which does not have an adjoint.

For each positive integer n, let e_n be the basis vector on l^2 with n^{th} component 1 and all other components 0. Let $\{a_n\}$ be a binary sequence containing at most one 1, and define a linear mapping C of l^2 into itself by

$$Cx \equiv \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m}{\sqrt{2^n}} x_m e_n$$

Then we have

$$\|Cx\|^2 = \sum_{n=1}^{\infty} \frac{1}{2^n} |\sum_{m=1}^{\infty} a_m x_m|^2 \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\|^2 = \|x\|^2,$$

and hence C is an operator. However, for

$$y \equiv \sum_{n=1}^{\infty} \frac{1}{\sqrt{2^n}} e_n,$$

we have

$$(Cx,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m}{2^n} x_m = \sum_{m=1}^{\infty} a_m x_m$$

and as we have seen in Brouwerian example 3.31, the bounded linear functional $x \to (Cx,y)$ is not normable. Therefore, C does not have an adjoint.

4.18 Definition: Let $|||.|||$ be the double norm on a separable Hilbert space H, defined by

$$|||x||| \equiv \sum_{n=1}^{\infty} \frac{|(x,y_n)|}{2^n} \quad (x \in H),$$

where $y_1, y_2, \ldots$ is a dense sequence of a closed unit sphere S of H. Then we say that a bounded linear mapping A of a normed linear space E into H is *weakly compact* if A(S) is totally bounded with respect to the double norm.

4.19 Theorem (Ishihara [16]): An operator on a separable Hilbert space has an adjoint if and only if it is weakly compact.

Proof:

Let A be an operator on a separable Hilbert space H, let y_1,

y_2, ... be a dense sequence in a closed unit sphere S, and define the double norm by

$$|||x||| \equiv \sum_{n=1}^{\infty} \frac{|(x,y_n)|}{2^n} \quad (x \in H).$$

We may assume without loss of generality that the bound of A is 1.

Suppose that A is weakly compact. In view of lemma 4.15, it will suffice to prove that, for each $y \in S$, $\{(Ax,y) : x \in A\}$ is a totally bounded subset of $\mathbb{C}$. Let $y \in S$ and let $\varepsilon > 0$. Then there exists $n \geq 1$ such that $||y-y_n|| < \varepsilon/4$; also there exists an $\varepsilon/2^{n+1}$-approximation $\{Ax_1, \ldots, Ax_m\}$ to A(S) in the double norm. For each $x \in S$ there exists k such that

$$\frac{1}{2}|(Ax-Ax_k,y_n)| \leq |||Ax-Ax_k||| < \frac{\varepsilon}{2^{n+1}}$$

Hence we have

$$|(Ax-Ax_k,y)| < \frac{\varepsilon}{2}+|||Ax-Ax_k|||\,\|y-y_n\|$$

$$\leq \frac{\varepsilon}{2}+2.\frac{\varepsilon}{4}=\varepsilon,$$

and therefore $\{(Ax,y) : x \in S\}$ is totally bounded.

Conversely, suppose that A* exists. For each $\varepsilon > 0$, choose N such that

and define the bounded linear mapping $u : H \to \mathbb{C}^N$ by

$$u(x) \equiv ((x, A^*y_1), \ldots, (x, A^*y_N)).$$

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{4},$$

Then, by proposition 4.11, u is a compact mapping, and hence there exists an $\varepsilon/2$-approximation $\{u(x_1), \ldots, u(x_m)\}$ to $u(S)$. For each $x \in S$ there exists k such that

$$\|u(x) - u(x_k)\| = \sum_{n=1}^{N} |(x - x_k, A^* y_n)| < \frac{\varepsilon}{2}$$

Hence we have

$$\|Ax - Ax_k\| = \sum_{n=1}^{\infty} \frac{1}{2^n} |(Ax - Ax_k, y_n)|$$

$$\leq \sum_{n=1}^{N} \frac{1}{2^n} |(x - x_k, A^* y_n)| + \frac{\varepsilon}{2} \leq \varepsilon$$

Therefore A is weakly compact.

CHAPTER FIVE

CONSTRUCTIVE COMPACT LINEAR MAPPINGS

5.1 Compact Linear Mappings

Here, we deal with compact linear mappings of a normed linear space, within the framework of Bishop's constructive mathematics.

From the constructive investigation of compact linear mappings in the last chapter, we have seen that some elementary classical facts on linear mappings have not been proved constructively: for example, the compactness of the sum of compact linear mappings, which is classically proved by means of the constructively unacceptable property of sequential compactness, even in the special case where the compact linear mappings are normable linear functionals (see Bouwerian example 4.14).

