A STUDY OF LIBRATION POINTS IN THE PHOTOGRAVITATIONAL CIRCULAR RESTRICTED THREE-BODY PROBLEM WITH ASPHERICITY AND POTENTIAL FROM A BELT

BY

Joel John TAURA
(B.Sc. 2000, M.Sc. 2011A.B.U ZARIA)

Ph.D/SCI/45641/2012-2013

A THESIS SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES,
AHMADU BELLO UNIVERSITY, ZARIA

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF
DOCTOR OF PHILOSOPHY IN MATHEMATICS

DEPARTMENT OF MATHEMATICS,
FACULTY OF SCIENCE
AHMADU BELLO UNIVERSITY,
ZARIA NIGERIA.

APRIL, 2016

DECLARATION

I declare that the work in this thesis entitled A STUDY OF LIBRATION POINTS IN THE PHOTOGRAVITATIONAL CIRCULAR RESTRICTED THREE-BODY PROBLEM WITH ASPHERICITY AND POTENTIAL FROM A BELT has been performed by me in the Department of Mathematics, Ahmadu Bello University, Zaria. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at this or any other Institution.

Joel John Taura Date.______________.

CERTIFICATION

This thesis entitled A STUDY OF LIBRATION POINTS IN THE PHOTOGRAVITATIONAL CIRCULAR RESTRICTED THREE-BODY PROBLEM WITH ASPHERICITYAND POTENTIAL FROM A BELT by JOEL JOHN TAURA meets the regulations governing the award of the degree of Doctor of Philosophy in Mathematicsof Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

Prof. J. Singh Date.______________.

Chairman, Supervisory Committee

Prof. B. K. Jha Date.______________.

Member, Supervisory Committee

Prof. D. Singh Date.______________.

Member, Supervisory Committee

Prof. A. A. Tijjani Date.______________.

Head of Department

Prof. K. Bala Date.______________.

Dean, School of Postgraduate Studies

DEDICATION

I dedicate this work to the memory of my late father, Mr. John Taura Kungkung.

ACKNOWLEDGMENTS

First and foremost, I bow my heart, in all earnest and profound sense of gratitude to the Almighty God in whom I live, walk and have my being for the attainment of this feat.

My heartfelt gratitude is addressed to the Chairman of my Supervisory Committee, Prof. J. Singh for his continued interest and support to my work.I am eternally grateful to have had the opportunity to work with him. I also wish to emphasize his teaching capacity that got me initialized to the exciting world of Space Dynamics.

My hearty thanks to our fathers Prof. A.A. Tijjani who is the head of Department, Prof. D. Singh, Prof. B.K. Jha,Prof. B. Sani andProf. S.B. Junaid. Equally I would like to appreciate the following lecturers in the department: Dr. A.O. Ajibade, Dr. A.M. Ibrahim, Dr. H.M. Jibril, Dr. A. Yahaya, Dr. A.A. Obiniyi, Dr. H.G.Dikko, Dr. A. Mohammed, Dr. I.A. Fulatan, Dr. Y.M. Baraya, Dr. A.J. Alkali and Dr.A.F.D. Kana, to whom I owe more than I can ever pay. My kudos also goes to othermembers of staff (academic and non-academic) of the Department of Mathematics.

I reserve special appreciation to the *Space Team* headed by Dr. (Mrs). A. Umar, for all the fruitful collaborations we had.

I am most grateful to the Authorities of Federal University Kashere for the opportunity granted me to pursue this program. I extend sincere thanks to those who in one way or the other have contributed to the success of this program, especially Dr. I.U. Andiyawa.

Lastly,I wouldwant to thank my family for supporting me through the difficult and stressful times ever since I started school at the age six. They have always been there for me; I cannot thank them enough, because there are no enough words to express my feelingstowards them.I would like to use this opportunity to thank my wife and all my friends for their prayers and encouragement.

To all my teachers over the years, I am forever grateful.

My love to you all!

Once again, I appreciate God for His infinite mercies.

ABSTRACT

This research has investigated an improved version of the classic restricted three-body problem where both primaries are considered as non-spheroids as well as sources of radiation, and are enclosed by a belt of homogeneous cluster of material points centered at the mass center of the system. It studied the effects of the gravitational potential created by the belt on the existence of libration points and their linear stability when the primaries are:(a) radiating-oblate and (b) radiating-triaxial bodies. To this end, the equations that govern the motion of the infinitesimal body have been derived, and the positions of the libration points areobtained. It has been established that the equations and the positions of the libration points are affected by the aforementioned perturbations, such that in addition to the usual five (three collinear, two triangular) libration points of the classicrestricted three-body problem, there exist four new collinear points in the case of radiating-oblate primaries and up to two in the case of radiating-triaxial primaries. The triangular points are seen to be stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$, where μ_c is the critical mass parameter influenced by the radiation and oblateness/triaxiality of the primaries and potential from the belt. The combined effect of these perturbations is a stabilizing tendency when the primaries are oblate and adestabilizing tendency when they are triaxial. The classical three collinear libration points $(L_{1,} L_2, L_3)$ remain unstable, whereas some of the new libration points have been found to be stable. Furthermore, periodic orbits around the stable triangular points owing to the amalgamated effect of the radiation and oblateness up to the coefficient J_4 of the primaries and potential from the belt are found to be ellipses.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

DEFINITIONS OF TERMS

(i) **Infinitesimal mass**: A small body having negligible gravitational influence on massive bodies.

(ii) **Primaries:** Suppose an infinitesimal mass is moving in the gravitational field of massive bodies, then, the massive bodies are known as the primaries.

(iii) **Belt**: Circular cluster of material points in space.

(iv) **Radiation*:*** A light energy that comes from a star and travels through space.

(v) **Asphericity**: Not perfectly spherical.

(vi) **Photogravitational**: At least one of the primaries is an intense emitter of radiation.

(vii) **Problem of three bodies (general):** The dynamics of three gravitationally interacting point masses.

(viii) **Problem of three bodies (restricted):** The modification of the general problem in the case when one of the three bodies, because of its small mass, is not influencing the motions of the other two massive bodies.

(ix) **Gravitational potential:** A function, such that its derivative gives the gravitational force.

LIST OF SYMBOLS

m	Mass of the infinitesimal body
m_1	Mass of the bigger primary
m_2	Mass of the smaller primary
M_b	Mass of the belt
μ	Mass parameter $= \dfrac{m_2}{m_1 + m_2}$
n	Mean motion of the primaries
R	The dimensional distance between the primaries
r_1	Distance of the infinitesimal mass from the bigger primary
r_2	Distance of the infinitesimal mass from the smaller primary
r	Radial distance of the infinitesimal mass
r_c	Radial distance of the infinitesimal mass with respect to the classical problem
$q_1 = 1 - p_1, p_1 << 1$	Radiation coefficient of the bigger primary
$q_2 = 1 - p_2, p_2 << 1$	Radiation coefficient of the smaller primary
$A_i (i = 1, 2)$	Oblateness coefficients of the bigger primary
$B_i (i = 1, 2)$	Oblateness coefficients of the smaller primary
T	Constant

CR3BP	Circular Restricted Three-Body Problem
σ_1, σ_2	Triaxiality of the bigger primary
σ_1', σ_2'	Triaxiality of the smaller primary

CHAPTER ONE

1.0 GENERAL INTRODUCTION

1.1 Introduction

The study of the motion of three bodies of finite masses, moving under their mutual gravitational attraction, is called the three-body problem (3BP). It has been the subject of several studies for many years. However, the solution of the problem remains a formidable challenge. The problem simplifies to the restricted three-body problem (R3BP) if one of the bodies is assumed to have infinitesimal mass and does not, therefore, affect the motion of the remaining two massive bodies, called the primaries. If further, the primaries move on circular orbits relative to their barycenter, it is known as the circular restricted three-body problem (CR3BP). Then, the goal in this CR3BP is the solution for the motion of the body of infinitesimal mass. A complete analytic solution to the CR3BP does not exist; nevertheless, libration (equilibrium) solutions satisfying precise conditions are obtainable. Three libration solutions L_1, L_2, L_3 are collinear with the primaries; and two others, called the triangular libration solutions L_4, L_5, form equilateral triangles with the primaries. The infinitesimal mass would remain fixed if placed with zero velocity at any of these libration points. Thus, they provide ideal locations for space based telescopes and other space science missions. The collinear points are unstable; while the triangular points are conditionally stable. The CR3BP is a stimulating and active research field that has been receiving considerable attention of mathematicians and astronomers since it was initially considered by L. Euler in 1772 (Szebehely, 1967). The most evident reason for this sustained interest is that the model of the circular restricted three-body problem can serve as a first approximation in a number of real situations in astronomy (dynamics of the solar and stellar systems, lunar theory and artificial satellites). A deeper motivation emanates most likely from the fact that no general solution exist, in spite of the apparent simplicity of the problem. The nearly circular motion of the planets about the sun and the small masses of asteroids and satellites of planets compared to the planets' masses, initially advocated the invention of the CR3BP.

1.2 Statement of the Problem

The classical CR3BP considers only the effects of the gravitational attraction on the infinitesimal body, but in reality there are other perturbing forces that affect the motion of the infinitesimal body. In the presence of a luminous body, it is altogether inadequate to consider merely the gravitational interaction force. Poynting (1903) pointed out that particles, such as small meteors or cosmic dust are comparably affected by the gravitation and light radiation force as they approach luminous celestial bodies. Radzievskii (1950) found that an allowance for direct solar radiation pressure results in a change in the positions of the libration points. There are rings of dust particles in the stellar systems which are regarded as the young analogues of the Kuiper belt (Greaves *et al.* 1998).Trilling *et al.* (2007) detected debris rings in many main-sequence stellar binary systems using the *Spitzer Space Telescope.* Out of an observed 69 A3-F8 main sequence binarystar systems nearly 42 (60%)showed dust rings surrounding binary stars. The gravitational potential due to these belts have great influence on the infinitesimal body (Jiang and Yeh, 2004a). In the classical CR3BP, the sphericity of the bodies is presumed whereas in reality numerous celestial bodies are non-spherical in shape. For instance, Jupiter and Saturn are oblate while Pluto and its moon Charon are triaxial. Singh and Ishwar (1999) revealed that lack of sphericity of the primaries affects the motion of the infinitesimal body.

Thus, the classical CR3BP does not adequately represent the motion of the infinitesimal body in the presence of any perturbing agent such as radiation pressure, non-sphericity of a body and gravitational potential from a belt. Therefore, the problem in this work is to modify the classical CR3BP to include the effects of radiation of the primaries, asphericity of the bodies and gravitational potential from a belt.

1.3 Justification of the Study

According to Henri Poincare:

> The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living (Poincare, 1958, p.8).

Great minds throughout the ages have been fascinated in the celestial motions they observed in the night sky. The human species stands on the brink of a new frontier, the evolution from a planet-bound to a space-faring advancement. Just as the transition from hunter-gatherer to farmer necessitated new techniques, so the expansion into the space, in terms of dynamics of artificial satellite, requires the formulation of new models that include the effects of some of the perturbing forces on the satellite. More importantly, the study of effect of these forces on its stationary (libration) positions in the space; since they provide ideal locations for space based telescopes and other space science missions. This study is one of such models that incorporated the effects of radiation of the primaries, asphericity of the bodies and gravitational potential from a belt on the satellite.

1.4 Aim and Objectives of the Study

The aim of this research is to examine the existence of libration points and their stability behaviors under the combined effect of radiation, asphericity of the primaries and gravitational potential from the belt within the framework of the restricted three-body problem. Moreover, the aim will be achieved through the following specified objectives, which are to:

i. derive equations of motion of the infinitesimal body in the modified restricted three-body problem.

ii. establish the existence of libration points in the modified restricted three-body problem.

iii. analyze the effects of the modification of the restricted three-body problem on the locations of the libration points.

iv. investigate the linear stability of the libration points in the modified restricted three-body problem.

v. determine periodic orbits about triangular libration points in the modified restricted three-body problem.

1.5 Research Problems

The research problems to be addressed by the study are:

i. Influence of zonal harmonics (oblateness up to J_4) and potential from a belt on the stability of libration points in the CR3BP.

ii. Combined effect of zonal harmonics, radiation and potential from a belt on the stability of libration points in the CR3BP.

iii. Effects of triaxiality and potential from a belt on the stability of libration points in the CR3BP.

iv. Combined effect of triaxiality,radiation and potential from a belton the stability of libration points in the CR3BP.

v. Periodic orbits around the libration points in photogravitationalCR3BP with zonal harmonics and potential from the belt.

1.6 Methodology

The methodology employed in this investigation involves:

i. Extension and modification of existing mathematical models of the CR3BP (Szebehely , 1967; Singh and Taura 2013).

ii. Obtaining analytical solutions by considering only linear terms in very small quantities (quantities strictly very less than one), in all analytical computations.

iii. Validating the obtained results by comparing them with well-established results of CR3BP.

iv. Studying the effects of the perturbations analytically and numerically on the obtained results.

v. Examining periodic orbits emanating from the triangular libration solutions obtained.

vi. Interpretation, discussion and drawing conclusions.

1.7 Theoretical Framework

The outlines of the theoretical bases on which the problem is built are first presented:

1.7.1 Circular restricted three-body problem

It describes the motion of an infinitesimal mass moving under the gravitational influence of two bodies, called primaries which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not affecting the motion of the primaries. The primaries are fixed on the x-axis in the coordinate system ($oxyz$) rotating at an orbital angular velocity n, with the origin (axis of rotation) at the centre of their masses. The (x, y) plane is the plane of motion of the primaries and the z-axis is orthogonal to the (x, y) plane.

In the circular restricted three body problem, the units are chosen in such a way that the properties of the system depend on a single parameter.

i. The sum of the masses of the primaries is taken as the unit of mass. The mass of the smaller primary is denoted by μ, whence the mass of the bigger primary $1-\mu$, where $\mu = \frac{m_2}{m_1 + m_2}$ is the mass parameter and m_1, m_2 are the masses of the bigger and smaller primaries, respectively.

ii. The distance between the primaries is a unity. The distances of the smaller primary and bigger primary from the centre of mass are then $1-\mu$ and μ, respectively.

iii. The unit of time is chosen so that the mean motion of the primaries n is equal to 1. From these it follows that the gravitational constant is unity. The only remaining parameter is μ.

Then, the equations of motion of the infinitesimal mass m with the coordinates (x, y, z) relative to the frame (Figure 1.1) that rotates with the primaries are:

$$
\begin{aligned}
&\ddot{x} - 2\dot{y} = \Omega_x, \\
&\ddot{y} + 2\dot{x} = \Omega_y, \\
&\ddot{z} = \Omega_z
\end{aligned}
\tag{1.1}
$$

where $\Omega(x, y, z)$ is

$$\Omega = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

with $r_1^2 = (x+\mu)^2 + y^2 + z^2$, $r_2^2 = (x+\mu-1)^2 + y^2 + z^2$, r_1, r_2 are the distances of the infinitesimal mass m from the bigger primary m_1 and smaller primary m_2, respectively.

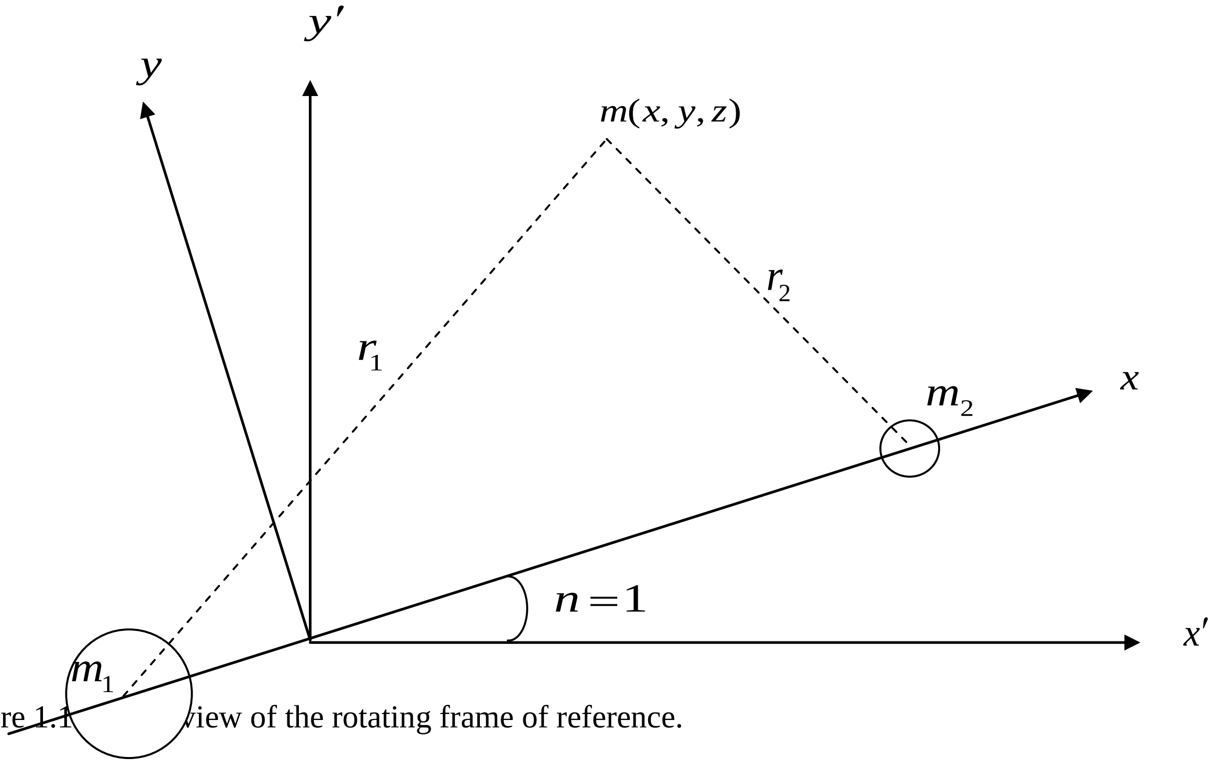


Figure 1.1 [illegible] view of the rotating frame of reference.

A complete analytic solution of the CR3BP does not exist; nevertheless, libration (equilibrium) solutions satisfying precise condition are obtainable. Three libration solutions L_1, L_2, L_3 are collinear with the primaries; and two others, called the triangular libration solutions L_4, L_5, form equilateral triangles with the primaries. L_4 and L_5 located at $(1/2-\mu, \pm\sqrt{3}/2)$ in the *xy*-plane. The infinitesimal mass would remain fixed if placed with zero velocity at any of these libration points. The collinear points are unstable; while the triangular points are conditionally stable.If the mass ratio $\mu = 0.25$, for example, we have the coordinates of the libration points $L_1(1.2659, 0), L_2(0.3607, 0), L_3(\text{-}1.1032, 0), L_4(0.25, 0.8660), L_5(0.25, -0.8660)$ as shown in Figure 1.2.

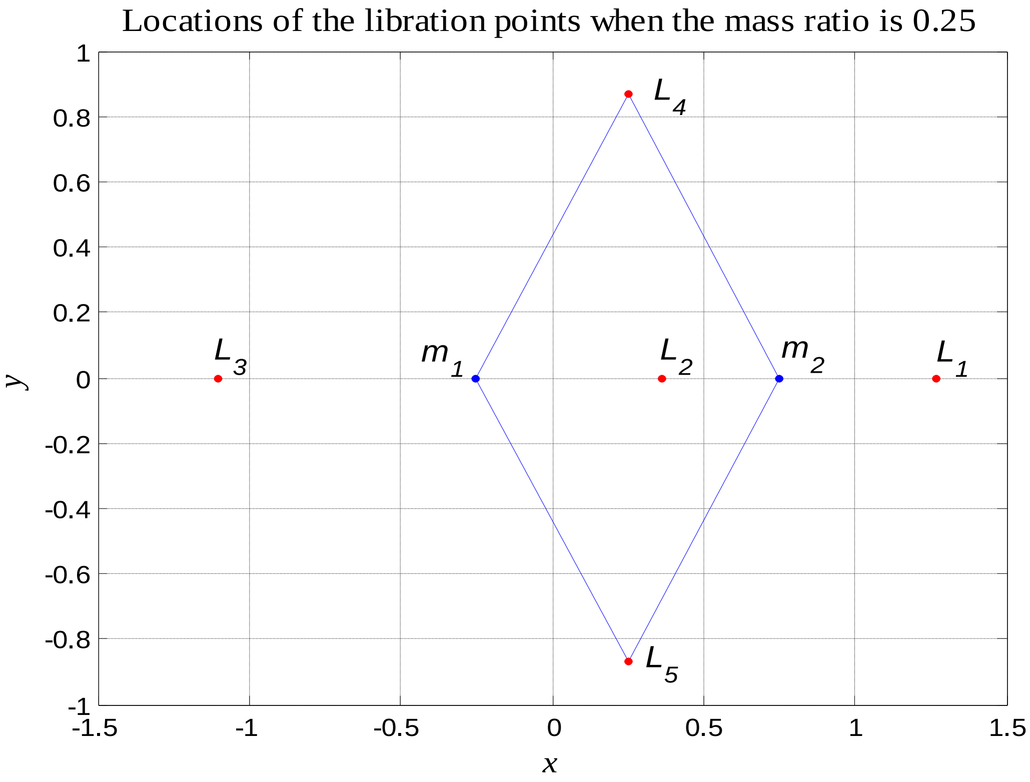


Figure 1.2: Locations of libration points in the CR3BP when $\mu = 0.25$.

1.7.2 Ellipsoid

An ellipsoid is a closed second-order surface; it is a three-dimensional equivalent of an ellipse. The canonical equation of an ellipsoid in the Cartesian coordinate system has the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

where a and b are the equatorial radii (along x and y axes) and c is the polar radius, determining the shape of the ellipsoid. If

- $a>b>c$: triaxial ellipsoid;
- $a=b>c$: oblate ellipsoid (oblate spheroid);
- $a=b<c$: prolate ellipsoid;
- $a=b=c$: sphere ellipsoid(spheroid).

1.7.2.1 *Potential of an oblate body*

Peter and Lissauer (2007) showed that, in free space, the gravitational potential exterior to an oblate body with its mass distributed symmetrically about its equator can be expanded in terms of Legendre polynomials in the form:

$$V_o(r_o,\phi,\theta)=-\frac{Gm_o}{r_o}\left[1-\sum_{n=1}^{\infty}J_{2n}P_{2n}(\cos\theta)\left(\frac{R_o}{r_o}\right)^{2n}\right]. \qquad (1.2)$$

This is expressed in standard spherical coordinates, with ϕ the longitude and θ representing the angle between the body's symmetry axis and the vector to a particle r_o (i.e., the colatitudes). R_o is the mean radius of the body.

J_{2n} are the gravitational moments determined by the body's mass distribution. They are called zonal harmonic coefficients. Higher order zonal harmonics are very small compared to J_2.

The terms P_{2n} $(\cos\theta)$ are the Legendre polynomials, given by

$$P_{2n}(x) = \frac{1}{2^{2n}(2n)!}\frac{d^{2n}}{dx^{2n}}\left(x^2-1\right)^{2n}. \tag{1.3}$$

Now, if the particle is moving in a circular orbit in the equatorial plane ($\theta=90^0$) the potential (Equation (1.2)) becomes

$$V_o(r_o,\phi,\theta) = -Gm_o\left[\frac{1}{r_o}+\frac{J_2R_o{}^2}{2r_o^3}-\frac{3J_4R_o{}^4}{8r_o^5}+\frac{5J_6R_o{}^6}{16r_o^7}+\ldots\right] \tag{1.4}$$

1.7.2.2 *Potential of a triaxial body*

The gravitational potential due to a triaxial body of mass m_t to an external particle is given by (McCuskii, 1963)

$$V_t = -\frac{Gm_t}{r_t}\left[1+\frac{1}{2m_tr_t^2}\left(I_1+I_2+I_3-3I\right)\right], \tag{1.5}$$

where $I_1 = \frac{\left(b^2+c^2\right)m_t}{5}$, $I_2 = \frac{\left(c^2+a^2\right)m_t}{5}$, $I_3 = \frac{\left(b^2+a^2\right)m_t}{5}$, are the principal moments of inertia of m_t about its centre of mass,

$I = I_1l'^2 + I_2m'^2 + I_3n'^2$ is the moment of inertia about the line joining the centre of the mass of the body m_t and the particle. l', m', n' are the direction cosines of the vector from the particle relative to the principle axes of m_t.

1.7.3 Radiation

Radzievskii (1950) formulated that the effect of radiation pressure of a source on a small particle, is expressed by means of a reduction factor $q = 1 - p$. Parameter p, that is often mentioned as the radiation coefficient, is the ratio of force F_r which is caused by radiation to force F_g which results from gravitation, that is $p = \frac{F_r}{F_g}$. Since in most cases the gravitational force exceeds radiation, we shall consider that $q > 0$ and consequently $0 < p << 1$. This formulation is based on some assumptions that can be summarized as follows:

i. The radiation emitted from the primaries (stars) influences the motion of the small body but does not affect the motion of the others.
ii. The mass and the temperature of the radiating star are almost constant.
iii. Light has no fluctuations, it propagates in straight lines and screening or shadowing effects produced by other bodies of the system are negligible.
iv. The force of light pressure is inversely proportional to the square of the distance between the particle and the illuminating body.

1.7.4 Potential of a belt

The gravitational potential from a belt (circular cluster of material points) centered at the origin of coordinates system *oxyz*, as specified by Miyamoto and Nagai (1975) is

$$V(r,z) = -\frac{M_b}{\sqrt{r^2 + \left(a + \sqrt{z^2 + b^2}\right)^2}}, \tag{1.6}$$

where M_b is the total mass of the belt, $r^2 = x^2 + y^2$, a and b are parameters which determine the density profile of the belt. The parameter *a* controls the flatness of the profile and is known as the *flatness parameter*. The parameter *b* controls the size of the core of the density profile and is called the *core parameter*. When *a* = *b* = 0, the potential reduces to the one by a point mass. Restricting ourselves to the $xy-plane$, Equation (1.6) becomes

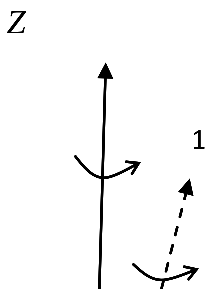
$$V(r,0) = \frac{M_b}{\left(r^2 + T^2\right)^{1/2}},\ where\ T = a + b. \tag{1.7}$$

1.8 Some Basic Concepts

Fundamental ideas which are crucial to the study are velocity in the rotating frame, Hamiltonian equations of motion and Lyapunov stability.

1.8.1 Velocity in the rotating frame

Let *OX, OY, OZ* constitute a rectangular frame, the point *O* being fixed for the moment, and the frame rotating instantaneously about an axis through O with angular velocity $\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$, as shown in Figure 1.3.



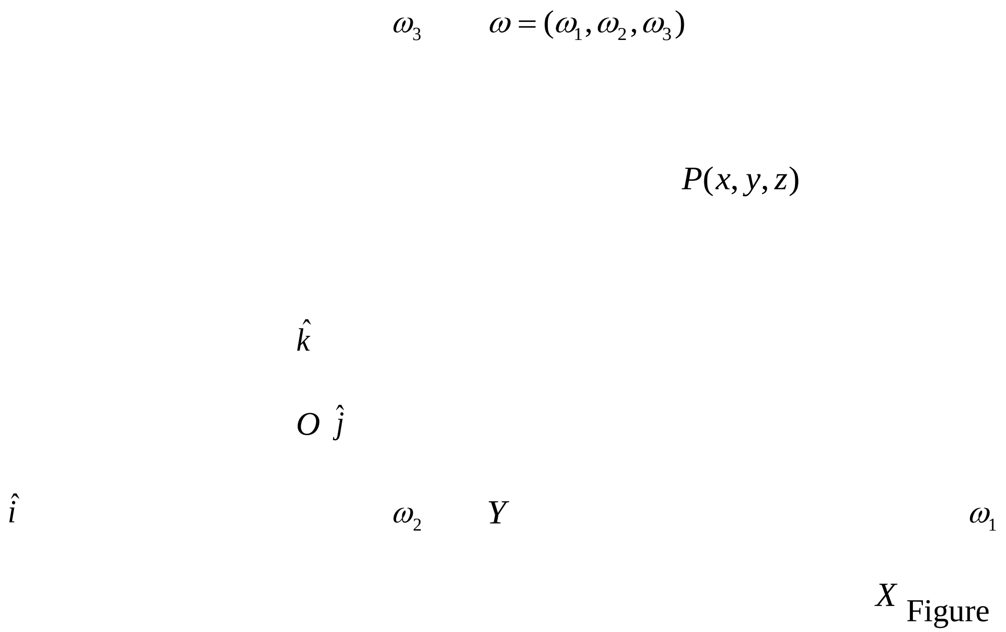


Figure 1.3: Rotating coordinate frame

$P(x, y, z)$ is any point in the space which can move relative to the frame with a position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$ then the velocity of P is

$$\dot{\vec{r}} = \dot{x}\hat{i} + x\frac{d\hat{i}}{dt} + \dot{y}\hat{j} + y\frac{d\hat{j}}{dt} + \dot{z}\hat{k} + z\frac{d\hat{k}}{dt}. \qquad (1.8)$$

Now, since the axes are rotating $\frac{d\hat{i}}{dt}, \frac{d\hat{j}}{dt}, \frac{d\hat{k}}{dt}$ are non-zero. $\frac{d\hat{i}}{dt}$ is the velocity of the point (1,0,0), this is due to the spins of ω_2 about OY and ω_3 about OZ. That is the angular velocity components rotate the system around unit axis, so that for example, ω_2 tends to rotate $\hat{i}$ towards the $-\hat{k}$ direction and ω_3 tends to rotate $\hat{i}$ towards the $+\hat{j}$ direction. Thus, $\frac{d\hat{i}}{dt} = \omega_3\hat{j} - \omega_2\hat{k},$ and we have

$$\vec{r} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} + (\omega_3 \hat{j} - \omega_2 \hat{k})x + (\omega_1 \hat{k} - \omega_3 \hat{i})y + (\omega_2 \hat{i} - \omega_1 \hat{j})z\,.$$

Or

$$\vec{r} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} + \vec{\omega} \times \vec{r}\ . \tag{1.9}$$

1.8.2 Hamiltonian equations of motion

A system with *s* degrees of freedom possesses *s* Lagrange's equations of motion of the form

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0\,, \tag{1.10}$$

where *L*=kinetic energy(K.E)- potential energy(*V*) of the system , q_i *and* $\dot{q}_i$, $(i = 1, 2, 3..., s)$ are the generalized co-ordinates and generalized velocities respectively and *t* is time. Thus the Lagrangian formulation is a description of mechanics in terms of the generalized co-ordinates and generalized velocities with time *t* as a parameter. Hamilton's formulation is an alternative to the Lagrangian formulation in which the independent variables are generalized coordinates q_i and momenta p_i defined as

$$p_i = \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i}. \tag{1.11}$$

The alteration in basis from the set $\{q_i, \dot{q}_i, t\}$ *to* $\{q_i, p_i, t\}$ is achieved by Legendretransformation.

The Hamiltonian function

We start by expressing the differential of the Lagrangian $L(q_i, \dot{q}_i, t)$, as

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt\,.$$

Or

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt, \qquad (1.12)$$

where

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \qquad (1.13)$$

Using Equation (1.13) in Equation (1.10), yields

$$\dot{p}_i = \frac{\partial L}{\partial q_i},$$

so that Equation (1.12) can be rewritten as

$$dL = \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt. \qquad (1.14)$$

The classical Hamiltonian function $H(q_i, p_i, t)$, is generated by the Legendretransformation

$$H = H(q_i, p_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t), \qquad (1.15)$$

which has the differential

$$dH = \sum_i \dot{q}_i d p_i + \sum_i p_i d\dot{q}_i - dL. \qquad (1.16)$$

Substituting Equation (1.14) in Equation (1.16), we have

$$dH = \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt\ . \qquad (1.17)$$

Since dH can also be expressed as

$$dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt, \qquad (1.18)$$

by comparing Equations (1.17) and (1.18), we obtain the relations

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}, \end{aligned}\right\} \tag{1.19}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{1.20}$$

The equations of Equation (1.19) are known as Hamilton's canonical equations. They form a set of *2s* first-order equations of motion substituting the *s* second order Lagrange's equations of motion.

1.8.3 Lyapunov stability

Consider the systems of autonomous ordinary differential equations, written in vector form as $\dot{x} = f(x)$, that have an isolated equilibrium point $x = x^*$ at the origin. Then, x^* is said to be stable in the sense of Lyapunov orLyapunov stableif for all $\varepsilon > 0$ there exists a δ >0 (depending on ε) such that for all $x(t_0)$ with $\left|x(t_0) - x^*\right| < \delta$, *implies* $\left|x(t) - x^*\right| < \varepsilon$ for all $t > 0$, where $x(t_0)$ represents the system at some initial time t_0. That is *x** is Lyapunov stable if for any ε-neighborhood of *x** a smaller δ - neighborhood can be found such that if the system is perturbed within the δ -neighborhood, it never leaves the ε-neighborhood.

If *x** Lyapunov stable and $\left|x(t) - x^*\right| \to 0$ as $t \to \infty$, then *x** is called an asymptotically stable equilibrium point. That is, a solution that starts sufficiently close to an asymptotically stable equilibrium point will not only remain nearby, but will actually be "attracted" by it, as illustrated in Figure 1.4. If *x**is not Lyapunov stable, it is said to be unstable in the sense of Lyapunov.

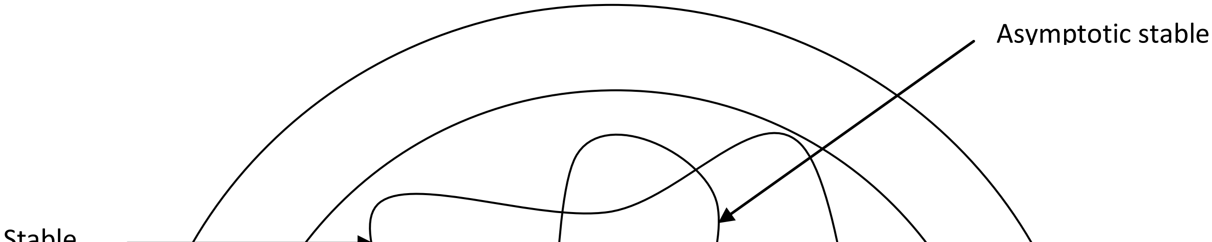


Figure 1.4: Graphical representation of Lyapunov stability.

Stability of an equilibrium point

One of the techniques to determine stability of an equilibrium x^* of a system of differential equations is to analyze the linearized system around it. Then, the solutions to linearized differential equations with constant coefficients can be expressed as linear combination of the product of a polynomial and an exponential function. Therefore solving a linear differential equation with constant coefficients can be transformed to solving a linear algebraic problem. Stability or instability then may follow from the roots of the characteristic equation of the linearized system.

Lyapunov proved that an equilibrium point x^* is stable if all the roots of the characteristic equation of the linearized system are either negative real numbers or distinct pure imaginary numbers or real parts of the complex numbers are negative.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Introduction

The R3BP has been studied by many researchers under various modifications. However, this chapter is devoted to the available relevant literature related with the present study. Work on R3BP when the primary bodies are radiating, non-spherical, and/or are encompassed by a dust particles will be reviewed. We resume with a brief look at the R3BP.

2.2 The Restricted Three-Body Problem

The general three-body problem asks for the motion of three masses in space under their mutual gravitational interaction. It is an old problem and logically follows from the two-body problem which was solved by IsaacNewton in his *Principia* in 1687. Newton also considered the three-body problem in connection with the motion of the Moon under the influences of the Sun and the Earth, but faced with some problems that remained unsolved.

Almost 100 years later, in 1772, Euler proposed the restricted three-body problem: one of the bodies is assumed to have infinitesimal mass and does not, therefore, affect the motion of the remaining two massive bodies. He studied the lunar theory using the restricted problem of three bodies; and found that no general closed form solution exists. However, at about the same time, the first special libration solutions L_4 and L_5 of the restricted three-body problem were discovered by J. L Lagrange and later, the collinear libration points L_1, L_2 and L_3 by L. Euler (Szebehely, 1967).

Szebehely (1967) studied the stability of the libration points of the restricted three-body problem. He established that, in linear sense the collinear libration points L_1, L_2 and L_3 are unstable for any value of the mass ratio of the primaries μ, and the triangular libration points L_4 and L_5 are stable for $0 < \mu < \mu_0 = 0.03852$..

2.3 The Restricted Three-Body Problem with Radiation

Poynting (1903) reported that particles, such as small meteors or cosmic dust are comparably affected by gravitation and light radiation force as they approach luminous celestial bodies. Radzievskii (1950) devised the photogravitational restricted three-body problem. It results from the classical problem when one or both of the masses of the primaries are intense emitters of radiation. He discussed it for three specific bodies: the sun, a planet and a dust particle. He studied the libration points of the photogravitational problem and found that their locations depend on the radiation pressure factor. Bhatnagar and Chawla (1979) investigated the stability of motion around triangular libration points in the photogravitational restricted three body problem. They found that the size of stability decreases due to the radiation pressure. Simmons *et al.* (1985) obtained a comprehensive solution of the modified restricted three-body problem. They discussed the existence and linear stability of the libration points for all values of radiation pressures of both luminous bodies and all values of mass ratios. An investigation of the positions of libration points, when the more massive primary is a source of radiation and the smaller one is an oblate spheroid, was carried out by Sharma (1987). He showed that the triangular libration points are linearly stable for the mass parameter $\mu \in (0, \mu_c)$ and the critical mass value μ_C decreases with the increase in oblateness and radiation coefficients. Singh and Ishwar (1999) considered both primaries as sources of radiation as well as oblate spheroids in the restricted three body problem. They found that the critical value of the mass parameter depends on the radiation and oblateness factors. Das *et al*. (2008) examined the stability of equilibrium points of a passive micro size particle in the field of radiating binary stellar systems within the framework of the CR3BP.Singh (2009) investigated the nonlinear stability of the triangular libration point L_4, when both of the primaries are oblate spheroids as well as sources of radiation. He found that L_4 is stable for all mass ratios in the range of linear stability except for three mass ratios depending upon oblateness coefficients and mass reduction factors.Abouelmagd (2013) examined the effects of photogravitational force and oblateness in the perturbed restricted three-body problem on the locations of the out of plane equilibrium points. He established that there is no explicit effect of Coriolis force on the positions of the out of plane equilibrium points.

2.4 The Restricted Three-Body Problem with Asphericity

The bodies in the R3BP are generally considered to be spherical, but in nature most celestial bodies are not spherical, they are either oblate or triaxial. Asphericity of the bodies can yield perturbation deviation from the two-body orbit.An outstanding example of perturbation deviation caused by oblateness in the solar system is the orbit of the fifth satellite of Jupiter, Amalthea (Moulton 1914). This enables researchers to study the restricted problem by taking into account the shapes of the bodies.

Khanna and Bhatnagar (1999) examined the existence and stability of libration points in the CR3BP when the smaller primary is an axis-symmetric rigid body and the bigger one an oblate spheroid. Sharma *et al.* (2001*a*) examined the existence and stability of libration points in the restricted three body problem when the bigger primary is a triaxial rigid body and source of radiation with one of the axes as the axis of symmetry and its equatorial plane coinciding with the plane of motion. They found five libration points, two triangular and three collinear. The collinear points are unstable, while the triangular points are stable for the mass parameter $0 < \mu < \mu_C$ and that they have long or short periodic elliptical orbits in the same range of the mass parameter. In the same problem, when the primaries are triaxial rigid bodies and source of radiations, Sharma *et al*. (2001*b*, 2001c)) arrived at similar conclusions. Jain *et al.* (2006) have analyzed the periodic orbits around the collinear libration points in the restricted three body problem when the smaller primary is a triaxial rigid body. They discussed the stability of the periodic orbits and concluded that, the periodic orbits are unstable. Singh and Begha (2011a) investigated the CR3BP when the bigger primary is triaxial rigid body and the smaller one an oblate spheroid, together with small perturbations in the Coriolis and centrifugal forces.Singh (2013) has considered both primaries as radiating and triaxial rigid bodies under the influence of small perturbations in the Coriolis and centrifugal forces in his studies of the CR3BP. He determined the positions of libration points and examined their linear stability. It is found that the positions are significantly affected by a small change in the centrifugal force, radiation and triaxiality factors of the primaries.

Abouelmagd (2012) proved that the locations of the triangular points and their linear stability are affected by the oblateness up to J_4 of the bigger primary in the CR3BP. Some researchers: Ishwar (1997, 1998); Elshaboury and Mostafa (2011); Singh and Umar (2013a); Abouelmagd, *et al.* (2013) and Singh and Haruna (2014a) have considered at least one of the primaries and the infinitesimal body as oblate spheroids in their investigations of the CR3BP.

2.5 The Restricted Three-Body Problem with Belt

Many extra solar planetary systems have been discovered, and most of them exhibit interesting features. Some of the discovered extrasolar planetary systems are belts of dust particles that are regarded as the young analogues of the Kuiper belt (Aumann, *et al.*1984). For instance, Greaves *et al.* (1998) found a dust ring around a nearby star, e Eridani. Trilling *et al.* (2007) detected debris belts in many main-sequence stellar binary systems using the *Spitzer Space Telescope.* Out of an observed 69 A3-F8 main sequence binarystars systems, nearly 60% showed dust surrounding binary stars. Jiang and Yeh (2003) studied the effect of the belts on planetary orbits and concluded that the planets might prefer to stay near the inner part rather than the outer part of the belt. They in their later paper (2004*a*) modified the restricted three-body problem by considering the effects of additional gravitational force from the belt on the infinitesimal mass. The influence from the belt makes the structure of the dynamical system quite different, such that new libration points exist under certain condition. In a further paper (2004*b*), they focused on the chaotic orbits of belt-star-planet systems for the modified restricted three-body problem. They found that chaotic boundary does not depend much on the belt mass for type I initial conditions, but can change a lot for different belt masses for type *II* initial conditions. They also found that the influence from the belt can change the locations of equilibrium points and the orbital behaviors for both types of initial conditions. Jiang and Yeh (2006) and Yeh and Jiang (2006) examined the conditions for the existence of equilibrium points in the Chermnykh-like problems for different values of mass parameter μ, including the potential of the belt, they found three collinear points, two triangular points, and two other new equilibrium points. Kushvah (2008) investigated the linear stability of equilibrium points in the generalizedphotogravitational Chermnykh's problem**.** The bigger primary was considered as a source of radiation and small primary as an oblate spheroid. He studied analytically and numerically the effect of radiation pressure, oblateness, and gravitational potential from the belt. The stability of the equilibrium points was also examined using Lypunov‘s method, and it was concluded that the collinear equilibrium points are linearly unstable while the triangular points are conditionally stable. In his recentpaper, Kusvah (2011), outlined in the same problem, a design of the trajectory and analysis of the stability of collinear point L_2 in the Sun- Earth system. It was found that, the point L_2 was asymptotically stable up to a specific value of time *t* corresponding to each set of values of parameters and initial conditions.

In the light of Kushvah (2008), Singh and Taura (2013) constructed a model of restricted three-body problem when the two primaries are oblate spheroids and radiating as well as the effect of gravitational potential from the belt. They established that, in addition to the usual five libration points, there are two new collinear points due to the potential from the belt. They concluded that, the collinear libration points remain unstable, while the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \le \mu \le \frac{1}{2}$, where μ_c is the critical mass ratio influenced by the oblateness and radiation of the primaries and the potential from the belt.

2.6 Periodic Orbits in the Restricted Three-Body Problem

Rabe and Schanzle (1962) have obtained periodic librations about the triangular solutions of the restricted earth-moon problem and their orbital stability. Lanzano (1965) developed an algorithm for the determination of the trigonometric series representing periodic orbits about a triangular libration point and made some remarks on the stability of the solutions. Guillaume (1969, 1973) has done work on symmetric and doubly symmetric orbits in the restricted problem. Bhatnagar (1972) considered restricted problem in a three-dimensional co-ordinate system and studied the existence of periodic orbits of the third kind. He also showed the existence of periodic orbits of collision. Meyer and Schmidt (1971) found periodic orbits in the neighborhood of the triangular libration point L_4 for mass ratios near the critical mass ratio of Routh. Markellos *et al.* (1974) outlined a method for the numerical determination of families of periodic orbits in the planer restricted three-body problem. Sharma (1976) proved the existence of periodic orbits of the first kind in the CR3BP when the bigger primary is an oblate spheroid.Hadjidemetriou (1984) considered various types of periodic orbits in connection with the study of problems in planetary and stellar systems. Gomez and Noguera (1985) presented various numerical results regarding natural families of periodic orbits, which emanate from limiting orbits around the equilateral libration points of the R3BP when the mass ratio is more than Routh's critical one.

Sharma (1987) studied the stationary solutions of the planar restricted three–body problem when the bigger primary is radiating as well as the smaller primary is an oblate spheroid with its equatorial plane coinciding with the plane of motion. Perdios and Zagouras (1991) examined the vertical stability character of the families of short and long periodic solutions around the triangular equilibrium points of the restricted three-body problem. Zagouras (1991) considered the effect of radiation pressure on the periodic motion of small particle in the vicinity of the triangular equilibrium points of the restricted three-body problem. He also constructed the second order parametric expansion and determined the family of the

periodic orbits numerically for two sets of values of the mass and radiation parameters corresponding to the non-resonant and the resonant case. Perdios *et al*. (1991) discussed three-dimensional bifurcations of periodic solution around the triangular equilibrium points of the restricted three-body problem. Elipe and Lara (1997) studied the periodic orbits when both the primaries are radiating in the R3BP. Many families of periodic orbits in two and three dimensions were obtained. Perdios *et al*. (1998) have examined the bifurcation of three dimensional periodic orbits from the plane of motion of the primaries in the restricted three-body problem with oblateness.Khanna and Bhatnagar (1999) have studied the short and long periodic orbits around the triangular libration point L_4in the restricted three-body.
Taqvi *et al.* (2002a, 2002b) have proved the existence of periodic orbits in the restricted problem when the smaller is a triaxial rigid body and the bigger primary is a source of radiation.Poddar *et. al.* (2002) confirmed the existence of periodic orbits, symmetric as well as doubly symmetric in the restricted problem of three bodies when the bigger primary is a source of radiation and the smaller primary a triaxial rigid body.Perdios (2003) studied critical symmetric periodic orbits in the photogravitational CR3BP in which the first primary is a source of radiation.Bruno and Varin (2008) have studied the families of periodic solutions of the restricted three-body problem.

AbdulRaheem and Singh (2008) examined the periodic orbits around the triangular points when both primaries are oblate spheroid as well as sources of radiation, under the effects of small perturbations in the Coriolis and centrifugal forces. Mittal *et al.* (2009) studied periodic orbits generated by Lagrangian solutions of the restricted three–body problem when one of the primaries is an oblate spheroid. Papadakis and Rodi (2010) presented a systematic numerical exploration of the families of asymmetric periodic orbits of the restricted three-body problem when the primary bodies are equal and for the Earth-Moon mass ratio.Singh and Begha (2011b) studied the existence of periodic orbits around the triangular points in the restricted three–body problem when the bigger primary is a triaxial, the smaller primary is considered as an oblate spheroid, working in the range of linear stability with the perturbed forces of Coriolis and centrifugal. Baltagiannis and Papadakis (2011) have found the families of periodic orbits in the restricted four-body problem. Abouelmagd and El–Shaboury (2012) studied periodic orbits around the triangular points when the three participating bodies are oblate spheroids and the primaries are radiating.

Abouelmagd and Sharaf (2013) examined orbits around the libration points when the more massive primary is radiating and the smaller is an oblate spheroid. Their study included the effects of oblateness up to 10^{-6} of the main term. Singh and Haruna (2014b) performed a semi-analytic study of the periodic orbits around stable triangular libration points when the three participating bodies are modeled as oblate spheroids, under effect of, radiation of the main masses and small change in the Coriolis and centrifugal forces. Singh and Leke (2014) investigated the motion around the triangular libration points, of a passively gravitating dust particle in the gravitational field of a low-mass post-AGB binary system, surrounded by circumbinary disc. Umar and Singh (2014)in the framework of the elliptic restricted three-body problem investigated the effects of oblateness, radiation and eccentricity of both primaries on the periodic orbits around the triangular libration points of oblate and luminous binary systems.

CHAPTER THREE

3.0 OBLATENESS AND RADIATION OF THE PRIMARIES WITH POTENTIAL FROM A BELT

3.1 Introduction

In this chapter, we shall examine the existence of libration points and their behaviours under the combined effect of radiation, oblateness up to the coefficients J_4 of the primaries and gravitational potential from the belt within the framework of the restricted three-body problem.We derive the equations of motion and the zero velocity surfaces/ curves of the infinitesimal body, establish the existence of libration points and determine their positions, discuss the linear stability of the libration points and then examine periodic orbits around linearly stable triangular points.

3.2 Derivation of Equations of Motion

The equations that govern the motion of the infinitesimal body in the gravitational field of radiating oblate primaries together with the influence of gravitational potential from a belt in the restricted three-body problem is derived in this section. We begin with mathematical formulations of the problem, the derivation of the equations of motion of the infinitesimal body, and then state the Jacobian integral.

3.2.1 Mathematical formulations of the problem

Let m_1 and m_2 $(m_1 > m_2)$ be the masses of the primaries moving in the circular orbits around their center of mass *O*. An infinitesimal mass m is moving in the plane of motion of m_1 and m_2. Let us take a coordinate system *oxyz* with origin at the centre of mass of the primaries and the *x*-axis is the line joining the primaries; while *y*-axis is perpendicular to it, the *z*-axis is perpendicular to the orbital plane of the primaries. The distances of m from m_1, m_2 are r_1, r_2 respectively, and the distance between the primaries is R. The coordinates of m_1, m_2 and m are (x_1, 0), (x_2, 0) and (x, y), respectively. We desire to find the equations of motion of m under the resultant influence of radiation and oblateness up to the zonal harmonic J_4 of the primariesand a circumbinary belt centered at the origin of the coordinate system *oxyz* as illustrated in Figure 3.1.

3.2.2 The kinetic energy

The kinetic energy (K.E) of the infinitesimal body in the barycentric coordinate system oxyz rotating about *z*-axis with uniform angular velocity n shown in Figure 3.1, *is* given in view of Equation (1.9) as

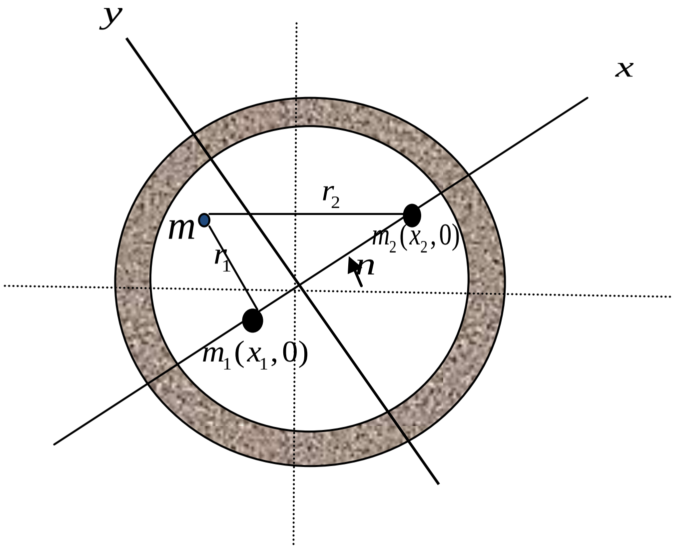


Figure 3.1: The planar configuration of the problem

$$\begin{aligned} K &= \frac{1}{2}m\left(\dot{x}\hat{i}+\dot{y}\hat{j}+n(x\hat{j}-y\hat{i})\right)^2 \\ &= \frac{1}{2}m\left[(\dot{x}\hat{i}+\dot{y}\hat{j})^2+2n(\dot{x}\hat{i}+\dot{y}\hat{j})(x\hat{j}-y\hat{i})+n^2(x\hat{j}-y\hat{i})^2\right] \\ &= \frac{1}{2}m\left[\dot{x}^2+\dot{y}^2+2n(-\dot{x}y+x\dot{y})+n^2(x^2+y^2)\right] \\ &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+mn\left(x\dot{y}-\dot{x}y\right)+\frac{1}{2}mn^2\left(x^2+y^2\right). \end{aligned} \tag{3.1}$$

3.2.3 The potential energy

Assuming that the primaries have their equatorial planes coinciding with the plane of motion, in view of Equation (1.4), we denote the oblateness coefficients for the bigger primary as A$_{i}$, $0<A_i=J_{2i}R_1^{2i}<<1$ and for the smaller primary as B$_{i}$, $0<B_i=J_{2i}R_2^{2i}<<1$, $i=1,2$, where J_{2i} are zonal harmonic coefficients; R_1 , R_2 are the mean radii of m_1 and m_2 respectively.

Then, the potential energy of the infinitesimal mass, under the combined influence of the radiation and oblateness up to the coefficients J_4 of the primaries and the circumbinary belt, is given by

$$V=-Gm\left\langle m_1q_1\left(\frac{1}{r_1}+\frac{A_1}{2r_1^3}-\frac{3A_2}{8r_1^5}\right)+m_2q_2\left(\frac{1}{r_2}+\frac{B_1}{2r_2^3}-\frac{3B_2}{8r_2^5}\right)+\frac{M_b}{\left(r^2+T^2\right)^{1/2}}\right\rangle \tag{3.2}$$

with

$$r_1^2=\left(x-x_1\right)^2+y^2,\ r_2^2=\left(x-x_2\right)^2+y^2, \tag{3.3}$$

G is the gravitational constant;

$q_i (i = 1, 2)$ are the radiation factors of the more massive and less massive primaries respectively; and are given by $q_i = 1 - p_i$ such that $0 < p_i = \frac{F_{r_i}}{F_{g_i}} << 1,$ where F_r is the force caused by radiation pressure force and F_g results due to gravitational force(Radzievskii, 1950).

$$\frac{M_b}{\left(r^2 + T^2\right)^{1/2}} \tag{3.4}$$

is the potential due to the circular cluster of material points (Miyamoto and Nagai ,1975) where M_b is the total mass of the circular cluster of material points, *r* is the radial distance of the infinitesimal mass and is given by $r^2 = x^2 + y^2$, $T = a + b$, a and b are parameters which determine the density profile of the circular cluster of material points. The parameter *a* controls the flatness of the profile and is known as the *flatness parameter*. The parameter *b* controls the size of the core of the density profile and is called the *core parameter*. If *a* = *b* = 0, the potential is the same as the one by a point mass.

3.2.4 Equations of motion in dimensional variables

We begin from the Lagrangian (L) of the problem, which is the kinetic energy (Equation 3.1) minus the potential energy (Equation 3.2) of the system. That is

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + mn\left(x\dot{y} - \dot{x}y\right) + \frac{1}{2} mn^2\left(x^2 + y^2\right) - V.$$

Let P_x and P_y be the momenta, conjugate to *x* and *y* respectively defined as

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\left(\dot{x} - ny\right),$$
$$P_y = \frac{\partial L}{\partial \dot{y}} = m\left(\dot{y} + nx\right). \tag{3.5}$$

Hence, we can write

$$\begin{aligned} \dot{x} &= \frac{P_x}{m} + ny, \\ \dot{y} &= \frac{P_y}{m} - nx. \end{aligned} \tag{3.6}$$

In view of Section 1.8.2, the Hamiltonian function *H* analogues to the motion of the infinitesimal mass *m*is given as

$$\begin{aligned} H &= \dot{x}P_x + \dot{y}P_y - L. \\ &= \dot{x}P_x + \dot{y}P_y - \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mn\left(x\dot{y} - \dot{x}y\right) - \frac{1}{2}mn^2\left(x^2 + y^2\right) + V. \end{aligned}$$

Or

$$H = \dot{x}P_x + \dot{y}P_y - \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mn\left(x\dot{y} - \dot{x}y\right) + \Psi, \tag{3.7}$$

$$where \quad \Psi = V - \frac{1}{2}mn^2\left(x^2 + y^2\right).$$

Employing Equation (3.6) in Equation (3.7) we have

$$\begin{aligned} H &= \left(\frac{P_x}{m} + ny\right)P_x + \left(\frac{P_y}{m} - nx\right)P_y - \frac{1}{2}m\left(\left(\frac{P_x}{m} + ny\right)^2 + \left(\frac{P_y}{m} - nx\right)^2 \right) - mn\left(x\left(\frac{P_y}{m} - nx\right) - \left(\frac{P_x}{m} + ny\right)y \right) + \Psi. \\ &= \frac{P_x^{\,2}}{m} + nyP_x + \frac{P_y^{\,2}}{m} - nxP_y - \frac{P_x^{\,2}}{2m} - \frac{P_y^{\,2}}{2m} + \frac{1}{2}mn^2x^2 + \frac{1}{2}mn^2y^2 + \Psi. \end{aligned}$$

$$= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + n\left(yp_x - xp_y\right) + \frac{1}{2}mn^2\left(x^2 + y^2\right) + \Psi. \tag{3.8}$$

In view of Equation (1.19), the Hamilton's canonical equations of motion are given by

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x}, & \dot{y} &= \frac{\partial H}{\partial p_y}, \\ \dot{p}_x &= -\frac{\partial H}{\partial x}, & \dot{p}_y &= -\frac{\partial H}{\partial y}. \end{aligned}$$

These yield four first order ordinary differential equations which govern the motion of the infinitesimal mass in terms of the Hamiltonian variables:

$$\left.\begin{aligned} \dot{x} &= \frac{P_x}{m} + ny, \\ \dot{y} &= \frac{P_y}{m} - nx, \\ \dot{p}_x &= np_y - mn^2 x - \frac{\partial \Psi}{\partial x}, \\ \dot{p}_y &= -np_x - mn^2 y - \frac{\partial \Psi}{\partial y}. \end{aligned}\right\} \tag{3.9}$$

Next, we eliminate the momenta from the equations of Equation (3.9). Now, from the first two equations of Equation (3.9), we obtain

$$\begin{aligned} P_x &= m\left(\dot{x} - ny\right), \\ P_y &= m\left(\dot{y} + nx\right). \end{aligned} \tag{3.10}$$

On, differentiating (3.10) with respect to time *t*, we have

$$\begin{aligned} \dot{P}_x &= m(\ddot{x} - n\dot{y}), \\ \dot{P}_y &= m(\ddot{y} + n\dot{x}). \end{aligned} \tag{3.11}$$

Utilizing Equations (3.10) and (3.11) in the last two equations of Equation (3.9), we arrive at

$$\begin{aligned} m(\ddot{x} - n\dot{y}) &= nm(\dot{y} + nx) - mn^2 x - \frac{\partial \Psi}{\partial x}, \\ m(\ddot{y} + n\dot{x}) &= -nm(\dot{x} - ny) - mn^2 y - \frac{\partial \Psi}{\partial y}. \end{aligned} \tag{3.12}$$

Multiplying through Equation (3.12) by 1/*m*, we get

$$\begin{aligned} \ddot{x} - n\dot{y} &= n(\dot{y} + nx) - n^2 x - \frac{\partial \Psi}{m\partial x}, \\ \ddot{y} + n\dot{x} &= -n(\dot{x} - ny) - n^2 y - \frac{\partial \Psi}{m\partial y}. \end{aligned}$$

Or

$$\ddot{x}-2n\dot{y}=-\frac{\partial\Psi}{m\partial x}, \tag{3.13}$$

$$\ddot{y}+2n\dot{x}=-\frac{\partial\Psi}{m\partial y}. \tag{3.14}$$

Equations (3.13) and (3.14) provide the equations that govern the motion of the infinitesimal mass in dimensional variables.

We now evaluate the partial derivatives $\frac{\partial\Psi}{\partial x}\, and \,\frac{\partial\Psi}{\partial y}$ of Equations (3.13) and (3.14) respectively.

Considering Equations (3.7) and (3.2) results to

$$\begin{aligned}\frac{\partial\Psi}{\partial x}&=-Gm\frac{\partial}{\partial x}\left\langle m_1q_1\left(\frac{1}{r_1}+\frac{A_1}{2r_1^3}-\frac{3A_2}{8r_1^5}\right)+m_2q_2\left(\frac{1}{r_2}+\frac{B_1}{2r_2^3}-\frac{3B_2}{8r_2^5}\right)+\frac{M_b}{\left(r^2+T^2\right)^{1/2}}\right\rangle-\frac{\partial}{\partial x}\left(\frac{1}{2}mn^2\left(x^2+y^2\right)\right).\\&=-Gm\left\langle m_1\left(q_1\frac{\partial}{\partial x}(\frac{1}{r_1})+\frac{A_1q_1}{2}\frac{\partial}{\partial x}(\frac{1}{r_1^3})-\frac{3A_2}{8}\frac{\partial}{\partial x}(\frac{1}{r_1^5})\right)+m_2\left(\frac{q_2\partial}{\partial x}(\frac{1}{r_2})+\frac{B_1q_2}{2}\frac{\partial}{\partial x}(\frac{1}{r_2^3})-\frac{3B_2}{8}\frac{\partial}{\partial x}(\frac{1}{r_2^5})\right)\right.\\&\qquad\left.+\frac{\partial}{\partial x}\left(\frac{M_b}{\left(r^2+T^2\right)^{1/2}}\right)\right\rangle-mn^2x,\end{aligned} \tag{3.15}$$

$$\frac{\partial \Psi}{\partial y} = -Gm\frac{\partial}{\partial y}\left\langle m_1q_1\left(\frac{1}{r_1}+\frac{A_1}{2r_1^3}-\frac{3A_2}{8r_1^5}\right)+m_2q_2\left(\frac{1}{r_2}+\frac{B_1}{2r_2^3}-\frac{3B_2}{8r_2^5}\right)+\frac{M_b}{\left(r^2+T^2\right)^{1/2}}\right\rangle-\frac{\partial}{\partial y}\left(\frac{1}{2}mn^2\left(x^2+y^2\right)\right).$$

$$= -Gm\left\langle m_1q_1\frac{\partial}{\partial y}\left(\frac{1}{r_1}+\frac{A_1}{2r_1^3}-\frac{3A_2}{8r_1^5}\right)+m_2q_2\frac{\partial}{\partial y}\left(\frac{1}{r_2}+\frac{B_1}{2r_2^3}-\frac{3B_2}{8r_2^5}\right)+\frac{\partial}{\partial y}\left(\frac{M_b}{\left(r^2+T^2\right)^{1/2}}\right)\right\rangle-mn^2y. \tag{3.16}$$

In view of Equations (3.3) and (3.4), we find:

$$2r_1\frac{\partial r_1}{\partial x}=2\left(x-x_1\right) \qquad\qquad 2r_2\frac{\partial r_2}{\partial x}=2\left(x-x_2\right)$$

$$\Rightarrow\frac{\partial r_1}{\partial x}=\frac{\left(x-x_1\right)}{r_1}, \qquad\qquad \Rightarrow\frac{\partial r_2}{\partial x}=\frac{\left(x-x_2\right)}{r_2},$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right)=-\frac{1}{r_1^2}\frac{\partial r_1}{\partial x} \qquad\qquad \frac{\partial}{\partial x}\left(\frac{1}{r_2}\right)=-\frac{1}{r_2^2}\frac{\partial r_2}{\partial x}$$

$$=-\frac{\left(x-x_1\right)}{r_1^3}, \qquad\qquad =-\frac{\left(x-x_2\right)}{r_2^3},$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_1^3}\right)=-\frac{3}{r_1^4}\frac{\partial r_1}{\partial x} \qquad\qquad \frac{\partial}{\partial x}\left(\frac{1}{r_2^3}\right)=-\frac{3}{r_2^4}\frac{\partial r_2}{\partial x}$$

$$=-\frac{3\left(x-x_1\right)}{r_1^5}, \qquad\qquad =-\frac{3\left(x-x_2\right)}{r_2^5},$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_1^5}\right)=-\frac{5}{r_1^6}\frac{\partial r_1}{\partial x} \qquad\qquad \frac{\partial}{\partial x}\left(\frac{1}{r_2^5}\right)=-\frac{5}{r_2^6}\frac{\partial r_1}{\partial x}$$

$$=-\frac{5\left(x-x_1\right)}{r_1^7}, \qquad\qquad =-\frac{5\left(x-x_2\right)}{r_2^7},$$

and

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{M_b}{\left(r^2+T^2\right)^{1/2}}\right)&=\frac{\partial}{\partial x}\left(\frac{M_b}{\left(x^2+y^2+T^2\right)^{1/2}}\right)\\&=-\frac{M_b x}{\left(r^2+T^2\right)^{3/2}}.\end{aligned}$$

Making use of the above relations in Equation (3.15), produces

$$\begin{aligned}\frac{\partial\Psi}{\partial x}&=-Gm\left\langle m_1\left(-\frac{q_1\left(x-x_1\right)}{r_1^3}-\frac{3A_1q_1}{2}\frac{\left(x-x_1\right)}{r_1^5}+\frac{15A_2q_1}{8}\frac{\left(x-x_1\right)}{r_1^7}\right)+m_2\left(-\frac{q_2\left(x-x_2\right)}{r_2^3}-\frac{3B_1q_2\left(x-x_2\right)}{2r_2^5}\right.\right.\\&\qquad\left.\left.+\frac{15B_2q_2\left(x-x_2\right)}{8r_2^7}\right)-\frac{M_b x}{\left(r^2+T^2\right)^{3/2}}\right\rangle-mn^2x.\\&=Gm\left\langle m_1\left(\frac{q_1\left(x-x_1\right)}{r_1^3}+\frac{3A_1q_1}{2}\frac{\left(x-x_1\right)}{r_1^5}-\frac{15A_2q_1}{8}\frac{\left(x-x_1\right)}{r_1^7}\right)+m_2\left(\frac{q_2\left(x-x_2\right)}{r_2^3}+\frac{3B_1q_2\left(x-x_2\right)}{2r_2^5}\right.\right.\\&\qquad\left.\left.-\frac{15B_2q_2\left(x-x_2\right)}{8r_2^7}\right)+\frac{M_b x}{\left(r^2+T^2\right)^{3/2}}\right\rangle-mn^2x.\end{aligned}\tag{3.17}$$

Still from Equations (3.3) and (3.4), we have:

$$2r_1\frac{\partial r_1}{\partial y}=2y \qquad\qquad 2r_2\frac{\partial r_2}{\partial y}=2y$$

$$\Rightarrow\frac{\partial r_1}{\partial y}=\frac{y}{r_1}, \qquad\qquad \Rightarrow\frac{\partial r_2}{\partial y}=\frac{y}{r_2},$$

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{1}{r_1}\right)&=-\frac{1}{r_1^2}\frac{\partial r_1}{\partial y}\\&=-\frac{y}{r_1^3},\end{aligned} \qquad\qquad \begin{aligned}\frac{\partial}{\partial y}\left(\frac{1}{r_2}\right)&=-\frac{1}{r_2^2}\frac{\partial r_2}{\partial y}\\&=-\frac{y}{r_2^3},\end{aligned}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r_1^3}\right)=-\frac{3}{r_1^4}\frac{\partial r_1}{\partial y} \qquad\qquad \frac{\partial}{\partial y}\left(\frac{1}{r_2^3}\right)=-\frac{3}{r_2^4}\frac{\partial r_2}{\partial y}$$

$$=-\frac{3y}{r_1^5}, \qquad\qquad =-\frac{3y}{r_2^5},$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r_1^5}\right)=-\frac{5}{r_1^6}\frac{\partial r_1}{\partial y} \qquad\qquad \frac{\partial}{\partial y}\left(\frac{1}{r_2^5}\right)=-\frac{5}{r_2^6}\frac{\partial r_1}{\partial y}$$

$$=-\frac{5y}{r_1^7}, \qquad\qquad =-\frac{5y}{r_2^7},$$

and

$$\frac{\partial}{\partial y}\left(\frac{M_b}{\left(r^2+T^2\right)^{1/2}}\right)=\frac{\partial}{\partial y}\left(\frac{M_b}{\left(x^2+y^2+T^2\right)^{1/2}}\right)$$

$$=-\frac{M_b y}{\left(r^2+T^2\right)^{3/2}}.$$

Utilizing the above relations in Equation (3.16), yield

$$\frac{\partial\Psi}{\partial y}=Gm\left\langle m_1\left(\frac{q_1 y}{r_1^3}+\frac{3A_1q_1y}{2r_1^5}-\frac{15A_2q_1y}{8r_1^7}\right)+m_2\left(\frac{q_2y}{r_2^3}+\frac{3B_1q_2y}{2r_2^5}-\frac{15B_2q_2y}{8r_2^7}\right)+\frac{M_by}{\left(r^2+T^2\right)^{3/2}}\right\rangle-mn^2y. \tag{3.18}$$

Inserting Equations (3.17) in Equation (3.13) and (3.18) in Equation (3.14), we get in a simplified form, the equations of motion of the infinitesimal mass in the dimensional variables as

$$\ddot{x}-2n\dot{y}=-G\left\langle \begin{array}{l} m_1\left(\frac{q_1\left(x-x_1\right)}{r_1^3}+\frac{3A_1q_1}{2}\frac{\left(x-x_1\right)}{r_1^5}-\frac{15A_2q_1}{8}\frac{\left(x-x_1\right)}{r_1^7}\right) \\ +m_2\left(\frac{q_2\left(x-x_2\right)}{r_2^3}+\frac{3B_1q_2\left(x-x_2\right)}{2r_2^5}-\frac{15B_2q_2\left(x-x_2\right)}{8r_2^7}\right)+\frac{M_bx}{\left(r^2+T^2\right)^{3/2}} \end{array}\right\rangle +n^2x,$$

$$\ddot{y}+2n\dot{x}=-G\left\langle m_1\left(\frac{q_1y}{r_1^3}+\frac{3A_1q_1y}{2r_1^5}-\frac{15A_2q_1y}{8r_1^7}\right)+m_2\left(\frac{q_2y}{r_2^3}+\frac{3B_1q_2y}{2r_2^5}-\frac{15B_2q_2y}{8r_2^7}\right)+\frac{M_by}{\left(r^2+T^2\right)^{3/2}}\right\rangle +n^2y.$$

(3.19)

3.2.5 The mean motion *n*

At this point, let us find out the mean motion *n*. In view of Figure (3.1), the distances of $m_1$and $m_2$from their common centre of mass O are $|x_1|$ *and* x_2, respectively. Since the primaries are moving in circular orbits around O, the mean motion *n* satisfies

$$m_1n^2\left|x_1\right|=m_2n^2x_2=F,$$

where F is the common force of gravity between m_1 *and* m_2 which is given with respect to Equation (1.4) as

$$F=-Gm_1m_2\frac{\partial}{\partial R}\left[\frac{1}{R}+\frac{A_1}{2R^3}-\frac{3A_2}{8R^5}+\frac{B_1}{2R^3}-\frac{3B_2}{8R^5}\right].$$

Thus,

$$m_1n^2\left|x_1\right|=Gm_1m_2\left[\frac{1}{R^2}+\frac{3A_1}{2R^4}-\frac{15A_2}{8R^6}+\frac{3B_1}{2R^4}-\frac{15B_2}{8R^6}\right],$$

$$m_2n^2x_2=Gm_1m_2\left[\frac{1}{R^2}+\frac{3A_1}{2R^4}-\frac{15A_2}{8R^6}+\frac{3B_1}{2R^4}-\frac{15B_2}{8R^6}\right].$$

Or

$$n^2 |x_1| = Gm_2 \left[\frac{1}{R^2} + \frac{3A_1}{2R^4} - \frac{15A_2}{8R^6} + \frac{3B_1}{2R^4} - \frac{15B_2}{8R^6} \right],$$
$$n^2 x_2 = Gm_1 \left[\frac{1}{R^2} + \frac{3A_1}{2R^4} - \frac{15A_2}{8R^6} + \frac{3B_1}{2R^4} - \frac{15B_2}{8R^6} \right]. \qquad (3.20)$$

Adding the equations of Equation (3.20) gives

$$n^2(|x_1| + x_2) = G(m_1 + m_2)\left[\frac{1}{R^2} + \frac{3A_1}{2R^4} - \frac{15A_2}{8R^6} + \frac{3B_1}{2R^4} - \frac{15B_2}{8R^6} \right].$$

Since $|x_1| + x_2 = R$, we have

$$n^2 = G(m_1 + m_2)\left[\frac{1}{R^3} + \frac{3A_1}{2R^5} - \frac{15A_2}{8R^7} + \frac{3B_1}{2R^5} - \frac{15B_2}{8R^7} \right]. \qquad (3.21)$$

Now, due to the additional gravitational force from the belt, the mean motion n given by Equation (3.21) of the primaries is modified as

$$n^2 = G(m_1 + m_2)\left[\frac{1}{R^3} + \frac{3A_1}{2R^5} - \frac{15A_2}{8R^7} + \frac{3B_1}{2R^5} - \frac{15B_2}{8R^7} \right] + 2 f_b(r),$$

where $f_b(r)$ is the gravitational force from the belt defined as $f_b(r) = -\frac{\partial}{\partial r} \frac{M_b}{\left(r^2 + T^2\right)^{1/2}}$.

$$n^2 = G(m_1 + m_2)\left[\frac{1}{R^3} + \frac{3A_1}{2R^5} - \frac{15A_2}{8R^7} + \frac{3B_1}{2R^5} - \frac{15B_2}{8R^7} \right] + \frac{2M_b r}{\left(r^2 + T^2\right)^{3/2}}. \qquad (3.22)$$

3.2.6 Equations of motion in dimensionless variables

In order to simplify and generalize the equations of motion specified by Equation (3.19), it is suitable to make the variables dimensionless such that, the properties of the system rely on a particular parameter as in Section 1.7.1:

- The sum of the masses of the primaries is taken as a unit mass. For this, we let $m_2 = \mu \ and \ m_1 = 1 - \mu,$ where $0 < \mu = \dfrac{m_2}{m_1 + m_2} \le \dfrac{1}{2}$, is the mass ratio.
- For the unit of length, the distance between the primaries is taken as one.
- The unit of time is so chosen that the gravitational constant *G* is unity i.e., G=1.

Now since the origin is at the barycentre of m_1 and m_2, we have

$$m_1 x_1 + m_2 x_2 = 0.$$

That is

$$\begin{aligned}
&(1-\mu)x_1 + \mu x_2 = 0 \\
&x_1 - \mu x_1 + \mu x_2 = 0 \\
&x_1 + \mu\left(-x_1 + x_2\right) = 0 \\
&x_1 + \mu\left(1\right) = 0 \\
&x_1 = -\mu.
\end{aligned}$$

And hence, we get $x_2 = 1 - \mu.$

After normalization of the physical quantities, we obtain the following dimensionless second order differential equations which describe the motion of the infinitesimal mass. That is by making use of $m_1 = 1 - \mu$, $m_2 = \mu,\ x_1 = -\mu,\ x_2 = 1 - \mu$ and *G*=1 in Equation (3.19), yield

$$\ddot{x}-2n\dot{y}=n^{2}x-\left\{(1-\mu)\left(\frac{q_{1}\left(x+\mu\right)}{r_{1}^{3}}+\frac{3A_{1}q_{1}\left(x+\mu\right)}{2r_{1}^{5}}-\frac{15A_{2}q_{1}}{8}\frac{\left(x+\mu\right)}{r_{1}^{7}}\right)+\mu\left(\frac{q_{2}\left(x+\mu-1\right)}{r_{2}^{3}}\right.\right.$$

$$\left.\left.+\frac{3B_{1}q_{2}\left(x+\mu-1\right)}{2r_{2}^{5}}-\frac{15B_{2}q_{2}\left(x+\mu-1\right)}{8r_{2}^{7}}\right)+\frac{M_{b}x}{\left(r^{2}+T^{2}\right)^{3/2}}\right\},$$

$$\ddot{y}+2n\dot{x}=n^{2}y-\left\{(1-\mu)\left(\frac{q_{1}y}{r_{1}^{3}}+\frac{3A_{1}q_{1}y}{2r_{1}^{5}}-\frac{15A_{2}q_{1}y}{8r_{1}^{7}}\right)+\mu\left(\frac{q_{2}y}{r_{2}^{3}}+\frac{3B_{1}q_{2}y}{2r_{2}^{5}}-\frac{15B_{2}q_{2}y}{8r_{2}^{7}}\right)+\frac{M_{b}y}{\left(r^{2}+T^{2}\right)^{3/2}}\right\}.$$

Or

$$\ddot{x}-2n\dot{y}=\Omega_{x},$$

$$\ddot{y}+2n\dot{x}=\Omega_{y}, \tag{3.23}$$

where

$$\Omega=\frac{n^{2}(x^{2}+y^{2})}{2}+\frac{(1-\mu)q_{1}}{r_{1}}+\frac{(1-\mu)A_{1}q_{1}}{2r_{1}^{3}}-\frac{3(1-\mu)A_{2}q_{1}}{8r_{1}^{5}}+\frac{\mu q_{2}}{r_{2}}+\frac{\mu B_{1}q_{2}}{2r_{2}^{3}}-\frac{3\mu B_{2}q_{2}}{8r_{2}^{5}}+\frac{M_{b}}{(r^{2}+T^{2})^{1/2}},$$

$$\Omega_{x}=n^{2}x-\frac{(1-\mu)\left(x+\mu\right)q_{1}}{r_{1}^{3}}-\frac{3(1-\mu)\left(x+\mu\right)A_{1}q_{1}}{2r_{1}^{5}}+\frac{15(1-\mu)\left(x+\mu\right)A_{2}q_{1}}{8r_{1}^{7}}-\frac{\mu\left(x+\mu-1\right)q_{2}}{r_{2}^{3}}$$

$$-\frac{3\mu\left(x+\mu-1\right)B_{1}q_{2}}{2r_{2}^{5}}+\frac{15\mu\left(x+\mu-1\right)B_{2}q_{2}}{8r_{2}^{7}}-\frac{M_{b}x}{\left(r^{2}+T^{2}\right)^{3/2}},$$

$$\Omega_{y}=n^{2}y-\frac{(1-\mu)q_{1}y}{r_{1}^{3}}-\frac{3(1-\mu)A_{1}q_{1}y}{2r_{1}^{5}}+\frac{15(1-\mu)A_{2}q_{1}y}{8r_{1}^{7}}-\frac{\mu q_{2}y}{r_{2}^{3}}-\frac{3\mu B_{1}q_{2}y}{2r_{2}^{5}}+\frac{15\mu B_{2}q_{2}y}{8r_{2}^{7}}-\frac{M_{b}y}{(r^{2}+T^{2})^{3/2}},$$

with

$$r_{1}^{2}=\left(x+\mu\right)^{2}+y^{2},\qquad r_{2}^{2}=\left(x+\mu-1\right)^{2}+y^{2}. \tag{3.24}$$

Also in dimensionless variables, the mean motion Equation (3.22) becomes

$$n^2 = 1 + \frac{3}{2}\left\langle A_1 + B_1 - \frac{5}{4}\left(A_2 + B_2\right)\right\rangle + \frac{2M_b r_c}{(r_c^2 + T^2)^{3/2}}, \tag{3.25}$$

where $r_c^{\,2} = 1 - \mu + \mu^2$ is the radial distance in the classical problem.

3.3 The Jacobian Integral and Zero Velocity Surfaces

The equations of motion (3.23) are unintuitive and it is difficult to interpret the motion of the infinitesimal mass from them. Therefore, it is essential to discover as much about the system as possible. Being able to identify regions that display predictable behavior is an important objective. Hence to this end, additional analysis can be made to pinpoint boundaries for the motion of the infinitesimal mass.

3.3.1 The Jacobian integral

Multiply the first and second equations of (3.23) by $\dot{x}$ and $\dot{y}$ respectively, we get

$$\dot{x}\ddot{x} - n\dot{x}\dot{y} = \dot{x}\Omega_x,$$

$$\dot{y}\ddot{y} + n\dot{y}\dot{x} = \dot{y}\Omega_y.$$

Adding them together, gives

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = \dot{x}\Omega_x + \dot{y}\Omega_y.$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{d}{dt}\Omega\left(x, y\right)$$

$$\Rightarrow \frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2\right) = 2\frac{d}{dt}\Omega\left(x, y\right) \tag{3.26}$$

Integrating Equation (3.26) with respect to time produces

$$\dot{x}^2 + \dot{y}^2 = 2\Omega\left(x, y\right) - C, \tag{3.27}$$

where C is the constant of integration, also known as Jacobi's constant.

Equation (3.27) is known as the Jacobian integral. Observing the left hand side of Equation (3.27), we find that it is the square of the velocity (v) of the infinitesimal mass. That is $v^2 = 2\Omega(x, y) - C$ or $2\Omega(x, y) - v^2 = C,$ which means that the quantity $2\Omega(x, y) - v^2$ is constant of the problem. From this, surfaces of zero velocity can be determined in the plane for the infinitesimal mass.

3.3.2 Zero velocity surfaces

Since the Jacobean integral (3.27) is a relation between the velocity and the coordinates of the infinitesimal particle with respect to the synodic system, so by setting velocity term to zero, we obtainthe equation of zero velocity curves/surfaces as:

$$2\Omega(x, y) = C.$$

Or

$$n^2(x^2+y^2) + \frac{2(1-\mu)q_1}{r_1} + \frac{(1-\mu)A_1q_1}{r_1^3} - \frac{3(1-\mu)A_2q_1}{4r_1^5} + \frac{2\mu q_2}{r_2} + \frac{\mu B_1q_2}{r_2^3} - \frac{3\mu B_2q_2}{4r_2^5} + \frac{2M_b}{(r^2+T^2)^{1/2}} = C. \tag{3.28}$$

Now, we observe that with larger values of x and y, the curves of (3.28) approximate very close to a circle $x^2 + y^2 = \left(\frac{\sqrt{C-\varepsilon}}{n}\right)^2$ of radius $\frac{\sqrt{C-\varepsilon}}{n}$, where

$$\varepsilon = \frac{2(1-\mu)q_1}{r_1} + \frac{(1-\mu)A_1q_1}{r_1^3} - \frac{3(1-\mu)A_2q_1}{4r_1^5} + \frac{2\mu q_2}{r_2} + \frac{\mu B_1q_2}{r_2^3} - \frac{3\mu B_2q_2}{4r_2^5} + \frac{2M_b}{(r^2+T^2)^{1/2}} \tag{3.29}$$

and is very small.

If the values of x and y are very small, the terms $n^2(x^2+y^2)$ will be very small with respect to the other terms on L.H.S. of (3.28). Thus, the curves of (3.28) will approximate closely to the following equipotential surface

$$\frac{2(1-\mu)q_1}{\sqrt{(x+\mu)^2+y^2}}+\frac{(1-\mu)A_1q_1}{((x+\mu)^2+y^2)^{\frac{3}{2}}}-\frac{3(1-\mu)A_2q_1}{4((x+\mu)^2+y^2)^{\frac{5}{2}}}+\frac{2\mu q_2}{\sqrt{(x+\mu-1)^2+y^2}}$$
$$+\frac{\mu B_1q_2}{((x+\mu-1)^2+y^2)^{\frac{3}{2}}}-\frac{3\mu B_2q_2}{4((x+\mu-1)^2+y^2)^{\frac{5}{2}}}+\frac{2M_b}{(x^2+y^2+T^2)^{1/2}}=C-\varepsilon_s, \tag{3.30}$$

where $\varepsilon_s = n^2(x^2+y^2)$ and is also very small.

Now, using Equation (3.28) and with the help of the MATLAB (R2007b) software package, we obtained the zero velocity curves and zero velocity surfaces as shown in Figures 3.2 and 3.3 respectively.

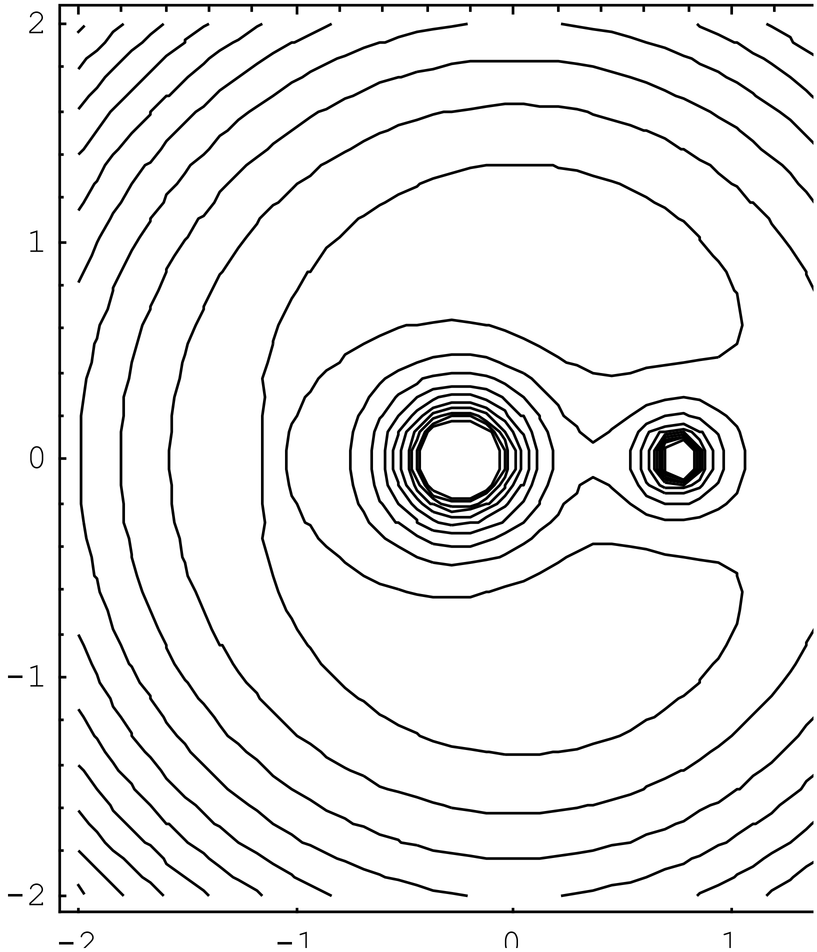


Figure 3.2: Zero velocity curves when $\mu = 0.25, A_1 = B_1 = 0.01,$ $A_2 = B_2 = 0.005, M_b = T = 0.01,\ q_1 = 0.98, q_2 = 0.95.$

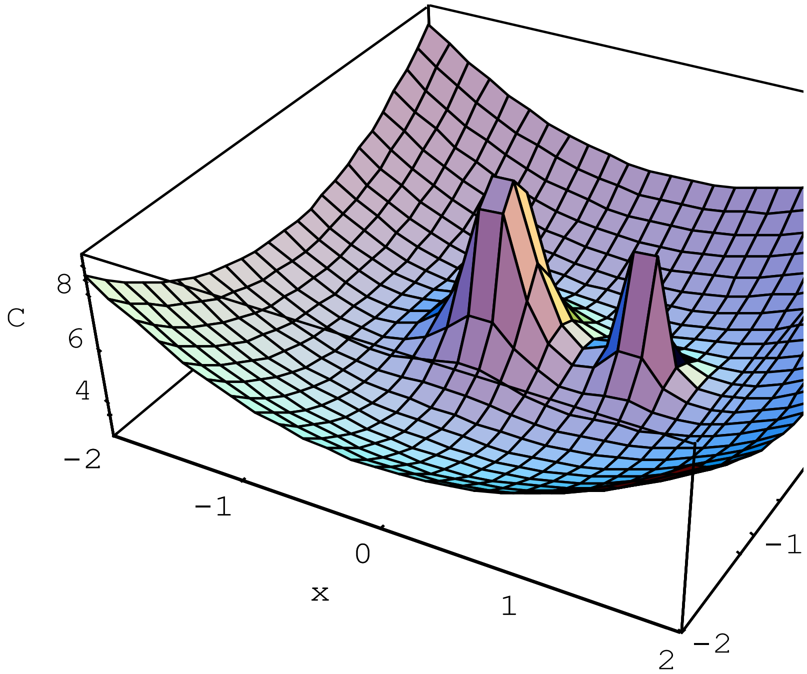


Figure 3.3: Zero velocity surfaces when $\mu = 0.25, A_1 = B_1 = 0.01, A_2 = B_2 = 0.005,$ $M_b = T = 0.01,\ q_1 = 0.98, q_2 = 0.95.$

Studying Figure 3.2 shows that any smaller values of x and y yields a pair of ovals surrounding $1-\mu$ and μ. Here motion can take place only if the particle is within the ovals or out- side an almost circular contour located at external boundary. That is motion cannot occur in the area between the ovals and the outer contour. The two singular regions are shown by conic shape in the zero velocity surfaces Figure 3.3.

Zero velocity curves/surfaces are critical because they create the boundary of regions from which the third body is dynamically excluded. Since the velocity must always be real

$2\Omega > C$, producing this constraint. The points where the velocity is zero are the libration points and hence, the existence of libration points in the problem under consideration.

3.4 Locations of Libration Points

We now search for possible libration pointsof the infinitesimal mass in our problem. The libration points are positions of gravitational balance between the primaries. At these points the two finite masses would exert zero net force on the infinitesimal mass, in effect, allowing the infinitesimal mass to have zero velocity in the rotating frame of reference. That is the libration points satisfy $\ddot{x} = \ddot{y} = \dot{x} = \dot{y} = 0$. It thus follows, fromEquation (3.23), that the libration points are the solutions of

$$n^2 x - \frac{(1-\mu)\left(x+\mu\right)q_1}{r_1^3} - \frac{\mu\left(x+\mu-1\right)q_2}{r_2^3} - \frac{3(1-\mu)\left(x+\mu\right)q_1A_1}{2r_1^5} + \frac{15(1-\mu)\left(x+\mu\right)q_1A_2}{8r_1^7} - \frac{3\mu\left(x+\mu-1\right)q_2B_1}{2r_2^5} + \frac{15\mu\left(x+\mu-1\right)q_2B_2}{8r_2^7} - \frac{M_b x}{\left(r^2+T^2\right)^{3/2}} = 0 \qquad 3.31)$$

and

$$n^2 y - \frac{(1-\mu)q_1 y}{r_1^3} - \frac{\mu y q_2}{r_2^3} - \frac{3(1-\mu)q_1A_1 y}{2r_1^5} + \frac{15(1-\mu)q_1A_2 y}{8r_1^7} - \frac{3\mu q_2B_1 y}{2r_2^5} + \frac{15\mu q_2B_2 y}{8r_2^7} - \frac{M_b y}{(r^2+T^2)^{3/2}} = 0. \qquad (3.32)$$

Now, one obvious solution of Equation (3.32) is $y = 0$, corresponding to libration points which lie on the *x*-axis. We first hunt for libration points which do not lieon the *x*-axis: triangular points.