In the last chapter, we have proved the following theorem:

Let A and B be compact operators on a Hilbert space H and let C be an operator on H. Then A + B is compact, A* is compact, and if C* exists, then AC is compact.

Since the proof of this theorem depends on the geometric properties of Hilbert spaces, we cannot expect to generalise the theorem to arbitrary normed linear spaces in the same manner.

Now, we shall prove similar results in a more general setting, and to do this, we have to deal with quasinormed linear spaces. As mentioned earlier in the latter part of

chapter 2, quasinormed linear spaces enable us to deal with linear spaces, such as Hom(E,F) (the set of bounded linear mappings), on which the classical norm cannot be computed everywhere. To see this, we consider the space C[a,b]. This is the space of continuous mappings on [a,b], which is a subspace of the space of all bounded mappings on [a,b]. We observe that the norm on C[a,b] at $x \in C[a,b]$, given by

$$\| x \| \equiv \max_{t\in[a,b]} |x(t)|,$$

is not computable everywhere.

Let E and F be quasinormed linear spaces with quasinorms $\{||.||_\alpha : \alpha \in A\}$ and $\{||.||_\beta : \beta\in B\}$ respectively, and let $S_E \equiv \{x\in E: ||x||_\alpha \le 1, \alpha\in A\}$ be the closed unit sphere on E. Then we say that a linear mapping T of E into F is *bounded* if there exists $c > 0$, called a bound for T, such that if $x\in E$ and $||Tx||_\beta > c$ for some $\beta\in B$, then $||x||_\alpha > 1$ for some $\alpha\in A$;

Normable if $||T|| \equiv \text{Sup}\{||Tx||_\beta : \beta\in B, x\in S_E\}$ exists and $||T||$ is a bound for T;

Compact if for each $\varepsilon > 0$ there exists a finite subset Y of $T(S_E)$, called an ε-approximation to $T(S_E)$, such that if $x\in E$ and for each $y\in Y$ there exists $\beta\in B$ with $||Tx-y||_\beta > \varepsilon$, then $||x||_\alpha > 1$ for some $\alpha\in A$.

It is easy to see that if T is bounded then $T(S_E)$ is bounded, in the sense that there exists $c>0$ such that $||Tx||_\beta < c$ for all $\beta\in B$ and $x\in S_E$; that if E is a normed linear space and $T(S_E)$ is bounded, then T is bounded; that if E is a normed linear space and $\text{Sup}\{||Tx|| : x\in S_E\}$ exists, then T is

normable; that if E is a normed linear space and T is compact, then $T(S_E)$ is totally bounded, in the sense that for each $\varepsilon > 0$ there exists a finite subset Y of $T(S_E)$, called an ε-approximation to $T(S_E)$, such that for each $x \in S_E$ there exists $y \in Y$ with $||Tx-y||_\beta < \varepsilon$ for all $\beta \in B$, and if each element of E (or F) is normable, then T is compact if and only if T is compact as a mapping from a normed (quasinormed) linear space E into a quasinormed (normed) linear space F, respectively; (Ishihara [14]).

5.1.1 **Proposition** (Ishihara [14]): If T is a compact linear mapping of a quasinormed linear space E with a quasinorm $\{||.||_\alpha : \alpha \in A\}$ into a quasinormed linear space F with a quasinorm $\{||.||_\beta : \beta \in B\}$, then T is bounded. If, in addition, for each $x \in E$, Tx is a normable element of F, then T is normable.

Proof:

Let $\{y_1, \ldots, y_n\}$ be a 1-approximation to $T(S_E)$. Then there exists positive numbers $c_1, \ldots c_n$ such that $||y_k||_\beta < c_k$ for all $\beta \in B$ and $k=1, \ldots, n$. Let $c \equiv \max\{c_1, \ldots c_n\} + 1$, and suppose that $||Tx||_\beta > c$ for some $\beta \in B$.

Then $||Tx-y_k||_\beta \geq ||Tx||_\beta - ||y_k||_\beta > c - c_k \geq 1$ $(k=1, \ldots, n)$,

and hence $||x||_\alpha > 1$ for some $\alpha \in A$. Therefore, T is bounded.