3.4.1Triangular libration points

The triangular libration points are obtained by solving Equations (3.31) and (3.32) when $y \neq 0.$

Starting from Equation (3.31), we have

$$x\left(n^2-\frac{(1-\mu)q_1}{r_1^3}-\frac{\mu q_2}{r_2^3}-\frac{3}{2}\frac{(1-\mu)q_1A_1}{r_1^5}+\frac{15}{8}\frac{(1-\mu)q_1A_2}{r_1^7}-\frac{3}{2}\frac{\mu q_2B_1}{r_2^5}+\frac{15}{8}\frac{\mu q_2B_2}{r_2^7}-\frac{M_b}{(r^2+T^2)^{3/2}}\right)$$
$$-\frac{(1-\mu)\mu q_1}{r_1^3}-\frac{\mu\left(\mu-1\right)q_2}{r_2^3}-\frac{3(1-\mu)\mu q_1A_1}{2r_1^5}+\frac{15(1-\mu)\mu q_1A_2}{8r_1^7}-\frac{3\mu(\mu-1)q_2B_1}{2r_2^5}+\frac{15\mu\left(\mu-1\right)q_2B_2}{8r_2^7}=0, \tag{3.33}$$

and from (3.32) with $y \neq 0$ yields

$$n^2-\frac{(1-\mu)q_1}{r_1^3}-\frac{\mu q_2}{r_2^3}-\frac{3}{2}\frac{(1-\mu)q_1A_1}{r_1^5}+\frac{15}{8}\frac{(1-\mu)q_1A_2}{r_1^7}-\frac{3}{2}\frac{\mu q_2B_1}{r_2^5}+\frac{15}{8}\frac{\mu q_2B_2}{r_2^7}-\frac{M_b}{(r^2+T^2)^{3/2}}=0. \tag{3.34}$$

Plugging Equation (3.34) in Equation (3.33) produces

$$\frac{q_1}{r_1^3}-\frac{q_2}{r_2^3}+\frac{3A_1q_1}{2r_1^5}-\frac{3B_1q_2}{2r_2^5}-\frac{15q_1A_2}{8r_1^7}+\frac{15q_2B_2}{8r_2^7}=0. \tag{3.35}$$

Equation (3.34) can be rearranged as

$$n^2-\frac{q_1}{r_1^3}-\frac{3A_1q_1}{2r_1^5}+\frac{15q_1A_2}{8r_1^7}+\mu\left(\frac{q_1}{r_1^3}-\frac{q_2}{r_2^3}+\frac{3A_1q_1}{2r_1^5}-\frac{3B_1q_2}{2r_2^5}-\frac{15q_1A_2}{8r_1^7}+\frac{15q_2B_2}{8r_2^7}\right)-\frac{M_b}{(r^2+T^2)^{3/2}}=0. \tag{3.36}$$

Again, putting Equation (3.35) in Equation (3.36) results to

$$n^2-\frac{q_1}{r_1^3}-\frac{3q_1A_1}{2r_1^5}+\frac{15q_1A_2}{8r_1^7}-\frac{M_b}{(r^2+T^2)^{3/2}}=0. \tag{3.37}$$

Merging Equations (3.35) and (3.37) returns

$$n^2 - \frac{q_2}{r_2^3} - \frac{3q_2B_1}{2r_2^5} + \frac{15q_2B_2}{8r_2^7} - \frac{M_b}{(r^2+T^2)^{3/2}} = 0. \tag{3.38}$$

Now, if the effects of radiation and oblateness up to J_4 of the primaries and the potential from the circular cluster of material points are neglected (*i.e.* $q_1=q_2=1$, $A_1= A_2= B_1= B_2 =M_b=0$), Equations (3.37) and (3.38) are reduced to

$$1-\frac{1}{r_1^3}=0 \, and \;\; 1-\frac{1}{r_2^3}=0$$

respectively, with the solutions $r_1 = r_2 = 1.$ Therefore, due to the perturbations, the values of $r_1 \, and \, r_2$ may change slightly, by $\varepsilon_1 \, and \, \varepsilon_2,$ say, so that

$$r_1 = 1+\varepsilon_1 \, and \;\; r_2 = 1+\varepsilon_2, \;\; where \; \varepsilon_i << 1 \;\; (i=1,2), \tag{3.39}$$

be the solutions of Equations (3.37) and (3.38) . That is ε_1 and ε_2 are very small quantities and are due to the combined effect of the perturbations.

Now, the exact coordinates of the triangular libration points are obtained by solving equations of (3.24) for x and y. Therefore, we substitute Equation (3.39) in Equation (3.24), to get

$$\begin{aligned}(1+\varepsilon_1)^2 &= \left(x+\mu\right)^2 + y^2, \\ (1+\varepsilon_2)^2 &= \left(x+\mu-1\right)^2 + y^2.\end{aligned} \tag{3.40}$$

Subtracting the second equation of Equation (3.40) from the first results to

$$(1+\varepsilon_1)^2 - (1+\varepsilon_2)^2 = \left(x+\mu\right)^2 - \left(x+\mu-1\right)^2,$$

and by considering only linear terms in ε_1 and ε_2, we get

$$x = \frac{1}{2} - \mu + \varepsilon_1 - \varepsilon_2. \tag{3.41}$$

Substituting Equation (3.41) in the first equation of Equation (3.40), gives

$$y^2 = (1+\varepsilon_1)^2 - \left(\frac{1}{2} + \varepsilon_1 - \varepsilon_2\right)^2,$$

and by taking into account only linear terms in *ε*

$$\begin{aligned} y^2 &= 1 + 2\varepsilon_1 - \left(\frac{1}{4} + \varepsilon_1 - \varepsilon_2\right) \\ &= \frac{3}{4} + \varepsilon_1 + \varepsilon_2 \\ &= \frac{3}{4}\left(1 + \frac{4}{3}(\varepsilon_1 + \varepsilon_2)\right) \\ &= \frac{3}{4}\left(1 + \frac{2}{3}(\varepsilon_1 + \varepsilon_2)\right)^2 \\ \therefore y &= \pm\sqrt{\frac{3}{4}\left(1 + \frac{2}{3}(\varepsilon_1 + \varepsilon_2)\right)^2} \end{aligned} \tag{3.42}$$

$$y = \pm\sqrt{3}\left(\frac{1}{2} + \frac{1}{3}(\varepsilon_1 + \varepsilon_2)\right). \tag{3.43}$$

Using Equations (3.41) and (3.42) in $r^2 = x^2 + y^2$ of Equation (3.4) produces

$$r^2 = \left(\frac{1}{2} - \mu + \varepsilon_1 - \varepsilon_2\right)^2 + \frac{3}{4} + \varepsilon_1 + \varepsilon_2,$$

which simplifies to

$$r^2 = r_c^{\,2} + 2(1-\mu)\varepsilon_1 + 2\mu\varepsilon_1, \tag{3.44}$$

where $r_c^{\,2} = 1 - \mu + \mu^2$. For simplicity, since $0.9 \le r_c < 1,\ \mu \in (0, 1/2]$, we assume the variation in r_c to be negligible by regarding r_c as a constant.

At this point, we apply Equations (3.25), (3.39), (3.44) in Equation (3.37), using $q_i = 1 - p_i,\ i = 1,2$ where $p_i = 1 - q_i << 1$ and considering only linear terms in $\varepsilon_i, A_i, B_i, p_i, M_b\ ,\ i = 1,2$ yield

$$\varepsilon_1 = -\frac{p_1}{3} - \frac{B_1}{2} + \frac{5B_2}{8} - \frac{M_b(2r_c - 1)}{3(r_c^2 + T^2)^{3/2}}. \qquad (3.45)$$

Likewise, from Equation (3.38) we have

$$\varepsilon_2 = -\frac{p_2}{3} - \frac{A_1}{2} + \frac{5A_2}{8} - \frac{M_b(2r_c - 1)}{3(r_c^2 + T^2)^{3/2}}. \qquad (3.46)$$

Inserting Equations (3.45) and (3.46) in Equations (3.41) and (3.43), we obtain the triangular libration points designatedas *L₄(x, y)* and *L₅(x, - y)*:

$$\begin{aligned} x &= \frac{1}{2}\left[1 - 2\mu - \frac{2p_1}{3} + \frac{2p_2}{3} + A_1 - \frac{5A_2}{4} - B_1 + \frac{5B_2}{4}\right], \\ y &= \pm\frac{\sqrt{3}}{2}\left(1 - \frac{2(p_1 + p_2)}{9} - \frac{A_1}{3} + \frac{5A_2}{12} - \frac{B_1}{3} + \frac{5B_2}{12} - \frac{4M_b(2r_c - 1)}{9(r_c^2 + T^2)^{3/2}}\right). \end{aligned} \qquad (3.47)$$

3.4.2 Locations of collinear libration points

The positions of collinear points are obtained by solving Equations (3.31) and (3.32) when $y = 0.$

From Equation (3.32)*y* must be zero, and substituting it in equation (3.31), gives

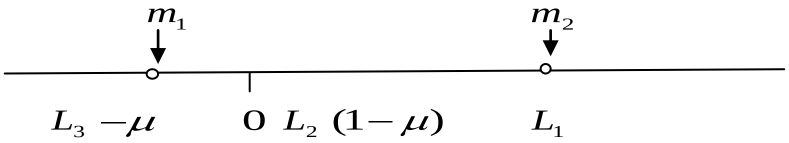
$$\begin{aligned} n^2x &- \frac{(1-\mu)\left(x+\mu\right)q_1}{\left|x+\mu\right|^3} - \frac{\mu\left(x+\mu-1\right)q_2}{\left|x+\mu-1\right|^3} - \frac{3(1-\mu)\left(x+\mu\right)q_1A_1}{2\left|x+\mu\right|^5} + \frac{15(1-\mu)\left(x+\mu\right)q_1A_2}{8\left|x+\mu\right|^7} \\ &- \frac{3\mu\left(x+\mu-1\right)q_2B_1}{2\left|x+\mu-1\right|^5} + \frac{15\mu\left(x+\mu-1\right)q_2B_2}{8\left|x+\mu-1\right|^7} - \frac{M_bx}{\left(x^2+T^2\right)^{3/2}} = 0. \end{aligned}$$

(3.48)

Now, if we neglect the effects of radiation and oblateness up to J_4 of the primaries and the potential from the belt (*i.e.* $q_1=q_2=1$, $A_1= A_2= B_1= B_2 =M_b=0$), Equation (3.48) resembles to classical case of Szebehely (1967)

$$x-\frac{(1-\mu)(x+\mu)}{|x+\mu|^3}-\frac{\mu(x+\mu-1)}{|x+\mu-1|^3}=0, \tag{3.49}$$

with three collinear points L_1, L_2 and L_3. The collinear point L_2 is disposed between the primaries, while L_1 and L_3 are beyond them as shown in Figure 1.2.



Now, using Equation (3.48) and with the help of the MATLAB (R2007b) software package, we examine the effects of the perturbations on the collinear libration points using the binary system Xi Bootis. The masses of the binary are $0.9M_0$ and $0.66M_0$, which produce the mass parameter $\mu = 0.4$ approximately. The coordinates of the collinear libration points for different cases as classified in the following order are presented in Table 3.1:

1 The classical case (absence of the perturbations).
2 Potential from the belt only.
3 Oblateness of the bigger primary up to $J_2$with potential from the belt only.
4 Oblateness of the bigger primary up to J_4 with potential from the belt only.
5 Oblateness of the smaller primary up to J_2 with potential from the belt only.
6 Oblateness of the smaller primary up to J_4 with potential from the belt only.
7 Radiation of the bigger primarywith potential from the belt only.

8 Radiation of the smaller primarywith potential from the belt only.

9 Radiation and oblateness of the primaries up to J_4 with potential from the belt.

Table 3.1: Positions of the collinear points when μ=0.4, $A_1 = B_1$ =0.01, $A_2 = B_2$ =0.005, q_1=0.98, q_2=0.95 and M_b =T=0.01.

Case	L_1	L_2	L_3	L_{n1}	L_{n2}	L_{n3}	L_{n4}
1	1.230814	0.141618	-1.162045				
2	1.225145	0.163243	-1.156152	-0.000265	-0.048710		
3	1.221462	0.167120	-1.158632	-0.000300	-0.045812		
4	1.223870	0.159016	-1.153827	-0.000162	-0.704629		
5	1.229484	0.156143	-1.151977	-0.000260	-0.048891		
6	1.218052	0.190087	-1.154651	-0.000268	-0.048620	0.909793	0.273997
7	1.224156	0.161508	-1.150821	-0.000257	-0.049246		
8	1.213816	0.168169	-1.154516	-0.000270	-0.048454		
9	1.203598	0.189788	-1.145168	-0.000165	-0.704932	0.910718	0.273293

The existence and positions of the libration points for $\mu \in (0,1/2)$ in the (a) absence of the perturbations, (b) presence of all perturbations and (c) presence of oblateness of the smaller primary only are shown in Figures 3.4, 3.5 and 3.6 respectively.

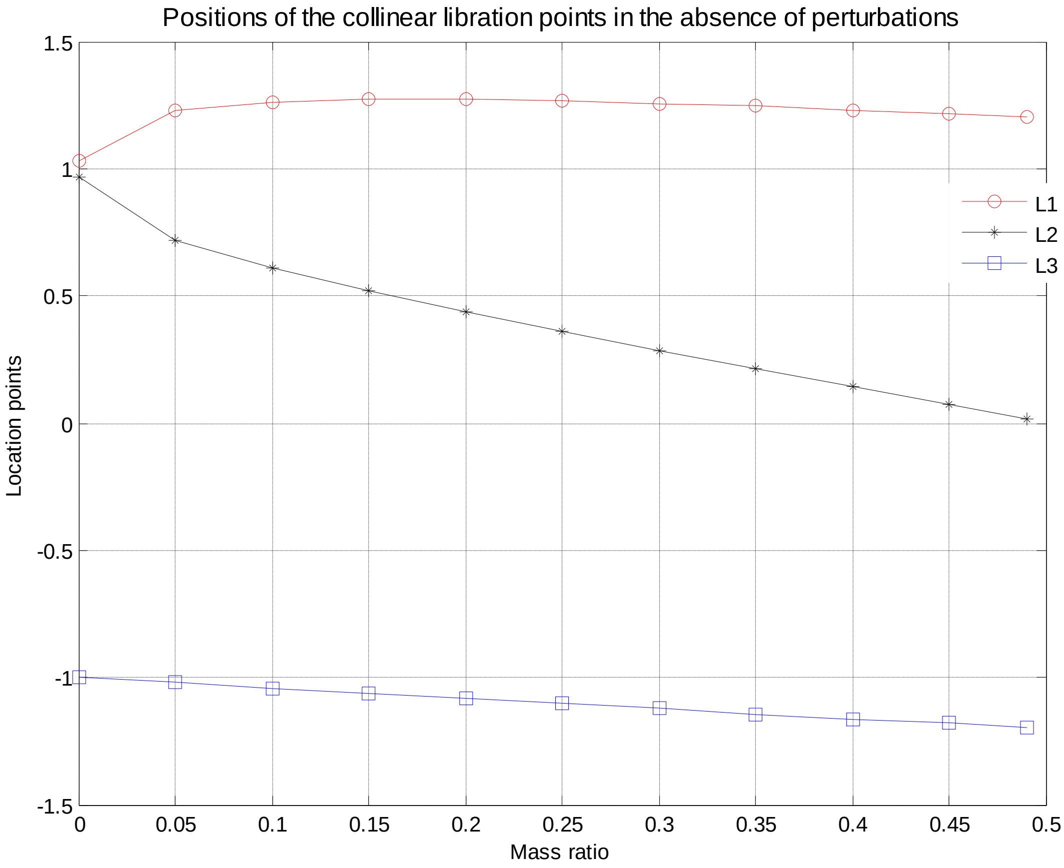


Figure 3.4: The collinear points in the classical case (i.e. $A_1 = A_2 = B_1 = B_2 = M_b = 0,\ q_1 = q_2 = 1$).

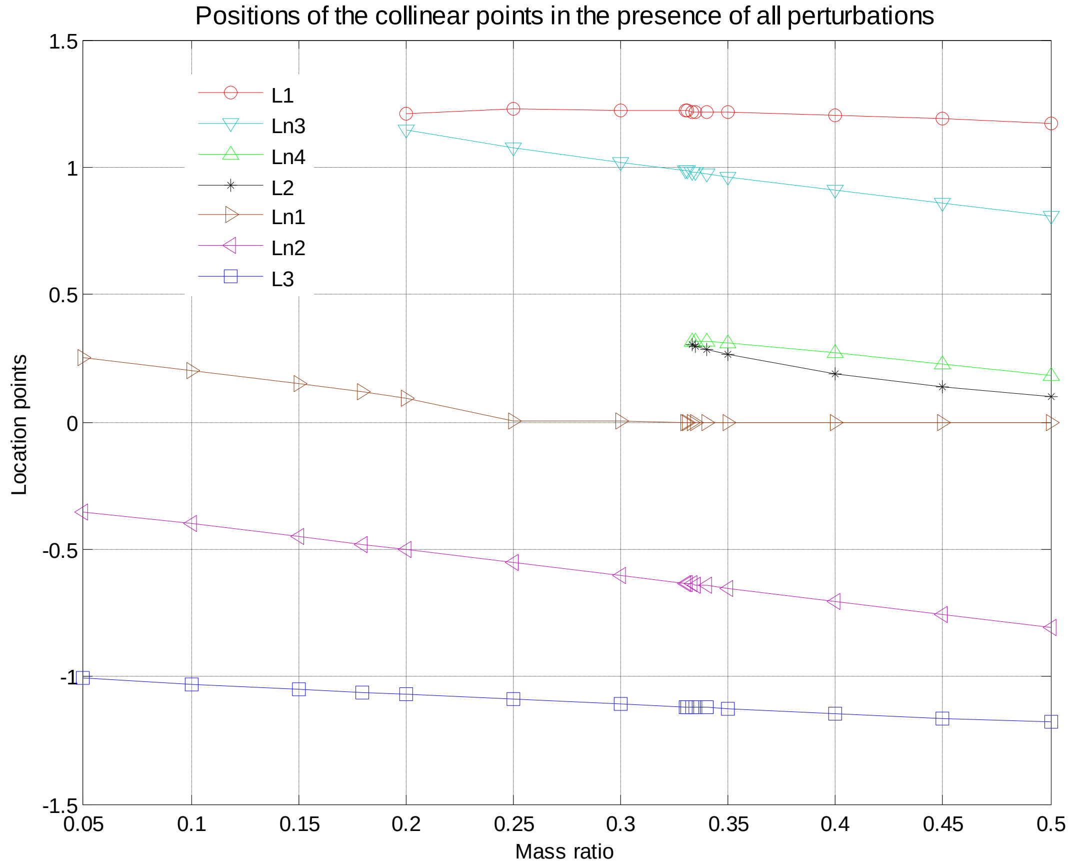


Figure 3.5: The collinear points under the combined effect of the perturbations (i.e. $A_1 = B_1 = 0.01, A_2 = B_2 = 0.005, \quad q_1 = 0.98,\ q_2 = 0.95,\ M_b = 0.01$).

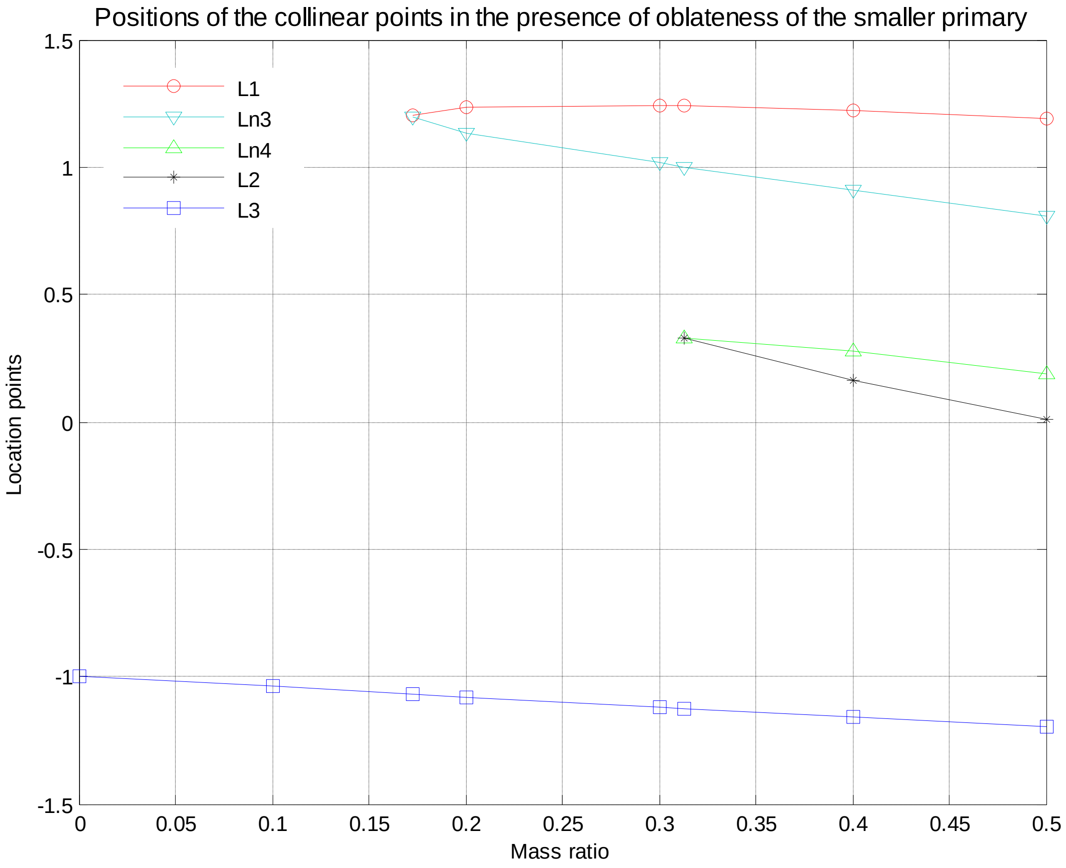


Figure 3.6: The collinear points under the effect of the oblateness of the smaller primary (i.e. $A_1 = A_2 = M_b = 0, q_1 = q_2 = 1,\ B_1 = 0.01, B_2 = 0.005$).

3.5 Stability of Libration Points

In order to study the stability of a libration point (x_0, y_0), we obtain the linear variational equations of motion corresponding to the equations of motion (3.23). Determine the characteristic equation and its roots; and then, converse the stability according to the nature of the roots.

3.5.1 Variational equations

We apply small displacement η, ξ to the coordinates (x_0, y_0) of the libration point. The variations η and ξ can be expressed as

$$\eta = x - x_0 \ \xi = y - y_0, \tag{3.50}$$

which produce the velocity

$$\dot{x} = \dot{\eta} \ \dot{y} = \dot{\xi}$$

and acceleration

$$\ddot{x} = \ddot{\eta} \ \ddot{y} = \ddot{\xi}.$$

Now, the functions $\Omega(x, y)$, $\Omega_x(x, y)$, $\Omega_y(x, y)$ in Equation (3.23) become

$$\begin{aligned}
\Omega(x,y) &= \Omega(x_0+\eta, y_0+\xi), \\
\Omega_x(x,y) &= \Omega(x_0+\eta, y_0+\xi), \\
\Omega_y(x,y) &= \Omega(x_0+\eta, y_0+\xi).
\end{aligned} \tag{3.51}$$

On expanding equation (3.51) by Taylor's series and considering only the linear terms in η and ξ, we have

$$\begin{aligned}
\Omega(x_0+\eta, y_0+\xi) &= \Omega(x_0,y_0) + \eta\Omega_x(x_0,y_0) + \xi\Omega_y(x_0,y_0) + \ldots \\
\Omega_x(x_0+\eta, y_0+\xi) &= \Omega_x(x_0,y_0) + \eta\Omega_{xx}(x_0,y_0) + \xi\Omega_{xy}(x_0,y_0) + \ldots \\
\Omega_y(x_0+\eta, y_0+\xi) &= \Omega_y(x_0,y_0) + \eta\Omega_{yx}(x_0,y_0) + \xi\Omega_{yy}(x_0,y_0) + \ldots
\end{aligned} \tag{3.52}$$

Since at the libration point (x_0, y_0),

$\Omega_x(x_0,y_0) = \Omega_y(x_0,y_0) = 0$, the last two equations of (3.52) become

$$\Omega_x(x_0+\eta, y_0+\xi) = \eta\Omega_{xx}(x_0,y_0) + \xi\Omega_{xy}(x_0,y_0)$$

$$\Omega_y(x_0+\eta, y_0+\xi) = \eta\Omega_{yx}(x_0,y_0) + \xi\Omega_{yy}(x_0,y_0) \tag{3.53}$$

Substituting Equations (3.50) and (3.53) in the equations of motion (3.23), we obtain the variational equations of motion as

$$
\begin{aligned}
&\ddot{\eta} - 2n\dot{\xi} = \eta\Omega_{xx}\left(x_{0,}y_0\right) + \xi\Omega_{xy}\left(x_{0,}y_0\right),\\
&\ddot{\xi} + 2n\dot{\eta} = \eta\Omega_{yx}\left(x_{0,}y_0\right) + \xi\Omega_{yy}\left(x_{0,}y_0\right),
\end{aligned}
\tag{3.54}
$$

We now obtain the second partial derivatives of Ω.

From Equation (3.23) we have the second partial derivatives of Ω with respect to *x* as

$$
\Omega_{xx} = \frac{\partial}{\partial x}\left(
\begin{aligned}
&n^2 x - \frac{(1-\mu)\left(x+\mu\right)q_1}{r_1^3} - \frac{\mu\left(x+\mu-1\right)q_2}{r_2^3} - \frac{3(1-\mu)\left(x+\mu\right)q_1A_1}{2r_1^5} + \frac{15(1-\mu)\left(x+\mu\right)q_1A_2}{8r_1^7}\\
&-\frac{3\mu\left(x+\mu-1\right)q_2B_1}{2r_2^5} + \frac{15\mu\left(x+\mu-1\right)q_2B_2}{8r_2^7} - \frac{M_b x}{\left(r^2+T^2\right)^{3/2}}
\end{aligned}
\right)
$$

$$
\begin{aligned}
= n^2 &- \frac{(1-\mu)q_1}{r_1^3} - (1-\mu)\left(x+\mu\right)q_1\frac{\partial}{\partial x}\left(\frac{1}{r_1^3}\right) - \frac{\mu q_2}{r_2^3} - \mu\left(x+\mu-1\right)q_2\frac{\partial}{\partial x}\left(\frac{1}{r_2^3}\right) - \frac{3(1-\mu)A_1q_1}{2r_1^5}\\
&- \frac{3(1-\mu)\left(x+\mu\right)A_1q_1}{2}\frac{\partial}{\partial x}\left(\frac{1}{r_1^5}\right) + \frac{15(1-\mu)q_1A_2}{8r_1^7} + \frac{15(1-\mu)\left(x+\mu\right)q_1A_2}{8}\frac{\partial}{\partial x}\left(\frac{1}{r_1^7}\right)\\
&- \frac{3\mu B_1q_2}{2r_2^5} - \frac{3\mu\left(x+\mu-1\right)B_1q_2}{2}\frac{\partial}{\partial x}\left(\frac{1}{r_2^5}\right) + \frac{15\mu q_2B_2}{8r_2^7} + \frac{15\mu\left(x+\mu-1\right)q_2B_2}{8}\frac{\partial}{\partial x}\left(\frac{1}{r_2^7}\right)\\
&- \frac{M_b}{\left(r^2+T^2\right)^{3/2}} - M_b x\frac{\partial}{\partial x}\frac{1}{\left(r^2+T^2\right)^{3/2}}.
\end{aligned}
\tag{3.55}
$$

In view of Equation (3.24) that is

$$r_1^2 = \left(x+\mu\right)^2 + y^2,\ r_2^2 = \left(x+\mu-1\right)^2 + y^2,$$

we obtain the following :

$$2r_1\frac{\partial r_1}{\partial x}=2\left(x+\mu\right)$$

$$\Rightarrow\frac{\partial r_1}{\partial x}=\frac{\left(x+\mu\right)}{r_1},$$

$$2r_2\frac{\partial r_2}{\partial x}=2\left(x+\mu-1\right)$$

$$\Rightarrow\frac{\partial r_2}{\partial x}=\frac{\left(x+\mu-1\right)}{r_2},$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_1^3}\right)=-\frac{3}{r_1^4}\frac{\partial r_1}{\partial x}$$

$$\Rightarrow\frac{\partial}{\partial x}\left(\frac{1}{r_1^3}\right)=-\frac{3\left(x+\mu\right)}{r_1^5},$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_2^3}\right)=-\frac{3}{r_2^4}\frac{\partial r_2}{\partial x}$$

$$\Rightarrow\frac{\partial}{\partial x}\left(\frac{1}{r_2^3}\right)=-\frac{3\left(x+\mu-1\right)}{r_2^5},$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_1^5}\right)=-\frac{5}{r_1^6}\frac{\partial r_1}{\partial x}$$

$$\Rightarrow\frac{\partial}{\partial x}\left(\frac{1}{r_1^5}\right)=-\frac{5\left(x+\mu\right)}{r_1^7},$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_2^5}\right)=-\frac{5}{r_2^6}\frac{\partial r_2}{\partial x}$$

$$\Rightarrow\frac{\partial}{\partial x}\left(\frac{1}{r_2^5}\right)=-\frac{5\left(x+\mu-1\right)}{r_2^7},$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_1^7}\right)=-\frac{7}{r_1^8}\frac{\partial r_1}{\partial x}$$

$$=-\frac{7\left(x+\mu\right)}{r_1^9},$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_2^7}\right)=-\frac{7}{r_2^8}\frac{\partial r_2}{\partial x}$$

$$=-\frac{7\left(x+\mu-1\right)}{r_2^9}.$$

From Equation (3.4), we have

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{1}{\left(r^2+T^2\right)^{3/2}}\right)&=-\frac{3}{2}\frac{1}{\left(r^2+T^2\right)^{5/2}}\frac{\partial r^2}{\partial x}\\&=-\frac{3}{2}\frac{1}{\left(r^2+T^2\right)^{5/2}}2x\ \left(\frac{\partial r^2}{\partial x}=\frac{\partial}{\partial x}\left(x^2+y^2\right)=2x\right)\\&=-\frac{3x}{\left(r^2+T^2\right)^{5/2}}.\end{aligned}$$

So, the second partial derivative $\mathbf{\Omega}_{xx}$ of Equation (3.55) becomes

$$\begin{aligned}\Omega_{xx} = n^2 &- \frac{(1-\mu)q_1}{r_1^3} + \frac{3(1-\mu)\left(x+\mu\right)^2 q_1}{r_1^5} - \frac{\mu q_2}{r_2^3} + \frac{3\mu\left(x+\mu-1\right)^2 q_2}{r_2^5} - \frac{3(1-\mu)q_1 A_1}{2r_1^5} \\ &+ \frac{15(1-\mu)\left(x+\mu\right)^2 q_1 A_1}{2r_1^7} + \frac{15(1-\mu)q_1 A_2}{8r_1^7} - \frac{105(1-\mu)\left(x+\mu\right)^2 q_1 A_2}{8r_1^9} - \frac{3\mu q_2 B_1}{2r_2^5} \\ &+ \frac{15\mu\left(x+\mu-1\right)^2 q_2 B_1}{2r_2^7} + \frac{15\mu q_2 B_2}{8r_2^7} - \frac{105\mu\left(x+\mu-1\right)^2 q_2 B_2}{8r_2^9} - \frac{M_b}{\left(r^2+T^2\right)^{3/2}} + \frac{3M_b x^2}{\left(r^2+T^2\right)^{5/2}}.\end{aligned} \tag{3.56}$$

Next, is the second partial derivative of Ω with respect to y *i.e.* Ω_{yy}.

Again from Equation (3.23) we have

$$\begin{aligned}\Omega_{yy} = n^2 &- \frac{(1-\mu)q_1}{r_1^3} - (1-\mu)q_1 y\frac{\partial}{\partial y}\left(\frac{1}{r_1^3}\right) - \frac{3(1-\mu)A_1 q_1}{2r_1^5} - \frac{3(1-\mu)A_1 q_1 y}{2}\frac{\partial}{\partial y}\left(\frac{1}{r_1^5}\right) + \frac{15(1-\mu)A_2 q_1}{8r_1^7} \\ &+ \frac{15(1-\mu)A_2 q_1 y}{8}\frac{\partial}{\partial y}\left(\frac{1}{r_1^7}\right) - \frac{\mu q_2}{r_2^3} - \mu q_2 y\frac{\partial}{\partial y}\left(\frac{1}{r_2^3}\right) - \frac{3\mu B_1 q_2}{2r_2^5} - \frac{3\mu B_1 q_2 y}{2}\frac{\partial}{\partial y}\left(\frac{1}{r_2^5}\right) \\ &+ \frac{15\mu B_2 q_2}{8r_2^7} + \frac{15\mu B_2 q_2 y}{8}\frac{\partial}{\partial y}\left(\frac{1}{r_2^7}\right) - \frac{M_b}{(r^2+T^2)^{3/2}} - M_b y\frac{\partial}{\partial y}\frac{1}{(r^2+T^2)^{3/2}}.\end{aligned}$$

We deduce the following from $r_1^2 = \left(x+\mu\right)^2 + y^2$ *and* $r_2^2 = \left(x+\mu-1\right)^2 + y^2$:

$$\frac{\partial}{\partial y} r_1^2 = \frac{\partial}{\partial y}\left((x+\mu)^2 + y^2\right)$$

$$2r_1\frac{\partial r_1}{\partial y} = 2y$$

$$\Rightarrow \frac{\partial r_1}{\partial y} = \frac{y}{r_1},$$

$$\frac{\partial}{\partial y} r_2^2 = \frac{\partial}{\partial y}\left((x+\mu-1)^2 + y^2\right)$$

$$2r_2\frac{\partial r_2}{\partial y} = 2y$$

$$\Rightarrow \frac{\partial r_2}{\partial y} = \frac{y}{r_2},$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r_1^3}\right)=-\frac{3}{r_1^4}\frac{\partial r_1}{\partial y} \qquad \frac{\partial}{\partial y}\left(\frac{1}{r_2^3}\right)=-\frac{3}{r_2^4}\frac{\partial r_2}{\partial y}$$

$$\Rightarrow\frac{\partial}{\partial y}\left(\frac{1}{r_1^3}\right)=-\frac{3y}{r_1^5}, \qquad \Rightarrow\frac{\partial}{\partial y}\left(\frac{1}{r_2^3}\right)=-\frac{3y}{r_2^5},$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r_1^5}\right)=-\frac{5}{r_1^6}\frac{\partial r_1}{\partial y} \qquad \frac{\partial}{\partial y}\left(\frac{1}{r_2^5}\right)=-\frac{5}{r_2^6}\frac{\partial r_2}{\partial y}$$

$$\Rightarrow\frac{\partial}{\partial y}\left(\frac{1}{r_1^5}\right)=-\frac{5y}{r_1^7}, \qquad \Rightarrow\frac{\partial}{\partial y}\left(\frac{1}{r_2^5}\right)=-\frac{5y}{r_2^7},$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r_1^7}\right)=-\frac{7}{r_1^8}\frac{\partial r_1}{\partial y} \qquad \frac{\partial}{\partial y}\left(\frac{1}{r_2^7}\right)=-\frac{7}{r_2^8}\frac{\partial r_2}{\partial y}$$

$$=-\frac{7y}{r_1^9}, \qquad =-\frac{7y}{r_2^9}.$$

And from $\frac{1}{\left(r^2+T^2\right)^{3/2}}$ with $r^2=x^2+y^2$ we obtain

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{1}{\left(r^2+T^2\right)^{3/2}}\right)&=-\frac{3}{2}\frac{1}{\left(r^2+T^2\right)^{5/2}}\frac{\partial r^2}{\partial y}\\&=-\frac{3}{2}\frac{1}{\left(r^2+T^2\right)^{5/2}}2y\\&=-\frac{3y}{\left(r^2+T^2\right)^{5/2}}.\end{aligned}$$

Thus, the second partial derivative Ω_{yy} simplifies to

$$\Omega_{yy} = n^2 - \frac{(1-\mu)q_1}{r_1^3} + \frac{3(1-\mu)y^2q_1}{r_1^5} - \frac{\mu q_2}{r_2^3} + \frac{3\mu y^2 q_2}{r_2^5} - \frac{3(1-\mu)q_1A_1}{2r_1^5} + \frac{15(1-\mu)y^2q_1A_1}{2r_1^7}$$
$$+ \frac{15(1-\mu)q_1A_2}{8r_1^7} - \frac{105(1-\mu)y^2q_1A_2}{8r_1^9} - \frac{3\mu q_2B_1}{2r_2^5} + \frac{15\mu y^2q_2B_1}{2r_2^7} + \frac{15\mu q_2B_2}{8r_2^7}$$
$$- \frac{105\mu y^2q_2B_2}{8r_2^9} - \frac{M_b}{\left(r^2+T^2\right)^{3/2}} + \frac{3M_b y^2}{\left(r^2+T^2\right)^{5/2}}.$$

(3.57)

In a similar manner, the second partial derivatives Ω_{xy} *and* Ω_{yx} in view of Equation (3.23) are obtain as

$$\Omega_{xy} = \Omega_{yx} = \frac{3(1-\mu)\left(x+\mu\right)yq_1}{r_1^5} + \frac{3\mu\left(x+\mu-1\right)yq_2}{r_2^5} + \frac{15(1-\mu)\left(x+\mu\right)yq_1A_1}{2r_1^7} - \frac{105(1-\mu)\left(x+\mu\right)yq_1A_2}{8r_1^9}$$
$$+ \frac{15\mu\left(x+\mu-1\right)yq_2B_1}{2r_2^7} - \frac{105\mu\left(x+\mu-1\right)yq_2B_2}{8r_2^9} + \frac{3M_b xy}{\left(r^2+T^2\right)^{5/2}}.$$

(3.58)

3.5.2 Characteristic equation

Consider trial solutions for the variational equations (3.54) to be in the form:

$$\eta = Ae^{\lambda t},\ \xi = Be^{\lambda t},$$

where *A*, *B* and λ are parameters to be determined.

Now,

$$\dot{\eta} = A\lambda e^{\lambda t} \qquad \dot{\xi} = B\lambda e^{\lambda t}$$

$$\ddot{\eta} = A\lambda^2 e^{\lambda t} \qquad \ddot{\xi} = B\lambda^2 e^{\lambda t}.$$

Substituting these values in Equation (3.54), we have

$$A\lambda^2 e^{\lambda t} - 2n\lambda Be^{\lambda t} = Ae^{\lambda t}\Omega_{xx}(x_0, y_0) + Be^{\lambda t}\Omega_{xy}(x_0, y_0),$$

$$B\lambda^2 e^{\lambda t} + 2n\lambda Ae^{\lambda t} = Ae^{\lambda t}\Omega_{yx}(x_0, y_0) + Be^{\lambda t}\Omega_{yy}(x_0, y_0).$$

Multiplying through by $e^{-\lambda t}$, we have

$$A\lambda^2 - 2n\lambda B = A\Omega_{xx}(x_0, y_0) + B\Omega_{xy}(x_0, y_0),$$

$$B\lambda^2 + 2n\lambda A = A\Omega_{yx}(x_0, y_0) + B\Omega_{yy}(x_0, y_0).$$

Or

$$A\lambda^2 - 2n\lambda B - A\Omega_{xx}(x_0, y_0) - B\Omega_{xy}(x_0, y_0) = 0,$$

$$B\lambda^2 + 2n\lambda A - A\Omega_{yx}(x_0, y_0) - B\Omega_{yy}(x_0, y_0) = 0.$$

Or

$$\left(\lambda^2 - \Omega_{xx}(x_0, y_0)\right)A + \left(-2n\lambda - \Omega_{xy}(x_0, y_0)\right)B = 0,$$

$$\left(2n\lambda - \Omega_{yx}(x_0, y_0)\right)A + \left(\lambda^2 - \Omega_{yy}(x_0, y_0)\right)B = 0.$$

These will have a non –trivial solution for *A* and *B* if

$$\begin{vmatrix} \lambda^2 - \Omega_{xx}(x_0, y_0) & -2n\lambda - \Omega_{xy}(x_0, y_0) \\ 2n\lambda - \Omega_{yx}(x_0, y_0) & \lambda^2 - \Omega_{yy}(x_0, y_0) \end{vmatrix} = 0.$$

Expanding the determinant yields

$$\left(\lambda^2 - \Omega_{xx}(x_0, y_0)\right)\left(\lambda^2 - \Omega_{yy}(x_0, y_0)\right) + \left(2n\lambda + \Omega_{xy}(x_0, y_0)\right)\left(2n\lambda - \Omega_{yx}(x_0, y_0)\right) = 0.$$

Or

$$\begin{aligned} &\lambda^4 - \Omega_{xx}(x_0, y_0)\lambda^2 - \Omega_{yy}(x_0, y_0)\lambda^2 + \Omega_{xx}(x_0, y_0)\Omega_{yy}(x_0, y_0) + 4n^2\lambda^2 \\ &\qquad - 2n\lambda\Omega_{yx}(x_0, y_0) + 2n\lambda\Omega_{xy}(x_0, y_0) - \Omega_{xy}(x_0, y_0)\Omega_{yx}(x_0, y_0) = 0. \end{aligned}$$

In view of Equation (3.58) we have

$$\lambda^4 - \Omega_{xx}(x_0, y_0)\lambda^2 - \Omega_{yy}(x_0, y_0)\lambda^2 + \Omega_{xx}(x_0, y_0)\Omega_{yy}(x_0, y_0) + 4n^2\lambda^2 - \Omega_{xy}(x_0, y_0)^2 = 0.$$

Or

$$\lambda^4 + (4n^2 - \Omega_{xx}(x_0, y_0) - \Omega_{yy}(x_0, y_0))\lambda^2 + \Omega_{xx}(x_0, y_0)\Omega_{yy}(x_0, y_0) - \Omega_{xy}(x_0, y_0)^2 = 0. \quad (3.59)$$

Equation (3.59) is the characteristic equation corresponding to the variational equations of motion (3.54).

3.5.3 Stability of triangular libration points

To establish the stability of the triangular points Equation (3.47), we evaluate the second partial derivatives equations (3.56), (3.57) and (3.58) at the triangular point, substitute the values of the second partial derivatives in the characteristic equation (3.59), then solve the characteristic equation.

Now, the partial derivatives Equations (3.56), (3.57) and (3.58) computed at any of the triangular libration points (Equation 3.47), neglecting second and higher order terms of p_i, B_i, A_i, M_b (i=1,2) and their products are:

$$\begin{aligned}
\Omega_{xx}(x_0, y_0) &= \frac{3}{4} + a_1 + \mu b_1 + \frac{5M_b(2r_c - 1)}{4(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{1}{4} - \mu + \mu^2)}{(r_c^2 + T^2)^{5/2}}, \\
\Omega_{yy}(x_0, y_0) &= \frac{9}{4} + a_2 + \mu b_2 + \frac{7M_b(2r_c - 1)}{4(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{3}{4})}{(r_c^2 + T^2)^{5/2}}, \\
\Omega_{xy}(x_0, y_0) &= \Omega_{yx}(x_0, y_0) = \sqrt{3}\left\{ \begin{aligned} &\frac{3}{4} + a_3 + \mu\left\langle b_3 - \tfrac{3}{2} - \frac{11M_b(2r_c - 1)}{6(r_c^2 + T^2)^{3/2}} \right\rangle \\ &+ \frac{11M_b(2r_c - 1)}{12(r_c^2 + T^2)^{3/2}} + \frac{\frac{3}{2}M_b(\frac{1}{2} - \mu)}{(r_c^2 + T^2)^{5/2}} \end{aligned} \right\},
\end{aligned} \tag{3.60}$$

where

$$a_1 = -\frac{p_1}{2} + p_2 + \frac{27A_1}{8} - \frac{165A_2}{32} + \frac{3B_1}{8} - \frac{15B_2}{32}, \ b_1 = \frac{3p_1}{2} - \frac{3p_2}{2} - 3A_1 + \frac{75A_2}{16} + 3B_1 - \frac{75B_2}{16},$$

$$a_2 = \frac{p_1}{2} - p_2 + \frac{33A_1}{8} - \frac{255A_2}{32} + \frac{33B_1}{8} - \frac{165B_2}{32}, \ b_2 = -\frac{3p_1}{2} + \frac{3p_2}{2} + \frac{45A_2}{16} - \frac{45B_2}{16},$$

$$a_3 = -\frac{p_1}{6} + \frac{p_2}{3} + \frac{19A_1}{8} - \frac{125A_2}{32} + \frac{7B_1}{8} - \frac{35B_2}{32}, \ b_3 = -\frac{p_1}{6} - \frac{p_2}{6} - \frac{13A_1}{4} + 5A_2 - \frac{13B_1}{4} + 5B_2.$$

Here each of $|a_k|$, $|b_k|$ $(k=1,2,3)$ is very small since $|A_i| << 1$, $|B_i| << 1$, $|p_i| << 1$ $(i=1,2)$. Utilizing the values of $\Omega_{xx}(x_0, y_0)$, $\Omega_{yy}(x_0, y_0)$, $\Omega_{xy}(x_0, y_0)$ and the value of n^2 from (3.25) in the characteristic equation (3.59) yield

$$\Lambda^2 + Q\Lambda + N = 0, \tag{3.61}$$

where

$$\Lambda = \lambda^2,$$

$$Q = 1 + 6(A_1 + B_1) - \frac{15(A_2 + B_2)}{2} - (a_1 + a_2) - \mu(b_1 + b_2) + \frac{M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{3M_b r_c^2}{(r_c^2 + T^2)^{5/2}} > 0,$$

$$\begin{aligned} N = & \left(-\frac{27}{4} + 9b_3 - \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} - \frac{27M_b}{4(r_c^2 + T^2)^{5/2}} \right)\mu^2 \\ & + \left(\frac{27}{4} + \frac{9b_1}{4} + \frac{3b_2}{4} - \frac{9b_3}{2} + 9a_3 + \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{4(r_c^2 + T^2)^{5/2}} \right)\mu \\ & + \frac{9a_1}{4} + \frac{3a_2}{4} - \frac{9a_3}{2}. \end{aligned}$$

The roots of Equation (3.61) are:

$$\Lambda = \frac{1}{2}\left[-Q \pm \sqrt{\Delta} \right],$$

where $\Delta = Q^2 - 4N$ is called the discriminant, which is given by

$$\Delta=\left(27-36b_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)\mu^2-\left(27+11b_1+5b_2-18b_3+36a_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)\mu$$
$$+1+12\left[A_1+B_1-\frac{5(A_2+B_2)}{4}\right]-11a_1-5a_2+18a_3+\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_br_c^2}{(r_c^2+T^2)^{5/2}}. \quad (3.62)$$

The nature of the roots

$$\lambda_{1,2}=\pm\sqrt{\Lambda_1},\ \lambda_{3,4}=\pm\sqrt{\Lambda_2} \quad (3.63)$$

depends on the discriminant Δ which is a quadratic function of the mass parameter μ. Here, in our problem $0<\mu<\frac{1}{2}$ so, we are interested in the behavior of Δ in the interval $(0,\frac{1}{2})$.