Moreover, suppose that for each $x \in E$, Tx is a normable element of F. Let a and b be real numbers with $a < b$ and let $\varepsilon = \frac{1}{4}(b-a)$. Let $\{y_1, \ldots, y_n\}$ be an ε-approximation to $T(S_E)$, and choose k such that $||y_k|| > \max\{||y_1||, \ldots, ||y_n||\} - \varepsilon$. Either $a < ||y_k||$ or $||y_k|| < a + \varepsilon$; in the former case, there

exists $\beta\in B$ such that $a < ||y_k||_\beta$; in the latter case, if $x\in S_E$ and there exists $\beta\in B$ such that $||Tx||_\beta > b-\varepsilon$, then

$$||Tx - y_i||_\beta \geq ||Tx||_\beta - ||y_i||$$
$$> b-\varepsilon - \max\{||y_1||, \ldots, ||y_n||\}$$
$$> b - \varepsilon - ||y_k|| - \varepsilon$$
$$> b-a - 3\varepsilon = \varepsilon,$$

and hence $||x||_\alpha > 1$ for some $\alpha\in A$, a contradiction; hence $||Tx||_\beta \leq b-\varepsilon < b$ for all $\beta\in B$ and all $x\in S_E$. Therefore $||T|| \equiv \mathrm{Sup}\{||Tx||_\beta : \beta\in B, x\in S_E\}$ exists.

5.1.2 **Proposition** (Ishihara [14]): Let T be a compact linear mapping of a quasinormed linear space E with quasinorm $\{||.||_\alpha : \alpha\in A\}$ into a quasinormed linear space F with a quasinormed $\{||.||_\beta : \beta\in B\}$, and let S be a bounded linear mapping of a quasinormed linear space F into a quasinormed linear space G with a quasinorm $\{||.||_\gamma : \gamma\in\Gamma\}$. Then ST is compact.

Proof:

Let $c > 0$ be a bound for S and let Y be an ε/c-approximation to $T(S_E)$. If $x\in E$ and for each $y\in Y$ there exists $\gamma\in\Gamma$ such that $||STx - Sy||_\gamma > \varepsilon$, then there exists $\beta\in B$ such that $||Tx-y||_\beta > \varepsilon/c$, and hence $||x||_\alpha > 1$ for some $\alpha\in A$. Therefore $\{Sy : y\in Y\}$ is an ε-approximation to $ST(S_E)$. Thus ST is compact.

Classically, the composition of a compact linear operator and a bounded linear operator yields compact linear operators. This can be easily seen from the following classical result.

Let $T : X \to X$ be a compact linear operator and $S : X \to X$ be a bounded linear operator on a normed space X. Then TS and

ST are compact; Kreyszig [20].

As we have seen in theorem 4.16 (4) ST is constructively compact. But this is not the case with TS, which is classically proved to be compact by means of the constructively unacceptable property of sequential compactness. This can easily be seen from the following example:

5.1.3 **Brouwerian Example** (Ishihara [14]): Bounded linear mappings S and T of a Hilbert space into itself such that T is compact, but TS is not compact.

Let l^2 be the Hilbert space of square-summable sequences of real numbers, for each positive integer n, let e_n be the basis vector with n^{th} component 1 and all other components 0, and for each $x \in l^2$, let x_n be the n^{th} component of x. Let $\{a_n\}$ be a binary sequence such that $a_n=1$ for at most one value of n and define linear mappings S, T : $l^2 \to l^2$ by

$$Sx \equiv \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m}{\sqrt{2^n}}x_m e_n,$$

$$Tx \equiv \sum_{n=1}^{\infty}\frac{1}{\sqrt{2^n}}x_n e_n$$

Then it is easy to see that T is compact; also we have

$$\|Sx\|^2 = \sum_{n=1}^{\infty}\frac{1}{2^n}\left|\sum_{m=1}^{\infty}a_m x_m\right|^2 \le \sum_{n=1}^{\infty}\frac{1}{2^n}\|x\|^2 = \|x\|^2$$

and hence S is bounded. However, we have

$$\|TSx\|^2 = \left\|\sum_{n=1}^{\infty}\frac{1}{2^n}\sum_{m=1}^{\infty}a_m x_m e_n\right\|^2 = \sum_{n=1}^{\infty}\frac{1}{4^n}\left|\sum_{m=1}^{\infty}a_m x_m\right|^2 = \frac{1}{3}\left|\sum_{m=1}^{\infty}a_m x_m\right|^2.$$

If TS is compact (hence, normable), then either $||TS|| > 0$, in which case there exists n such that $a_n=1$; or else $||TS|| < \frac{1}{3}$, and $a_n=0$ for all n. Thus we have arrived at the alternatives stated in LPO.

With a reformulation of the above classical result, certain conditions become fulfilled and TS can be shown to be constructively compact (Theorem 5.1.6).