In the absence of the perturbations ($i.e.\ p_i=A_i=B_i=M_b=0$, i=1,2), the discriminant Δ boils down to the classical case of Szebehely (1967) $\Delta=27\mu^2-27\mu+1$, with the roots $\mu=\frac{1}{2}\pm\frac{\sqrt{69}}{18}$ ($\mu=\frac{1}{2}-\frac{\sqrt{69}}{18}=0.0385209...$ or $\mu=\frac{1}{2}+\frac{\sqrt{69}}{18}=0.961479...$).

Now, using Equation (3.62) we find that when $\mu=0$,

$$\Delta=1+\left(12(A_1+B_1)-15(A_2+B_2)-11a_1-5a_2+18a_3+\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_br_c^2}{(r_c^2+T^2)^{5/2}}\right) \quad (3.64)$$
$$>0,$$

and when $\mu=1/2$

$$\Delta=\left(27-36b_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)\frac{1}{4}-\left(27+11b_1+5b_2-18b_3+36a_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)\frac{1}{2}$$
$$+1+12\left[A_1+B_1-\frac{5(A_2+B_2)}{4}\right]-11a_1-5a_2+18a_3+\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_br_c^2}{(r_c^2+T^2)^{5/2}}.$$

$$=\frac{27}{4}-9b_3+\frac{33M_b(2r_c-1)}{2(r_c^2+T^2)^{3/2}}+\frac{27M_b}{4(r_c^2+T^2)^{5/2}}-\frac{27}{2}-\frac{11b_1}{2}-\frac{5b_2}{2}+9b_3-18a_3-\frac{33M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}-\frac{27M_b}{2(r_c^2+T^2}$$

$$+1+12(A_1+B_1)-15(A_2+B_2)-11a_1-5a_2+18a_3+\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_b r_c^2}{(r_c^2+T^2)^{5/2}}.$$

$$\therefore(\Delta)_{\mu=\frac{1}{2}}=-\frac{23}{4}+\left\langle\begin{array}{l}12(A_1+B_1)-15(A_2+B_2)-11a_1-5a_2-\frac{11b_1}{2}-\frac{5b_2}{2}\\ -\frac{M_b(58r_c-45)}{2(r_c^2+T^2)^{3/2}}-\frac{3M_b(8r_c^{\,2}+9)}{4(r_c^2+T^2)^{5/2}}\end{array}\right\rangle$$

$$<0. \tag{3.65}$$

Also, when μ=1,

$$\Delta=\left(27-36b_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)$$

$$-\left(27+11b_1+5b_2-18b_3+36a_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)$$

$$+1+12(A_1+B_1)-15(A_2+B_2)-11a_1-5a_2+18a_3+\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_b r_c^2}{(r_c^2+T^2)^{5/2}}.$$

$$\Delta_{\mu=1}=1+\left\langle\begin{array}{l}12(A_1+B_1)-15(A_2+B_2)-11(a_1+b_1)-5(a_2+b_2)-18(a_3+b_3)\\ +\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_b r_c^2}{(r_c^2+T^2)^{5/2}}\end{array}\right\rangle$$

$$>0. \tag{3.66}$$

Now, since Δ is a polynomial we conclude with respect to Equations (3.64), (3.65) and (3.66) that there is only one value of μ in the interval (0, 1/2)for which $\Delta=0$. This value of μ for which the discriminant vanishes is termed the critical value of mass parameter represented as μ_c. Thus, three regions of values of μ can be considered: $0<\mu<\mu_c$, $\mu_c<\mu<\frac{1}{2}$ *and* $\mu=\mu_c$.

Region 1: When $0 < \mu < \mu_c$ implies $\Delta = Q^2 - 4N > 0$ and since Δ is decreasing in the interval $(0, \frac{1}{2})$ suggests $N > 0.$ Thus,

$Q^2 - 4N < Q^2 \Rightarrow \sqrt{Q^2 - 4N} < Q$ [since $Q > 0$ Equation (3.61)]

$$\Rightarrow \Lambda = \frac{1}{2}\left[-Q \pm \sqrt{\Delta}\right] < 0.$$

And consequently the four values of λ in Equation (3.63)are distinct pure imaginary numbers. Hence, in view of Section 1.8.3, the triangular point is stable in this region.

Region2: when $\mu_c < \mu < \frac{1}{2}$ in this region, the discriminant of the characteristic equation is negative *i.e.* $\Delta = Q^2 - 4N < 0.$

Therefore,

$$\Lambda = \frac{-Q \pm i\sqrt{|\Delta|}}{2} \text{ , where } i = \sqrt{-1}.$$

Or

$$\Lambda_1 = \frac{-Q + i\delta}{2}, \qquad \Lambda_2 = \frac{-Q - i\delta}{2}$$

where,

$$\delta = \left|\sqrt{|\Delta|}\right|.$$

Thus, the roots (3.64) of the characteristic equation (3.61) are:

$$\lambda_{1,2} = \pm\sqrt{\frac{-Q+i\delta}{2}}, \qquad \lambda_{3,4} = \pm\sqrt{\frac{-Q-i\delta}{2}}.$$

These indicate that the real parts of two of the characteristic roots are positive; hence by Section 1.8.3, the triangular point is unstable in this region.

Region3: when $\mu = \mu_c$ in this region, the discriminant of the characteristic equation is zero i.e. *the value of* Δ *is zero* $\Rightarrow \sqrt{\Delta} = 0.$

And so, we have

$$\Lambda = \frac{-Q \pm 0}{2}.$$

Or

$$\Lambda_1 = \Lambda_2 = \frac{-Q}{2}.$$

The roots (3.64) of the characteristic equation (3.61) now become:

$$\lambda_{1,2} = \pm\sqrt{\frac{-Q}{2}} \qquad \lambda_{3,4} = \pm\sqrt{\frac{-Q}{2}}$$

or

$$\lambda_1 = \lambda_3 = i\sqrt{\frac{Q}{2}} \; \lambda_2 = \lambda_4 = -i\sqrt{\frac{Q}{2}}.$$

Thus we have double roots and so, in view of Section 1.8.3 the triangular point is unstable when $\mu = \mu_c$.

3.5.3.1 *The critical mass ratio* μ_c

The critical value μ_c of the mass parameter is the value of the mass ratio μ when the discriminant (3.62) is equal to zero. That is,

$$\left(27-36b_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)\mu^2-\left(27+11b_1+5b_2-18b_3+36a_3+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)\mu$$
$$+1+12\left[A_1+B_1-\frac{5(A_2+B_2)}{4}\right]-11a_1-5a_2+18a_3+\frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}-\frac{6M_br_c^2}{(r_c^2+T^2)^{5/2}}=0.$$

Hence, the mass ratio $\mu_c$is given by

$$\mu_c=\frac{27+11b_1+5b_2-18b_3+36a_3+66M_3+27M_4}{2\left(27-36b_3+66M_3+27M_4\right)}\pm\frac{1}{2}\sqrt{\frac{d_2^2-4d_1d_3}{d_1^2}},$$

where

$$d_1=27-36b_3+66M_3+27M_4,$$
$$d_2=-(27+11b_1+5b_2-18b_3+36a_3+66M_3+27M_4),$$
$$d_3=1+12\left(A_1+B_1\right)-15(A_2+B_2)-11a_1-5a_2+18a_3+2M_1-6M_2;$$

and

$$M_1=\frac{M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}},\qquad M_2=\frac{M_br_c^2}{(r_c^2+T^2)^{5/2}},$$
$$M_3=\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}},\qquad M_4=\frac{M_b}{(r_c^2+T^2)^{5/2}}.$$

$$\mu_c=\frac{27+11b_1+5b_2-18b_3+36a_3+66M_3+27M_4}{54\left(1-\frac{1}{27}\left(36b_3-66M_3-27M_4\right)\right)}\pm\frac{1}{2}\sqrt{\frac{d_2^2-4d_1d_3}{d_1^2}}.$$

$$\mu_c = \frac{\left(27+11b_1+5b_2-18b_3+36a_3+66M_3+27M_4\right)\left(1-\frac{1}{27}\left(36b_3-66M_3-27M_4\right)\right)^{-1}}{54}$$

$$\pm\frac{1}{2}\sqrt{\frac{d_2^2-4d_1d_3}{d_1^2}}.$$

Employing binomial expansion and retaining only linear terms in very small values, we obtain

$$\mu_c = \frac{27+11b_1+5b_2-18b_3+36a_3+66M_3+27M_4+36b_3-66M_3-27M_4}{54}\pm\frac{1}{2}\sqrt{\frac{d_2^2-4d_1d_3}{d_1^2}}.$$

$$= \frac{27+11b_1+5b_2+18b_3+36a_3}{54}\pm\frac{1}{2}\sqrt{\frac{d_2^2-4d_1d_3}{d_1^2}}.$$

(3.67)

Now,

$$\frac{d_2^2-4d_1d_3}{d_1^2} = \frac{\left(27+11b_1+5b_2-18b_3+36a_3+66M_3+27M_4\right)^2}{\left(27-36b_3+66M_3+27M_4\right)^2}-\frac{4d_3}{d_1}.$$

$$= \frac{27^2+2(27)\left(11b_1+5b_2-18b_3+36a_3+66M_3+27M_4\right)}{27^2-2\left(27\right)\left(36b_3-66M_3-27M_4\right)}-\frac{4d_3}{27-36b_3+66M_3+27M_4}.$$

$$= \frac{\left[27^2+54\left(11b_1+5b_2-18b_3+36a_3+66M_3+27M_4\right)\right]}{27^2\left[1-\frac{2}{27}\left(36b_3-66M_3-27M_4\right)\right]}-\frac{4d_3}{27\left[1-\frac{1}{27}\left(36b_3-66M_3-27M_4\right)\right]}.$$

$$= \frac{\left[27^2+54\left(11b_1+5b_2-18b_3+36a_3+66M_3+27M_4\right)\right]\left[1-\frac{2}{27}\left(36b_3-66M_3-27M_4\right)\right]^{-1}}{27^2}$$

$$-\frac{4\left(1+12\left(A_1+B_1\right)-15(A_2+B_2)-11a_1-5a_2+18a_3+2M_1-6M_2\right)\left[1-\frac{1}{27}\left(36b_3-66M_3-27M_4\right)\right]^{-1}}{27}.$$

Applying binomial expansion and taking into consideration only linear terms in small quantities, we get

$$\frac{d_2^2-4d_1d_3}{d_1^2}=\frac{\left[27^2+54\left(11b_1+5b_2-18b_3+36a_3+66M_3+27M_4\right)\right]\left[1+\frac{2}{27}\left(36b_3-66M_3-27M_4\right)\right]}{27^2}$$

$$-\frac{4\left(1+12\left(A_1+B_1\right)-15(A_2+B_2)-11a_1-5a_2+18a_3+2M_1-6M_2\right)\left[1+\frac{1}{27}\left(36b_3-66M_3-27M_4\right)\right]}{27}.$$

$$=\frac{27^2+54\left(11b_1+5b_2-18b_3+36a_3+66M_3+27M_4\right)+54\left(36b_3-66M_3-27M_4\right)}{27^2}$$

$$-\frac{4+4\left(12\left(A_1+B_1\right)-15(A_2+B_2)-11a_1-5a_2+18a_3+2M_1-6M_2\right)+\frac{4}{27}\left(36b_3-66M_3-27M_4\right)}{27}.$$

$$=\frac{27^2+54\left(11b_1+5b_2+18b_3+36a_3\right)}{27^2}-\frac{4}{27}-\frac{4\left(12\left(A_1+B_1\right)-15(A_2+B_2)-11a_1-5a_2+18a_3+2M_1-6M_2\right)}{27}$$

$$-\frac{4}{27^2}\left(36b_3-66M_3-27M_4\right).$$

$$=\frac{27^2\left[1+\frac{2}{27}\left(11b_1+5b_2+18b_3+36a_3\right)\right]}{27^2}-\frac{4}{27}-\frac{4\left(12\left(A_1+B_1\right)-15(A_2+B_2)-11a_1-5a_2+18a_3+2M_1-6M_2\right)}{27}$$

$$-\frac{4}{27^2}\left(36b_3-66M_3-27M_4\right).$$

$$=1+\frac{2\left(11b_1+5b_2+18b_3+36a_3\right)}{27}-\frac{4}{27}-\frac{4\left(12\left(A_1+B_1\right)-15(A_2+B_2)-11a_1-5a_2+18a_3+2M_1-6M_2\right)}{27}$$

$$-\frac{4}{27^2}\left(36b_3-66M_3-27M_4\right).$$

$$=\frac{23}{27}+\frac{2}{27}\left[\begin{array}{l}\left(11b_1+5b_2+18b_3+36a_3\right)-2\left(12\left(A_1+B_1\right)-15(A_2+B_2)-11a_1-5a_2+18a_3+2M_1-6M_2\right)\\-\frac{2}{27}\left(36b_3-66M_3-27M_4\right)\end{array}\right].$$

$$=\frac{23}{27}+\frac{2}{27}\left[11b_1+5b_2+\frac{46b_3}{3}-24\left(A_1+B_1\right)+30(A_2+B_2)+22a_1+10a_2-4M_1+12M_2+\frac{44M_3}{9}+2M_4\right].$$

(3.68)

Thus, plugging Equation (3.68) in Equation (3.67), yields

$$\mu_c = \frac{27+11b_1+5b_2+18b_3+36a_3}{54} \pm \frac{1}{2}\sqrt{\frac{23}{27}+\frac{2}{27}e_1},$$

where

$$e_1 = 11b_1+5b_2+\frac{46b_3}{3}-24\left(A_1+B_1\right)+30(A_2+B_2)+22a_1+10a_2-4M_1+12M_2+\frac{44M_3}{9}+2M_4.$$

$$\mu_c = \frac{27+11b_1+5b_2+18b_3+36a_3}{54} \pm \frac{1}{2}\sqrt{\frac{23}{27}\left(1+\frac{27}{23}\frac{2}{27}e_1\right)}.$$

$$= \frac{27+11b_1+5b_2+18b_3+36a_3}{54} \pm \frac{1}{2}\sqrt{\frac{23}{27}}\sqrt{\left(1+\frac{2}{23}e_1\right)}.$$

$$= \frac{27+11b_1+5b_2+18b_3+36a_3}{54} \pm \frac{1}{2}\sqrt{\frac{23}{27}}\left(1+\frac{2}{23}e_1\right)^{\frac{1}{2}}.$$

Using binomial expansion and considering only linear term in e_1, we obtain

$$\mu_c = \frac{27+11b_1+5b_2+18b_3+36a_3}{54} \pm \frac{1}{2}\sqrt{\frac{23}{27}}\left[1+\frac{1}{2}\frac{2}{23}e_1\right].$$

$$= \frac{27+11b_1+5b_2+18b_3+36a_3}{54} \pm \frac{1}{2}\sqrt{\frac{23}{27}}\left[1+\frac{1}{23}e_1\right].$$

We consider only the minus sign since the mass ratio is less than ½.

$$\begin{aligned}\mu_c &= \frac{1}{2}+\frac{11b_1+5b_2+18b_3+36a_3}{54}-\frac{1}{2}\sqrt{\frac{23}{27}}-\frac{1}{2}\sqrt{\frac{23}{27}}\,\frac{1}{23}e_1.\\
&= \frac{1}{2}+\frac{11b_1+5b_2+18b_3+36a_3}{54}-\frac{1}{2}\sqrt{\frac{23}{27}}-\frac{1}{2}\frac{\sqrt{23}}{\sqrt{27}}\,\frac{1}{\sqrt{23}^{2}}\,\frac{9}{9}e_1.\\
&= \frac{1}{2}+\frac{11b_1+5b_2+18b_3+36a_3}{54}-\frac{1}{2}\sqrt{\frac{23}{27}}-\frac{1}{2}\frac{1}{\sqrt{27}}\,\frac{1}{\sqrt{23}}\,\frac{9}{\sqrt{3}\times\sqrt{3}\times\sqrt{9}}e_1.\\
&= \frac{1}{2}+\frac{11b_1+5b_2+18b_3+36a_3}{54}-\frac{1}{2}\sqrt{\frac{23}{27}}-\frac{1}{2}\frac{1}{\sqrt{27}}\,\frac{1}{\sqrt{69}}\,\frac{9}{\sqrt{27}}e_1.\\
&= \frac{1}{2}+\frac{11b_1+5b_2+18b_3+36a_3}{54}-\frac{1}{2}\sqrt{\frac{23}{27}}-\frac{1}{2}\frac{9}{27\sqrt{69}}e_1.\\
&= \frac{1}{2}+\frac{11b_1+5b_2+18b_3+36a_3}{54}-\frac{1}{2}\sqrt{\frac{23}{27}}-\frac{1}{6\sqrt{69}}e_1.\end{aligned}$$

Substituting the value of e_1, produces

$$\begin{aligned}\mu_c = \frac{1}{2}-\frac{1}{2}\sqrt{\frac{23}{27}}+\frac{11b_1+5b_2+18b_3+36a_3}{54}-\frac{1}{6\sqrt{69}}\Bigg[&11b_1+5b_2+\frac{46b_3}{3}-24\left(A_1+B_1\right)+30(A_2+B_2)\\
&+22a_1+10a_2-4M_1+12M_2+\frac{44M_3}{9}+2M_4\Bigg].\end{aligned}$$

$$\begin{aligned}= \frac{1}{2}-\frac{1}{2}\sqrt{\frac{23}{27}}+\frac{11b_1+5b_2+18b_3+36a_3}{54}-\frac{1}{6\sqrt{69}}\Bigg[&11b_1+5b_2+\frac{46b_3}{3}-24\left(A_1+B_1\right)+30(A_2+B_2)\\
&+22a_1+10a_2-\frac{4M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}+\frac{12M_br_c^2}{(r_c^2+T^2)^{5/2}}+\frac{44}{9}\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{2M_b}{(r_c^2+T^2)^{5/2}}\Bigg].\end{aligned}$$

In view of the definitions of $a_i, b_i\left(i=1,2,3\right)$ Equation (3.60), we arrive at

$$\mu_c = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{23}{27}} - \frac{2p_1}{27\sqrt{69}} - \frac{2p_2}{27\sqrt{69}} - \frac{A_1}{9} - \frac{13A_1}{9\sqrt{69}} + \frac{5A_2}{18} + \frac{125A_2}{36\sqrt{69}} + \frac{B_1}{9} - \frac{13B_1}{\sqrt{69}}$$

$$- \frac{5B_2}{18} + \frac{125B_2}{36\sqrt{69}} + \left[\frac{2(2r_c+3)}{3(r_c^2+T^2)^{3/2}} - \frac{2r_c^2}{(r_c^2+T^2)^{5/2}} - \frac{22(2r_c-1)}{27(r_c^2+T^2)^{3/2}} - \frac{1}{3(r_c^2+T^2)^{5/2}}\right]\frac{M_b}{\sqrt{69}}.$$

$$= \frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right) - 2\frac{\left(p_1+p_2\right)}{27\sqrt{69}} - \frac{1}{9}\left(1+\frac{13}{\sqrt{69}}\right)A_1 + \frac{5}{18}\left(1+\frac{25}{2\sqrt{69}}\right)A_2 + \frac{1}{9}\left(1-\frac{13}{\sqrt{69}}\right)B_1$$

$$- \frac{5}{18}\left(1-\frac{25}{2\sqrt{69}}\right)B_2 + \left[\frac{76-8r_c}{27(r_c^2+T^2)^{3/2}} - \frac{6r_c^2+1}{(r_c^2+T^2)^{5/2}}\right]\frac{M_b}{\sqrt{69}}.$$

$$= \frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right) - 2\frac{\left(p_1+p_2\right)}{27\sqrt{69}} - \frac{1}{9}\left(1+\frac{13}{\sqrt{69}}\right)A_1 + \frac{5}{18}\left(1+\frac{25}{2\sqrt{69}}\right)A_2 + \frac{1}{9}\left(1-\frac{13}{\sqrt{69}}\right)B_1$$

$$- \frac{5}{18}\left(1-\frac{25}{2\sqrt{69}}\right)B_2 + \left[\frac{4\left(19-2r_c\right)\left(r_c^2+T^2\right)^{5/2} - 9(1+6r_c^2)\left(r_c^2+T^2\right)^{3/2}}{27\sqrt{69}\left(r_c^2+T^2\right)^4}\right]M_b. \qquad (3.69)$$

3.5.4 Stability of collinear libration points

As in the case of the triangular libration points, to investigate the stability of any collinear libration point obtained in Table 3.1 we:

- evaluate the second partial derivatives Equations (3.56), (3.57) and (3.58) at the collinear point;
- substitute the values of the second partial derivatives in the characteristic equation (3.59);
- then determine the roots of the characteristic equation .

Now, for any collinear libration point (x_0, 0) of Table 3.1, we have

$$\Omega_{xx}(x_0,0)=n^2+\frac{2(1-\mu)q_1}{|x_0+\mu|^3}+\frac{2\mu q_2}{|x_0+\mu-1|^3}+\frac{6(1-\mu)A_1q_1}{|x_0+\mu|^5}-\frac{45(1-\mu)A_2q_1}{4|x_0+\mu|^7}+\frac{6\mu B_1q_2}{|x_0+\mu-1|^5}$$
$$-\frac{45\mu B_2q_2}{4|x_0+\mu-1|^7}+\left(\frac{3{x_0}^2}{({x_0}^2+T^2)^{5/2}}-\frac{1}{({x_0}^2+T^2)^{3/2}}\right)M_b, \tag{3.70}$$

$$\Omega_{yy}(x_0,0)=n^2-\frac{(1-\mu)q_1}{|x_0+\mu|^3}-\frac{3(1-\mu)A_1q_1}{2|x_0+\mu|^5}+\frac{15(1-\mu)A_2q_1}{8|x_0+\mu|^7}-\frac{\mu q_2}{|x_0+\mu-1|^3}-\frac{3\mu B_1q_2}{2|x_0+\mu-1|^5}$$
$$+\frac{15\mu B_2q_2}{8|x_0+\mu-1|^7}-\frac{M_b}{({x_0}^2+T^2)^{3/2}}, \tag{3.71}$$

$$\Omega_{xy}(x_0,0)=\Omega_{yx}(x_0,0)=0. \tag{3.72}$$

Substituting Equations (3.70), (3.71), (3.72) and (3.22) in Equation (3.59), the characteristic equation becomes

$$\lambda^4+W\lambda^2+C=0, \tag{3.73}$$

where $W=4n^2-\Omega^0_{xx}-\Omega^0_{yy}$, $C=\Omega^0_{xx}\Omega^0_{yy}$.

The roots of the characteristic equation (3.73) for L_i (i = 1, 2,3), L_{nj} *(j=1,2,3,4)* of Table 3.1 are given in the Tables 3.2 to 3.8 respectively.

Table 3.2: Stability of the libration point L_1

Case	L_1	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	1.230814	4.4637	-0.7319	±1.2955	±1.3951i	Unstable
2	1.225145	4.5913	-0.7562	±1.3165	±1.4153i	Unstable
3	1.221462	4.6701	-0.7723	±1.3304	±1.4274i	Unstable
4	1.223870	4.6198	-0.7616	±1.3213	±1.4196i	Unstable
5	1.229484	4.7795	-0.7671	±1.3564	±1.4116i	Unstable
6	1.218052	4.3278	-0.7672	±1.2492	±1.4586i	Unstable
7	1.224156	4.6018	-0.7614	±1.3207	±1.4174i	Unstable
8	1.213816	4.6092	-0.7652	±1.3236	±1.4188i	Unstable
9	1.203598	4.3446	-0.7885	±1.2560	±1.4736	Unstable

Table 3.3: Stability of the libration point L_2

Case	L_2	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	0.141618	16.8590	-6.9295	±3.7646	±2.8711i	Unstable
2	0.163243	21.8904	-9.4185	±4.3748	±3.2821i	Unstable
3	0.167120	22.3356	-9.4638	±4.4193	±3.2899i	Unstable
4	0.159016	20.8264	-9.3745	±4.2552	±3.2837i	Unstable
5	0.156143	23.7489	-9.9802	±4.5748	±3.3653i	Unstable
6	0.190087	11.8785	-7.7356	±3.0973	±3.0949i	Unstable
7	0.161508	21.8507	-9.3994	±4.3703	±3.2792i	Unstable
8	0.168169	21.1677	-9.0554	± 4.2913	±3.2262i	Unstable
9	0.189788	10.8703	-7.3599	±2.9377	±3.0447i	Unstable

Table 3.4: Stability of the libration point L_3

Case	L_3	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	-1.162045	3.8579	-0.4290	±1.0160	±1.2662i	Unstable
2	-1.156152	3.9625	-0.4418	± 1.0309	±1.2835i	Unstable
3	-1.158632	4.0930	-0.4487	±1.0581	±1.2808i	Unstable
4	-1.153827	3.8985	-0.4457	±1.0115	±1.3032i	Unstable
5	-1.151977	4.0267	-0.4510	±1.0415	±1.2939i	Unstable
6	-1.154651	3.9862	-0.4450	±1.0346	±1.2872i	Unstable
7	-1.150821	3.9669	-0.4440	±1.0332	±1.2845i	Unstable
8	-1.154516	3.9737	-0.4474	±1.0368	±1.2860	Unstable
9	-1.145168	3.9313	-0.4571	±1.0220	±1.3117	Unstable

Table 3.5: Stability of the libration point L_{n1}

Case	L_{n1}	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	-	-	-	-	-	-
2	-0.000265	-9944.9	-9999.7	±98.8089i	±100.9248i	Unstable
3	-0.000300	-9932.5	-9997.6	±98.8397	±100.8202i	Unstable
4	-0.000162	-9981.8	-10004.0	±98.9081	±101.0306	Unstable
5	-0.000260	-9945.8	-10000.0	±64.2677	±155.1784	Unstable
6	-0.000268	-9944.7	-9999.4	± 98.8676i	±100.8622i	Stable
7	-0.000257	-9947.2	-10000.0	±98.9019i	±100.8435i	Stable
8	-0.000270	-9943.9	-9999.2	±98.8648i	±100.8601i	Stable
9	-0.000165	-9982.1	-10003.0	±99.0054i	±100.9303i	Stable

Table 3.6: Stability of the libration point L_{n2}

Case	L_{n2}	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	-	-	-	-	-	-
2	-0.048710	184.4310	-95.6098	± 13.4811	$\pm 9.8501i$	Unstable
3	-0.045812	218.2288	-112.5548	± 14.6797	$\pm 10.6764i$	Unstable
4	-0.704629	-81.0144	-0.7261	$\pm 0.8310i$	$\pm 9.2292i$	Stable
5	-0.048891	183.1413	-94.8033	± 13.4317	$\pm 9.8100i$	Unstable
6	-0.048620	184.9400	-96.0029	± 13.4995	$\pm 9.8706i$	Unstable
7	-0.049246	179.4529	-92.8683	± 13.2951	$\pm 9.7098i$	Unstable
8	-0.048454	186.4604	-96.7508	± 13.5562	$\pm 9.9079i$	Unstable
9	-0.704932	-78.6235	-0.7597	$\pm 0.8494i$	$\pm 9.1001i$	Stable

Table 3.7: Stability of the libration point L_{n3}

Case	L_{n3}	Ω_{xx}^{0}	Ω_{yy}^{0}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	-	-	-	-	-	-
2	-	-	-	-	-	-
3	-	-	-	-	-	-
4	-	-	-	-	-	-
5	-	-	-	-	-	-
6	0.909793	-45.2527	-1.1110	$\pm 1.0080i$	$\pm 7.0339i$	Stable
7	-	-	-	-	-	-
8	-	-	-	-	-	-
9	0.910718	-41.6463	-1.1330	$\pm 1.0139i$	$\pm 6.7751i$	Stable

Table 3.8: Stability of the libration point L_{n4}

Case	L_{n4}	Ω_{xx}^{0}	Ω_{yy}^{0}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	-	-	-	-	-	-
2	-	-	-	-	-	-
3	-	-	-	-	-	-
4	-	-	-	-	-	-
5	-	-	-	-	-	-
6	0.273997	-21.9676	-5.0048	$\pm 2.0167i$	$\pm 5.1993i$	Stable
7	-	-	-	-	-	-
8	-	-	-	-	-	-
9	0.273293	-20.2882	-4.8151	$\pm 1.9608i$	$\pm 5.0407i$	Stable

3.6 Periodic Orbits around the Triangular Libration Points

The orbits around triangular libration points for the mass ratio $\mu \in (0, \mu_c)$ are bounded,but unbounded in the interval $\mu_c \le \mu \le 1/2$, where $\mu_c \in (0, 1/2)$ is the critical mass ratio influenced by the oblateness and radiation of the primaries and potential from the belt. We shall study the nature of these orbits.

3.6.1 The frequencies of the periodic orbits

As shown in Section 3.5.3, the triangular libration points Equation (3.47) are linearly stable in the region $0 < \mu < \mu_c$ and the characteristic equation

$$\lambda^4 + Q\lambda^2 + N = 0,$$

where

$$Q = 1 + 3(\mu - \frac{1}{2})A_1 + \frac{15}{2}(\frac{3}{4} - \mu)A_2 + 3(\frac{1}{2} - \mu)B_1 + \frac{15}{2}(\mu - \frac{1}{4})B_2 + \frac{M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{3M_b r_c^2}{(r_c^2 + T^2)^{5/2}},$$

$$\begin{aligned} N = &\left(\frac{27}{4} + \frac{3p_1}{2} + \frac{3p_2}{2} + \frac{117A_1}{4} - 45A_2 + \frac{117B_1}{4} - 45B_2 + \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{4(r_c^2 + T^2)^{5/2}} \right)\mu \\ &- \left(\frac{27}{4} + \frac{3p_1}{2} + \frac{3p_2}{2} + \frac{117A_1}{4} - 45A_2 + \frac{117B_1}{4} - 45B_2 + \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{4(r_c^2 + T^2)^{5/2}} \right)\mu^2, \end{aligned}$$

has four pure imaginary roots. The roots can be expressed as

$$\lambda_{1,2} = \pm i s_1, \; \lambda_{3,4} = \pm i s_2, \tag{3.74}$$

with

$$s_1 = \sqrt{\frac{Q - \sqrt{\Delta}}{2}}, \; s_2 = \sqrt{\frac{Q + \sqrt{\Delta}}{2}} \tag{3.75}$$

and

$$\Delta = \left(27 + 6p_1 + 6p_2 + 117A_1 - 180A_2 + 117B_1 - 180B_2 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right)\mu^2$$
$$- \left(27 + 6p_1 + 6p_2 + 111A_1 - 165A_2 + 123B_1 - 195B_2 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right)\mu$$
$$+ 1 - 3A_1 + \frac{75A_2}{4} + 3B_1 + \frac{15B_2}{4} + \frac{2M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{6M_b r_c^2}{(r_c^2 + T^2)^{5/2}}.$$

Hence, the motion around the triangular points is bounded and is composed of two harmonic motions given by the variations η and ξ as:

$$\begin{aligned} \eta &= D_1 \cos s_1 t + E_1 \sin s_1 t + D_2 \cos s_2 t + E_2 \sin s_2 t, \\ \xi &= \bar{D}_1 \cos s_1 t + \bar{E}_1 \sin s_1 t + \bar{D}_2 \cos s_2 t + \bar{E}_2 \sin s_2 t, \end{aligned} \tag{3.76}$$

where $s_1 \; and \; s_2$ are the frequencies of the orbits, $D_1, E_1, \bar{D}_1, \bar{E}_1$ are terms associated to s_1 while $D_2, E_2, \bar{D}_2, \bar{E}_2$ are terms associated with s_2.

D_1, E_1, D_2, E_2 are arbitrary constants of integration and $\bar{D}_1, \bar{E}_1, \bar{D}_2, \bar{E}_2$ are related to the constants of integration. In order to determine these relationships, we substitute Equation (3.76) in Equation (3.54) and then equate coefficients of $\cos s_i, \sin s_i \; (i = 1, 2)$.

Therefore, differentiating Equation (3.76) twice gives

$$\left. \begin{aligned} \eta &= D_1 \cos s_1 t + E_1 \sin s_1 t + D_2 \cos s_2 t + E_2 \sin s_2 t, \\ \dot{\eta} &= -D_1 s_1 \sin s_1 t + E_1 s_1 \cos s_1 t - D_2 s_2 \sin s_2 t + E_2 s_2 \cos s_2 t, \\ \ddot{\eta} &= -D_1 s_1^2 \cos s_1 t - E_1 s_1^2 \sin s_1 t - D_2 s_2^2 \cos s_2 t - E_2 s_2^2 \sin s_2 t. \\ \\ \xi &= \bar{D}_1 \cos s_1 t + \bar{E}_1 \sin s_1 t + \bar{D}_2 \cos s_2 t + \bar{E}_2 \sin s_2 t, \\ \dot{\xi} &= -\bar{D}_1 s_1 \sin s_1 t + \bar{E}_1 s_1 \cos s_1 t - \bar{D}_2 s_2 \sin s_2 t + \bar{E}_2 s_2 \cos s_2 t, \\ \ddot{\xi} &= -\bar{D}_1 s_1^2 \cos s_1 t - \bar{E}_1 s_1^2 \sin s_1 t - \bar{D}_2 s_2^2 \cos s_2 t - \bar{E}_2 s_2^2 \sin s_2 t \end{aligned} \right\}. \tag{3.77}$$

Using Equation (3.77) in Equation (3.54) produces

$$\begin{bmatrix} (-D_1s_1^2\cos s_1t - E_1s_1^2\sin s_1t - D_2s_2^2\cos s_2t - E_2s_2^2\sin s_2t) \\ -2n(-\bar{D}_1s_1\sin s_1t + \bar{E}_1s_1\cos s_1t - \bar{D}_2s_2\sin s_2t + \bar{E}_2s_2\cos s_2t) \end{bmatrix} = \begin{bmatrix} (D_1\cos s_1t + E_1\sin s_1t + D_2\cos s_2t + E_2\sin s_2t)\Omega^0_{xx} \\ +(\bar{D}_1\cos s_1t + \bar{E}_1\sin s_1t + \bar{D}_2\cos s_2t + \bar{E}_2\sin s_2t)\Omega^0_{xy} \end{bmatrix}, \tag{3.78}$$

$$\begin{bmatrix} (-\bar{D}_1s_1^2\cos s_1t - \bar{E}_1s_1^2\sin s_1t - \bar{D}_2s_2^2\cos s_2t - \bar{E}_2s_2^2\sin s_2t) + \\ 2n(-D_1s_1\sin s_1t + E_1s_1\cos s_1t - D_2s_2\sin s_2t + E_2s_2\cos s_2t) \end{bmatrix} = \begin{bmatrix} (D_1\cos s_1t + E_1\sin s_1t + D_2\cos s_2t + E_2\sin s_2t)\Omega^0_{yx} \\ +(\bar{D}_1\cos s_1t + \bar{E}_1\sin s_1t + \bar{D}_2\cos s_2t + \bar{E}_2\sin s_2t)\Omega^0_{yy} \end{bmatrix}. \tag{3.79}$$

From Equation (3.78) equating the coefficients of $\cos s_1$, yields

$$-D_1s_1^2 - 2n\bar{E}_1s_1 = D_1\Omega^0_{xx} + \bar{D}_1\Omega^0_{xy}. \tag{3.80}$$

Equating the coefficients of $\sin s_1$,

$$-E_1s_1^2 + 2n\bar{D}_1s_1 = E_1\Omega^0_{xx} + \bar{E}_1\Omega^0_{xy}. \tag{3.81}$$

Make $\bar{E}_1$ the subject of Equation (3.80)

$$\bar{E}_1 = \frac{D_1s_1^2 + D_1\Omega^0_{xx} + \bar{D}_1\Omega^0_{xy}}{-2ns_1}, \tag{3.82}$$

and substituting (3.82) in Equation (3.81, produces

$$2nE_1s_1^3 - 4n\bar{D}_1s_1^2 = -2nE_1s_1\Omega^0_{xx} + D_1\Omega^0_{xx}\Omega^0_{xy} + \bar{D}_1\Omega^{0\ 2}_{xy} + D_1s_1^2\Omega^0_{xy}.$$

Collecting terms in $\bar{D}_1$

$$2nE_1s_1^3 - D_1s_1^2\Omega_{xy}^0 + 2nE_1s_1\Omega_{xx}^0 - D_1\Omega_{xx}^0\Omega_{xy}^0 = 4n\bar{D}_1s_1^2 + \bar{D}_1\Omega_{xy}^{0\ 2}.$$

$$s_1^2\left(2nE_1s_1 - D_1\Omega_{xy}^0\right) + \Omega_{xx}^0\left(2nE_1s_1 - D_1\Omega_{xy}^0\right) = \left(4n^2s_1^2 + \Omega_{xy}^{0\ 2}\right)\bar{D}_1.$$

$$\bar{D}_1 = \frac{\left(s_1^2 + \Omega_{xx}^0\right)\left(2nE_1s_1 - D_1\Omega_{xy}^0\right)}{\left(4n^2s_1^2 + \Omega_{xy}^{0\ 2}\right)}. \tag{3.83}$$

Or

$$\bar{D}_1 = \Gamma\left(2nE_1s_1 - D_1\Omega_{xy}^0\right),$$
$$where \quad \Gamma = \frac{\left(s_1^2 + \Omega_{xx}^0\right)}{\left(4n^2s_1^2 + \Omega_{xy}^{0\ 2}\right)}. \tag{3.84}$$

Plugging Equation (3.83) in Equation (3.82), yields

$$\bar{E}_1 = \frac{-\left(s_1^2 + \Omega_{xx}^0\right)\left(2nD_1s_1 + E_1\Omega_{xy}^0\right)}{\left(4n^2s_1^2 + \Omega_{xy}^{0\ 2}\right)}.$$

Or

$$\bar{E}_1 = -\Gamma\left(2nD_1s_1 + E_1\Omega_{xy}^0\right), \tag{3.85}$$

$$where \quad \Gamma = \frac{\left(s_1^2 + \Omega_{xx}^0\right)}{\left(4n^2s_1^2 + \Omega_{xy}^{0\ 2}\right)}.$$

Alternatively, equating the coefficients of $\cos s_1$ in Equation (3.79), yields

$$-\bar{D}_1s_1^2 + 2nE_1s_1 = D_1\Omega_{yx}^0 + \bar{D}_1\Omega_{yy}^0$$

$$\Rightarrow \bar{D}_1 s_1^2 + \bar{D}_1 \Omega_{yy}^0 = 2nE_1 s_1 - D_1 \Omega_{yx}^0$$

$$\Rightarrow \bar{D}_1 = \frac{2nE_1 s_1 - D_1 \Omega_{yx}^0}{s_1^2 + \Omega_{yy}^0}. \qquad (3.86)$$

And equating the coefficients of $\sin s_1$, produces

$$-\bar{E}_1 s_1^2 - 2nD_1 s_1 = E_1 \Omega_{yx}^0 + \bar{E}_1 \Omega_{yy}^0$$

$$\Rightarrow \bar{E}_1 s_1^2 + \bar{E} \Omega_{yy}^0 = -2nD_1 s_1 - E_1 \Omega_{yx}^0.$$

$$\Rightarrow \bar{E}_1 = \frac{-\left(2nD_1 s_1 + E_1 \Omega_{yx}^0\right)}{s_1^2 + \Omega_{yy}^0}. \qquad (3.87)$$

Examining Equations (3.84) and (3.86), (3.85) and (3.87) and in view of Equation (3.60), we conclude that

$$\Gamma_1 = \frac{1}{s_1^2 + \Omega_{yy}^0} .$$

Similarly, equating the coefficients of $\cos s_2$, $\sin s_2$, yields the same conclusion.

Thus, in general, the relationships among the constants can be stated as

$$\begin{aligned} \bar{D}_i &= \Gamma_i (2ns_i E_i - \Omega_{xy}^0 D_i), \\ \bar{E}_i &= -\Gamma_i (2ns_i D_i + \Omega_{xy}^0 E_i), \\ \textit{where} \quad \Gamma_i &= \frac{s_1^2 + \Omega_{xx}^0}{4n^2 s_1^2 + \Omega_{xy}^{0\;2}} = \frac{1}{s_i^2 + \Omega_{yy}^0} \quad (i = 1, 2). \end{aligned} \qquad (3.88)$$

Next we determine the values of the frequencies s_1 and s_2 as stated in Equation (3.75).

Now, from $s_1 = \sqrt{\dfrac{Q-\sqrt{\Delta}}{2}}$, we have

$$s_1 = \left\langle \frac{1}{2}\left\{ Q - \left[\begin{array}{l} 1-3A_1+\dfrac{75}{4}A_2+3B_1+\dfrac{15}{4}B_2+2M_{b1}-6M_{b2}+27\mu(\mu-1)+6\mu(\mu-1)p_1 \\ +6\mu(\mu-1)p_2+(117\mu^2-111\mu)A_1+(-180\mu^2+165\mu)A_2+(117\mu^2-123\mu)B_1 \\ +(-180\mu^2+195\mu)B_2+66M_{b3}\mu(\mu-1)+27M_{b4}\mu(\mu-1) \end{array} \right]^{1/2} \right\} \right\rangle^{\frac{1}{2}}$$

where

$$M_{b1} = \frac{M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}}, \qquad M_{b2} = \frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}},$$
$$M_{b3} = \frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}, \qquad M_{b4} = \frac{M_b}{(r_c^2+T^2)^{5/2}}. \tag{3.89}$$

Employing Binomial expansion and considering only linear terms in very small quantities, we have

$$s_1 = \left\langle \frac{1}{2}\left\{ Q - \left[\begin{array}{l} 1-\dfrac{3A_1}{2}+\dfrac{75}{8}A_2+\dfrac{3B_1}{2}+\dfrac{15}{8}B_2+M_{b1}-3M_{b2}+\dfrac{27}{2}\mu(\mu-1)+3\mu(\mu-1)p_1 \\ +3\mu(\mu-1)p_2+(117\mu^2-111\mu)\dfrac{A_1}{2}+(-180\mu^2+165\mu)\dfrac{A_2}{2}+(117\mu^2-123\mu)\dfrac{B_1}{2} \\ +(-180\mu^2+195\mu)\dfrac{B_2}{2}+33M_{b3}\mu(\mu-1)+\dfrac{27}{2}M_{b4}\mu(\mu-1) \end{array} \right] \right\} \right\rangle^{\frac{1}{2}}$$

Simplifying yields

$$s_1 = \left\langle \frac{1}{2}\left\{ \begin{array}{l} -\dfrac{27}{2}\mu(\mu-1) - 3\mu(\mu-1)p_1 - 3\mu(\mu-1)p_2 + (\mu-\mu^2)\dfrac{117A_1}{2} - (\dfrac{15}{4} + 90\mu - 90\mu^2)A_2 \\ +(\mu-\mu^2)\dfrac{117B_1}{2} - (\dfrac{15}{4} + 90\mu - 90\mu^2)B_2 - 33M_{b3}\mu(\mu-1) - \dfrac{27}{2}M_{b4}\mu(\mu-1) \end{array} \right\} \right\rangle^{\frac{1}{2}}$$

or

$$s_1 = \left\langle \begin{array}{l} \dfrac{27}{4}\mu(1-\mu) + \mu(1-\mu)\dfrac{3p_1}{2} + \mu(1-\mu)\dfrac{3p_2}{2} + \mu(1-\mu)\dfrac{117A_1}{4} - (\mu-\mu^2)45A_2 - \dfrac{15A_2}{8} \\ +(\mu-\mu^2)\dfrac{117B_1}{4} - (\mu-\mu^2)45B_2 - \dfrac{15B_2}{8} + \dfrac{33}{2}M_{b3}\mu(1-\mu) + \dfrac{27}{4}M_{b4}\mu(1-\mu) \end{array} \right\rangle^{\frac{1}{2}}.$$

Or

$$s_1 = \left[\mu(1-\mu)\left(\frac{27}{4} + \frac{3p_1}{2} + \frac{3p_2}{2} + \frac{117A_1}{4} - 45A_2 + \frac{117B_1}{4} - 45B_2 + \frac{33}{2}M_{b3} + \frac{27}{4}M_{b4} \right) - \frac{15A_2}{8} - \frac{15B_2}{8} \right]^{\frac{1}{2}}. \tag{3.90}$$

Or

$$s_1 = \left[\begin{array}{l} \mu(1-\mu)\left(\dfrac{27}{4} + \dfrac{3p_1}{2} + \dfrac{3p_2}{2} + \dfrac{117A_1}{4} - 45A_2 + \dfrac{117B_1}{4} - 45B_2 + \dfrac{33M_b(2r_c-1)}{2(r_c^2+T^2)^{3/2}} + \dfrac{27M_b}{4(r_c^2+T^2)^{5/2}} \right) \\ -\dfrac{15A_2}{8} - \dfrac{15B_2}{8} \end{array} \right]^{\frac{1}{2}}.$$

Similarly,

$$s_2 = \left\langle \frac{1}{2}\left\{ Q + \left[\begin{array}{l} 1 - \dfrac{3A_1}{2} + \dfrac{75}{8}A_2 + \dfrac{3B_1}{2} + \dfrac{15}{8}B_2 + M_{b1} - 3M_{b2} + \dfrac{27}{2}\mu(\mu-1) + 3\mu(\mu-1)p_1 \\ +3\mu(\mu-1)p_2 + (117\mu^2 - 111\mu)\dfrac{A_1}{2} + (-180\mu^2 + 165\mu)\dfrac{A_2}{2} + (117\mu^2 - 123\mu)\dfrac{B_1}{2} \\ +(-180\mu^2 + 195\mu)\dfrac{B_2}{2} + 33M_{b3}\mu(\mu-1) + \dfrac{27}{2}M_{b4}\mu(\mu-1) \end{array} \right] \right\} \right\rangle^{\frac{1}{2}}$$

$$= \left\langle \frac{1}{2}\left\{ \begin{array}{l} 2+\frac{27}{2}\mu(\mu-1)+3\mu(\mu-1)p_1+3\mu(\mu-1)p_2+(-6+117\mu^2-105\mu)\frac{A_1}{2} \\ +(15+75\mu-90\mu^2)A_2+(6+117\mu^2-129\mu)\frac{B_1}{2}+(105\mu-90\mu^2)B_2 \\ +2M_{b1}-6M_{b2}+33M_{b3}\mu(\mu-1)+\frac{27}{2}M_{b4}\mu(\mu-1) \end{array}\right\}\right\rangle^{\frac{1}{2}}.$$

$$= \left[\begin{array}{l} 1+\frac{27}{4}\mu(\mu-1)+\mu(\mu-1)\frac{3p_1}{2}+\mu(\mu-1)\frac{3p_2}{2}+(-6+117\mu^2-105\mu)\frac{A_1}{4}+(15+75\mu-90\mu^2)\frac{A_2}{2} \\ +(6+117\mu^2-129\mu)\frac{B_1}{4}+(105\mu-90\mu^2)\frac{B_2}{2}+M_{b1}-3M_{b2}+\frac{33}{2}M_{b3}\mu(\mu-1)+\frac{27}{4}M_{b4}\mu(\mu-1) \end{array}\right]^{\frac{1}{2}}.$$

(3.91)

Or

$$s_2 = \left[1+\frac{27}{4}\mu(\mu-1)+\mu(\mu-1)\frac{3\left(p_1+p_2\right)}{2}+(-6+117\mu^2-105\mu)\frac{A_1}{4}+(15+75\mu-90\mu^2)\frac{A_2}{2}\right.$$
$$+(6+117\mu^2-129\mu)\frac{B_1}{4}+(105\mu-90\mu^2)\frac{B_2}{2}+\left((2r_c+3)-\frac{33(2r_c-1)\mu}{2}+\frac{33(2r_c-1)\mu^2}{2}\right)\frac{M_b}{(r_c^2+T^2)^{3/2}}$$
$$\left.-\left(3r_c^2+\frac{27\mu}{4}-\frac{27\mu^2}{4}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}\right]^{\frac{1}{2}}.$$

It could be observed from Equations (3.90) and (3.91) that $s_2 \geq s_1$, so s_1 is the frequency of long periodic orbit while s_2 is for short periodic orbit. The frequencies s_1 and s_2 are functions of the mass ratio and all the perturbations. Using Equations (3.90) and (3.91), we analyze the effects of the perturbations on the frequencies which are presented in Table 3.9.

Table 3.9: Effects of the perturbations on the frequencies (μ=0.03 and T=0.01, r_c=0.99)

case	q_1	q_2	A_1	A_2	B_1	B_2	M_b	s_1	s_2
1	1	1	0	0	0	0	0	0.4431986010	0.8964234490
2	0.98	1	0	0	0	0	0	0.4441823949	0.8959363816
	0.96	1	0	0	0	0	0	0.4451640147	0.8954490493
	0.94	1	0	0	0	0	0	0.4461434746	0.8949614516
3	1	0.97	0	0	0	0	0	0.4446734757	0.8956927486
	1	0.95	0	0	0	0	0	0.4456540137	0.8952052837
	1	0.93	0	0	0	0	0	0.4466323991	0.8947175531
4	1	1	0.01	0	0	0	0	0.4526994035	0.8837212513
	1	1	0.02	0	0	0	0	0.4620048701	0.8708337958
	1	1	0.03	0	0	0	0	0.4711265753	0.8577527324
	1	1	0.01	0.005	0	0	0	0.4360340816	0.9076815245
	1	1	0.02	0.01	0	0	0	0.4288586460	0.9188016652

	1	1	0.03	0.015	0	0	0	0.4216724544	0.9297888201
5	1	1	0	0	0.01	0	0	0.4526994035	0.8995350187
	1	1	0	0	0.02	0	0	0.4620048701	0.9026358623
	1	1	0	0	0.03	0	0	0.4711265753	0.9057260899
	1	1	0	0	0.01	0.005	0	0.4360340816	0.9037896602
	1	1	0	0	0.02	0.01	0	0.4288586460	0.9110963176
	1	1	0	0	0.03	0.015	0	0.4216724544	0.9183448426
6	1	1	0	0	0	0	0.01	0.4509309429	0.9039178042
	1	1	0	0	0	0	0.02	0.4585329111	0.9113505328
	1	1	0	0	0	0	0.03	0.4660108860	0.9187231304
7	0.99	0.98	0.01	0.005	0.01	0.005	0.01	0.4389067658	0.9211169289
	0.98	0.97	0.02	0.01	0.02	0.01	0.02	0.4337090094	0.9453963685
	0.97	0.96	0.03	0.015	0.03	0.015	0.03	0.4281322312	0.9690676913

3.6.2 The nature of periodic orbits

The expansion of the potential Ω around the triangular libration points up to second order of η, ξ is expressed as

$$\begin{aligned}\Omega &= \Omega^0 + \eta\Omega_x^0 + \xi\Omega_y^0 + \frac{\eta^2}{2}\Omega_{xx}^0 + \frac{\eta\xi}{2}\Omega_{xy}^0 + \frac{\xi\eta}{2}\Omega_{yx}^0 + \frac{\xi^2}{2}\Omega_{yy}^0 + (0)3 \\ &= \Omega^0 + \frac{\eta^2}{2}\Omega_{xx}^0 + \eta\xi\Omega_{xy}^0 + \frac{\xi^2}{2}\Omega_{yy}^0,\end{aligned} \tag{3.92}$$

where

$$\Omega^0 = \frac{3}{2} - (1-\mu)p_1 - \mu p_2 + (\frac{5}{4} - \frac{\mu}{2})A_1 + (\frac{3\mu}{8} - \frac{21}{16})A_2 + (\frac{3}{4} + \frac{\mu}{2})B_1 - (\frac{3\mu}{8} + \frac{15}{16})B_2$$
$$+ (\frac{r_c}{(r_c^2 + T^2)^{3/2}} + \frac{1}{(r_c^2 + T^2)^{1/2}})M_b,$$
$$\Omega_x^0 = \Omega_y^0 = 0,$$

$$\Omega_{xx}^0 = \frac{3}{4} - (\frac{1}{2} - \frac{3\mu}{2})p_1 + (1 - \frac{3\mu}{2})p_2 + (\frac{27}{8} - 3\mu)A_1 - (\frac{165}{32} - \frac{75\mu}{16})A_2 + (\frac{3}{8} + 3\mu)B_1 - (\frac{15}{32} + \frac{75\mu}{16})B_2$$
$$+ \frac{5M_b(2r_c - 1)}{4(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{1}{4} - \mu + \mu^2)}{(r_c^2 + T^2)^{5/2}},$$
$$\Omega_{yy}^0 = \frac{9}{4} + (\frac{1}{2} - \frac{3\mu}{2})p_1 - (1 - \frac{3\mu}{2})p_2 + \frac{33A_1}{8} - (\frac{255}{32} - \frac{45\mu}{16})A_2 + \frac{33B_1}{8} - (\frac{165}{32} + \frac{45\mu}{16})B_2$$
$$+ \frac{7M_b(2r_c - 1)}{4(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{3}{4})}{(r_c^2 + T^2)^{5/2}},$$
$$\Omega_{xy}^0 = \Omega_{yx}^0 = \sqrt{3}\left\{\begin{array}{l} \frac{3}{4} - \frac{3\mu}{2} - (\frac{1}{6} + \frac{\mu}{6})p_1 + (\frac{1}{3} - \frac{\mu}{6})p_2 + (\frac{19}{8} - \frac{13\mu}{4})A_1 + (5\mu - \frac{125}{32})A_2 + (\frac{7}{8} - \frac{13\mu}{4})B_1 \\ +(5\mu - \frac{35}{32})B_2 + \left\langle \frac{11}{12} - \frac{11\mu}{6} \right\rangle \frac{M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{\frac{3}{2}M_b(\frac{1}{2} - \mu)}{(r_c^2 + T^2)^{5/2}} \end{array}\right\}.$$

These derivatives are given in view of Equation (3.60) the superscript 0 indicates that, thepotential Ω and its derivatives have been evaluated at any of the libration points Equation (3.47).

Now, the potential Ω satisfies the Jacobi integral $2\Omega = C$ of Section 3.3, that is

$$\eta^2\Omega_{xx}^0 + 2\eta\xi\Omega_{xy}^0 + \xi^2\Omega_{yy}^0 = C - 2\Omega^0. \tag{3.93}$$

The discriminant of the quadratic part of Equation (3.93) is

$$\begin{vmatrix} \Omega_{xx}^0 & \Omega_{xy}^0 \\ \Omega_{xy}^0 & \Omega_{yy}^0 \end{vmatrix} = \Omega_{xx}^0\Omega_{yy}^0 - \left(\Omega_{xy}^0\right)^2$$
$$= \left(\mu - \mu^2\right)\left(\frac{27}{4} + \frac{3}{2}p_1 + \frac{3}{2}p_2 + \frac{117A_1}{4} - 45A_2 + \frac{117B_1}{4} - 45B_2 + \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{4(r_c^2 + T^2)^{5/2}}\right)$$
$$> 0,$$

which indicates that the periodic orbits around the triangular libration pointsare elliptical.Therefore, eccentricities, inclination and semi major axes of the orbits are used to determine the shapes, orientation and sizes of the orbits.

3.6.2.1*Orientation of the elliptical orbit*

Equation (3.92) contains a bilinear term $\eta\xi$ which causes the rotation of the principal axes of the ellipse with respect to the coordinates system $0\eta\xi$ by an angle β Figure 3.7. This advices the introduction of a normal reference frame of coordinates system $0\bar{\eta}\bar{\xi}$ such that the bilinear term is zero.

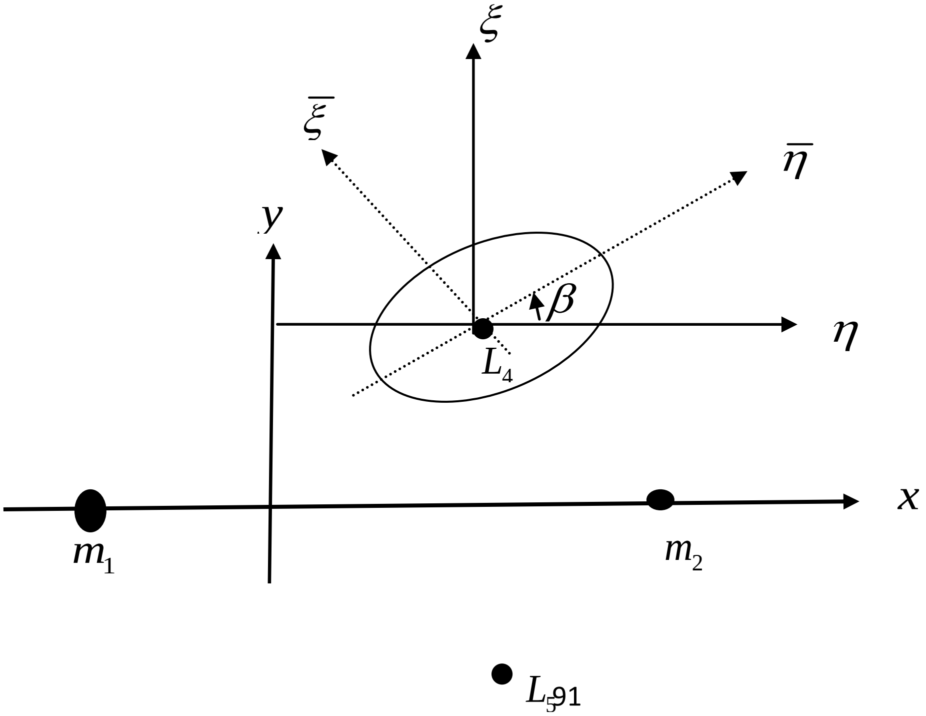


Figure 3.7: Orientation of the principal axes

The relations between the normal coordinates $(\bar{\eta}, \bar{\xi})$ and the previous coordinates system (η, ξ) are given by

$$\begin{aligned}\bar{\eta} &= \eta \cos\beta + \xi \sin\beta, \\ \bar{\xi} &= -\eta \sin\beta + \xi \cos\beta.\end{aligned}$$

Or

$$\begin{aligned}\eta &= \bar{\eta} \cos\beta - \bar{\xi} \sin\beta, \\ \xi &= \bar{\eta} \sin\beta + \bar{\xi} \cos\beta.\end{aligned} \tag{3.94}$$

Using Equation (3.94) in Equation (3.92), we obtain the potential function under this transformation as

$$\begin{aligned}\bar{\Omega} = {} & w_1\left(\bar{\eta}^2 \cos^2\beta + \bar{\xi}^2 \sin^2\beta - 2\bar{\eta}\bar{\xi} \cos\beta \sin\beta\right) + w_2\left(\bar{\eta}^2 \sin^2\beta + \bar{\xi}^2 \cos^2\beta + 2\bar{\eta}\bar{\xi} \cos\beta \sin\beta\right) \\ & + w_3\left(\bar{\eta}^2 \cos\beta \sin\beta - \bar{\xi}^2 \cos\beta \sin\beta + \bar{\eta}\bar{\xi}(\cos^2\beta - \sin^2\beta)\right) + w_4,\end{aligned} \tag{3.95}$$

where

$$w_1 = \frac{\Omega_{xx}^0}{2} = \frac{3}{8} - (\frac{1}{4} - \frac{3\mu}{4}) p_1 + (\frac{1}{2} - \frac{3\mu}{4}) p_2 + (\frac{27}{16} - \frac{3}{2}\mu) A_1 - (\frac{165}{64} - \frac{75\mu}{32}) A_2$$
$$+ (\frac{3}{16} + \frac{3}{2}\mu) B_1 - (\frac{15}{64} + \frac{75\mu}{32}) B_2 + \frac{5M_b(2r_c - 1)}{8(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{1}{4} - \mu + \mu^2)}{2(r_c^2 + T^2)^{5/2}},$$

$$w_2 = \frac{\Omega_{yy}^0}{2} = \frac{9}{8} + (\frac{1}{4} - \frac{3\mu}{4}) p_1 - (\frac{1}{2} - \frac{3\mu}{4}) p_2 + \frac{33A_1}{16} - (\frac{255}{64} - \frac{45\mu}{32}) A_2 + \frac{33B_1}{16}$$
$$- (\frac{165}{64} + \frac{45\mu}{32}) B_2 + \frac{7M_b(2r_c - 1)}{8(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{3}{4})}{2(r_c^2 + T^2)^{5/2}},$$

$$w_3 = \Omega_{xy}^0 = \sqrt{3} \left\{ \begin{array}{l} \frac{3}{4} - \frac{3\mu}{2} - (\frac{1}{6} + \frac{\mu}{6}) p_1 + (\frac{1}{3} - \frac{\mu}{6}) p_2 + (\frac{19}{8} - \frac{13\mu}{4}) A_1 + (5\mu - \frac{125}{32}) A_2 \\ +(\frac{7}{8} - \frac{13\mu}{4}) B_1 + (5\mu - \frac{35}{32}) B_2 + \left\langle \frac{11}{12} - \frac{11\mu}{6} \right\rangle \frac{M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{3M_b(\frac{1}{2} - \mu)}{2(r_c^2 + T^2)^{5/2}} \end{array} \right\},$$

$$w_4 = \Omega^0 = \frac{3}{2} - (1 - \mu) p_1 - \mu p_2 + (\frac{5}{4} - \frac{\mu}{2}) A_1 + (\frac{3\mu}{8} - \frac{21}{16}) A_2 + (\frac{3}{4} + \frac{\mu}{2}) B_1 - (\frac{3\mu}{8} + \frac{15}{16}) B_2$$
$$+ (\frac{r_c}{(r_c^2 + T^2)^{3/2}} + \frac{1}{(r_c^2 + T^2)^{1/2}}) M_b.$$

Now, setting the bilinear terms in Equation (3.95) to zero we have,

$$w_1(-\bar{\eta}\bar{\xi}\sin 2\beta) + w_2\left(\bar{\eta}\bar{\xi}\sin 2\beta\right) + w_3\left(\bar{\eta}\bar{\xi}\cos 2\beta\right) = 0.$$

Or

$$(w_2 - w_1)\sin 2\beta + w_3\cos 2\beta = 0.$$

$$\frac{\sin 2\beta}{\cos 2\beta} = \frac{w_3}{(w_1 - w_2)}.$$

$$\tan 2\beta = \frac{w_3}{(w_1 - w_2)} = \frac{2\Omega_{xy}^0}{\Omega_{xx}^0 - \Omega_{yy}^0}. \tag{3.96}$$

Substituting the values of the partial derivatives in Equation (3.96), we have

$$\tan 2\beta = \frac{2\sqrt{3}\left\{\begin{array}{l} \frac{3}{4}-\frac{3\mu}{2}-(\frac{1}{6}+\frac{\mu}{6})p_1+(\frac{1}{3}-\frac{\mu}{6})p_2+(\frac{19}{8}-\frac{13\mu}{4})A_1+(5\mu-\frac{125}{32})A_2 \\ +(\frac{7}{8}-\frac{13\mu}{4})B_1+(5\mu-\frac{35}{32})B_2+\left\langle\frac{11}{12}-\frac{11\mu}{6}\right\rangle\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{3M_b(\frac{1}{2}-\mu)}{2(r_c^2+T^2)^{5/2}} \end{array}\right\}}{\left\{\begin{array}{l} -\frac{3}{2}-(1-3\mu)p_1+(2-3\mu)p_2-(\frac{3}{4}+3\mu)A_1+(\frac{45}{16}+\frac{15\mu}{8})A_2 \\ +(-\frac{15}{4}+3\mu)B_1+(\frac{75}{16}-\frac{15\mu}{8})B_2-\frac{M_b(2r_c-1)}{2(r_c^2+T^2)^{3/2}}+\frac{3M_b(-\frac{1}{2}-\mu+\mu^2)}{(r_c^2+T^2)^{5/2}} \end{array}\right\}}$$

$$= \frac{2\sqrt{3}\left\{\begin{array}{l} \frac{3}{4}-\frac{3\mu}{2}-(\frac{1}{6}+\frac{\mu}{6})p_1+(\frac{1}{3}-\frac{\mu}{6})p_2+(\frac{19}{8}-\frac{13\mu}{4})A_1+(5\mu-\frac{125}{32})A_2 \\ +(\frac{7}{8}-\frac{13\mu}{4})B_1+(5\mu-\frac{35}{32})B_2+\left\langle\frac{11}{12}-\frac{11\mu}{6}\right\rangle\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{\frac{3}{2}M_b(\frac{1}{2}-\mu)}{(r_c^2+T^2)^{5/2}} \end{array}\right\}}{-\frac{3}{2}\left\{1-\frac{2}{3}\left(\begin{array}{l} -(1-3\mu)p_1+(2-3\mu)p_2-(\frac{3}{4}+3\mu)A_1+(\frac{45}{16}+\frac{15\mu}{8})A_2 \\ +(-\frac{15}{4}+3\mu)B_1+(\frac{75}{16}-\frac{15\mu}{8})B_2-\frac{M_b(2r_c-1)}{2(r_c^2+T^2)^{3/2}}+\frac{3M_b(-\frac{1}{2}-\mu+\mu^2)}{(r_c^2+T^2)^{5/2}} \end{array}\right)\right\}}$$

Applying Binomial expansion and ignoring higher powers in small quantities, we get

$$\tan 2\beta = -\frac{4}{3}\sqrt{3}\left\{\begin{array}{l} \frac{3}{4}-\frac{3\mu}{2}-(\frac{1}{6}+\frac{\mu}{6})p_1+(\frac{1}{3}-\frac{\mu}{6})p_2+(\frac{19}{8}-\frac{13\mu}{4})A_1+(5\mu-\frac{125}{32})A_2 \\ +(\frac{7}{8}-\frac{13\mu}{4})B_1+(5\mu-\frac{35}{32})B_2+\left\langle\frac{11}{12}-\frac{11\mu}{6}\right\rangle\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{3M_b(\frac{1}{2}-\mu)}{2(r_c^2+T^2)^{5/2}} \end{array}\right\}$$

$$\times\left\{1+\frac{2}{3}\left(\begin{array}{l} -(1-3\mu)p_1+(2-3\mu)p_2-(\frac{3}{4}+3\mu)A_1+(\frac{45}{16}+\frac{15\mu}{8})A_2 \\ +(-\frac{15}{4}+3\mu)B_1+(\frac{75}{16}-\frac{15\mu}{8})B_2-\frac{M_b(2r_c-1)}{2(r_c^2+T^2)^{3/2}}+\frac{3M_b(-\frac{1}{2}-\mu+\mu^2)}{(r_c^2+T^2)^{5/2}} \end{array}\right)\right\}$$

$$\begin{aligned} &= -\frac{4}{3}\sqrt{3}\left[\frac{3}{4}-\frac{3\mu}{2}+\left(-\frac{2}{3}+\frac{7\mu}{3}-3\mu^2\right)p_1+(\frac{4}{3}-\frac{11\mu}{3}+3\mu^2)p_2\right. \\ &\quad +\left(2-4\mu+3\mu^2\right)A_1+(-\frac{5}{2}+\frac{25\mu}{8}-\frac{15\mu^2}{8})A_2+\left(-1+2\mu-3\mu^2\right)B_1 \\ &\quad \left.+(\frac{5}{4}-\frac{5\mu}{8}+\frac{15\mu^2}{8})B_2+\left(\frac{2}{3}-\frac{4\mu}{3}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{3M_b(-\mu+3\mu^2-2\mu^3)}{2(r_c^2+T^2)^{5/2}}\right]. \end{aligned}$$

$$
\begin{aligned}
\tan 2\beta = \pm\frac{4}{3}\sqrt{3}\Bigg[& \frac{3}{4}-\frac{3\mu}{2}+\left(-\frac{2}{3}+\frac{7\mu}{3}-3\mu^{2}\right)p_{1}+\left(\frac{4}{3}-\frac{11\mu}{3}+3\mu^{2}\right)p_{2} \\
& +\left(2-4\mu+3\mu^{2}\right)A_{1}+\left(-\frac{5}{2}+\frac{25\mu}{8}-\frac{15\mu^{2}}{8}\right)A_{2}+\left(-1+2\mu-3\mu^{2}\right)B_{1} \\
& +\left(\frac{5}{4}-\frac{5\mu}{8}+\frac{15\mu^{2}}{8}\right)B_{2}+\left(\frac{2}{3}-\frac{4\mu}{3}\right)\frac{M_{b}(2r_{c}-1)}{(r_{c}^{2}+T^{2})^{3/2}}+\frac{3M_{b}(-\mu+3\mu^{2}-2\mu^{3})}{2(r_{c}^{2}+T^{2})^{5/2}}\Bigg]. \qquad (3.97)
\end{aligned}
$$

Thus, Equation (3.97) gives the orientation of the elliptical orbits. Note that the positive sign refers to the periodic motion around L_4 while negative one gives the motion around L_5. The effects of the perturbations on this angle of rotation when μ=0.03, T=0.01 and r_c=0.99 are outlined in Table 3.10.

Table 3.10: Effects of the perturbations on the angle of rotation of the principal axis

case	q_1	q_2	A_1	A_2	B_1	B_2	M_b	$\tan 2\beta$	β
1	1	1	0	0	0	0	0	1.6281	29.2208
	0.98	1	0	0	0	0	0	1.6004	29.0009
2	0.96	1	0	0	0	0	0	1.5728	28.7754

	0.94	1	0	0	0	0	0	1.5451	28.5442
3	1	0.97	0	0	0	0	0	1.7131	29.8629
	1	0.95	0	0	0	0	0	1.7697	30.2653
	1	0.93	0	0	0	0	0	1.8263	30.6486
4	1	1	0.01	0	0	0	0	1.6716	29.5555
	1	1	0.02	0	0	0	0	1.7151	29.8776
	1	1	0.03	0	0	0	0	1.7586	30.1877
	1	1	0.01	0.005	0	0	0	1.6438	29.3430
	1	1	0.02	0.01	0	0	0	1.6595	29.4634
	1	1	0.03	0.015	0	0	0	1.6752	29.5822
5	1	1	0	0	0.01	0	0	1.6064	29.0483
	1	1	0	0	0.02	0	0	1.5846	28.8724
	1	1	0	0	0.03	0	0	1.5628	28.6930
	1	1	0	0	0.01	0.005	0	1.6206	29.1615
	1	1	0	0	0.02	0.01	0	1.6131	29.1018
	1	1	0	0	0.03	0.015	0	1.6055	29.0417
6	1	1	0	0	0	0	0.01	1.6417	29.3271
	1	1	0	0	0	0	0.02	1.6554	29.4320
	1	1	0	0	0	0	0.03	1.6690	29.5357
7	0.99	0.98	0.01	0.005	0.01	0.005	0.01	1.7071	29.8197
	0.98	0.97	0.02	0.01	0.02	0.01	0.02	1.7579	30.1828
	0.97	0.96	0.03	0.015	0.03	0.015	0.03	1.8086	30.5303

3.6.2.2 *Equations of the ellipses*

The quadratic part of Equation (3.93) admits the characteristic equation

$$\bar{\lambda}^2 - \left(\Omega^0_{xx} + \Omega^0_{yy}\right)\bar{\lambda} + \Omega^0_{xx}\Omega^0_{yy} - \left(\Omega^0_{xy}\right)^2 = 0. \tag{3.98}$$

Utilizing the values of the partial derivatives of Equation (3.92) in Equation(3.98), yields

$$\bar{\lambda}^2 - d_1\bar{\lambda} + d_2 = 0\,, \tag{3.99}$$

where

$$d_1 = \left(3 + (\frac{15}{2} - 3\mu)A_1 - (\frac{105}{8} - \frac{15\mu}{2})A_2 + (\frac{9}{2} + 3\mu)B_1 - (\frac{45}{8} + \frac{15\mu}{2})B_2 + \frac{3M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{3M_b r_c^2}{(r_c^2 + T^2)^{5/2}} \right),$$

$$d_2 = \left(\mu - \mu^2\right)\left(\frac{27}{4} + \frac{3}{2}p_1 + \frac{3}{2}p_2 + \frac{117A_1}{4} - 45A_2 + \frac{117B_1}{4} - 45B_2 + \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{4(r_c^2 + T^2)^{5/2}} \right).$$

The roots of Equation(3.99) are

$$\bar{\lambda}_{1,2} = \frac{1}{2}\left(d_1 \pm \sqrt{d_1^{\,2} - 4d_2}\right). \tag{3.100}$$

Now,

$$\sqrt{d_1^{\,2} - 4d_2} = \left(d_1^{\,2} - 4d_2\right)^{\frac{1}{2}}.$$

$$
\begin{aligned}
\sqrt{{d_1}^2-4d_2} = \Bigg(\Bigg(& 3+(\frac{15}{2}-3\mu)A_1-(\frac{105}{8}-\frac{15\mu}{2})A_2+(\frac{9}{2}+3\mu)B_1-(\frac{45}{8}+\frac{15\mu}{2})B_2 \\
& +\frac{3M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{3M_br_c^2}{(r_c^2+T^2)^{5/2}}\Bigg)^2-4d_2\Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

$$
=\left(\begin{aligned}
& 9+(45-18\mu)A_1-(\frac{315}{4}-45\mu)A_2+(27+18\mu)B_1-(\frac{135}{4}+45\mu)B_2+\frac{18M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{18M_br_c^2}{(r_c^2+T^2)^{5/2}} \\
& -4\left(\left(\mu-\mu^2\right)\left(\frac{27}{4}+\frac{3p_1}{2}+\frac{3p_2}{2}+\frac{117A_1}{4}-45A_2+\frac{117B_1}{4}-45B_2+\frac{33M_b(2r_c-1)}{2(r_c^2+T^2)^{3/2}}+\frac{27M_b}{4(r_c^2+T^2)^{5/2}}\right)\right)
\end{aligned}\right)^{\frac{1}{2}}.
$$

$$
\begin{aligned}
=\Bigg\{& 9-27\left(\mu-\mu^2\right)-6\left(\mu-\mu^2\right)(p_1+p_2)+\left(45-135\mu+117\mu^2\right)A_1-\left(\frac{315}{4}-225\mu+180\mu^2\right)A_2 \\
& +\left(27-99\mu+117\mu^2\right)B_1-\left(\frac{135}{4}-135\mu+180\mu^2\right)B_2+\left(18-66(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} \\
& +\left(18-27(\mu-\mu^2)\right)\frac{M_br_c^2}{(r_c^2+T^2)^{5/2}}\Bigg\}^{\frac{1}{2}}.
\end{aligned}
$$

$$
\begin{aligned}
=\Bigg\{& 9-27\left(\mu-\mu^2\right)-6\left(\mu-\mu^2\right)(p_1+p_2)+9\left(5-15\mu+13\mu^2\right)A_1-9\left(\frac{35}{4}-25\mu+20\mu^2\right)A_2 \\
& +9\left(3-11\mu+13\mu^2\right)B_1-9\left(\frac{15}{4}-15\mu+20\mu^2\right)B_2+9\left(2-\frac{22}{3}(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} \\
& +9\left(2-3(\mu-\mu^2)\right)\frac{M_br_c^2}{(r_c^2+T^2)^{5/2}}\Bigg\}^{\frac{1}{2}}.
\end{aligned}
$$

$$
\begin{aligned}
=3\Bigg\{& 1-3\left(\mu-\mu^2\right)-\frac{2}{3}\left(\mu-\mu^2\right)(p_1+p_2)+\left(5-15\mu+13\mu^2\right)A_1-\left(\frac{35}{4}-25\mu+20\mu^2\right)A_2 \\
& +\left(3-11\mu+13\mu^2\right)B_1-\left(\frac{15}{4}-15\mu+20\mu^2\right)B_2+\left(2-\frac{22}{3}(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} \\
& +\left(2-3(\mu-\mu^2)\right)\frac{M_br_c^2}{(r_c^2+T^2)^{5/2}}\Bigg\}^{\frac{1}{2}}.
\end{aligned}
$$

$$
\begin{aligned}
&= 3\Bigg\{1-\frac{3}{2}\left(\mu-\mu^2\right)-\frac{1}{3}\left(\mu-\mu^2\right)(p_1+p_2)+\left(\frac{5}{2}-\frac{15}{2}\mu+\frac{13}{2}\mu^2\right)A_1-\left(\frac{35}{8}-\frac{25}{2}\mu+10\mu^2\right)A_2 \\
&\quad +\left(\frac{3}{2}-\frac{11}{2}\mu+\frac{13}{2}\mu^2\right)B_1-\left(\frac{15}{8}-\frac{15}{2}\mu+10\mu^2\right)B_2+\left(1-\frac{11}{3}(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} \\
&\quad +\left(1-\frac{3}{2}(\mu-\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\Bigg\}.
\end{aligned}
$$

$$
\begin{aligned}
&= 3-\frac{9}{2}\left(\mu-\mu^2\right)-\left(\mu-\mu^2\right)(p_1+p_2)+\left(\frac{15}{2}-\frac{45}{2}\mu+\frac{39}{2}\mu^2\right)A_1-\left(\frac{105}{8}-\frac{75}{2}\mu+30\mu^2\right)A_2 \\
&\quad +\left(\frac{9}{2}-\frac{33}{2}\mu+\frac{39}{2}\mu^2\right)B_1-\left(\frac{45}{8}-\frac{45}{2}\mu+30\mu^2\right)B_2+\left(3-11(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} \\
&\quad +\left(3-\frac{9}{2}(\mu-\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}.
\end{aligned}
$$

Therefore, the roots of Equation (3.100) become

$$
\bar{\lambda}_{1,2}=\frac{1}{2}\left\{\begin{array}{l}
\left(3+(\frac{15}{2}-3\mu)A_1-(\frac{105}{8}-\frac{15\mu}{2})A_2+(\frac{9}{2}+3\mu)B_1-(\frac{45}{8}+\frac{15\mu}{2})B_2+\frac{3M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{3M_b r_c^2}{(r_c^2+T^2)^{5/2}}\right) \\
\mp\left(\begin{array}{l}
3-\frac{9}{2}\left(\mu-\mu^2\right)-\left(\mu-\mu^2\right)(p_1+p_2)+\left(\frac{15}{2}-\frac{45}{2}\mu+\frac{39}{2}\mu^2\right)A_1-\left(\frac{105}{8}-\frac{75}{2}\mu+30\mu^2\right)A_2 \\
+\left(\frac{9}{2}-\frac{33}{2}\mu+\frac{39}{2}\mu^2\right)B_1-\left(\frac{45}{8}-\frac{45}{2}\mu+30\mu^2\right)B_2+\left(3-11(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} \\
+\left(3-\frac{9}{2}(\mu-\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}.
\end{array}\right)
\end{array}\right\}.
$$

$$
\bar{\lambda}_1=\frac{1}{2}\left\{\begin{array}{l}
\frac{9}{2}\left(\mu-\mu^2\right)+\left(\mu-\mu^2\right)(p_1+p_2)+\left(\frac{39}{2}\mu-\frac{39}{2}\mu^2\right)A_1-\left(30\mu-30\mu^2\right)A_2+\left(\frac{39}{2}\mu-\frac{39}{2}\mu^2\right)B_1 \\
-\left(30\mu-30\mu^2\right)B_2+11(\mu-\mu^2)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{9}{2}(\mu-\mu^2)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}
\end{array}\right\}.
$$

$$
\begin{aligned}
&=\frac{9}{4}\left(\mu-\mu^{2}\right)+\frac{1}{2}\left(\mu-\mu^{2}\right)(p_{1}+p_{2})+\left(\frac{39}{4}\mu-\frac{39}{4}\mu^{2}\right)A_{1}-\left(15\mu-15\mu^{2}\right)A_{2}+\left(\frac{39}{4}\mu-\frac{39}{4}\mu^{2}\right)B_{1}\\
&\quad-\left(15\mu-15\mu^{2}\right)B_{2}+\frac{11}{2}(\mu-\mu^{2})\frac{M_{b}(2r_{c}-1)}{(r_{c}^{2}+T^{2})^{3/2}}+\frac{9}{4}(\mu-\mu^{2})\frac{M_{b}r_{c}^{2}}{(r_{c}^{2}+T^{2})^{5/2}}.
\end{aligned}
\tag{3.101}
$$

$$
\bar{\lambda}_{2}=\frac{1}{2}\left\{
\begin{array}{l}
6-\frac{9}{2}\left(\mu-\mu^{2}\right)-\left(\mu-\mu^{2}\right)(p_{1}+p_{2})+\left(15-\frac{51}{2}\mu+\frac{39}{2}\mu^{2}\right)A_{1}-\left(\frac{105}{4}-45\mu+30\mu^{2}\right)A_{2}\\
+\left(9-\frac{27}{2}\mu+\frac{39}{2}\mu^{2}\right)B_{1}-\left(\frac{45}{4}-15\mu+30\mu^{2}\right)B_{2}+\left(6-11(\mu-\mu^{2})\right)\frac{M_{b}(2r_{c}-1)}{(r_{c}^{2}+T^{2})^{3/2}}\\
+\left(6-\frac{9}{2}(\mu-\mu^{2})\right)\frac{M_{b}r_{c}^{2}}{(r_{c}^{2}+T^{2})^{5/2}}
\end{array}
\right\}.
$$

$$
\begin{aligned}
&=3-\frac{9}{4}\left(\mu-\mu^{2}\right)-\frac{1}{2}\left(\mu-\mu^{2}\right)(p_{1}+p_{2})+\left(\frac{15}{2}-\frac{51}{4}\mu+\frac{39}{4}\mu^{2}\right)A_{1}-\left(\frac{105}{8}-\frac{45}{2}\mu+15\mu^{2}\right)A_{2}\\
&\quad+\left(\frac{9}{2}-\frac{27}{4}\mu+\frac{39}{4}\mu^{2}\right)B_{1}-\left(\frac{45}{8}-\frac{15}{2}\mu+15\mu^{2}\right)B_{2}+\left(3-\frac{11}{2}(\mu-\mu^{2})\right)\frac{M_{b}(2r_{c}-1)}{(r_{c}^{2}+T^{2})^{3/2}}\\
&\quad+\left(3-\frac{9}{4}(\mu-\mu^{2})\right)\frac{M_{b}r_{c}^{2}}{(r_{c}^{2}+T^{2})^{5/2}}.
\end{aligned}
\tag{3.102}
$$

Now, the equations of motion of the infinitesimal mass can be expressed as

$$
\begin{aligned}
&\ddot{\bar{\eta}}-2\dot{\bar{\xi}}=\bar{\lambda}_{1}\bar{\eta},\\
&\ddot{\bar{\xi}}+2\dot{\bar{\eta}}=\bar{\lambda}_{2}\bar{\xi};
\end{aligned}
\tag{3.103}
$$

and the new potential function takes the form

$$
\bar{\Omega}=\Omega^{0}+\frac{1}{2}\bar{\lambda}_{1}\bar{\eta}^{2}+\frac{1}{2}\bar{\lambda}_{2}\bar{\xi}^{2}.
$$

So that the periodic solutions Equation (3.76) can be written in the new coordinates as

$$
\begin{aligned}
&\bar{\eta}=G_{1}\cos s_{1}t+H_{1}\sin s_{1}t+G_{2}\cos s_{2}t+H_{2}\sin s_{2}t,\\
&\bar{\xi}=\bar{G}_{1}\cos s_{1}t+\bar{H}_{1}\sin s_{1}t+\bar{G}_{2}\cos s_{2}t+\bar{H}_{2}\sin s_{2}t.
\end{aligned}
\tag{3.104}
$$

As in Section 5.1, we use Equation (3.104) in Equation (3.103) in order to obtain the relations among the coefficients $G_i, H_i, \bar{G}_i, \bar{H}_i \; (i = 1, 2)$. Hence, from the first and the second equations of Equation (3.103), we have respectively

$$\begin{bmatrix} (-G_1 s_1^2 \cos s_1 t - H_1 s_1^2 \sin s_1 t - G_2 s_2^2 \cos s_2 t - H_2 s_2^2 \sin s_2 t) \\ -2(-\bar{G}_1 s_1 \sin s_1 t + \bar{H}_1 s_1 \cos s_1 t - \bar{G}_2 s_2 \sin s_2 t + \bar{H}_2 s_2 \cos s_2 t) \end{bmatrix} = \bar{\lambda}_1 \begin{bmatrix} G_1 \cos s_1 t + H_1 \sin s_1 t \\ +G_2 \cos s_2 t + H_2 \sin s_2 t \end{bmatrix}$$

and

$$\begin{bmatrix} (-\bar{G}_1 s_1^2 \cos s_1 t - \bar{H}_1 s_1^2 \sin s_1 t - \bar{G}_2 s_2^2 \cos s_2 t - \bar{H}_2 s_2^2 \sin s_2 t) \\ +2(-G_1 s_1 \sin s_1 t + H_1 s_1 \cos s_1 t - G_2 s_2 \sin s_2 t + H_2 s_2 \cos s_2 t) \end{bmatrix} = \bar{\lambda}_2 \begin{bmatrix} \bar{G}_1 \cos s_1 t + \bar{H}_1 \sin s_1 t \\ +\bar{G}_2 \cos s_2 t + \bar{H}_2 \sin s_2 t \end{bmatrix}.$$

(3.105)

Equating the coefficients of $\cos s_1$, from the first equation of Equation (3.105), yields

$$-G_1 s_1^2 - 2\bar{H}_1 s_1 = \bar{\lambda}_1 G_1$$

$$\bar{H}_1 = \frac{-\left(s_1^2 + \bar{\lambda}_1\right) G_1}{2 s_1}.$$

Or

$$\bar{H}_1 = -\sigma_1 G_1,$$

$$\textit{where} \quad \sigma_1 = \frac{(s_1^2 + \bar{\lambda}_1)}{2 s_1}. \qquad (3.106a)$$

Equating the coefficients of $\sin s_1$, produces

$$-H_1 s_1^2 + 2\bar{G}_1 s_1 = \bar{\lambda}_1 H_1$$

$$\bar{G}_1 = \frac{(s_1^2 + \bar{\lambda}_1) H_1}{2 s_1}.$$

Or

$$\bar{G}_1 = \sigma_1 H_1,$$

$$where \quad \sigma_1 = \frac{(s_1^2 + \bar{\lambda}_1)}{2s_1}. \tag{3.106b}$$

Alternatively, using the second equation of Equation (3.105) we get the coefficients of $\cos s_1$ as

$$\begin{aligned} -\bar{G}_1 s_1^2 + 2H_1 s_1 &= \bar{\lambda}_2 \bar{G}_1 \\ \bar{G}_1 &= \frac{2s_1}{s_1^2 + \bar{\lambda}_2} H_1, \end{aligned} \tag{3.107}$$

and the coefficients of $\sin s_1$ as

$$\begin{aligned} -\bar{H}_1 s_1^2 + 2G_1 s_1 &= \bar{\lambda}_2 \bar{H}_1 \\ \bar{H}_1 &= -\frac{2s_1}{s_1^2 + \bar{\lambda}_2} G_1. \end{aligned} \tag{3.108}$$

Observing Equations (3.106b) and (3.107), (3.106a) and (3.108), it is inferred that

$$\sigma_1 = \frac{2s_1}{s_1^2 + \bar{\lambda}_2}.$$

In the same manner $\bar{G}_2$ *and* $\bar{H}_2$ can be obtained, so that, in general the relationships between the constants are controlled by

$$\begin{aligned} &\bar{G}_i = \sigma_i H_i, \\ &\bar{H}_i = -\sigma_i G_i, \\ &where \quad \sigma_i = \frac{(s_i^2 + \bar{\lambda}_1)}{2s_i} = \frac{2s_i}{s_i^2 + \bar{\lambda}_2} \quad (i = 1, 2). \end{aligned} \tag{3.109}$$

From Equation (3.104) we observe that by choosing suitable initial conditions, the short or the long periodic terms can be eliminated from the solution. Now, let us assume that the constants associated to the short periodic terms are zeros, so that Equation (3.104) reduces to

$$\begin{aligned}\bar{\eta} &= G_1 \cos s_1 t + H_1 \sin s_1 t,\\ \bar{\xi} &= \bar{G}_1 \cos s_1 t + \bar{H}_1 \sin s_1 t.\end{aligned} \tag{3.110}$$

If we let $\bar{\eta}_0$ *and* $\bar{\xi}_0$ be the initial conditions at the initial time *t*, then Equation (3.110) produces

$$\begin{aligned}G_1 &= \bar{\eta}_0,\\ \bar{G}_1 &= \bar{\xi}_0.\end{aligned} \tag{3.111}$$

Using Equation (3.111) in Equation (3.110) yields

$$\begin{aligned}H_1 &= \frac{\bar{G}_1}{\sigma_1} = \frac{\bar{\xi}_0}{\sigma_1},\\ \bar{H}_1 &= -\sigma_1 G_1 = -\sigma_1 \bar{\eta}_0.\end{aligned} \tag{3.112}$$

Substituting Equations (3.111) and (3.112) in Equation (3.110), we have

$$\begin{aligned}\bar{\eta} &= \bar{\eta}_0 \cos s_1 t + \frac{\bar{\xi}_0}{\sigma_1} \sin s_1 t,\\ \bar{\xi} &= \bar{\xi}_0 \cos s_1 t - \sigma_1 \bar{\eta}_0 \sin s_1 t.\end{aligned} \tag{3.113}$$

Equation (3.113) signifies a particular solution with only two arbitrary constants, which cannot be freely selected. Thus, we need to eliminate sine and cosine from the equations of (3.113).

Now, to eliminate cosine, multiply the first equation of Equation (3.113) by $\bar{\xi}_0$ and the second by $-\bar{\eta}_0$ then, adding produces

$$\bar{\eta}\bar{\xi}_0 - \bar{\xi}\bar{\eta}_0 = \left(\frac{\bar{\xi}_0^2}{\sigma_1} + \sigma_1\bar{\eta}_0^2\right)\sin s_1 t. \tag{3.114}$$

To remove sine, we multiply the first equation of Equation (3.113) by $\sigma_1\bar{\eta}_0$ and the second by $\frac{\bar{\xi}_0}{\sigma_1}$, then adding yields

$$\sigma_1\bar{\eta}\bar{\eta}_0 + \frac{\bar{\xi}\bar{\xi}_0}{\sigma_1} = \left(\sigma_1\bar{\eta}_0^2 + \frac{\bar{\xi}_0^2}{\sigma_1}\right)\cos s_1 t. \tag{3.115}$$

Squaring and adding Equations (3.114) and (3.115), we obtain

$$\left(\bar{\eta}\bar{\xi}_0 - \bar{\xi}\bar{\eta}_0\right)^2 + \left(\sigma_1\bar{\eta}\bar{\eta}_0 + \frac{\bar{\xi}\bar{\xi}_0}{\sigma_1}\right)^2 = \left(\sigma_1\bar{\eta}_0^{\,2} + \frac{\bar{\xi}_0^2}{\sigma_1}\right)^2.$$

$$\left(\bar{\eta}\bar{\xi}_0\right)^2 - 2\bar{\eta}\bar{\xi}\bar{\eta}_0\bar{\xi}_0 + \left(\bar{\xi}\bar{\eta}_0\right)^2 + \left(\sigma_1\bar{\eta}\bar{\eta}_0\right)^2 + 2\bar{\eta}\bar{\eta}_0\bar{\xi}\bar{\xi}_0 + \left(\frac{\bar{\xi}\bar{\xi}_0}{\sigma_1}\right)^2 = \left(\sigma_1\bar{\eta}_0^{\,2} + \frac{\bar{\xi}_0^2}{\sigma_1}\right)^2.$$

$$\left(\bar{\eta}\bar{\xi}_0\right)^2 + \left(\bar{\xi}\bar{\eta}_0\right)^2 + \left(\sigma_1\bar{\eta}\bar{\eta}_0\right)^2 + \left(\frac{\bar{\xi}\bar{\xi}_0}{\sigma_1}\right)^2 = \left(\sigma_1\bar{\eta}_0^{\,2} + \frac{\bar{\xi}_0^2}{\sigma_1}\right)^2.$$

$$\bar{\eta}^2\left(\bar{\xi}_0^2 + \sigma_1^2\bar{\eta}_0^2\right) + \bar{\xi}^2\left(\frac{\sigma_1^2\bar{\eta}_0^2 + \bar{\xi}_0^2}{\sigma_1^2}\right) = \frac{\left(\sigma_1^2\bar{\eta}_0^{\,2} + \bar{\xi}_0^2\right)^2}{\sigma_1^2}.$$

$$\bar{\eta}^2 + \frac{\bar{\xi}^2}{\sigma_1^2} = \frac{\sigma_1^2\bar{\eta}_0^{\,2} + \bar{\xi}_0^2}{\sigma_1^2}.$$

$$\frac{\sigma_1^2\bar{\eta}^2}{\sigma_1^2\bar{\eta}_0^{\,2} + \bar{\xi}_0^2} + \frac{\bar{\xi}^2}{\left(\sigma_1^2\bar{\eta}_0^{\,2} + \bar{\xi}_0^2\right)} = 1.$$

$$\frac{\bar{\eta}^2}{\bar{\eta}_0^{\,2} + \frac{\bar{\xi}_0^2}{\sigma_1^2}} + \frac{\bar{\xi}^2}{\left(\sigma_1^2\bar{\eta}_0^{\,2} + \bar{\xi}_0^2\right)} = 1.$$

Or

$$\frac{\bar{\eta}^2}{a_1^2}+\frac{\bar{\xi}^2}{b_1^2}=1,$$

$$where\ \ a_1^2=\bar{\eta}_0{}^2+\frac{\bar{\xi}_0^2}{\sigma_1^2}\quad and\quad b_1^2=\sigma_1^2\bar{\eta}_0{}^2+\bar{\xi}_0^2. \tag{3.116}$$

Equation (3.116) is the equation of the elliptical orbits of the long periodic orbit, a_1 its length of the semi-major axis and $b_1$the semi-minor axis. Therefore, its eccentricity $e_1$isgiven by

$$\begin{aligned} e_1^2 &= 1-\frac{b_1^2}{a_1^2} \\ &= 1-\sigma_1^2. \end{aligned}$$

In the same manner, the constants associated with the long periodic terms can be set to zero in Equation (3.104), in order to obtain the eccentricity of the short periodic orbit e_2.