Let E be a normed linear space. Then we use the notations E* and E′ to denote the set of all bounded functionals on E, and the set of all normable linear functionals on E, respectively. We consider E* as a quasinormed linear space with the quasinorm $\{||.||_x : x \in S_E\}$ given by $||f||_x \equiv |f(x)|$ for all $f \in E^*$.

Let T be a bounded linear mapping of a normed linear space E into a normed linear space F. Then the mapping $f \to foT$ of F* into E* is called the adjoint of T, and is written T^x; if c is a bound for T and $|(T^x f)x| > c$ for some $x \in S_E$, then $|f(Tx)| > c$ and hence $|f(Tx/c)| > 1$; whence T^x is a bounded linear mapping with the same bound of T. It can also be easily seen that if F is a separable normed space, then T is normable if and only if T^x is normable; Ishihara [14].

We note that the definition of this adjoint is slightly different from that of the Hilbert adjoint operator T* in chapter 3, in the sense that the adjoint in this definition always exists, but one on a Hilbert space exists when $T^x(F') \subseteq E'$.

5.1.4 **Definition:** Let $T : E \to F$ be a bounded linear mapping, where E and F are normed linear spaces. Then the

adjoint operator $T^x : F^* \to E^*$ of T is defined by

$$F(x) \equiv (T^x g)(x) = g(Tx) \qquad (g \in F')$$

where F' is the normable linear functional of F.

If E' is a linear space, then since we can find normable g, x is also a normable linear functional on a subspace E' of E^*.

We denote the range of a mapping T by ran(T)

With the aid of the above definition, we shall state the following constructive substitute for the classically well-known theorem that a bounded linear mapping is compact if and only if the adjoint is compact.

5.1.5 **Theorem** (Ishihara [14]): If T is a linear mapping of a normed linear space E into a separable normed linear space F, then T is compact if and only if $ran(T^x) \subseteq E'$ and T^x is compact.

As a consequence of the above theorem, we shall be able to prove the compactness of composition of compact and bounded linear mappings, which is constructively non-trivial.

5.1.6 **Theorem** (Ishihara [14]): If S is a bounded linear mapping of a normed linear space E into a normed linear space F with $S^x(F') \subseteq E'$ and if T is a compact linear mapping of F into a separable normed linear space G, then TS is compact.

Proof:

Since S^x is bounded and T^x is compact, by theorem 5.1.5, $S^xT^x: G^* \to E^*$ is compact by proposition 5.1.2. Also, since $ran(T^x) \subset F'$ and $S^x(F') \subset E'$, we have ran $(S^xT^x) \subset E'$. Therefore, TS is compact (by theorem 5.1.5).

Brouwerian example 5.1.3 shows that the above theorem is the best we can hope for in the constructive setting, in the sense that the condition $S^{x}(F') \subset E'$ cannot be dropped.

5.1.7 **Corollary** (Ishihara [14]): Let S and T be compact linear mappings of a normed linear space E into a separable normed linear space F. If E' is a linear space, then S +T is compact.

Proof:

Since $ran(S^{x}) \subset E'$, $ran(T^{x}) \subset E'$ and E' is linear, we have $ran(S^{x}T^{x}) \subset E'$. Hence S + T is compact.

Brouwerian example 4.14 shows not only that there exists a Banach space l^{1} such that E' is not a linear space, but also that we cannot drop the condition $ran(S^{x}T^{x}) \subset E'$ in the constructive setting.

5.2 Conclusion:

The widespread application of mathematics in various fields, including computer science and numerical analysis, requires us to clarify the meaning of *existence* phrased by, for example, "there exists" or "we can find", etc, to distinguish between computational existence and idealistic existence, and to develop mathematics with a computational interpretation. Constructive mathematics is a hopeful candidate for mathematics which meets those requirements.

No scientist would be contented to learn that a solution to his mathematical problem exists, but that there is no way to calculate it. When a constructivist says that a number is either this or that, he is prepared to say which, or at least

to give a finite procedure by which it may be determined. Thus, all experience tends to indicate that any mathematics that is applicable must be constructive, especially when experimental verification of a theory is required.

Although most mathematicians make no effort to produce constructive results, nevertheless their results often have a very large (and largely hidden) constructive content. It is this hidden constructive content that is useful and applicable, and a major goal of the constructivist program is to make it explicit. The constructive content of a classical theorem, even of its proof, is often sufficient for applications.