In summary, the eccentricities of the long periodic orbit e_1 and the short periodic orbit e_2 are

$$e_i=\sqrt{\left(1-\sigma_i^2\right)}, \tag{3.117}$$

where

$$\sigma_i=\frac{(s_i^2+\bar{\lambda}_1)}{2s_i}=\frac{2s_i}{s_i^2+\bar{\lambda}_2}\quad (i=1,2).$$

We will determine the eccentricities, semi-major and semi-minor axes of the long and short periodic orbits:

Eccentricity of the ellipses

In view of Equation (3.117), the eccentricities of the long and short periodic orbits are given correspondingly as

$$e_i = \sqrt{\left(1-\sigma_i^2\right)} \quad (3.118)$$

where

$$\sigma_i = \frac{2s_i}{s_i^2+\bar{\lambda}_2} \quad \left(i=1,2\right).$$

Here, s_1, s_2 and $\bar{\lambda}_2$ are specified in Equations (3.90), (3.91) and (3.102) respectively:

The eccentricity of the long period orbit is obtained using Equation (3.118) when i =1. That is

$$\sigma_1^2 = \frac{4s_1^2}{\left(s_1^2+\bar{\lambda}_2\right)^2} = 4s_1^2 \times \frac{1}{\left(s_1^2+\bar{\lambda}_2\right)^2} \quad (3.119)$$

Now,

$$s_1^2+\bar{\lambda}_2=\left[\mu(1-\mu)\left(\frac{27}{4}+\frac{3(p_1+p_2)}{2}+\frac{117A_1}{4}-45A_2+\frac{117B_1}{4}-45B_2+\frac{33}{2}M_{b3}+\frac{27}{4}M_{b4}\right)-\frac{15A_2}{8}-\frac{15B_2}{8}\right]$$

$$+\left(\frac{9}{2}-\frac{27}{4}\mu+\frac{39}{4}\mu^2\right)B_1+\left[3-\frac{9}{4}\left(\mu-\mu^2\right)-\frac{1}{2}\left(\mu-\mu^2\right)(p_1+p_2)+\left(\frac{15}{2}-\frac{51}{4}\mu+\frac{39}{4}\mu^2\right)A_1-\left(\frac{105}{8}-\frac{45}{2}\mu+15\mu^2\right)A_2\right.$$

$$\left.-\left(\frac{45}{8}-\frac{15}{2}\mu+15\mu^2\right)B_2+\left(3-\frac{11}{2}(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(3-\frac{9}{4}(\mu-\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\right].$$

$$s_1^2+\bar{\lambda}_2=3+\frac{9}{2}\left(\mu-\mu^2\right)+\left(\mu-\mu^2\right)(p_1+p_2)+\left(\frac{15}{2}+\frac{33}{2}\mu-\frac{39}{2}\mu^2\right)A_1-\left(15+\frac{45}{2}\mu-30\mu^2\right)A_2+\left(\frac{9}{2}+\frac{45}{2}\mu-\frac{39}{2}\mu^2\right)B_1$$

$$-\left(\frac{15}{2}+\frac{75}{2}\mu-30\mu^2\right)B_2+\left(\frac{39}{2}-\frac{11}{2}(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27}{4}M_{b4}+\left(3-\frac{9}{4}(\mu-\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}.$$

$$s_1^2+\bar{\lambda}_2=3\left[1+\frac{3}{2}\left(\mu-\mu^2\right)+\frac{1}{3}\left(\mu-\mu^2\right)(p_1+p_2)+\left(\frac{5}{2}+\frac{11}{2}\mu-\frac{13}{2}\mu^2\right)A_1-\left(5+\frac{15}{2}\mu-10\mu^2\right)A_2\right.$$

$$+\left(\frac{3}{2}+\frac{15}{2}\mu-\frac{13}{2}\mu^2\right)B_1-\left(\frac{5}{2}+\frac{25}{2}\mu-10\mu^2\right)B_2+\left(\frac{13}{2}-\frac{11}{6}(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{9}{4}M_{b4}$$

$$\left.+\left(1-\frac{2}{4}(\mu-\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\right]$$

Taking reciprocal of both sides and utilizing Binomial expansion, we get

$$\frac{1}{s_1^2+\bar{\lambda}_2}=\frac{1}{3}\left[1-\frac{3}{2}\left(\mu-\mu^2\right)-\frac{1}{3}\left(\mu-\mu^2\right)(p_1+p_2)-\left(\frac{5}{2}+\frac{11}{2}\mu-\frac{13}{2}\mu^2\right)A_1+\left(5+\frac{15}{2}\mu-10\mu^2\right)A_2\right.$$

$$-\left(\frac{3}{2}+\frac{15}{2}\mu-\frac{13}{2}\mu^2\right)B_1+\left(\frac{5}{2}+\frac{25}{2}\mu-10\mu^2\right)B_2-\left(\frac{13}{2}-\frac{11}{6}(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}-\frac{9}{4}M_{b4}$$

$$\left.-\left(1-\frac{3}{4}(\mu-\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\right].$$

Squaring both sides and applying Binomial expansion, yield

$$\frac{1}{\left(s_1^2+\bar{\lambda}_2\right)^2}=\frac{1}{9}1-3\left(\mu-\mu^2\right)-\frac{2}{3}\left(\mu-\mu^2\right)(p_1+p_2)-\left(5+11\mu-13\mu^2\right)A_1$$

$$+\left(10+15\mu-20\mu^2\right)A_2-\left(3+15\mu-13\mu^2\right)B_1+\left(5+25\mu-20\mu^2\right)B_2$$

$$-\left(13-\frac{11}{3}(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}-\frac{9}{2}M_{b4}-\left(2-\frac{3}{2}(\mu-\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}.$$

(3.120)

Using Equation (3.120) in Equation (3.119), gives

$$\sigma_1^2 = 4\left[\mu(1-\mu)\left(\frac{27}{4}+\frac{3(p_1+p_2)}{2}+\frac{117A_1}{4}-45A_2+\frac{117B_1}{4}-45B_2+\frac{33}{2}M_{b3}+\frac{27}{4}M_{b4}\right)-\frac{15A_2}{8}-\frac{15B_2}{8}\right]$$

$$\times\frac{1}{9}\left[\begin{array}{l} 1-3\left(\mu-\mu^2\right)-\frac{2}{3}\left(\mu-\mu^2\right)(p_1+p_2)-\left(5+11\mu-13\mu^2\right)A_1+\left(10+15\mu-20\mu^2\right)A_2-\left(3+15\mu-13\mu^2\right)B_1 \\ +\left(5+25\mu-20\mu^2\right)B_2-\left(13-\frac{11}{3}(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}-\frac{9}{2}M_{b4}-\left(2-\frac{3}{2}(\mu-\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}} \end{array}\right]$$

$$=\left[\mu(1-\mu)\left(27+6(p_1+p_2)+117A_1-180A_2+117B_1-180B_2+66M_{b3}+27M_{b4}\right)-\frac{15A_2}{2}-\frac{15B_2}{2}\right]$$

$$\times\frac{1}{9}\left[\begin{array}{l} 1-3\left(\mu-\mu^2\right)-\frac{2}{3}\left(\mu-\mu^2\right)(p_1+p_2)-\left(5+11\mu-13\mu^2\right)A_1+\left(10+15\mu-20\mu^2\right)A_2-\left(3+15\mu-13\mu^2\right)B_1 \\ +\left(5+25\mu-20\mu^2\right)B_2-\left(13-\frac{11}{3}(\mu-\mu^2)\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}-\frac{9}{2}M_{b4}-\left(2-\frac{3}{2}(\mu-\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}} \end{array}\right].$$

Open out and considering only linear terms in small quantities, produces

$$\sigma_1^2 = \frac{1}{9}\left\{27\left(\mu-4\mu^2\right)+\left(6\mu-42\mu^2\right)(p_1+p_2)-\left(18\mu+630\mu^2\right)A_1+\left(-\frac{15}{2}+\frac{225}{2}\mu+\frac{1665}{2}\mu^2\right)A_2\right.$$
$$+\left(36\mu-792\mu^2\right)B_1+\left(-\frac{15}{2}-\frac{45}{2}\mu+\frac{2475}{2}\mu^2\right)B_2+\left(-\frac{189}{2}\mu-\frac{459}{2}\mu^2\right)M_{b4}$$
$$\left.+\left(-285\mu+186\mu^2\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(-54\mu+\frac{189}{2}\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\right\}.$$

Or

$$\sigma_1^2 = 3\left(\mu-4\mu^2\right)+\left(\frac{2}{3}\mu-\frac{14}{3}\mu^2\right)(p_1+p_2)-\left(2\mu+70\mu^2\right)A_1+\left(-\frac{5}{6}+\frac{25}{2}\mu+\frac{185}{2}\mu^2\right)A_2$$
$$+\left(4\mu-88\mu^2\right)B_1+\left(-\frac{5}{6}-\frac{5}{2}\mu+\frac{275}{2}\mu^2\right)B_2+\left(-\frac{21}{2}\mu-\frac{51}{2}\mu^2\right)M_{b4}$$
$$+\left(-\frac{95}{3}\mu+\frac{62}{3}\mu^2\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(-6\mu+\frac{21}{2}\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}.$$

(3.121)

Substituting Equation (3.121) in Equation (3.118) when *i* =1, turns out

$$e_1=\left[1-\left\{\begin{aligned}&\left(3\mu-12\mu^2\right)+\left(\frac{2}{3}\mu-\frac{14}{3}\mu^2\right)(p_1+p_2)-\left(2\mu+70\mu^2\right)A_1+\left(-\frac{5}{6}+\frac{25}{2}\mu+\frac{185}{2}\mu^2\right)A_2\\&+\left(4\mu-88\mu^2\right)B_1+\left(-\frac{5}{6}-\frac{5}{2}\mu+\frac{275}{2}\mu^2\right)B_2+\left(-\frac{21}{2}\mu-\frac{51}{2}\mu^2\right)M_{b4}\\&+\left(-\frac{95}{3}\mu+\frac{62}{3}\mu^2\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(-6\mu+\frac{21}{2}\mu^2\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\end{aligned}\right\}\right]^{\frac{1}{2}}.$$

Using binomial expansion and retaining only linear terms of very small parameters, yield the eccentricity of the long period orbits of the ellipse as

$$\begin{aligned}e_1=1&-\left(\frac{3}{2}\mu-6\mu^2\right)-\left(\frac{1}{3}\mu-\frac{7}{3}\mu^2\right)(p_1+p_2)+\left(\mu+35\mu^2\right)A_1-\left(-\frac{5}{12}+\frac{25}{4}\mu+\frac{185}{4}\mu^2\right)A_2\\&-\left(2\mu-44\mu^2\right)B_1-\left(-\frac{5}{12}-\frac{5}{4}\mu+\frac{275}{4}\mu^2\right)B_2-\left(-\frac{21}{4}\mu-\frac{51}{4}\mu^2\right)M_{b4}\\&-\left(-\frac{95}{6}\mu+\frac{31}{3}\mu^2\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}-\left(-3\mu+\frac{21}{4}\mu^2\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}.\end{aligned}$$

Or

$$\begin{aligned}e_1=1&-\left(\frac{3}{2}\mu-6\mu^2\right)-\left(\frac{1}{3}\mu-\frac{7}{3}\mu^2\right)(p_1+p_2)+\left(\mu+35\mu^2\right)A_1-\left(-\frac{5}{12}+\frac{25}{4}\mu+\frac{185}{4}\mu^2\right)A_2\\&-\left(2\mu-44\mu^2\right)B_1-\left(-\frac{5}{12}-\frac{5}{4}\mu+\frac{275}{4}\mu^2\right)B_2+\left(\frac{95}{6}\mu-\frac{31}{3}\mu^2\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}\\&+\left(\frac{(21+12r_c^2)\mu}{4}+\frac{(51-21r_c^2)\mu^2}{4}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}.\end{aligned}\tag{3.122}$$

In a similar manner the eccentricity of the short period orbit is obtained by using Equation (3.118) when *i*=2. That is

$$e_2=\sqrt{\left(1-\sigma_2^2\right)}\tag{3.123}$$

with

$$\sigma_2^2 = \frac{4s_2^2}{\left(s_2^2+\bar{\lambda}_2\right)^2} = 4s_2^2 \times \frac{1}{\left(s_2^2+\bar{\lambda}_2\right)^2}. \tag{3.124}$$

Now, from Equations (3.91) and (3.102), we have

$$s_2^2+\bar{\lambda}_2 = \begin{bmatrix} 1+\frac{27}{4}\mu(\mu-1)+(-6+117\mu^2-105\mu)\frac{A_1}{4}+(15+75\mu-90\mu^2)\frac{A_2}{2}+(6+117\mu^2-129\mu)\frac{B_1}{4} \\ +(105\mu-90\mu^2)\frac{B_2}{2}+M_{b1}-3M_{b2}+\frac{33}{2}M_{b3}\mu(\mu-1)+\frac{27}{4}M_{b4}\mu(\mu-1) \end{bmatrix}$$

$$+\begin{bmatrix} 3-\frac{9}{4}\left(\mu-\mu^2\right)+\left(\frac{15}{2}-\frac{51}{4}\mu+\frac{39}{4}\mu^2\right)A_1-\left(\frac{105}{8}-\frac{45}{2}\mu+15\mu^2\right)A_2+\left(\frac{9}{2}-\frac{27}{4}\mu+\frac{39}{4}\mu^2\right)B_1 \\ -\left(\frac{45}{8}-\frac{15}{2}\mu+15\mu^2\right)B_2+\left(3-\frac{11}{2}(\mu-\mu^2)\right)M_{b3}+\left(3-\frac{9}{4}(\mu-\mu^2)\right)M_{b2} \end{bmatrix}.$$

$$s_2^2+\bar{\lambda}_2 = 4-9\left(\mu-\mu^2\right)+\left(6-39\mu+39\mu^2\right)A_1+\left(-\frac{45}{8}+60\mu+60\mu^2\right)A_2+\left(6-39\mu+39\mu^2\right)B_1$$
$$-\left(\frac{45}{8}-60\mu+60\mu^2\right)B_2+M_{b1}-\frac{9}{4}\left(\mu-\mu^2\right)M_{b2}+\left(3-22(\mu-\mu^2)\right)M_{b3}-\frac{27}{4}(\mu-\mu^2)M_{b4}.$$

$$=4\Bigg[1-\frac{9}{4}\left(\mu-\mu^2\right)+\frac{1}{4}\left(6-39\mu+39\mu^2\right)A_1+\frac{1}{4}\left(-\frac{45}{8}+60\mu+60\mu^2\right)A_2+\frac{1}{4}\left(6-39\mu+39\mu^2\right)B_1$$
$$-\frac{1}{4}\left(\frac{45}{8}-60\mu+60\mu^2\right)B_2+\frac{1}{4}M_{b1}-\frac{9}{16}\left(\mu-\mu^2\right)M_{b2}+\frac{1}{4}\left(3-22(\mu-\mu^2)\right)M_{b3}-\frac{27}{16}(\mu-\mu^2)M_{b4}\Bigg].$$

Squaring and take reciprocal gives

$$\frac{1}{\left(s_2^2+\bar{\lambda}_2\right)^2} = \frac{1}{16}\begin{bmatrix} 1-\frac{9}{4}\left(\mu-\mu^2\right)+\frac{1}{4}\left(6-39\mu+39\mu^2\right)A_1+\frac{1}{4}\left(-\frac{45}{8}+60\mu+60\mu^2\right)A_2 \\ +\frac{1}{4}\left(6-39\mu+39\mu^2\right)B_1-\frac{1}{4}\left(\frac{45}{8}-60\mu+60\mu^2\right)B_2+\frac{1}{4}M_{b1}-\frac{9}{16}\left(\mu-\mu^2\right)M_{b2} \\ +\frac{1}{4}\left(3-22(\mu-\mu^2)\right)M_{b3}-\frac{27}{16}(\mu-\mu^2)M_{b4} \end{bmatrix}^{-2}.$$

$$=\frac{1}{16}\begin{bmatrix} 1+\frac{9}{2}\left(\mu-\mu^2\right)-\frac{1}{2}\left(6-39\mu+39\mu^2\right)A_1-\frac{1}{2}\left(-\frac{45}{8}+60\mu+60\mu^2\right)A_2-\frac{1}{2}\left(6-39\mu+39\mu^2\right)B_1 \\ +\frac{1}{2}\left(\frac{45}{8}-60\mu+60\mu^2\right)B_2-\frac{1}{2}M_{b1}+\frac{9}{8}\left(\mu-\mu^2\right)M_{b2}-\frac{1}{2}\left(3-22(\mu-\mu^2)\right)M_{b3}+\frac{27}{8}(\mu-\mu^2)M_{b4} \end{bmatrix} \tag{3.125}$$

Making use of Equation (3.125) in Equation (3.124), produces

$$\sigma_2^2=4\begin{bmatrix} 1+\frac{27}{4}\mu(\mu-1)+(-6+117\mu^2-105\mu)\frac{A_1}{4}+(15+75\mu-90\mu^2)\frac{A_2}{2}+(6+117\mu^2-129\mu)\frac{B_1}{4} \\ +(105\mu-90\mu^2)\frac{B_2}{2}+M_{b1}-3M_{b2}+\frac{33}{2}M_{b3}\mu(\mu-1)+\frac{27}{4}M_{b4}\mu(\mu-1) \end{bmatrix}$$

$$\times\frac{1}{16}\begin{bmatrix} 1+\frac{9}{2}\left(\mu-\mu^2\right)-\frac{1}{2}\left(6-39\mu+39\mu^2\right)A_1-\frac{1}{2}\left(-\frac{45}{8}+60\mu+60\mu^2\right)A_2-\frac{1}{2}\left(6-39\mu+39\mu^2\right)B_1 \\ +\frac{1}{2}\left(\frac{45}{8}-60\mu+60\mu^2\right)B_2-\frac{1}{2}M_{b1}+\frac{9}{8}\left(\mu-\mu^2\right)M_{b2}-\frac{1}{2}\left(3-22(\mu-\mu^2)\right)M_{b3}+\frac{27}{8}(\mu-\mu^2)M_{b4} \end{bmatrix}.$$

$$=\frac{1}{4}\begin{bmatrix} 1+\frac{27}{4}\mu(\mu-1)+(-6+117\mu^2-105\mu)\frac{A_1}{4}+(15+75\mu-90\mu^2)\frac{A_2}{2}+(6+117\mu^2-129\mu)\frac{B_1}{4} \\ +(105\mu-90\mu^2)\frac{B_2}{2}+M_{b1}-3M_{b2}+\frac{33}{2}M_{b3}\mu(\mu-1)+\frac{27}{4}M_{b4}\mu(\mu-1) \end{bmatrix}$$

$$\times\begin{bmatrix} 1+\frac{9}{2}\left(\mu-\mu^2\right)-\left(3-\frac{39}{2}\mu+\frac{39}{2}\mu^2\right)A_1-\left(-\frac{45}{16}+30\mu+30\mu^2\right)A_2-\left(3-\frac{39}{2}\mu+\frac{39}{2}\mu^2\right)B_1 \\ +\left(\frac{45}{16}-30\mu+30\mu^2\right)B_2-\frac{1}{2}M_{b1}+\frac{9}{8}\left(\mu-\mu^2\right)M_{b2}-\left(\frac{3}{2}-11(\mu-\mu^2)\right)M_{b3}+\frac{27}{8}(\mu-\mu^2)M_{b4} \end{bmatrix}$$

Expanding and consider only linear terms in small quantities, yield to

$$\sigma_2^2=\frac{1}{4}\left\{\begin{array}{l} 1-\dfrac{9}{4}\mu-\dfrac{225}{8}\mu^2+\left(-\dfrac{9}{2}+\dfrac{27}{4}\mu-\dfrac{507}{2}\mu^2\right)A_1+\left(\dfrac{165}{16}+\dfrac{3585}{64}\mu+\dfrac{2805}{4}\mu^2\right)A_2 \\ +\left(-\dfrac{3}{2}+\dfrac{57}{4}\mu-294\mu^2\right)B_1+\left(\dfrac{45}{16}+\dfrac{225}{64}\mu+\dfrac{28335}{64}\mu^2\right)B_2+\left(\dfrac{1}{2}+\dfrac{63}{8}\mu-\dfrac{63}{8}\mu^2\right)M_{b1} \\ +\left(-3-\dfrac{99}{8}\mu+\dfrac{153}{32}\mu^2\right)M_{b2}+\left(-\dfrac{3}{2}+\dfrac{37}{8}\mu-\dfrac{1225}{8}\mu^2\right)M_{b3}-\dfrac{1593}{32}\mu^2M_{b4} \end{array}\right\}.$$

$$\begin{aligned} &=\frac{1}{4}-\frac{9}{16}\mu-\frac{225}{32}\mu^2+\left(-\frac{9}{8}+\frac{27}{16}\mu-\frac{507}{8}\mu^2\right)A_1+\left(\frac{165}{64}+\frac{3585}{256}\mu+\frac{2805}{16}\mu^2\right)A_2 \\ &\quad+\left(-\frac{3}{8}+\frac{57}{16}\mu-\frac{147}{2}\mu^2\right)B_1+\left(\frac{45}{64}+\frac{225}{256}\mu+\frac{28335}{256}\mu^2\right)B_2+\left(\frac{1}{8}+\frac{63}{32}\mu-\frac{63}{32}\mu^2\right)M_{b1} \\ &\quad+\left(-\frac{3}{4}-\frac{99}{32}\mu+\frac{153}{128}\mu^2\right)M_{b2}+\left(-\frac{3}{8}+\frac{37}{32}\mu-\frac{1225}{32}\mu^2\right)M_{b3}-\frac{1593}{128}\mu^2M_{b4}. \end{aligned}$$

(3.126)

Plugging Equation (3.126) in Equation (3.123), produces

$$e_2=\left[1-\left(\begin{array}{l} \dfrac{1}{4}-\dfrac{9}{16}\mu-\dfrac{225}{32}\mu^2+\left(-\dfrac{9}{8}+\dfrac{27}{16}\mu-\dfrac{507}{8}\mu^2\right)A_1+\left(\dfrac{165}{64}+\dfrac{3585}{256}\mu+\dfrac{2805}{16}\mu^2\right)A_2 \\ +\left(-\dfrac{3}{8}+\dfrac{57}{16}\mu-\dfrac{147}{2}\mu^2\right)B_1+\left(\dfrac{45}{64}+\dfrac{225}{256}\mu+\dfrac{28335}{256}\mu^2\right)B_2+\left(\dfrac{1}{8}+\dfrac{63}{32}\mu-\dfrac{63}{32}\mu^2\right)M_{b1} \\ +\left(-\dfrac{3}{4}-\dfrac{99}{32}\mu+\dfrac{153}{128}\mu^2\right)M_{b2}+\left(-\dfrac{3}{8}+\dfrac{37}{32}\mu-\dfrac{1225}{32}\mu^2\right)M_{b3}-\dfrac{1593}{128}\mu^2M_{b4}. \end{array}\right)\right]^{\frac{1}{2}}$$

$$e_2=\left[\begin{array}{l}1-\frac{1}{4}+\frac{9}{16}\mu+\frac{225}{32}\mu^2-\left(-\frac{9}{8}+\frac{27}{16}\mu-\frac{507}{8}\mu^2\right)A_1-\left(\frac{165}{64}+\frac{3585}{256}\mu+\frac{2805}{16}\mu^2\right)A_2\\ -\left(-\frac{3}{8}+\frac{57}{16}\mu-\frac{147}{2}\mu^2\right)B_1-\left(\frac{45}{64}+\frac{225}{256}\mu+\frac{28335}{256}\mu^2\right)B_2-\left(\frac{1}{8}+\frac{63}{32}\mu-\frac{63}{32}\mu^2\right)M_{b1}\\ -\left(-\frac{3}{4}-\frac{99}{32}\mu+\frac{153}{128}\mu^2\right)M_{b2}-\left(-\frac{3}{8}+\frac{37}{32}\mu-\frac{1225}{32}\mu^2\right)M_{b3}+\frac{1593}{128}\mu^2M_{b4}.\end{array}\right]^{\frac{1}{2}}.$$

$$=\left[\begin{array}{l}\frac{3}{4}+\frac{9}{16}\mu+\frac{225}{32}\mu^2-\left(-\frac{9}{8}+\frac{27}{16}\mu-\frac{507}{8}\mu^2\right)A_1-\left(\frac{165}{64}+\frac{3585}{256}\mu+\frac{2805}{16}\mu^2\right)A_2\\ -\left(-\frac{3}{8}+\frac{57}{16}\mu-\frac{147}{2}\mu^2\right)B_1-\left(\frac{45}{64}+\frac{225}{256}\mu+\frac{28335}{256}\mu^2\right)B_2-\left(\frac{1}{8}+\frac{63}{32}\mu-\frac{63}{32}\mu^2\right)M_{b1}\\ -\left(-\frac{3}{4}-\frac{99}{32}\mu+\frac{153}{128}\mu^2\right)M_{b2}-\left(-\frac{3}{8}+\frac{37}{32}\mu-\frac{1225}{32}\mu^2\right)M_{b3}+\frac{1593}{128}\mu^2M_{b4}.\end{array}\right]^{\frac{1}{2}}$$

$$=\frac{\sqrt{3}}{2}\left[\begin{array}{l}1+\frac{3}{4}\mu+\frac{75}{8}\mu^2-\left(-\frac{3}{2}+\frac{9}{4}\mu-\frac{169}{2}\mu^2\right)A_1-\left(\frac{55}{16}+\frac{1195}{64}\mu+\frac{935}{4}\mu^2\right)A_2\\ -\left(-\frac{1}{2}+\frac{19}{4}\mu-98\mu^2\right)B_1-\left(\frac{15}{16}+\frac{75}{64}\mu+\frac{9445}{64}\mu^2\right)B_2-\left(\frac{1}{6}+\frac{21}{8}\mu-\frac{21}{8}\mu^2\right)M_{b1}\\ -\left(-1-\frac{33}{8}\mu+\frac{51}{32}\mu^2\right)M_{b2}-\left(-\frac{1}{2}+\frac{37}{24}\mu-\frac{1225}{24}\mu^2\right)M_{b3}+\frac{531}{32}\mu^2M_{b4}.\end{array}\right]^{\frac{1}{2}}$$

Applying Binomial expansion and consider only linear terms in small quantities, yield the eccentricity of the short period orbit as

$$\begin{aligned}e_2=\frac{\sqrt{3}}{2}\Bigg\{&1+\frac{3}{8}\mu+\frac{75}{16}\mu^2+\left(\frac{\mu}{12}+\frac{13\mu^2}{6}\right)(p_1+p_2)+\left(\frac{3}{4}-\frac{9}{8}\mu+\frac{169}{4}\mu^2\right)A_1-\left(\frac{55}{33}+\frac{1195}{128}\mu+\frac{935}{8}\mu^2\right)A_2\\ &+\left(\frac{1}{4}-\frac{19}{8}\mu+49\mu^2\right)B_1-\left(\frac{15}{32}+\frac{75}{128}\mu+\frac{9445}{128}\mu^2\right)B_2-\left(\frac{1}{12}+\frac{21}{16}\mu-\frac{21}{16}\mu^2\right)M_{b1}\\ &+\left(\frac{1}{2}+\frac{33}{16}\mu-\frac{51}{64}\mu^2\right)M_{b2}+\left(\frac{1}{4}-\frac{37}{48}\mu+\frac{1225}{48}\mu^2\right)M_{b3}+\frac{531}{64}\mu^2M_{b4}\Bigg\}.\end{aligned}$$

Or

$$
\begin{aligned}
e_2 = \frac{\sqrt{3}}{2}\Bigg\{ & 1+\frac{3}{8}\mu+\frac{75}{16}\mu^2+\left(\frac{\mu}{12}+\frac{13\mu^2}{6}\right)(p_1+p_2)+\left(\frac{3}{4}-\frac{9}{8}\mu+\frac{169}{4}\mu^2\right)A_1-\left(\frac{55}{33}+\frac{1195}{128}\mu+\frac{935}{8}\mu^2\right)A_2 \\
& +\left(\frac{1}{4}-\frac{19}{8}\mu+49\mu^2\right)B_1-\left(\frac{15}{32}+\frac{75}{128}\mu+\frac{9445}{128}\mu^2\right)B_2-\left(\frac{(2r_c+3)}{6}-\frac{(98r_c-73)\mu}{24}\right. \\
& \left.-\frac{(1285r_c-521)\mu^2}{24}\right)\frac{M_b}{(r_c^2+T^2)^{3/2}}+\left(\frac{r_c^2}{2}+\frac{33r_c^2\mu}{16}+\frac{(531-51r_c^2)\mu^2}{64}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}\Bigg\}.
\end{aligned}
\tag{3.127}
$$

Using Equations (3.122) and (3.127) we examine the effects of the perturbations on the eccentricities of the elliptic orbits which are presented in Table 3.11

Table 3.11: Effects of the perturbations on the eccentricities (μ=0.03,T=0.01, r_c=0.99)

case	q_1	q_2	A_1	A_2	B_1	B_2	M_b	e_1	e_2
1	1	1	0	0	0	0	0	0.9604000000	0.8794217342
	0.98	1	0	0	0	0	0	0.9602420000	0.8794988105
2	0.96	1	0	0	0	0	0	0.9600840000	0.8795758867

	0.94	1	0	0	0	0	0	0.9599260000	0.8796529630
3	1	0.97	0	0	0	0	0	0.9601630000	0.8795373486
	1	0.95	0	0	0	0	0	0.9600050000	0.8796144249
	1	0.93	0	0	0	0	0	0.9598470000	0.8796915011
4	1	1	0.01	0	0	0	0	0.9610150000	0.8859539473
	1	1	0.02	0	0	0	0	0.9616300000	0.8924861604
	1	1	0.03	0	0	0	0	0.9622450000	0.8990183735
	1	1	0.01	0.005	0	0	0	0.9619527083	0.8770688199
	1	1	0.02	0.01	0	0	0	0.9635054166	0.8747159055
	1	1	0.03	0.015	0	0	0	0.9650581250	0.8723629912
5	1	1	0	0	0.01	0	0	0.9601960000	0.8813516718
	1	1	0	0	0.02	0	0	0.9599920000	0.8832816094
	1	1	0	0	0.03	0	0	0.9597880000	0.8852115470
	1	1	0	0	0.01	0.005	0	0.9621574583	0.8789582448
	1	1	0	0	0.02	0.01	0	0.9639149166	0.8784947555
	1	1	0	0	0.03	0.015	0	0.9656723750	0.8780312661
6	1	1	0	0	0	0	0.01	0.9677578486	0.8776093784
	1	1	0	0	0	0	0.02	0.9751156973	0.8757970226
	1	1	0	0	0	0	0.03	0.9824735460	0.8739846668
7	0.99	0.98	0.01	0.005	0.01	0.005	0.01	0.9706730153	0.8749856654
	0.98	0.97	0.02	0.01	0.02	0.01	0.02	0.9810250306	0.8705110584
	0.97	0.96	0.03	0.015	0.03	0.015	0.03	0.9913770460	0.8660364514

In view of Equation (3.116), the lengths of the semi-major axes a_i and the semi-minor axes b_i of the long and short period orbits are determined correspondingly from the following expressions:

$$a_i = \sqrt{\eta_0^2 + \frac{\xi_0^2}{\sigma_i^2}}\,,\; b_i = \sqrt{\eta_0^2\sigma_i^2 + \xi_0^2}\;\left(i = 1, 2\right). \tag{3.128}$$

Here, η_0, ξ_0 are the coordinates of the triangular libration points taken as the initial conditions; σ_1, σ_2 are specified in Equations (3.121) and (3.126) respectively.

Now, the coordinates of the triangular points Equation (3.47) are

$$\eta_0 = \frac{1}{2}\left[1 - 2\mu - \frac{2p_1}{3} + \frac{2p_2}{3} + A_1 - \frac{5A_2}{4} - B_1 + \frac{5B_2}{4}\right],$$
$$\xi_0 = \pm\frac{\sqrt{3}}{2}\left(1 - \frac{2(p_1 + p_2)}{9} - \frac{A_1}{3} + \frac{5A_2}{12} - \frac{B_1}{3} + \frac{5B_2}{12} - \frac{4M_b(2r_c - 1)}{9(r_c^2 + T^2)^{3/2}}\right),$$

and squaring yields to

$$\eta_0^2 = \left\{\frac{1}{4} - \mu + \mu^2 - \frac{\left(1 - 2\mu\right)p_1}{3} + \frac{\left(1 - 2\mu\right)p_2}{3} + \frac{\left(1 - 2\mu\right)A_1}{2} - \frac{\left(1 - 2\mu\right)5A_2}{8} - \frac{\left(1 - 2\mu\right)B_1}{2} + \frac{5\left(1 - 2\mu\right)B_2}{8}\right\}.$$
$$\xi_0^2 = \left(\frac{3}{4} - \frac{(p_1 + p_2)}{3} - \frac{A_1}{2} + \frac{5A_2}{8} - \frac{B_1}{2} + \frac{5B_2}{8} - \frac{2M_b(2r_c - 1)}{3(r_c^2 + T^2)^{3/2}}\right). \tag{3.129}$$

From Equation (3.121), we have

$$\frac{1}{\sigma_1^2}=\frac{1}{\begin{bmatrix}3\mu-12\mu^2+\left(\frac{2\mu}{3}-\frac{14\mu^2}{3}\right)(p_1+p_2)-\left(2\mu+70\mu^2\right)A_1+\left(-\frac{5}{6}+\frac{25}{2}\mu+\frac{185}{2}\mu^2\right)A_2\\ +\left(4\mu-88\mu^2\right)B_1+\left(-\frac{5}{6}-\frac{5}{2}\mu+\frac{275}{2}\mu^2\right)B_2+\left(-\frac{21}{2}\mu-\frac{51}{2}\mu^2\right)M_{b4}\\ +\left(-\frac{95}{3}\mu+\frac{62}{3}\mu^2\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(-6\mu+\frac{21}{2}\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\end{bmatrix}}.$$

$$=\frac{1}{3\mu\begin{bmatrix}1-4\mu+\frac{1}{3\mu}\left(\frac{2\mu}{3}-\frac{14\mu^2}{3}\right)(p_1+p_2)-\frac{1}{3\mu}\left(2\mu+70\mu^2\right)A_1+\frac{1}{3\mu}\left(-\frac{5}{6}+\frac{25}{2}\mu+\frac{185}{2}\mu^2\right)A_2\\ +\frac{1}{3\mu}\left(4\mu-88\mu^2\right)B_1+\frac{1}{3\mu}\left(-\frac{5}{6}-\frac{5}{2}\mu+\frac{275}{2}\mu^2\right)B_2+\frac{1}{3\mu}\left(-\frac{21}{2}\mu-\frac{51}{2}\mu^2\right)M_{b4}\\ +\frac{1}{3\mu}\left(-\frac{95}{3}\mu+\frac{62}{3}\mu^2\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{1}{3\mu}\left(-6\mu+\frac{21}{2}\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\end{bmatrix}}.$$

$$=\frac{1}{3\mu}\begin{bmatrix}1-4\mu+\frac{1}{3\mu}\left(\frac{2\mu}{3}-\frac{14\mu^2}{3}\right)(p_1+p_2)-\frac{1}{3\mu}\left(2\mu+70\mu^2\right)A_1+\frac{1}{3\mu}\left(-\frac{5}{6}+\frac{25}{2}\mu+\frac{185}{2}\mu^2\right)A_2\\ +\frac{1}{3\mu}\left(4\mu-88\mu^2\right)B_1+\frac{1}{3\mu}\left(-\frac{5}{6}-\frac{5}{2}\mu+\frac{275}{2}\mu^2\right)B_2+\frac{1}{3\mu}\left(-\frac{21}{2}\mu-\frac{51}{2}\mu^2\right)M_{b4}\\ +\frac{1}{3\mu}\left(-\frac{95}{3}\mu+\frac{62}{3}\mu^2\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{1}{3\mu}\left(-6\mu+\frac{21}{2}\mu^2)\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\end{bmatrix}^{-1}.$$

$$=\frac{1}{3\mu}\left\{\begin{matrix}1+4\mu-\left(\frac{2}{9}-\frac{14\mu}{9}\right)(p_1+p_2)+\left(\frac{2}{3}+\frac{70}{3}\mu\right)A_1-\left(-\frac{5}{18\mu}+\frac{25}{6}+\frac{185}{6}\mu\right)A_2\\ -\left(\frac{4}{3}-\frac{88}{3}\mu\right)B_1-\left(-\frac{5}{18\mu}-\frac{5}{6}+\frac{275}{6}\mu\right)B_2-\left(-\frac{21}{6}-\frac{51}{6}\mu\right)M_{b4}\\ -\left(-\frac{95}{9}+\frac{62}{9}\mu\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}-\left(-2+\frac{21}{6}\mu\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\end{matrix}\right\}.$$

So that

$$\xi_0^2 \times \frac{1}{\sigma_1^2} = \left(\frac{3}{4} - \frac{(p_1+p_2)}{3} - \frac{A_1}{2} + \frac{5A_2}{8} - \frac{B_1}{2} + \frac{5B_2}{8} - \frac{2M_b(2r_c-1)}{3(r_c^2+T^2)^{3/2}}\right)$$

$$\times\left\{\begin{array}{l} \frac{1}{3\mu}+\frac{4}{3} - \left(\frac{2}{27\mu}-\frac{14}{27}\right)(p_1+p_2) + \left(\frac{2}{9\mu}+\frac{70}{9}\right)A_1 + \left(\frac{5}{54\mu^2}-\frac{25}{18\mu}-\frac{185}{18}\right)A_2 - \left(\frac{4}{9\mu}-\frac{88}{9}\right)B_1 \\ + \left(\frac{5}{54\mu^2}+\frac{5}{18\mu}-\frac{275}{18}\right)B_2 + \left(\frac{7}{6\mu}+\frac{17}{6}\right)M_{b4} + \left(\frac{95}{27\mu}-\frac{62}{27}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} + \left(\frac{2}{3\mu}-\frac{7}{6}\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}} \end{array}\right\}.$$

$$= 1 + \frac{1}{4\mu} - \left(\frac{1}{6\mu}+\frac{1}{18}\right)(p_1+p_2) + \frac{31A_1}{6} + \left(\frac{5}{72\mu^2}-\frac{5}{6\mu}-\frac{55}{8}\right)A_2 + \left(-\frac{1}{2\mu}+\frac{20}{3}\right)B_1$$

$$+\left(\frac{5}{72\mu^2}+\frac{5}{12\mu}-\frac{85}{8}\right)B_2 + \left(\frac{29}{12\mu}-\frac{47}{18}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} + \left(\frac{7}{8\mu}+\frac{51}{64}\right)M_{b4} + \left(\frac{1}{\mu}-\frac{7}{8}\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}.$$

and

$$\left(\eta_0^2+\xi_0^2\sigma_1^{-2}\right) = \left\{\begin{array}{l} \frac{1}{4} - \mu + \mu^2 - \frac{(1-2\mu)\,p_1}{3} + \frac{(1-2\mu)\,p_2}{3} + \frac{(1-2\mu)\,A_1}{2} - \frac{5(1-2\mu)\,A_2}{8} \\ -\frac{(1-2\mu)\,B_1}{2} + \frac{5(1-2\mu)\,B_2}{8} \end{array}\right\}$$

$$+\left[\begin{array}{l} 1 + \frac{1}{4\mu} - \left(\frac{1}{6\mu}+\frac{1}{18}\right)(p_1+p_2) + \frac{31A_1}{6} + \left(\frac{5}{72\mu^2}-\frac{5}{6\mu}-\frac{55}{8}\right)A_2 + \left(-\frac{1}{2\mu}+\frac{20}{3}\right)B_1 \\ +\left(\frac{5}{72\mu^2}+\frac{5}{12\mu}-\frac{85}{8}\right)B_2 + \left(\frac{29}{12\mu}-\frac{47}{18}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} + \left(\frac{7}{8\mu}+\frac{51}{64}\right)M_{b4} + \left(\frac{1}{\mu}-\frac{7}{8}\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}} \end{array}\right].$$

$$= \frac{5}{4} + \frac{1}{4\mu} - \mu + \mu^2 + \left(-\frac{7}{18}+\frac{2\mu}{3}-\frac{1}{6\mu}\right)p_1 + \left(\frac{5}{18}-\frac{2\mu}{3}-\frac{1}{6\mu}\right)p_2 + \left(\frac{17}{3}-\mu\right)A_1$$

$$+\left(-\frac{15}{2}+\frac{5\mu}{4}-\frac{5}{6\mu}+\frac{5}{72\mu^2}\right)A_2 + \left(\frac{37}{6}+\frac{1}{2\mu}+\mu\right)B_1 + \left(-10-\frac{5\mu}{4}+\frac{5}{12\mu}+\frac{5}{72\mu^2}\right)B_2$$

$$+\left(\frac{29}{12\mu}-\frac{47}{18}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} + \left(\frac{7}{8\mu}+\frac{51}{64}\right)M_{b4} + \left(\frac{1}{\mu}-\frac{7}{8}\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}.$$

Thus, the length of the semi-major axis of the long periodic orbit $a_1 = \sqrt{\eta_0^2+\xi_0^2\sigma_1^{-2}}$,

becomes

$$a_1 = \left[\begin{array}{l} \frac{5}{4}+\frac{1}{4\mu}-\mu+\mu^2+\left(-\frac{7}{18}+\frac{2\mu}{3}-\frac{1}{6\mu}\right)p_1+\left(\frac{5}{18}-\frac{2\mu}{3}-\frac{1}{6\mu}\right)p_2+\left(\frac{17}{3}-\mu\right)A_1 \\ +\left(-\frac{15}{2}+\frac{5\mu}{4}-\frac{5}{6\mu}+\frac{5}{72\mu^2}\right)A_2+\left(\frac{37}{6}+\frac{1}{2\mu}+\mu\right)B_1+\left(-10-\frac{5\mu}{4}+\frac{5}{12\mu}+\frac{5}{72\mu^2}\right)B_2 \\ +\left(\frac{29}{12\mu}-\frac{47}{18}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(\frac{7}{8\mu}+\frac{51}{64}\right)M_{b4}+\left(\frac{1}{\mu}-\frac{7}{8}\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}. \end{array}\right]^{\frac{1}{2}}$$

$$= \frac{\sqrt{5}}{2}\left[1+\frac{4}{5}\left(\begin{array}{l} \frac{1}{4\mu}-\mu+\mu^2+\left(-\frac{7}{18}+\frac{2\mu}{3}-\frac{1}{6\mu}\right)p_1+\left(\frac{5}{18}-\frac{2\mu}{3}-\frac{1}{6\mu}\right)p_2+\left(\frac{17}{3}-\mu\right)A_1 \\ +\left(-\frac{15}{2}+\frac{5\mu}{4}-\frac{5}{6\mu}+\frac{5}{72\mu^2}\right)A_2+\left(\frac{37}{6}+\frac{1}{2\mu}+\mu\right)B_1+\left(-10-\frac{5\mu}{4}+\frac{5}{12\mu}+\frac{5}{72\mu^2}\right)B_2 \\ +\left(\frac{29}{12\mu}-\frac{47}{18}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(\frac{7}{8\mu}+\frac{51}{64}\right)M_{b4}+\left(\frac{1}{\mu}-\frac{7}{8}\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}} \end{array}\right)\right]^{\frac{1}{2}}.$$

$$= \frac{\sqrt{5}}{2}\left[1+\frac{2}{5}\left(\begin{array}{l} \frac{1}{4\mu}-\mu+\mu^2+\left(-\frac{7}{18}+\frac{2\mu}{3}-\frac{1}{6\mu}\right)p_1+\left(\frac{5}{18}-\frac{2\mu}{3}-\frac{1}{6\mu}\right)p_2+\left(\frac{17}{3}-\mu\right)A_1 \\ +\left(-\frac{15}{2}+\frac{5\mu}{4}-\frac{5}{6\mu}+\frac{5}{72\mu^2}\right)A_2+\left(\frac{37}{6}+\frac{1}{2\mu}+\mu\right)B_1+\left(-10-\frac{5\mu}{4}+\frac{5}{12\mu}+\frac{5}{72\mu^2}\right)B_2 \\ +\left(\frac{29}{12\mu}-\frac{47}{18}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(\frac{7}{8\mu}+\frac{51}{64}\right)M_{b4}+\left(\frac{1}{\mu}-\frac{7}{8}\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}} \end{array}\right)\right].$$

Or

$$\begin{aligned} a_1 = \frac{\sqrt{5}}{2}\Bigg[& 1+\frac{1}{10\mu}-\frac{2}{5}\mu+\frac{2}{5}\mu^2+\left(-\frac{7}{45}+\frac{4\mu}{15}-\frac{1}{15\mu}\right)p_1+\left(\frac{1}{9}-\frac{4\mu}{15}-\frac{1}{15\mu}\right)p_2+\left(\frac{34}{15}-\frac{2}{5}\mu\right)A_1 \\ & +\left(-3+\frac{\mu}{2}-\frac{1}{3\mu}+\frac{1}{36\mu^2}\right)A_2+\left(\frac{37}{15}+\frac{1}{5\mu}+\frac{2}{5}\mu\right)B_1+\left(-4-\frac{\mu}{2}+\frac{1}{6\mu}+\frac{1}{36\mu^2}\right)B_2 \\ & +\left(\frac{29}{30\mu}-\frac{47}{45}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(\frac{7}{20\mu}+\frac{51}{185}\right)M_{b4}+\left(\frac{2}{5\mu}-\frac{7}{20}\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\Bigg]. \end{aligned}$$

Or

$$a_1 = \frac{\sqrt{5}}{2}\left[1 + \frac{1}{10\mu} - \frac{2}{5}\mu + \frac{2}{5}\mu^2 + \left(-\frac{7}{45} + \frac{4\mu}{15} - \frac{1}{15\mu}\right)p_1 + \left(\frac{1}{9} - \frac{4\mu}{15} - \frac{1}{15\mu}\right)p_2 + \left(\frac{34}{15} - \frac{2}{5}\mu\right)A_1\right.$$
$$+\left(-3 + \frac{\mu}{2} - \frac{1}{3\mu} + \frac{1}{36\mu^2}\right)A_2 + \left(\frac{37}{15} + \frac{1}{5\mu} + \frac{2}{5}\mu\right)B_1 + \left(-4 - \frac{\mu}{2} + \frac{1}{6\mu} + \frac{1}{36\mu^2}\right)B_2$$
$$\left. + \left(\frac{29}{30\mu} - \frac{47}{45}\right)\frac{M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \left(\frac{7 + 8r_c^2}{20\mu} + \frac{204 - 259r_c^2}{740}\right)\frac{M_b}{(r_c^2 + T^2)^{5/2}}\right].$$

(3.130)

To obtain the semi-minor axis $b_1 = \sqrt{\eta_0^2\sigma_1^2 + \xi_0^2}$ of the long periodic orbit, we first compute the following terms

$$\eta_0^2\sigma_1^2 = \left\{\frac{1}{4} - \mu + \mu^2 - \frac{(1-2\mu)p_1}{3} + \frac{(1-2\mu)p_2}{3} + \frac{(1-2\mu)A_1}{2} - \frac{(1-2\mu)5A_2}{8} - \frac{(1-2\mu)B_1}{2} + \frac{5(1-2\mu)B_2}{8}\right\}$$
$$\times\left(\begin{array}{l} (3\mu - 12\mu^2) + \left(\frac{2\mu}{3} - \frac{14\mu^2}{3}\right)(p_1 + p_2) - (2\mu + 70\mu^2)A_1 + \left(-\frac{5}{6} + \frac{25}{2}\mu + \frac{185}{2}\mu^2\right)A_2 + (4\mu - 88\mu^2)B_1 \\ + \left(-\frac{5}{6} - \frac{5}{2}\mu + \frac{275}{2}\mu^2\right)B_2 + \left(-\frac{21}{2}\mu - \frac{51}{2}\mu^2\right)M_{b4} + \left(-\frac{95}{3}\mu + \frac{62}{3}\mu^2\right)\frac{M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} \\ + \left(-6\mu + \frac{21}{2}\mu^2)\right)\frac{M_b r_c^2}{(r_c^2 + T^2)^{5/2}} \end{array}\right).$$

$$\eta_0^2\sigma_1^2 = \frac{3\mu}{4} - 6\mu^2 + \left(\frac{-5\mu}{6} + \frac{25\mu^2}{6}\right)p_1 + \left(\frac{7\mu}{6} - \frac{47\mu^2}{6}\right)p_2 + \left(\mu - \frac{49\mu^2}{2}\right)A_1$$
$$+\left(-\frac{5}{24} + \frac{25\mu}{12} + \frac{1085\mu^2}{12}\right)A_2 + \left(-\frac{\mu}{2} - 17\mu^2\right)B_1 + \left(-\frac{5}{24} + \frac{25\mu}{12} + \frac{595\mu^2}{24}\right)B_2$$
$$+\left(-\frac{21\mu}{8} + \frac{33\mu^2}{8}\right)M_{b4} + \left(-\frac{95\mu}{12} + \frac{221\mu^2}{6}\right)\frac{M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \left(-\frac{\mu}{8} - \frac{15\mu^2}{4}\right)\frac{M_b r_c^2}{(r_c^2 + T^2)^{5/2}},$$

and

$$\eta_0^2\sigma_1^2+\xi_0^2=\eta_0^2\sigma_1^2+\left[\frac{3}{4}-\frac{(p_1+p_2)}{3}-\frac{A_1}{2}+\frac{5A_2}{8}-\frac{B_1}{2}+\frac{5B_2}{8}-\frac{2M_b(2r_c-1)}{3(r_c^2+T^2)^{3/2}}\right],$$

$$=\frac{3}{4}+\frac{3\mu}{4}-6\mu^2-\left(\frac{1}{3}+\frac{5\mu}{6}-\frac{25\mu^2}{6}\right)p_1-\left(\frac{1}{3}-\frac{7\mu}{6}+\frac{47\mu^2}{6}\right)p_2+\left(-\frac{1}{2}+\mu-\frac{49\mu^2}{2}\right)A_1$$
$$+\left(\frac{5}{12}+\frac{25\mu}{12}+\frac{1085\mu^2}{12}\right)A_2-\left(\frac{1}{2}+\frac{\mu}{2}+17\mu^2\right)B_1+\left(\frac{5}{12}+\frac{25\mu}{12}+\frac{595\mu^2}{24}\right)B_2$$
$$+\left(-\frac{21\mu}{8}+\frac{33\mu^2}{8}\right)M_{b4}+\left(-\frac{2}{3}-\frac{95\mu}{12}+\frac{221\mu^2}{6}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(-\frac{\mu}{8}-\frac{15\mu^2}{4}\right)\frac{M_br_c^2}{(r_c^2+T^2)^{5/2}}.$$

Hence, the length of the semi-minor axis of the long periodic orbit is

$$b_1=\left[\frac{3}{4}+\frac{3\mu}{4}-6\mu^2-\left(\frac{1}{3}+\frac{5\mu}{6}-\frac{25\mu^2}{6}\right)p_1-\left(\frac{1}{3}-\frac{7\mu}{6}+\frac{47\mu^2}{6}\right)p_2+\left(-\frac{1}{2}+\mu-\frac{49\mu^2}{2}\right)A_1\right.$$
$$+\left(\frac{5}{12}+\frac{25\mu}{12}+\frac{1085\mu^2}{12}\right)A_2-\left(\frac{1}{2}+\frac{\mu}{2}+17\mu^2\right)B_1+\left(\frac{5}{12}+\frac{25\mu}{12}+\frac{595\mu^2}{24}\right)B_2+\left(-\frac{21\mu}{8}+\frac{33\mu^2}{8}\right)M_{b4}$$
$$\left.+\left(-\frac{2}{3}-\frac{95\mu}{12}+\frac{221\mu^2}{6}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(-\frac{\mu}{8}-\frac{15\mu^2}{4}\right)\frac{M_br_c^2}{(r_c^2+T^2)^{5/2}}\right]^{\frac{1}{2}}.$$

$$=\frac{\sqrt{3}}{2}\left[1+\frac{4}{3}\left\{\frac{3\mu}{4}-6\mu^2-\left(\frac{1}{3}+\frac{5\mu}{6}-\frac{25\mu^2}{6}\right)p_1-\left(\frac{1}{3}-\frac{7\mu}{6}+\frac{47\mu^2}{6}\right)p_2+\left(-\frac{1}{2}+\mu-\frac{49\mu^2}{2}\right)A_1\right.\right.$$
$$+\left(\frac{5}{12}+\frac{25\mu}{12}+\frac{1085\mu^2}{12}\right)A_2-\left(\frac{1}{2}+\frac{\mu}{2}+17\mu^2\right)B_1+\left(\frac{5}{12}+\frac{25\mu}{12}+\frac{595\mu^2}{24}\right)B_2+\left(-\frac{21\mu}{8}+\frac{33\mu^2}{8}\right)M_{b4}$$
$$\left.\left.+\left(-\frac{2}{3}-\frac{95\mu}{12}+\frac{221\mu^2}{6}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(-\frac{\mu}{8}-\frac{15\mu^2}{4}\right)\frac{M_br_c^2}{(r_c^2+T^2)^{5/2}}\right\}\right]^{\frac{1}{2}}.$$

$$\begin{aligned}
&=\frac{\sqrt{3}}{2}\left[1+\frac{2}{3}\left\{\frac{3\mu}{4}-6\mu^2-\left(\frac{1}{3}+\frac{5\mu}{6}-\frac{25\mu^2}{6}\right)p_1-\left(\frac{1}{3}-\frac{7\mu}{6}+\frac{47\mu^2}{6}\right)p_2+\left(-\frac{1}{2}+\mu-\frac{49\mu^2}{2}\right)A_1\right.\right.\\
&\quad+\left(\frac{5}{12}+\frac{25\mu}{12}+\frac{1085\mu^2}{12}\right)A_2-\left(\frac{1}{2}+\frac{\mu}{2}+17\mu^2\right)B_1+\left(\frac{5}{12}+\frac{25\mu}{12}+\frac{595\mu^2}{24}\right)B_2+\left(-\frac{21\mu}{8}+\frac{33\mu^2}{8}\right)M_{b4}\\
&\quad\left.\left.+\left(-\frac{2}{3}-\frac{95\mu}{12}+\frac{221\mu^2}{6}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(-\frac{\mu}{8}-\frac{15\mu^2}{4}\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\right\}\right].
\end{aligned}$$

$$\begin{aligned}
&=\frac{\sqrt{3}}{2}\left[1+\frac{\mu}{2}-4\mu^2-\left(\frac{2}{9}+\frac{5\mu}{9}-\frac{25\mu^2}{9}\right)p_1-\left(\frac{2}{9}-\frac{7\mu}{9}+\frac{47\mu^2}{9}\right)p_2+\left(-\frac{1}{3}+\frac{2\mu}{3}-\frac{49\mu^2}{3}\right)A_1\right.\\
&\quad+\left(\frac{5}{18}+\frac{25\mu}{18}+\frac{1085\mu^2}{18}\right)A_2-\left(\frac{1}{3}+\frac{\mu}{3}+\frac{34\mu^2}{3}\right)B_1+\left(\frac{5}{18}+\frac{25\mu}{18}+\frac{595\mu^2}{36}\right)B_2+\left(-\frac{7\mu}{4}+\frac{11\mu^2}{4}\right)M_{b4}\\
&\quad\left.+\left(-\frac{4}{9}-\frac{95\mu}{27}+\frac{221\mu^2}{9}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\left(-\frac{\mu}{12}-\frac{5\mu^2}{2}\right)\frac{M_b r_c^2}{(r_c^2+T^2)^{5/2}}\right].
\end{aligned}$$

Or

$$\begin{aligned}
b_1&=\frac{\sqrt{3}}{2}\left[1+\frac{\mu}{2}-4\mu^2-\left(\frac{2}{9}+\frac{5\mu}{9}-\frac{25\mu^2}{9}\right)p_1-\left(\frac{2}{9}-\frac{7\mu}{9}+\frac{47\mu^2}{9}\right)p_2-\left(\frac{1}{3}-\frac{2\mu}{3}+\frac{49\mu^2}{3}\right)A_1\right.\\
&\qquad+\left(\frac{5}{18}+\frac{25\mu}{18}+\frac{1085\mu^2}{18}\right)A_2-\left(\frac{1}{3}+\frac{\mu}{3}+\frac{34\mu^2}{3}\right)B_1+\left(\frac{5}{18}+\frac{25\mu}{18}+\frac{595\mu^2}{36}\right)B_2\\
&\qquad\left.-\left(\frac{4}{9}+\frac{95\mu}{27}-\frac{221\mu^2}{9}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}-\left(\frac{(21+r_c)\mu}{12}+\frac{(11-10r_c)\mu^2}{4}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}\right].
\end{aligned}$$

(3.131)

In a similar manner, we obtained the lengths of the semi-major axis $a_2=\left(x_0^2+y_0^2\sigma_2^{-2}\right)^{\frac{1}{2}}$ and semi-minor axis $b_2=\sqrt{\eta_0^2\sigma_2^2+\xi_0^2}$ of the short periodic orbit as

$$
\begin{aligned}
a_2 = \frac{\sqrt{13}}{2}\Bigg\{ & 1+\frac{23\mu}{26}+\frac{683\mu^2}{52}-\left(\frac{11}{39}+\frac{5\mu}{39}-\frac{3\mu^2}{13}\right)p_1-\left(\frac{5}{39}+\frac{\mu}{3}-\frac{3\mu^2}{13}\right)p_2+\left(\frac{24}{13}-\frac{103\mu}{26}+\frac{2817\mu^2}{26}\right)A_1 \\
& -\left(\frac{465}{104}+\frac{10315}{416}\mu-\frac{32535}{104}\mu^2\right)A_2+\left(\frac{4}{13}-\frac{185\mu}{26}+\frac{3303\mu^2}{26}\right)B_1-\left(\frac{85}{104}+\frac{395}{416}\mu+\frac{80505}{104}\mu^2\right)B_2 \\
& +\left(\frac{4r_c+38}{39}-\frac{(174r_c+102)\mu}{13}+\frac{(2422r_c-12867)\mu^2}{52}\right)\frac{M_b}{(r_c^2+T^2)^{3/2}} \\
& +\left(\frac{18r_c^2}{13}+\frac{297r_c^2\mu}{52}+\frac{(4779-459r_c^2)\mu^2}{208}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}\Bigg\}
\end{aligned}
\tag{3.132}
$$

and

$$
\begin{aligned}
b_2 = \frac{\sqrt{13}}{4}\Bigg\{ & 1-\frac{25\mu}{104}-\frac{121\mu^2}{208}+\left(\frac{-10}{39}+\frac{61\mu}{156}+\frac{41\mu^2}{52}\right)p_1+\left(\frac{-2}{13}-\frac{37\mu}{156}-\frac{85\mu^2}{52}\right)p_2 \\
& +\left(-\frac{21}{52}+\frac{5\mu}{8}-\frac{1383\mu^2}{104}\right)A_1+\left(\frac{285}{416}+\frac{125\mu}{128}+\frac{1155\mu^2}{52}\right)A_2+\left(-\frac{23}{52}+\frac{115\mu}{104}-\frac{1239\mu^2}{104}\right)B_1 \\
& +\left(\frac{245}{416}-\frac{1175\mu}{1664}+\frac{1875\mu^2}{128}\right)B_2+\left(-\frac{73}{156}+\frac{85\mu}{208}-\frac{1421\mu^2}{208}\right)\frac{M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} \\
& +\left(-\frac{3r_c^2}{26}-\frac{3r_c^2\mu}{208}+\frac{(1353r_c^2-1593)\mu^2}{832}\right)\frac{M_b}{(r_c^2+T^2)^{5/2}}\Bigg\}
\end{aligned}
\tag{3.133}
$$

respectively. Using Equations (3.130)-(3.133) we investigate the effects of the various perturbations on the semi axes of the long and short periodic orbits and are presented in Tables 3.12 and 3.13 correspondingly.

Table 3.12: Effects of the perturbations on the semi axes of the long periodic

case	q_1	q_2	A_1	A_2	B_1	B_2	M_b	a_1	b_1
1	1	1	0	0	0	0	0	4.8318000356	0.8758980933
2	0.98	1	0	0	0	0	0	4.7857668495	0.8718037177
	0.96	1	0	0	0	0	0	4.7397336634	0.8677093420
	0.94	1	0	0	0	0	0	4.6937004773	0.8636149664
3	1	0.97	0	0	0	0	0	4.7607228881	0.8706086988
	1	0.95	0	0	0	0	0	4.7133381232	0.8670824359
	1	0.93	0	0	0	0	0	4.6659533582	0.8635561729
4	1	1	0.01	0	0	0	0	4.8570079752	0.8730572413
	1	1	0.02	0	0	0	0	4.8822159149	0.8702163893
	1	1	0.03	0	0	0	0	4.9074238546	0.8673755373
	1	1	0.01	0.005	0	0	0	4.9507444280	0.8746753857
	1	1	0.02	0.01	0	0	0	5.0696888203	0.8734526782
	1	1	0.03	0.015	0	0	0	5.1886332127	0.8722299706
5	1	1	0	0	0.01	0	0	5.2789860253	0.8728364049
	1	1	0	0	0.02	0	0	5.7261720150	0.8697747164
	1	1	0	0	0.03	0	0	6.1733580047	0.8667130279
	1	1	0	0	0.01	0.005	0	5.4601341020	0.8742840505
	1	1	0	0	0.02	0.01	0	6.0884681685	0.8726700077
	1	1	0	0	0.03	0.015	0	6.7168022350	0.8710559649
6	1	1	0	0	0	0	0.01	5.3793997766	0.8707785846
	1	1	0	0	0	0	0.02	5.9269995176	0.8656590759
	1	1	0	0	0	0	0.03	6.4745992586	0.8605395671
	0.99	0.98	0.01	0.005	0.01	0.005	0.01	6.0095679019	0.8585580640

7	0.98	0.97	0.02	0.01	0.02	0.01	0.02	7.211028150	0.8429811663
	0.97	0.96	0.03	0.015	0.03	0.015	0.03	8.4124883994	0.8274042685

Table 3.13: Effects of the perturbations on the semi axes of the short periodic orbit.

case	q_1	q_2	A_1	A_2	B_1	B_2	M_b	a_2	b_2
1	1	1	0	0	0	0	0	1.8719294178	0.8944154974
2	0.98	1	0	0	0	0	0	1.8616287276	0.8900172680
	0.96	1	0	0	0	0	0	1.8513280374	0.8856190385
	0.94	1	0	0	0	0	0	1.8410273472	0.8812208091
3	1	0.97	0	0	0	0	0	1.8644660653	0.8900230519
	1	0.95	0	0	0	0	0	1.8594904970	0.8870947549
	1	0.93	0	0	0	0	0	1.8545149287	0.8841664578
4	1	1	0.01	0	0	0	0	1.9048268143	0.8908364070
	1	1	0.02	0	0	0	0	1.9377242108	0.8872573167
	1	1	0.03	0	0	0	0	1.9706216073	0.8836782263
	1	1	0.01	0.005	0	0	0	1.8603571063	0.8941462283
	1	1	0.02	0.01	0	0	0	1.8487847948	0.8938769593
	1	1	0.03	0.015	0	0	0	1.8372124833	0.8936076903
5	1	1	0	0	0.01	0	0	1.8756893837	0.8906309599
	1	1	0	0	0.02	0	0	1.8794493497	0.8868464225
	1	1	0	0	0.03	0	0	1.8832093156	0.8830618851
	1	1	0	0	0.01	0.005	0	1.8617857366	0.8934401792
	1	1	0	0	0.02	0.01	0	1.8516420555	0.8924648611
	1	1	0	0	0.03	0.015	0	1.8414983743	0.8914895429
6	1	1	0	0	0	0	0.01	1.9060494140	0.8891331907
	1	1	0	0	0	0	0.02	1.9401694102	0.8838508841
	1	1	0	0	0	0	0.03	1.9742894065	0.8785685775
	0.99	0.98	0.01	0.005	0.01	0.005	0.01	1.8665693787	0.8790979286

7	0.98	0.97	0.02	0.01	0.02	0.01	0.02	1.8636971238	0.8652445084
	0.97	0.96	0.03	0.015	0.03	0.015	0.03	1.8608248688	0.8513910882

CHAPTER FOUR

4.0 TRIAXIALITY AND RADIATION OF THE PRIMARIES WITH POTENTIAL FROM A BELT

4.1 Introduction

The present Chapter is devoted to the study of libration points and their behaviours under the combined effect of radiation and triaxiality of the primaries together with gravitational potential from the belt within the framework of the restricted three-body problem. The equations that govern the motion of the infinitesimal body, in the gravitational field of radiating-triaxial primaries; together with the influence of gravitational potential from the belt in the restricted three-body problem will be derived. The positions of the libration points will be identified, and their linear stability discussed.

4.2 Derivation of Equations of Motion

4.2.1 Mathematical formulations of the problem

Using the same set of axes and notations as in Section 3.2, we desire to find the equations of motion of m under the resultant influence of radiation and triaxiality of the primaries and a circumbinary belt centered at the origin of the rotating coordinate system *oxyz* as illustrated in Figure 3.1.

4.2.2 Energy of the infinitesimal mass

Proceeding as in chapter three, the kinetic energy (K.E) of the infinitesimal body *m* in the barycentric coordinate system *oxyz* is

$$K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mn\left(x\dot{y} - \dot{x}y\right) + \frac{1}{2}mn^2\left(x^2 + y^2\right); \tag{4.1}$$

and its total potential energy under the combined influence of the radiation and triaxiality of the primaries and the circumbinary belt, is given with respect to Section 1.7 as

$$V = -Gm\left\langle q_1\left(\frac{m_1}{r_1} + \frac{1}{2r_1^3}\left(I_1 + I_2 + I_3 - 3I\right)\right) + q_2\left(\frac{m_2}{r_2} + \frac{1}{2r_2^3}\left(I_1' + I_2' + I_3' - 3I'\right)\right) + \frac{M_b}{\left(r^2 + T^2\right)^{1/2}}\right\rangle \tag{4.2}$$

with

$$r_1^2 = \left(x - x_1\right)^2 + y^2,\ r_2^2 = \left(x - x_2\right)^2 + y^2,$$

where I_1, I_2, I_3 and I_1', I_2', I_3' are the principal moments of inertia of m_1 and m_2 about their centre of mass respectively, I is the moment of inertia of the vector from m_1 to m and I' is the moment of inertia of the vector from m_2 to m.

4.2.3 Equations of motion in dimensional variables

By defining the Lagrangian $L = K - V$ here, and proceeding as inSection 3.2.4, we obtain the equations of motion of *m* in dimensional form as

$$\ddot{x} - 2n\dot{y} = -\frac{\partial\Psi}{m\partial x}, \tag{4.3}$$

$$\ddot{y} + 2n\dot{x} = -\frac{\partial\Psi}{m\partial y}, \tag{4.4}$$

where

$$\frac{\partial \Psi}{\partial x} = -Gm\frac{\partial}{\partial x}\left\langle q_1\left(\frac{m_1}{r_1}+\frac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)\right)+q_2\left(\frac{m_2}{r_2}+\frac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right)\right)+\frac{M_b}{\left(r^2+T^2\right)^{1/2}}\right\rangle$$

$$-mn^2x,$$

$$=-Gm\left\langle \begin{array}{l} q_1\left(-\frac{m_1(x-x_1)}{r_1^3}+\frac{\partial}{\partial x}\frac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)\right) \\ +q_2\left(-\frac{m_2(x-x_1)}{r_2}+\frac{\partial}{\partial x}\frac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right)\right)-\frac{M_bx}{\left(r^2+T^2\right)^{3/2}} \end{array}\right\rangle - mn^2x,$$

and

$$\frac{\partial \Psi}{\partial y} = -Gm\frac{\partial}{\partial y}\left\langle q_1\left(\frac{m_1}{r_1}+\frac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)\right)+q_2\left(\frac{m_2}{r_2}+\frac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right)\right)+\frac{M_b}{\left(r^2+T^2\right)^{1/2}}\right\rangle$$

$$-\frac{1}{2}mn^2\frac{\partial}{\partial y}\left(x^2+y^2\right).$$

$$=-Gm\left\langle \begin{array}{l} q_1\left(-\frac{m_1(x-x_1)}{r_1}+\frac{\partial}{\partial y}\frac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)\right) \\ +q_2\left(-\frac{m_2(x-x_1)}{r_2}+\frac{\partial}{\partial y}\frac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right)\right)-\frac{M_by}{\left(r^2+T^2\right)^{3/2}} \end{array}\right\rangle - mn^2y.$$

Or

$$\ddot{x}-2n\dot{y}=G\left\langle \begin{array}{l} q_1\left(-\dfrac{m_1\left(x-x_1\right)}{r_1^3}+\dfrac{\partial}{\partial x}\dfrac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)\right) \\ +q_2\left(-\dfrac{m_2\left(x-x_2\right)}{r_2^3}+\dfrac{\partial}{\partial x}\dfrac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right)\right)-\dfrac{M_b x}{\left(r^2+T^2\right)^{3/2}} \end{array}\right\rangle+n^2x,$$

$$\ddot{y}+2n\dot{x}=G\left\langle \begin{array}{l} q_1\left(-\dfrac{m_1 y}{r_1^3}+\dfrac{\partial}{\partial y}\dfrac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)\right)+q_2\left(-\dfrac{m_2 y}{r_2^3}+\dfrac{\partial}{\partial y}\dfrac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right)\right) \\ -\dfrac{M_b y}{\left(r^2+T^2\right)^{3/2}} \end{array}\right\rangle+n^2y.$$

(4.5)

4.2.4 Equations of motion in dimensionless variables

In order to express the equations of motion specified by Equation (4.5) in dimensionlessvariables, we first evaluate the partial derivatives contained therein. We begin by simplifying $\left(I_1+I_2+I_3-3I\right) and \left(I_1'+I_2'+I_3'-3I'\right)$.

Now, in view of Equation (1.5), we have

$$\begin{aligned} I_1&=\frac{\left(b^2+c^2\right)m_1}{5}, \quad I_1'=\frac{\left(b'^2+c'^2\right)m_2}{5}, \\ I_2&=\frac{\left(c^2+a^2\right)m_1}{5}, \quad I_2'=\frac{\left(c'^2+a'^2\right)m_2}{5}, \quad (4.6) \\ I_3&=\frac{\left(b^2+a^2\right)m_1}{5}. \quad I_3'=\frac{\left(b'^2+a'^2\right)m_2}{5}. \end{aligned}$$

where, $\left(a,b,c\right) and \left(a',b',c'\right)$ are the semi- axes of the triaxial bodies m_1 *and* m_2 respectively.

So that

$$\begin{aligned} I_1+I_2+I_3&=\frac{\left(b^2+c^2\right)m_1}{5}+\frac{\left(c^2+a^2\right)m_1}{5}+\frac{\left(b^2+a^2\right)m_1}{5}, \\ &=\frac{2\left(a^2+b^2+c^2\right)m_1}{5}, \end{aligned} \qquad (4.7)$$

and

$$I'_1 + I'_2 + I'_3 = \frac{\left(b'^2 + c'^2\right)m_2}{5} + \frac{\left(c'^2 + a'^2\right)m_2}{5} + \frac{\left(b'^2 + a'^2\right)m_2}{5},$$

$$= \frac{2\left(a'^2 + b'^2 + c'^2\right)m_2}{5}. \qquad (4.8)$$

Making use of the dimensionless variables defined in Section 3.2.6 and with respect to Figure 4.1 produces

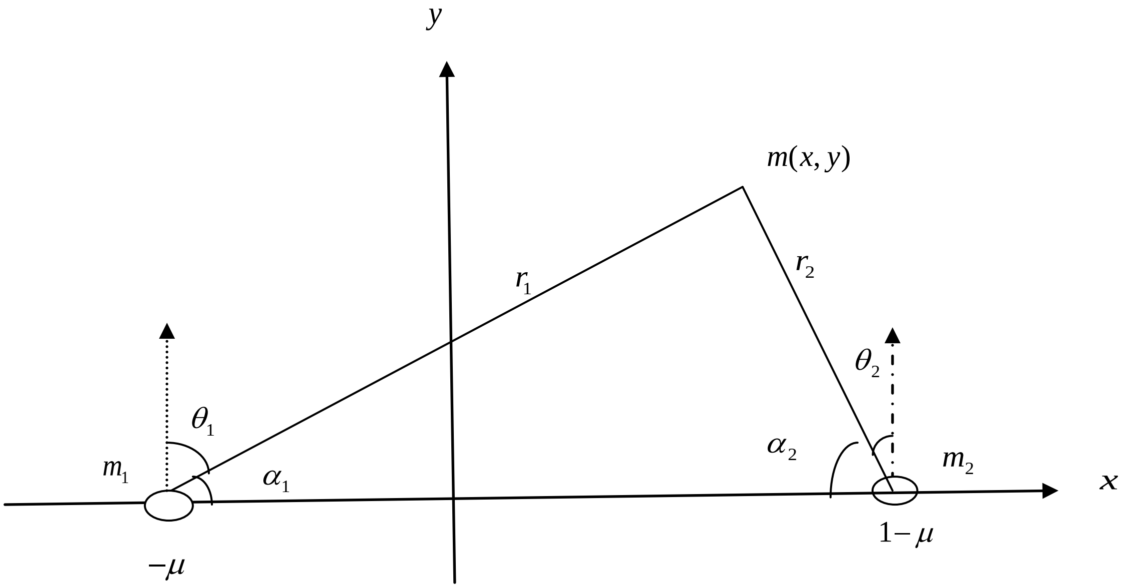


Figure 4.1: Configuration to determine direction cosines

$$I = I_1 l'^2 + I_2 m'^2 + I_3(0)$$

and

$$I' = I'_1 l''^2 + I'_2 m''^2 + I'_3\left(0\right)$$

where ,

$(l' = \cos\alpha_1 = \dfrac{x+\mu}{r_1},\ m' = \cos\theta_1 = \dfrac{y}{r_1}, 0)$ and $(l'' = \cos\alpha_2 = \dfrac{1-\mu-x}{r_2}, m'' = \cos\theta_2 = \dfrac{y}{r_2}, 0)$ are the direction cosines of $\overrightarrow{m_1 m}$ and $\overrightarrow{m_2 m}$ respectively. Here,

$$r_1^2 = (x+\mu)^2 + y^2\ , \qquad r_2^2 = (x+\mu-1)^2 + y^2. \tag{4.9}$$

Hence,

$$I = \frac{(b^2+c^2)m_1}{5}\left(\frac{x+\mu}{r_1}\right)^2 + \frac{(c^2+a^2)m_1}{5}\left(\frac{y}{r_1}\right)^2.$$

$$= \frac{m_1}{5r_1^2}\left[(b^2+c^2)(x+\mu)^2 + (c^2+a^2)y^2\right].$$

$$= \frac{m_1}{5r_1^2}\left[(a^2+b^2+c^2)\left\{(x+\mu)^2+y^2\right\} - a^2(x+\mu)^2 - b^2y^2\right].$$

So that

$$= \frac{m_1}{5r_1^2}\left[(a^2+b^2+c^2)r_1^2 - a^2\left\{(x+\mu)^2+y^2\right\} + a^2y^2 - b^2y^2\right].$$

$$= \frac{m_1}{5r_1^2}\left[(a^2+b^2+c^2)r_1^2 - a^2r_1^2 + (a^2-b^2)y^2\right].$$

$$= \frac{m_1}{5r_1^2}\left[(b^2+c^2)r_1^2 + (a^2-b^2)y^2\right].$$

$$(I_1+I_2+I_3-3I) = \frac{2m_1(a^2+b^2+c^2)}{5} - \frac{3m_1(b^2+c^2)}{5} - \frac{3m_1}{5r_1^2}(a^2-b^2)y^2.$$

$$= \frac{m_1}{5}(2a^2-b^2-c^2) - \frac{3m_1}{5r_1^2}(a^2-b^2)y^2.$$

$$= \frac{m_1}{5}\left[(a^2-b^2)+(a^2-c^2)\right] - \frac{3m_1}{5r_1^2}(a^2-b^2)y^2.$$

Two of the partial derivatives in Equation (4.5) now become

$$\begin{aligned}\frac{\partial}{\partial x}\frac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)&=\frac{\partial}{\partial x}\frac{m_1}{2r_1^3}\left[\frac{(a^2-b^2)}{5}+\frac{(a^2-c^2)}{5}\right]-\frac{\partial}{\partial x}\frac{3m_1}{2r_1^5}\frac{\left(a^2-b^2\right)}{5}y^2.\\&=-\frac{3m_1(x+\mu)}{2r_1^5}\left[\frac{(a^2-b^2)}{5}+\frac{(a^2-c^2)}{5}\right]+\frac{15m_1(x+\mu)y^2}{2r_1^7}\frac{\left(a^2-b^2\right)}{5}\end{aligned}$$

(4.10)

and

$$\begin{aligned}\frac{\partial}{\partial y}\frac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)&=\frac{\partial}{\partial y}\frac{m_1}{2r_1^3}\left[\frac{(a^2-b^2)}{5}+\frac{(a^2-c^2)}{5}\right]-\frac{\partial}{\partial y}\frac{3m_1}{2r_1^5}\frac{\left(a^2-b^2\right)}{5}y^2.\\&=-\frac{3m_1y}{2r_1^5}\left[\frac{(a^2-b^2)}{5}+\frac{(a^2-c^2)}{5}\right]-\frac{3m_1y}{r_1^5}\frac{\left(a^2-b^2\right)}{5}+\frac{15m_1y^3}{2r_1^7}\frac{\left(a^2-b^2\right)}{5}.\end{aligned}$$

(4.11)

In a similar manner,

$$\begin{aligned}I'&=\frac{\left(b'^2+c'^2\right)m_2}{5}\left(\frac{1-\mu-x}{r_2}\right)^2+\frac{\left(c'^2+a'^2\right)m_2}{5}\left(\frac{y}{r_2}\right)^2.\\&=\frac{m_2}{5r_2^2}\left[\left(b'^2+c'^2\right)\left(1-\mu-x\right)^2+\left(c'^2+a'^2\right)y^2\right].\\&=\frac{m_2}{5r_2^2}\left[\left(a'^2+b'^2+c'^2\right)\left\{\left(1-\mu-x\right)^2+y^2\right\}-a'^2\left(1-\mu-x\right)^2-b'^2y^2\right].\\&=\frac{m_2}{5r_2^2}\left[\left(a'^2+b'^2+c'^2\right)r_2^2-a'^2\left\{\left(1-\mu-x\right)^2+y^2\right\}+a'^2y^2-b'^2y^2\right].\\&=\frac{m_2}{5r_2^2}\left[\left(a'^2+b'^2+c'^2\right)r_2^2-a'^2r_2^2+\left(a'^2-b'^2\right)y^2\right].\\&=\frac{m_2}{5r_2^2}\left[\left(b'^2+c'^2\right)r_2^2+\left(a'^2-b'^2\right)y^2\right].\end{aligned}$$

So that

$$
\begin{aligned}
\left(I_1'+I_2'+I_3'-3I'\right) &= \frac{2\left(a'^2+b'^2+c'^2\right)m_2}{5}-\frac{3m_2}{5r_2^2}\left[\left(b'^2+c'^2\right)r_2^2+\left(a'^2-b'^2\right)y^2\right]. \\
&= \frac{2m_2\left(a'^2+b'^2+c'^2\right)}{5}-\frac{m_2\left(b'^2+c'^2\right)}{5}-\frac{3m_2}{5r_2^2}\left(a'^2-b'^2\right)y^2. \\
&= \frac{m_2}{5}\left[(a'^2-b'^2)+(a'^2-c'^2)\right]-\frac{3m_2}{5r_2^2}\left(a'^2-b'^2\right)y^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial}{\partial x}\frac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right) &= \frac{\partial}{\partial x}\frac{m_2}{2r_2^3}\left[\frac{(a'^2-b'^2)}{5}+\frac{(a'^2-c'^2)}{5}\right]-\frac{\partial}{\partial x}\frac{3m_2}{2r_2^5}\frac{\left(a'^2-b'^2\right)}{5}y^2. \\
&= -\frac{3m_2(x+\mu-1)}{2r_2^5}\left[\frac{(a'^2-b'^2)}{5}+\frac{(a'^2-c'^2)}{5}\right]+\frac{15m_2(x+\mu-1)y^2}{2r_2^7}\frac{\left(a'^2-b'^2\right)}{5}
\end{aligned}
\tag{4.12}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial y}\frac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right) &= \frac{\partial}{\partial y}\frac{m_2}{2r_2^3}\left[\frac{(a'^2-b'^2)}{5}+\frac{(a'^2-c'^2)}{5}\right]-\frac{\partial}{\partial y}\frac{3m_2}{2r_2^5}\frac{\left(a'^2-b'^2\right)}{5}y^2. \\
&= -\frac{3m_2 y}{2r_2^5}\left[\frac{(a'^2-b'^2)}{5}+\frac{(a'^2-c'^2)}{5}\right]-\frac{3m_2 y}{r_2^5}\frac{\left(a'^2-b'^2\right)}{5}+\frac{15m_2 y^3}{2r_2^7}\frac{\left(a'^2-b'^2\right)}{5}.
\end{aligned}
\tag{4.13}
$$

Utilizing Equations (4.10) to (4.13), normalized variables: $m_1 = 1-\mu$, $m_2 = \mu$, R=G=1 in Equation (4.5), yield the equations that govern the motion of the infinitesimal mass in dimensionless form as

$$\begin{aligned}\ddot{x}-2n\dot{y}=\Bigg\langle & q_1\left(-\frac{(1-\mu)\left(x+\mu\right)}{r_1^3}-\frac{3(1-\mu)(x+\mu)}{2r_1^5}\left[\frac{(a^2-b^2)}{5R^2}+\frac{(a^2-c^2)}{5R^2}\right]+\frac{15(1-\mu)(x+\mu)y^2}{2r_1^7}\frac{\left(a^2-b^2\right)}{5R^2}\right)\\ & +q_2\left(-\frac{\mu\left(x+\mu-1\right)}{r_2^3}-\frac{3\mu(x+\mu-1)}{2r_2^5}\left[\frac{(a'^2-b'^2)}{5R^2}+\frac{(a'^2-c'^2)}{5R^2}\right]+\frac{15\mu(x+\mu-1)y^2}{2r_2^7}\frac{\left(a'^2-b'^2\right)}{5R^2}\right)\\ & -\frac{M_b x}{\left(r^2+T^2\right)^{3/2}}\Bigg\rangle+n^2x,\end{aligned}$$

$$\begin{aligned}\ddot{y}+2n\dot{x}=\Bigg\langle & q_1\left(-\frac{(1-\mu)y}{r_1^3}-\frac{3(1-\mu)y}{2r_1^5}\left[\frac{(a^2-b^2)}{5R^2}+\frac{(a^2-c^2)}{5R^2}\right]-\frac{3(1-\mu)y}{r_1^5}\frac{\left(a^2-b^2\right)}{5R^2}+\frac{15(1-\mu)y^3}{2r_1^7}\frac{\left(a^2-b^2\right)}{5R^2}\right)\\ & +q_2\left(-\frac{\mu y}{r_2^3}-\frac{3\mu y}{2r_2^5}\left[\frac{(a'^2-b'^2)}{5R^2}+\frac{(a'^2-c'^2)}{5R^2}\right]-\frac{3\mu y}{r_2^5}\frac{\left(a'^2-b'^2\right)}{5R^2}+\frac{15\mu y^3}{2r_2^7}\frac{\left(a'^2-b'^2\right)}{5R^2}\right)-\frac{M_b y}{\left(r^2+T^2\right)^{3/2}}\Bigg\rangle+n^2y.\end{aligned}$$

Or

$$\begin{aligned}\ddot{x}-2n\dot{y}=n^2x & -\frac{q_1(1-\mu)\left(x+\mu\right)}{r_1^3}-\frac{3q_1(1-\mu)(x+\mu)(2\sigma_1-\sigma_2)}{2r_1^5}+\frac{15q_1(1-\mu)(x+\mu)(\sigma_1-\sigma_2)y^2}{2r_1^7}\\ & -\frac{\mu q_2\left(x+\mu-1\right)}{r_2^3}-\frac{3\mu q_2(x+\mu-1)(2\sigma_1'-\sigma_2')}{2r_2^5}+\frac{15\mu q_2(x+\mu-1)(\sigma_1'-\sigma_2')y^2}{2r_2^7}-\frac{M_b x}{\left(r^2+T^2\right)^{3/2}},\end{aligned}$$

$$\begin{aligned}\ddot{y}+2n\dot{x}=n^2y & -\frac{q_1(1-\mu)y}{r_1^3}-\frac{3(1-\mu)q_1(2\sigma_1-\sigma_2)y}{2r_1^5}-\frac{3(1-\mu)q_1(\sigma_1-\sigma_2)y}{r_1^5}+\frac{15(1-\mu)q_1(\sigma_1-\sigma_2)y^3}{2r_1^7}\\ & -\frac{\mu q_2 y}{r_2^3}-\frac{3\mu q_2(2\sigma_1'-\sigma_2')y}{2r_2^5}-\frac{3\mu q_2(\sigma_1'-\sigma_2')y}{r_2^5}+\frac{15\mu q_2(\sigma_1'-\sigma_2')y^3}{2r_2^7}-\frac{M_b y}{\left(r^2+T^2\right)^{3/2}},\end{aligned}$$

where,

$$\sigma_1 = \frac{a^2 - c^2}{5R^2}, \quad \sigma_2 = \frac{b^2 - c^2}{5R^2}, \; \sigma_1' = \frac{a'^2 - c'^2}{5R^2}, \quad \sigma_2' = \frac{b'^2 - c'^2}{5R^2}. \; (\sigma_1, \sigma_2, \sigma_1', \sigma_2' << 1)$$

Or

$$\begin{aligned} &\ddot{x} - 2n\dot{y} = \Omega_x, \\ &\ddot{y} + 2n\dot{x} = \Omega_y, \end{aligned} \tag{4.14}$$

with

$$\begin{aligned} \Omega = {} & \frac{n^2(x^2+y^2)}{2} + \frac{(1-\mu)q_1}{r_1} + \frac{(1-\mu)q_1(2\sigma_1 - \sigma_2)}{2r_1^3} - \frac{3(1-\mu)q_1(\sigma_1 - \sigma_2)y^2}{2r_1^5} + \frac{\mu q_2}{r_2} + \frac{\mu q_2(2\sigma_1' - \sigma_2')}{2r_2^3} \\ & - \frac{3\mu q_2(\sigma_1' - \sigma_2')y^2}{8r_2^5} + \frac{M_b}{(r^2+T^2)^{1/2}}, \end{aligned}$$

$$\begin{aligned} \Omega_x = {} & n^2 x - \frac{q_1(1-\mu)\left(x+\mu\right)}{r_1^3} - \frac{3q_1(1-\mu)(x+\mu)(2\sigma_1 - \sigma_2)}{2r_1^5} + \frac{15q_1(1-\mu)(x+\mu)(\sigma_1 - \sigma_2)y^2}{2r_1^7} \\ & - \frac{\mu q_2\left(x+\mu-1\right)}{r_2^3} - \frac{3\mu q_2(x+\mu-1)(2\sigma_1' - \sigma_2')}{2r_2^5} + \frac{15\mu q_2(x+\mu-1)(\sigma_1' - \sigma_2')y^2}{2r_2^7} - \frac{M_b x}{\left(r^2+T^2\right)^{3/2}}, \end{aligned}$$

$$\begin{aligned} \Omega_y = {} & n^2 y - \frac{q_1(1-\mu)y}{r_1^3} - \frac{3(1-\mu)q_1(4\sigma_1 - 3\sigma_2)y}{2r_1^5} + \frac{15(1-\mu)q_1(\sigma_1 - \sigma_2)y^3}{2r_1^7} - \frac{\mu q_2 y}{r_2^3} \\ & - \frac{3\mu q_2(4\sigma_1' - 3\sigma_2')y}{2r_2^5} + \frac{15\mu q_2(\sigma_1' - \sigma_2')y^3}{2r_2^7} - \frac{M_b y}{\left(r^2+T^2\right)^{3/2}}. \end{aligned}$$

4.2.5 The mean motion *n*

At this point, let us find out the mean motion *n*. In view of Figure (3.1), the distances of $m_1$and $m_2$from their common centre of mass O are $|x_1| \; and \; x_2$, respectively. So that the distance between the primaries $R = |x_1| + x_2$. Since the primaries are moving in circular orbits around O (see Figure 4.2), the mean motion *n* satisfies

$$m_1 n^2 |x_1| = m_2 n^2 x_2 = F,$$

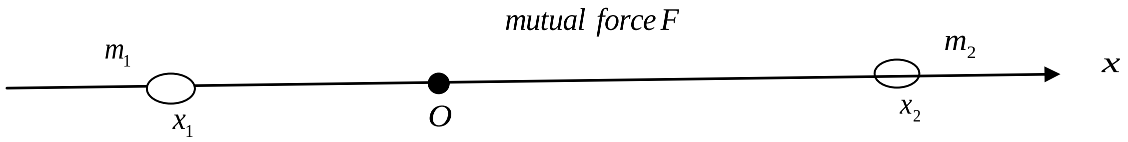


Figure 4.2: Configuration for the determination of mean motion

where F is the common force of gravity between m_1 *and* m_2 which is given with respect to Equation (1.5) as

$$F = -Gm_1m_2 \frac{\partial}{\partial R}\left[\frac{1}{R} + \frac{1}{2R^3m_1}\left(I_1 + I_2 + I_3 - 3I\right) + \frac{1}{2R^3m_2}\left(I_1' + I_2' + I_3' - 3I'\right)\right]$$