From the various aspects of constructive linear mappings, we can easily see that they behave as classically under suitable conditions. We have seen that a bounded linear mapping T from a normed linear space E into a normed linear space F behave as classically under the condition $T^{\times}(F') \subseteq E'$ (Theorem 5.1.6); and that when T is an operator on a Hilbert space H, the condition is equivalent to the existence of the adjoint operator; furthermore, if H is a separable Hilbert space, then the condition is equivalent to the weak compactness of T (Theorem 4.19).

The proper evaluation of constructive mathematics depends on the correct interpretation of its language. Since every constructive proof of a proposition is a classical proof of the same proposition, every classical result which does not entail an omniscience principle is constructive. As such, constructivism is not only limited to compact linear mappings, but involves other areas of mathematics.

REFERENCES

1. E. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York, Pg 294 (1967).

2. D. Bridges, *Constructive Functional Analysis*, Pitman, London, Ch. 1 (1981).

3. D. Bridges, *Locatedness, convexity, and Lebesgne Measurability*, Quart. J. Math., Oxford 39, 411-421 (1988).

4. D. Bridges, *Sequential, Pointwise and Uniform Continuity*: A Constructive Note, Math. Log. Quart 39, 55-61 (1993).

5. D. Bridges, *A Constructive look at the real number line*, P. Ehrlich (ed.), Real Numbers, Generalisations of the Reals, and Theories of continua, 29-92 (1994).

6. D. Bridges, A. Calder, W. Julian, R. Mines and F. Richman, *Bounded Linear Mappings of finite rank*, J. Func. Anal. 43, 143-148 (1981).

7. D. Bridges, A. Caldes, W. Julian, R. Mines and F. Richman, *Compactly generated Banach spaces*, Arch. Math. 36, 239-243 (1981).

8. D. Bridges and O. Demuth, *On the Lebsegne Measurability of Continuous Functions in Constructive Analysis*, Bull. (New Series) Amer. Math. Soc. 24, 259-276 (1991).

9. D. Bridges and H. Ishihara, *Locating the Range of an operator on a Hilbert space*, Bull. London Math. Soc. 24, 599-605 (1992).

10. D. Bridges and R. Mines, *What is Constructive Mathematics?*, Math. Intelligencer 6, 32-38 (1984).

11. D. Van Dalen, *Logic and Structure*, Springer-Verlag, New York (1980).

12. C. Goffman and G. Pedrick, *First Course in Functional Analysis*, Prentice-Hall of India Private Ltd., New Delhi (1987).

13. H. Ishihara, *Boundedness, Normability and Compactness of Constructive Linear mappings*, Hiroshima University, Japan (1989).

14. H. Ishihara, *Constructive compact linear mappings*, Bull. London Math. Soc. 21, 577-584 (1989).

15 H. Ishihara, *An Omniscience principle, the König's lemma and the Hahn-Banach theorem*, Z. Math. Logik Grundlagen Math. 36, S-237-240 (1990).

16. H. Ishihara, *Constructive compact operators on a Hilbert space*, Annals of Pure and Applied Logic 52, 31-37 (1991); North-Holland.

17. H. Ishihara, *Continuity and Nondiscontinuity in Constructive Mathematics*, J. Symbolic Logic 56, 1349-1354 (1991).

18. H. Ishihara, *Continuity properties in Constructive Mathematics*, J. Symbolic Logic 57, 557-565 (1992).

19. G.T. Kneebone, *Mathematical Logic and the Foundations of Mathematics*, D. Van Nostrand Co. Inc. (1963).

20. E. Kreyszig, *Introductory Functional Analysis with Applications*, John Willey & Sons Inc. (1978).

21. M. Mandelkern, Constructive Mathematics, Mathematics Magazine 58, 272-280 (1985).

22. M. Mandelkern, *Brouwerian Counterexamples*, Mathematics Magazine 62, 3-27 (1989).

23. J.T. Oden, *Applied Functional Analysis*, Prentice Hall, Inc., New Jersy (1979).

24. J.D. Pryce, *Basic Methods of Linear Functional Analysis*, Hutchinson & Co. Ltd. (1973).

25. F. Richman, *Meaning and Information in Constructive Mathematics*, Amer. Math. Monthly 89, 385-388 (1992).

26. W. Rudin, *Functional Analysis*, McGraw-Hill, New York (1973).

27. N.A. Shanin, *On the Constructive Interpretations of Math. Judgements*, Amer. Math. Soc. Translations, Series 2, 23, 109-189 (1963).

28. A.E. Taylor and D.C. Lay, *Introduction to Functional Analysis*, John Wiley & Sons Inc. (1980).

29. K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin (1968).