Hence,

$$\begin{aligned} m_1n^2\left|x_1\right| &= -Gm_1m_2 \frac{\partial}{\partial R}\left[\frac{1}{R} + \frac{1}{2R^3m_1}\left(-2I_1 + I_2 + I_3\right) + \frac{1}{2R^3m_2}\left(-2I_1' + I_2' + I_3'\right)\right] \\ &= \frac{Gm_1m_2}{R^2}\left\{1 + \frac{3}{2R^2m_1}\left(-2I_1 + I_2 + I_3\right) + \frac{3}{2R^2m_2}\left(-2I_1' + I_2' + I_3'\right)\right\}. \end{aligned}$$

Or

$$n^2\left|x_1\right| = \frac{Gm_2}{R^2}\left\{1 + \frac{3}{2R^2m_1}\left(-2I_1 + I_2 + I_3\right) + \frac{3}{2R^2m_2}\left(-2I_1' + I_2' + I_3'\right)\right\}.$$

In a similar manner, we have

$$n^2x_2 = \frac{Gm_1}{R^2}\left\{1 + \frac{3}{2R^2m_1}\left(-2I_1 + I_2 + I_3\right) + \frac{3}{2R^2m_2}\left(-2I_1' + I_2' + I_3'\right)\right\}.$$

Therefore,

$$n^2\left(\left|x_1\right| + x_2\right) = \frac{G\left(m_1 + m_2\right)}{R^2}\left\{1 + \frac{3}{2R^2m_1}\left(-2I_1 + I_2 + I_3\right) + \frac{3}{2R^2m_2}\left(-2I_1' + I_2' + I_3'\right)\right\}.$$

Or

$$n^2 = \frac{G\left(m_1+m_2\right)}{R^3}\left\{1+\frac{3}{2R^2 m_1}\left(-2I_1+I_2+I_3\right)+\frac{3}{2R^2 m_2}\left(-2I_1'+I_2'+I_3'\right)\right\}. \tag{4.15}$$

Utilizing Equation (4.6) in Equation (4.15), produces

$$n^2 = \frac{G\left(m_1+m_2\right)}{R^3}\left\{1+\frac{3}{2R^2}\left(\frac{(a^2-b^2)+(a^2-c'^2)}{5}\right)+\frac{3}{2R^2}\left(\frac{(a'^2-b'^2)+(a'^2-c'^2)}{5}\right)\right\}. \tag{4.16}$$

Under the normalized variables, Equation (4.16) transforms to

$$n^2 = 1+\frac{3}{2}\left(\frac{(a^2-b^2)+(a^2-c'^2)}{5R^2}\right)+\frac{3}{2}\left(\frac{(a'^2-b'^2)+(a'^2-c'^2)}{5R^2}\right).$$

Or

$$n^2 = 1+\frac{3}{2}\left(2\sigma_1-\sigma_2\right)+\frac{3}{2}\left(2\sigma_1'-\sigma_2'\right). \tag{4.17}$$

Now, as in Section 3.2.6 due to the additional gravitational force from the belt, the mean motion *n (*Equation (4.17))becomes

$$n^2 = 1+\frac{3}{2}\left(2\sigma_1-\sigma_2\right)+\frac{3}{2}\left(2\sigma_1'-\sigma_2'\right)+\frac{2M_b r_c}{(r_c^2+T^2)^{3/2}}. \tag{4.18}$$

4.3 Location of Libration Points

The libration points signify stationary solutions of the restricted three-body problem. They are determined by putting $\dot{x}=\dot{y}=\ddot{x}=\ddot{y}=0$ in the Equation (4.14). Hence, the libration points are the solutions of

$$\Omega_x = 0,$$
$$\Omega_y = 0\ .$$

That is

$$\begin{aligned} &n^2x-\frac{q_1(1-\mu)\left(x+\mu\right)}{r_1^3}-\frac{3q_1(1-\mu)(x+\mu)(2\sigma_1-\sigma_2)}{2r_1^5}+\frac{15q_1(1-\mu)(x+\mu)(\sigma_1-\sigma_2)y^2}{2r_1^7}-\frac{\mu q_2\left(x+\mu-1\right)}{r_2^3}\\ &\qquad -\frac{3\mu q_2(x+\mu-1)(2\sigma_1'-\sigma_2')}{2r_2^5}+\frac{15\mu q_2(x+\mu-1)(\sigma_1'-\sigma_2')y^2}{2r_2^7}-\frac{M_bx}{\left(r^2+T^2\right)^{3/2}}=0, \end{aligned} \tag{4.19}$$

$$\begin{aligned} &n^2y-\frac{q_1(1-\mu)y}{r_1^3}-\frac{3(1-\mu)q_1(4\sigma_1-3\sigma_2)y}{2r_1^5}+\frac{15(1-\mu)q_1(\sigma_1-\sigma_2)y^3}{2r_1^7}-\frac{\mu q_2y}{r_2^3}\\ &\qquad -\frac{3\mu q_2(4\sigma_1'-3\sigma_2')y}{2r_2^5}+\frac{15\mu q_2(\sigma_1'-\sigma_2')y^3}{2r_2^7}-\frac{M_by}{\left(r^2+T^2\right)^{3/2}}=0. \end{aligned} \tag{4.20}$$

These equations can be re written as

$$\begin{aligned} &x\left(n^2-\frac{q_1(1-\mu)}{r_1^3}-\frac{3q_1(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5}+\frac{15q_1(1-\mu)(\sigma_1-\sigma_2)y^2}{2r_1^7}-\frac{\mu q_2}{r_2^3}-\frac{3\mu q_2(2\sigma_1'-\sigma_2')}{2r_2^5}\right.\\ &\left.+\frac{15\mu q_2(\sigma_1'-\sigma_2')y^2}{2r_2^7}-\frac{M_b}{(r^2+T^2)^{3/2}}\right)-\frac{q_1(1-\mu)\mu}{r_1^3}-\frac{3q_1(1-\mu)\mu(2\sigma_1-\sigma_2)}{2r_1^5}+\frac{15q_1(1-\mu)\mu(\sigma_1-\sigma_2)y^2}{2r_1^7}\\ &\quad -\frac{\mu q_2(\mu-1)}{r_2^3}-\frac{3\mu q_2(\mu-1)(2\sigma_1'-\sigma_2')}{2r_2^5}+\frac{15\mu q_2(\mu-1)(\sigma_1'-\sigma_2')y^2}{2r_2^7}=0, \end{aligned} \tag{4.21}$$

$$\begin{aligned} &y\left(n^2-\frac{q_1(1-\mu)}{r_1^3}-\frac{3(1-\mu)q_1(4\sigma_1-3\sigma_2)}{2r_1^5}+\frac{15(1-\mu)q_1(\sigma_1-\sigma_2)y^2}{2r_1^7}-\frac{\mu q_2}{r_2^3}\right.\\ &\qquad\left.-\frac{3\mu q_2(4\sigma_1'-3\sigma_2')}{2r_2^5}+\frac{15\mu q_2(\sigma_1'-\sigma_2')y^2}{2r_2^7}-\frac{M_b}{\left(r^2+T^2\right)^{3/2}}\right)=0. \end{aligned} \tag{4.22}$$

Or

$$\begin{aligned} &y\left(n^2-\frac{q_1(1-\mu)}{r_1^3}-\frac{3(1-\mu)q_1(2\sigma_1-\sigma_2)}{2r_1^5}-\frac{3(1-\mu)q_1(\sigma_1-\sigma_2)}{r_1^5}+\frac{15(1-\mu)q_1(\sigma_1-\sigma_2)y^2}{2r_1^7}\right.\\ &\qquad\left.-\frac{\mu q_2}{r_2^3}-\frac{3\mu q_2(2\sigma_1'-\sigma_2')}{2r_2^5}-\frac{3\mu q_2(\sigma_1'-\sigma_2')}{r_2^5}+\frac{15\mu q_2(\sigma_1'-\sigma_2')y^2}{2r_2^7}-\frac{M_b}{\left(r^2+T^2\right)^{3/2}}\right)=0. \end{aligned}$$

Here two cases arise: $y \neq 0$ gives the triangular points and $y = 0$ the collinear points.

4.3.1 Location of triangular libration points

The triangular points are the solutions of Equations (4.21) and (4.22) when $y \neq 0$.

Now, Equation (4.22) with $y \neq 0$ implies

$$
\begin{aligned}
n^2 - \frac{q_1(1-\mu)}{r_1^3} - \frac{3(1-\mu)q_1(4\sigma_1 - 3\sigma_2)}{2r_1^5} + \frac{15(1-\mu)q_1(\sigma_1-\sigma_2)y^2}{2r_1^7} - \frac{\mu q_2}{r_2^3} - \frac{3\mu q_2(4\sigma_1' - 3\sigma_2')}{2r_2^5} \\
+ \frac{15\mu q_2(\sigma_1'-\sigma_2')y^2}{2r_2^7} - \frac{M_b}{\left(r^2+T^2\right)^{3/2}} = 0.
\end{aligned}
$$

(4.23)

Plugging Equation (4.23) in Equation (4.21) gives

$$
\begin{aligned}
x\left(\frac{3(1-\mu)q_1(\sigma_1-\sigma_2)}{r_1^5} + \frac{3\mu q_2(\sigma_1'-\sigma_2')}{r_2^5}\right) - \frac{q_1(1-\mu)\mu}{r_1^3} - \frac{3q_1(1-\mu)\mu(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{15q_1(1-\mu)\mu(\sigma_1-\sigma_2)y^2}{2r_1^7} \\
- \frac{\mu q_2(\mu-1)}{r_2^3} - \frac{3\mu q_2(\mu-1)(2\sigma_1'-\sigma_2')}{2r_2^5} + \frac{15\mu q_2(\mu-1)(\sigma_1'-\sigma_2')y^2}{2r_2^7} = 0.
\end{aligned}
$$

Or

$$
\begin{aligned}
x\left(\frac{3(1-\mu)q_1(\sigma_1-\sigma_2)}{r_1^5} + \frac{3\mu q_2(\sigma_1'-\sigma_2')}{r_2^5}\right) - \frac{q_1(1-\mu)\mu}{r_1^3} - \frac{3q_1(1-\mu)\mu(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{15q_1(1-\mu)\mu(\sigma_1-\sigma_2)y^2}{2r_1^7} \\
+ (\mu-1)\left(-\frac{q_2\mu}{r_2^3} - \frac{3\mu q_2(2\sigma_1'-\sigma_2')}{2r_2^5} + \frac{15\mu q_2(\sigma_1'-\sigma_2')y^2}{2r_2^7}\right) = 0.
\end{aligned}
$$

(4.24)

Equation (4.23) can be re-written as

$$
\begin{aligned}
-\frac{\mu q_2}{r_2^3} - \frac{3\mu q_2(4\sigma_1'-3\sigma_2')}{2r_2^5} + \frac{15\mu q_2(\sigma_1'-\sigma_2')y^2}{2r_2^7} = -n^2 + \frac{q_1(1-\mu)}{r_1^3} + \frac{3(1-\mu)q_1(4\sigma_1-3\sigma_2)}{2r_1^5} \\
- \frac{15(1-\mu)q_1(\sigma_1-\sigma_2)y^2}{2r_1^7} + \frac{M_b}{\left(r^2+T^2\right)^{3/2}}.
\end{aligned}
$$

Or

$$-\frac{\mu q_2}{r_2^3}-\frac{3\mu q_2(2\sigma_1'-\sigma_2')}{2r_2^5}+\frac{15\mu q_2(\sigma_1'-\sigma_2')y^2}{2r_2^7}=-n^2+\frac{q_1(1-\mu)}{r_1^3}+\frac{3(1-\mu)q_1(4\sigma_1-3\sigma_2)}{2r_1^5}$$
$$-\frac{15(1-\mu)q_1(\sigma_1-\sigma_2)y^2}{2r_1^7}+\frac{3\mu q_2(\sigma_1'-\sigma_2')}{r_2^5}+\frac{M_b}{\left(r^2+T^2\right)^{3/2}}.$$

(4.25)

Using Equation (4.25) in Equation (4.24) produces

$$x\left(\frac{3(1-\mu)q_1(\sigma_1-\sigma_2)}{r_1^5}+\frac{3\mu q_2(\sigma_1'-\sigma_2')}{r_2^5}\right)-\frac{q_1(1-\mu)\mu}{r_1^3}-\frac{3q_1(1-\mu)\mu(2\sigma_1-\sigma_2)}{2r_1^5}$$
$$+\frac{15q_1(1-\mu)\mu(\sigma_1-\sigma_2)y^2}{2r_1^7}+(\mu-1)\left(-n^2+\frac{q_1(1-\mu)}{r_1^3}+\frac{3(1-\mu)q_1(4\sigma_1-3\sigma_2)}{2r_1^5}\right.$$
$$\left.-\frac{15(1-\mu)q_1(\sigma_1-\sigma_2)y^2}{2r_1^7}+\frac{3\mu q_2(\sigma_1'-\sigma_2')}{r_2^5}+\frac{M_b}{\left(r^2+T^2\right)^{3/2}}\right)=0.$$

Or

$$x\left(\frac{3(1-\mu)q_1(\sigma_1-\sigma_2)}{r_1^5}+\frac{3\mu q_2(\sigma_1'-\sigma_2')}{r_2^5}\right)-n^2\mu+\frac{3(1-\mu)\mu q_1(\sigma_1-\sigma_2)}{r_1^5}+\frac{3\mu^2 q_2(\sigma_1'-\sigma_2')}{r_2^5}+\frac{\mu M_b}{\left(r^2+T^2\right)^{3/2}}$$
$$+n^2-\frac{q_1(1-\mu)}{r_1^3}-\frac{3(1-\mu)q_1(4\sigma_1-3\sigma_2)}{2r_1^5}+\frac{15(1-\mu)q_1(\sigma_1-\sigma_2)y^2}{2r_1^7}-\frac{3\mu q_2(\sigma_1'-\sigma_2')}{r_2^5}-\frac{M_b}{\left(r^2+T^2\right)^{3/2}}=0.$$

Or

$$\frac{3(1-\mu)q_1(x+\mu-1)(\sigma_1-\sigma_2)}{r_1^5}+\frac{3\mu q_2(x+\mu-1)(\sigma_1'-\sigma_2')}{r_2^5}+n^2-n^2\mu-\frac{q_1(1-\mu)}{r_1^3}-\frac{3(1-\mu)q_1(2\sigma_1-\sigma_2)}{2r_1^5}$$
$$+\frac{15(1-\mu)q_1(\sigma_1-\sigma_2)y^2}{2r_1^7}+\frac{\mu M_b}{\left(r^2+T^2\right)^{3/2}}-\frac{M_b}{\left(r^2+T^2\right)^{3/2}}=0.$$

Or

$$\frac{3(1-\mu)q_1(x+\mu-1)(\sigma_1-\sigma_2)}{r_1^5}+\frac{3\mu q_2(x+\mu-1)(\sigma_1'-\sigma_2')}{r_2^5}+n^2\left(1-\mu\right)-\frac{q_1(1-\mu)}{r_1^3}-\frac{3(1-\mu)q_1(2\sigma_1-\sigma_2)}{2r_1^5}$$
$$+\frac{15(1-\mu)q_1(\sigma_1-\sigma_2)y^2}{2r_1^7}-\frac{(1-\mu)M_b}{\left(r^2+T^2\right)^{3/2}}=0.$$

(4.26)

Equation

(4.26) can further be simplified as

$$(1-\mu)\left[\frac{3q_1(x+\mu-1)(\sigma_1-\sigma_2)}{r_1^5}+\frac{3\frac{\mu}{(1-\mu)}q_2(x+\mu-1)(\sigma_1'-\sigma_2')}{r_2^5}+n^2-\frac{q_1(1-\mu)}{r_1^3}\right.$$
$$\left.-\frac{3q_1(2\sigma_1-\sigma_2)}{2r_1^5}+\frac{15q_1(\sigma_1-\sigma_2)y^2}{2r_1^7}-\frac{M_b}{\left(r^2+T^2\right)^{3/2}}\right]=0.$$

This implies

$$\frac{3q_1(x+\mu-1)(\sigma_1-\sigma_2)}{r_1^5}+\frac{3\frac{\mu}{(1-\mu)}q_2(x+\mu-1)(\sigma_1'-\sigma_2')}{r_2^5}+n^2-\frac{q_1(1-\mu)}{r_1^3}$$
$$-\frac{3q_1(2\sigma_1-\sigma_2)}{2r_1^5}+\frac{15q_1(\sigma_1-\sigma_2)y^2}{2r_1^7}-\frac{M_b}{\left(r^2+T^2\right)^{3/2}}=0. \qquad (4.27)$$

Subtracting Equation (4.26) from Equation (4.22) produces

$$\mu n^2-\frac{\mu q_2}{r_2^3}-\frac{3\mu q_2(2\sigma_1'-\sigma_2')}{2r_2^5}+\frac{15\mu q_2(\sigma_1'-\sigma_2')y^2}{2r_2^7}-\frac{3(1-\mu)q_1(x+\mu)(\sigma_1-\sigma_2)}{r_1^5}$$
$$-\frac{3\mu q_2(x+\mu)(\sigma_1'-\sigma_2')}{r_2^5}-\frac{\mu M_b}{\left(r^2+T^2\right)^{3/2}}=0.$$

Or

$$n^2 - \frac{q_2}{r_2^3} - \frac{3q_2(2\sigma_1' - \sigma_2')}{2r_2^5} + \frac{15q_2(\sigma_1' - \sigma_2')y^2}{2r_2^7} - \frac{3(\frac{1-\mu}{\mu})q_1(x+\mu)(\sigma_1 - \sigma_2)}{r_1^5} - \frac{3q_2(x+\mu)(\sigma_1' - \sigma_2')}{r_2^5} - \frac{M_b}{\left(r^2 + T^2\right)^{3/2}} = 0. \tag{4.28}$$

Now, if the influences of triaxiality and radiation of the primaries and the potential from the belt are ignored $(i.e.\ \sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = M_b = 0,\ q_1 = q_2 = 1)$, Equations (4.27) and (4.28) boil down to

$$1 - \frac{1}{r_1^3} = 0 \, and \ 1 - \frac{1}{r_2^3} = 0$$

respectively, with the solutions $r_1 = r_2 = 1.$ Hence, the values of $r_1 \, and \, r_2$ may change slightly, owing to the perturbations . So, let

$$r_1 = 1 + \varepsilon_1 \, and \ r_2 = 1 + \varepsilon_2, \quad where \ \varepsilon_i << 1 \ (i = 1, 2), \tag{4.29}$$

be the solutions of Equations (4.27) and (4.28), where ε_1 and ε_2 are very small quantities and are due to the combined effect of the radiation and triaxiality of the primaries and the potential from the belt.

Proceeding as in as in Section 3.4.1, we obtain from Equations (4.29) and (4.9),

$$\left.\begin{array}{l} x = \frac{1}{2} - \mu + \varepsilon_1 - \varepsilon_2, \\ y = \pm \frac{\sqrt{3}}{2}\left(1 + \frac{2}{3}(\varepsilon_1 + \varepsilon_2)\right). \end{array}\right\} (4.30)$$

Making use of $q_i = 1 - p_i \,(where \, p_i << 1, i = 1, 2)$ and Equations (4.30), (4.29), (4.18) in Equations (4.27) and (4.28); and considering only linear terms in small quantities, produces

$$\left.\begin{aligned}\varepsilon_1 &= -\frac{p_1}{3}-\frac{11}{8}\sigma_1+\frac{11}{8}\sigma_2+\left(\frac{\mu}{2(1-\mu)}-1\right)\sigma_1'+\left(\frac{1}{2}-\frac{\mu}{2(1-\mu)}\right)\sigma_2'-\frac{M_b(2r_c-1)}{3(r_c^2+T^2)^{3/2}},\\ \varepsilon_2 &= -\frac{p_2}{3}-\left(\frac{3}{2}-\frac{1}{2\mu}\right)\sigma_1+\left(1-\frac{1}{2\mu}\right)\sigma_2-\frac{11}{8}\sigma_1'+\frac{11}{8}\sigma_2'-\frac{M_b(2r_c-1)}{3(r_c^2+T^2)^{3/2}}.\end{aligned}\right\}(4.31)$$

Thus, utilizing Equation (4.31) in (4.30) we get the coordinates of the triangular libration points $L_4(x,y)\ and\ L_5(x,-y)$ as

$$\left.\begin{aligned}x &= \frac{1}{2}\left[1-2\mu-\frac{2p_1}{3}+\frac{2p_2}{3}+(\frac{1}{4}-\frac{1}{\mu})\sigma_1+(\frac{3}{4}+\frac{1}{\mu})\sigma_2+(\frac{3}{4}+\frac{\mu}{1-\mu})\sigma_1'-(\frac{7}{4}+\frac{\mu}{1-\mu})\sigma_2'\right],\\ y &= \pm\frac{\sqrt{3}}{2}\left[1-\frac{2p_1}{9}-\frac{2p_2}{9}+\frac{1}{3}\left((\frac{1}{\mu}-\frac{23}{4})\sigma_1+(\frac{19}{4}-\frac{1}{\mu})\sigma_2+(\frac{\mu}{1-\mu}-\frac{19}{4})\sigma_1'\right.\right.\\ &\qquad\left.\left.+(\frac{15}{4}-\frac{\mu}{1-\mu})\sigma_2'-\frac{4M_b(2r_c-1)}{3(r_c^2+T^2)^{3/2}}\right)\right].\end{aligned}\right\}(4.32)$$

4.3.2 Locations of collinear libration points

The positions of collinear points are obtained by solving Equations (4.21) and (4.22) when $y=0.$

From Equation (4.22)y must be zero, and substituting it in equation (4.21), yields

$$n^2x-\frac{(1-\mu)\left(x+\mu\right)q_1}{\left|x+\mu\right|^3}-\frac{3(1-\mu)\left(x+\mu\right)q_1(2\sigma_1-\sigma_2)}{2\left|x+\mu\right|^5}-\frac{\mu\left(x+\mu-1\right)q_2}{\left|x+\mu-1\right|^3}-\frac{3\mu\left(x+\mu-1\right)q_2(2\sigma_1'-\sigma_2')}{2\left|x+\mu-1\right|^5}$$
$$-\frac{M_bx}{\left(x^2+T^2\right)^{3/2}}=0.$$

(4.33)

Now, neglecting the influences of triaxiality and radiation of the primaries and the potential from the belt $(i.e.\ \sigma_1=\sigma_2=\sigma_1'=\sigma_2'=M_b=0,\ q_1=q_2=1),$ Equation (4.33) reduces to

$$x-\frac{(1-\mu)\left(x+\mu\right)}{\left|x+\mu\right|^{3}}-\frac{\mu\left(x+\mu-1\right)}{\left|x+\mu-1\right|^{3}}=0, \qquad (4.34)$$

with three collinear points L_1, L_2 and L_3. The collinear point L_2 is disposed between the primaries, while L_1 and L_3 are beyond them as shown in Figure 1.2.

Now, using Equation (4.33) and with the help of the MATLAB (R2007) software package, we obtained the coordinates of the collinear libration points for different cases as classified in the following order, which are presented in Table 4.1:

1 The classical case (absence of the perturbations).
2 Potential from the belt only.
3 Triaxiality of the bigger primary with potential from the belt only.
4 Radiation and Triaxiality of the bigger primary with potential from the belt only.
5 Triaxiality of the smaller primary with potential from the belt only.
6 Radiation and Triaxiality of the smaller primarywith potential from the belt only.
7 Triaxiality of the primaries with potential from the belt only.
8 Radiation and Triaxiality of the primaries with potential from the belt only

Table 4.1 Positions of the collinear points when $\sigma_1=\sigma_1'=M_b$ =T=0.01, $\sigma_2=\sigma_2'=0.008,\ q_1=0.98,\ q_2=0.97, \mu=0.25.$

Case	L_1	L_2	L_3	L_{n1}	L_{n2}
1	1.265858	0.360743	-1.103167	-	-
2	1.261034	0.364839	-1.097847	-0.001192	-0.023389
3	1.25723	0.369905	-1.099561	-0.001583	-0.019520
4	1.255981	0.367537	-1.093774	-0.001548	-0.019774

5	1.268700	0.353408	-1.092499	-0.001191	-0.023400
6	1.263284	0.356370	-1.091890	-0.001193	-0.023388
7	1.265086	0.358440	-1.094384	-0.001582	-0.019529
8	1.258540	0.359093	-1.088040	-0.001548	-0.019773

The existence and positions of the libration points for $\mu \in (0,1/2)$ in the presence of all the perturbations are shown in Figure 4.3.

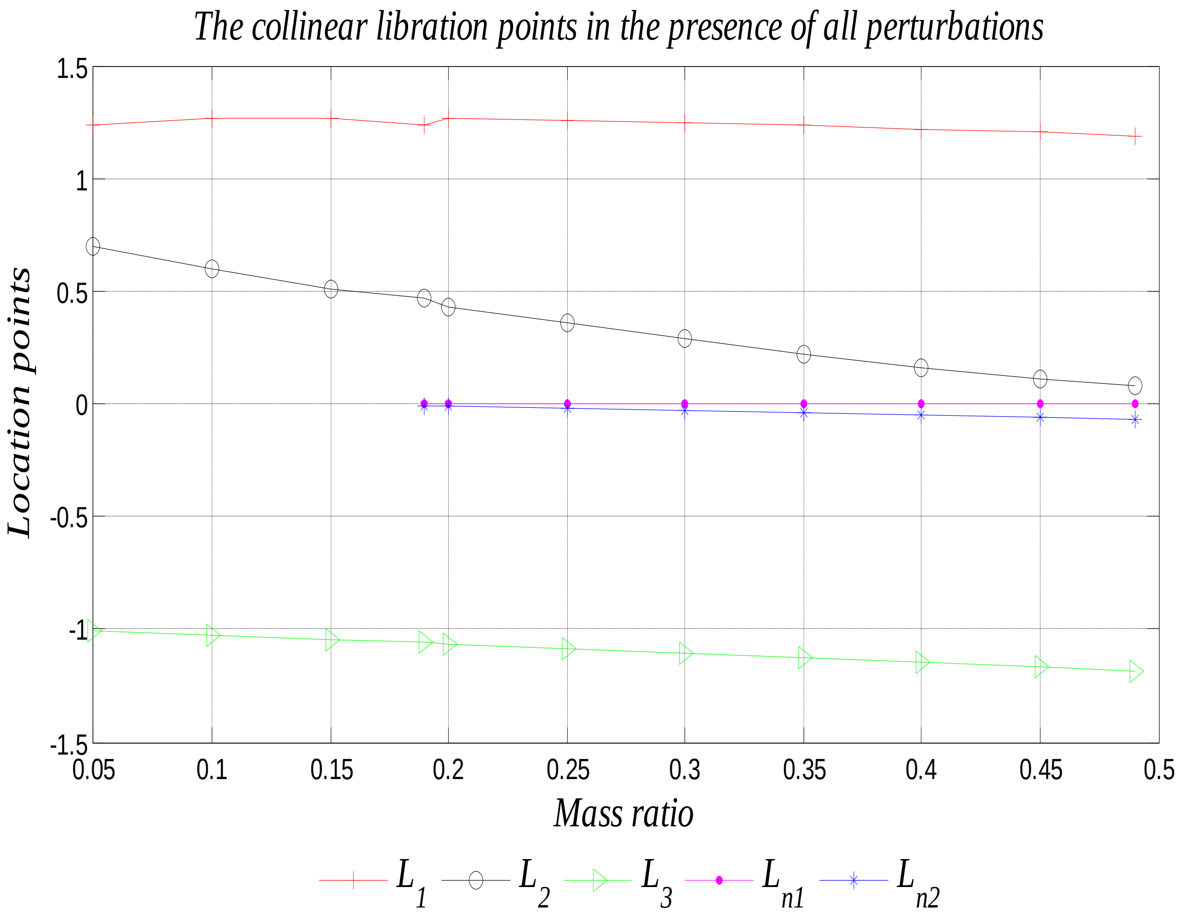


Figure 4.3: The collinear points under the combined effect of the perturbations (i.e. $\sigma_1 = \sigma_1' = M_b$ =T=0.01, $\sigma_2 = \sigma_2' = 0.008$, $q_1 = 0.98$, $q_2 = 0.97$).

4.4 Stability of Libration Points.

When a slight displacement of a particle away from a libration point will not produce unbounded motion, but rather an oscillation about the point, then that point is said to stable otherwise unstable.

4.4.1 Variational and characteristic equations

Let (x_0, y_0) be the coordinates of any of the libration points and let η, ξ *be* a small change in (x_0, y_0) such that $(x_0 + \eta, y_0 + \xi)$ is a point in the neighborhood of (x_0, y_0). Following the same linear stability analysis used in Section 3.5.1, the linear variational equations of motion of Equation (4.14) are

$$\begin{aligned} &\ddot{\eta}-2n\dot{\xi}=(\Omega_{xx}^{0})\eta+(\Omega_{xy}^{0})\xi, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.35) \\ &\ddot{\xi}+2n\dot{\eta}=(\Omega_{yx}^{0})\eta+(\Omega_{yy}^{0})\xi. \end{aligned}$$

Let $\eta=Ae^{\lambda t}$, $\xi=Be^{\lambda t}$ (where $A,\ B,\lambda$ are constants) be the solutions of Equation (4.35), then the equations will have a non –trivial solution for *A* and *B* if

$$\begin{vmatrix} \lambda^{2}-\Omega_{xx}^{0} & -2n\lambda-\Omega_{xy}^{0} \\ 2n\lambda-\Omega_{yx}^{0} & \lambda^{2}-\Omega_{yy}^{0} \end{vmatrix}=0. \tag{4.36}$$

On expanding the determinant we obtain the characteristic equation corresponding to the variational equations of motion (4.35) as

$$\lambda^{4}+(4n^{2}-\Omega_{xx}^{0}-\Omega_{yy}^{0})\lambda^{2}+\Omega_{xx}^{0}\Omega_{yy}^{0}-\Omega_{xy}^{0\ 2}=0 \tag{4.37}$$

where the superscript 0 indicates that the partial derivatives have been computed at the libration point.

Now, from Equation (4.14) the second partial derivatives of Ω with respect to *x* is

$$\begin{aligned} \Omega_{xx}={}&n^{2}-\frac{(1-\mu)q_{1}}{r_{1}^{3}}-(1-\mu)\left(x+\mu\right)q_{1}\frac{\partial}{\partial x}\left(\frac{1}{r_{1}^{3}}\right)-\frac{3q_{1}(1-\mu)(2\sigma_{1}-\sigma_{2})}{2r_{1}^{5}} \\ &-\frac{3q_{1}(1-\mu)(x+\mu)(2\sigma_{1}-\sigma_{2})}{2}\frac{\partial}{\partial x}\left(\frac{1}{r_{1}^{5}}\right)+\frac{15q_{1}(1-\mu)(\sigma_{1}-\sigma_{2})y^{2}}{2r_{1}^{7}}+ \\ &\frac{15q_{1}(1-\mu)(x+\mu)(\sigma_{1}-\sigma_{2})y^{2}}{2}\frac{\partial}{\partial x}\left(\frac{1}{r_{1}^{7}}\right)-\frac{\mu q_{2}}{r_{2}^{3}}-\mu\left(x+\mu-1\right)q_{2}\frac{\partial}{\partial x}\left(\frac{1}{r_{2}^{3}}\right) \\ &-\frac{3\mu q_{2}(2\sigma_{1}'-\sigma_{2}')}{2r_{2}^{5}}-\frac{3\mu q_{2}(x+\mu-1)(2\sigma_{1}'-\sigma_{2}')}{2}\frac{\partial}{\partial x}\left(\frac{1}{r_{2}^{5}}\right)+\frac{15\mu q_{2}(\sigma_{1}'-\sigma_{2}')y^{2}}{2r_{2}^{7}} \\ &+\frac{15\mu q_{2}(x+\mu-1)(\sigma_{1}'-\sigma_{2}')y^{2}}{2}\frac{\partial}{\partial x}\left(\frac{1}{r_{2}^{7}}\right)-\frac{M_{b}}{\left(r^{2}+T^{2}\right)^{3/2}}-M_{b}x\frac{\partial}{\partial x}\frac{1}{\left(r^{2}+T^{2}\right)^{3/2}}, \end{aligned}$$

and in view of Section 3.5.1 it becomes

$$
\begin{aligned}
\Omega_{xx} = n^2 &- \frac{(1-\mu)q_1}{r_1^3} + \frac{3(1-\mu)\left(x+\mu\right)^2 q_1}{r_1^5} - \frac{3q_1(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{15q_1(1-\mu)(x+\mu)^2(2\sigma_1-\sigma_2)}{2r_1^7} \\
&+ \frac{15q_1(1-\mu)(\sigma_1-\sigma_2)y^2}{2r_1^7} - \frac{105q_1(1-\mu)(x+\mu)^2(\sigma_1-\sigma_2)y^2}{2r_1^9} - \frac{\mu q_2}{r_2^3} + \frac{3\mu\left(x+\mu-1\right)^2 q_2}{r_2^5} \\
&- \frac{3\mu q_2(2\sigma_1'-\sigma_2')}{2r_2^5} + \frac{15\mu q_2(x+\mu-1)^2(2\sigma_1'-\sigma_2')}{2r_2^7} + \frac{15\mu q_2(\sigma_1'-\sigma_2')y^2}{2r_2^7} \\
&- \frac{105\mu q_2(x+\mu-1)^2(\sigma_1'-\sigma_2')y^2}{2r_2^9} - \frac{M_b}{\left(r^2+T^2\right)^{3/2}} + \frac{3M_b x^2}{\left(r^2+T^2\right)^{5/2}}.
\end{aligned}
\tag{4.38}
$$

In a similar manner, we obtain

$$
\begin{aligned}
\Omega_{yy} = n^2 &- \frac{(1-\mu)q_1}{r_1^3} + \frac{3(1-\mu)y^2 q_1}{r_1^5} - \frac{3(1-\mu)q_1(4\sigma_1-3\sigma_2)}{2r_1^5} + \frac{15(1-\mu)q_1 y^2(4\sigma_1-3\sigma_2)}{2r_1^7} \\
&+ \frac{45(1-\mu)q_1(\sigma_1-\sigma_2)y^2}{2r_1^7} - \frac{105(1-\mu)q_1(\sigma_1-\sigma_2)y^4}{2r_1^9} - \frac{\mu q_2}{r_2^3} + \frac{3\mu y^2 q_2}{r_2^5} - \frac{3\mu q_2(4\sigma_1'-3\sigma_2')}{2r_2^5} \\
&+ \frac{15\mu q_2(4\sigma_1'-3\sigma_2')y^2}{2r_2^7} - \frac{105\mu q_2(\sigma_1'-\sigma_2')y^4}{2r_2^9} - \frac{M_b}{\left(r^2+T^2\right)^{3/2}} + \frac{3M_b y^2}{\left(r^2+T^2\right)^{5/2}} = 0
\end{aligned}
\tag{4.39}
$$

and

$$\Omega_{xy}=\Omega_{yx}=\frac{3q_1(1-\mu)\left(x+\mu\right)y}{r_1^5}+\frac{15q_1(1-\mu)(x+\mu)y(2\sigma_1-\sigma_2)}{2r_1^7}+\frac{30q_1(1-\mu)(x+\mu)(\sigma_1-\sigma_2)y}{2r_1^7}$$

$$-\frac{105q_1(1-\mu)(x+\mu)(\sigma_1-\sigma_2)y^3}{2r_1^9}+\frac{3\mu q_2\left(x+\mu-1\right)y}{r_2^5}+\frac{15\mu q_2(x+\mu-1)y(2\sigma_1'-\sigma_2')}{2r_2^7}$$

$$+\frac{30\mu q_2(x+\mu-1)(\sigma_1'-\sigma_2')y}{2r_2^7}-\frac{105\mu q_2(x+\mu-1)(\sigma_1'-\sigma_2')y^3}{2r_2^9}-\frac{M_b xy}{\left(r^2+T^2\right)^{5/2}}.$$

$$=\frac{3q_1(1-\mu)\left(x+\mu\right)y}{r_1^5}+\frac{15(1-\mu)q_1(4\sigma_1-3\sigma_2)\left(x+\mu\right)y}{2r_1^7}-\frac{105(1-\mu)q_1(\sigma_1-\sigma_2)\left(x+\mu\right)y^3}{2r_1^9}$$

$$+\frac{3\mu q_2\left(x+\mu-1\right)y}{r_2^5}+\frac{15\mu q_2(4\sigma_1'-3\sigma_2')\left(x+\mu-1\right)y}{2r_2^7}-\frac{105\mu q_2(\sigma_1'-\sigma_2')y^3\left(x+\mu-1\right)}{2r_2^9}-\frac{M_b xy}{\left(r^2+T^2\right)^{5/2}}. \tag{4.40}$$

4.4.2 Stability of triangular libration points.

The second partial derivatives of Equations (4.38) to (4.40) evaluated at any of the triangular libration points Equation (4.32), considering only linear terms in small quantities, are:

$$\Omega_{xx}^0=\frac{3}{4}-\left(\frac{1}{2}-\frac{3\mu}{2}\right)p_1+\left(1-\frac{3\mu}{2}\right)p_2+\left(\frac{57}{16}+\frac{45\mu}{16}-\frac{3}{2\mu}\right)\sigma_1+\left(-\frac{3}{16}-\frac{93\mu}{16}+\frac{3}{2\mu}\right)\sigma_2$$

$$+\left(\frac{39}{8}-\frac{69}{16}-\frac{3\mu^2}{2(1-\mu)}\right)\sigma_1'+\left(-\frac{9}{2}+\frac{117\mu}{16}+\frac{3\mu^2}{2(1-\mu)}\right)\sigma_2'+\frac{5M_b(2r_c-1)}{4(r_c^2+T^2)^{3/2}}+\frac{3M_b(\tfrac{1}{4}-\mu+\mu^2)}{(r_c^2+T^2)^{5/2}}, \tag{4.41}$$

$$\Omega_{yy}^0=\frac{9}{4}+\left(\frac{1}{2}-\frac{3\mu}{2}\right)p_1-\left(1-\frac{3\mu}{2}\right)p_2+\left(\frac{87}{16}-\frac{45\mu}{16}+\frac{3}{2\mu}\right)\sigma_1+\left(-\frac{21}{16}+\frac{45\mu}{16}-\frac{3}{2\mu}\right)\sigma_2$$

$$+\left(\frac{33}{8}+\frac{135\mu}{16}+\frac{45\mu^2-11\mu}{8(1-\mu)}\right)\sigma_1'+\left(-\frac{135\mu}{16}+\frac{33\mu-45\mu^2}{8(1-\mu)}\right)\sigma_2'+\frac{7M_b(2r_c-1)}{4(r_c^2+T^2)^{3/2}}+\frac{3M_b(\tfrac{3}{4})}{(r_c^2+T^2)^{5/2}}, \tag{4.42}$$

$$\begin{aligned}\Omega^0_{xy} = \Omega^0_{yx} = \sqrt{3}\Bigg\{&\frac{3}{4}-\frac{3\mu}{2}-\left(\frac{1}{6}+\frac{\mu}{6}\right)p_1+\left(\frac{1}{3}-\frac{\mu}{6}\right)p_2+\left(\frac{47}{16}-\frac{89\mu}{16}-\frac{1}{2\mu}\right)\sigma_1+\left(-\frac{9}{16}+\frac{37\mu}{16}+\frac{1}{2\mu}\right)\sigma_2\\ &+\left(\frac{25}{8}-\frac{85\mu}{16}+\frac{\mu+\mu^2}{4(1-\mu)}\right)\sigma_1'+\left(-\frac{9}{4}+\frac{33\mu}{16}-\frac{\mu+\mu^2}{4(1-\mu)}\right)\sigma_2' \;-\frac{11\mu M_b(2r_c-1)}{6(r_c^2+T^2)^{3/2}}\\ &+\frac{11M_b(2r_c-1)}{12(r_c^2+T^2)^{3/2}}+\frac{\frac{3}{2}M_b(\frac{1}{2}-\mu)}{(r_c^2+T^2)^{5/2}}\Bigg\},\end{aligned}$$

(4.43)

respectively.

Applying these values of the partial derivatives: $\Omega^0_{xx}, \Omega^0_{yy},\ \Omega^0_{xy}$ in Equation (4.37) and neglecting powers of $\mu > 2$, the characteristic equation becomes

$$\lambda^4 + Q\lambda^2 + W = 0, \tag{4.44}$$

where

$$Q = 1 + 3\sigma_1 + 3(\mu-\frac{3}{2})\sigma_2 + 3\sigma_1' - (\frac{3}{2}+3\mu)\sigma_2' + \frac{M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}} - \frac{3M_b r_c^2}{(r_c^2+T^2)^{5/2}} > 0,$$

$$\begin{aligned}W = &\left(\frac{27\mu}{4}-\frac{27\mu^2}{4}\right)+\left(\frac{3\mu}{2}-\frac{3\mu^2}{2}\right)p_1+\left(\frac{3\mu}{2}-\frac{3\mu^2}{2}\right)p_2+\left(\frac{-45}{8}+\frac{891\mu}{16}-\frac{801\mu^2}{16}\right)\sigma_1\\ &+\left(\frac{45}{8}-\frac{423\mu}{16}+\frac{333\mu^2}{16}\right)\sigma_2+\left(\frac{711\mu}{16}-\frac{801\mu^2}{16}\right)\sigma_1'-\left(\frac{243\mu}{16}-\frac{333\mu^2}{16}\right)\sigma_2'\\ &+\left(\frac{33\mu(2r_c-1)}{2(r_c^2+T^2)^{3/2}}+\frac{27\mu}{4(r_c^2+T^2)^{5/2}}-\frac{33\mu^2(2r_c-1)}{2(r_c^2+T^2)^{3/2}}-\frac{27\mu^2}{4(r_c^2+T^2)^{5/2}}\right)M_b.\end{aligned}$$

The roots of Equation (4.44) are:

$$\lambda^2 = \frac{1}{2}\left[-Q \pm \sqrt{\Delta}\right], \tag{4.45}$$

where $\Delta = Q^2 - 4W$ is the discriminant, given by

$$\Delta = \left(27 + 6p_1 + 6p_2 + \frac{801\sigma_1}{4} - \frac{333\sigma_2}{4} + \frac{801\sigma_1'}{4} - \frac{333\sigma_2'}{4} + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right)\mu^2$$
$$- \left(27 + 6p_1 + 6p_2 + \frac{891\sigma_1}{4} - \frac{447\sigma_2}{4} + \frac{711\sigma_1'}{4} - \frac{219\sigma_2'}{4} + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right)\mu$$
$$+1 + \frac{57\sigma_1}{2} - \frac{63\sigma_2}{2} + 6\sigma_1' - 3\sigma_2' + \frac{2M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{6M_b r_c^2}{(r_c^2 + T^2)^{5/2}}. \qquad (4.46)$$

Now, Δ in Equation (4.46) is a quadratic function of the mass parameter μ. As in Section (3.5.3), we study the behavior of Δ in the interval $0 \le \mu \le \frac{1}{2}$.

When $\mu = 0$,

$$\Delta = 1 + \frac{57\sigma_1}{2} - \frac{63\sigma_2}{2} + 6\sigma_1' - 3\sigma_2' + \frac{2M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{6M_b r_c^2}{(r_c^2 + T^2)^{5/2}} > 0, \qquad (4.47)$$

and when $\mu = \frac{1}{2}$,

$$\Delta = \left(27 + 6p_1 + 6p_2 + \frac{801\sigma_1}{4} - \frac{333\sigma_2}{4} + \frac{801\sigma_1'}{4} - \frac{333\sigma_2'}{4} + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right)\frac{1}{4}$$
$$- \left(27 + 6p_1 + 6p_2 + \frac{891\sigma_1}{4} - \frac{447\sigma_2}{4} + \frac{711\sigma_1'}{4} - \frac{219\sigma_2'}{4} + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right)\frac{1}{2}$$
$$+1 + \frac{57\sigma_1}{2} - \frac{63\sigma_2}{2} + 6\sigma_1' - 3\sigma_2' + \frac{2M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{6M_b r_c^2}{(r_c^2 + T^2)^{5/2}}.$$

$$= -\frac{23}{4} - \left[\frac{3p_1}{2} + \frac{3p_2}{2} + \frac{525\sigma_1}{16} - \frac{57\sigma_2}{16} + \frac{525\sigma_1'}{16} - \frac{57\sigma_2'}{16} + \left(\frac{(58r_c - 45)}{2(r_c^2 + T^2)^{3/2}} + \frac{3(8r_c^2 + 9)}{4(r_c^2 + T^2)^{5/2}} \right) M_b \right] < 0. \qquad (4.48)$$

Differentiating the discriminant Δ with respect to the mass parameter μ, produces

$$\frac{d\Delta}{d\mu}=\left(27+6p_1+6p_2+\frac{801\sigma_1}{4}-\frac{333\sigma_2}{4}+\frac{801\sigma_1'}{4}-\frac{333\sigma_2'}{4}+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)2\mu$$
$$-27-6p_1-6p_2-\frac{891\sigma_1}{4}+\frac{447\sigma_2}{4}-\frac{711\sigma_1'}{4}+\frac{219\sigma_2'}{4}-\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}-\frac{27M_b}{(r_c^2+T^2)^{5/2}}.$$

Evaluating $\left(\frac{d\Delta}{d\mu}\right)$ when $\mu=0$ and $\mu=\frac{1}{2}$, yield

$-\left(27+6p_1+6p_2+\frac{891\sigma_1}{4}-\frac{447\sigma_2}{4}+\frac{711\sigma_1'}{4}-\frac{219\sigma_2'}{4}+\frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}}+\frac{27M_b}{(r_c^2+T^2)^{5/2}}\right)<0$ and $\left(-\frac{45\sigma_1}{2}+\frac{57\sigma_2}{2}+\frac{45\sigma_1'}{2}-\frac{57\sigma_2'}{2}\right)\approx 0$ respectively.These imply that $\frac{d\Delta}{d\mu}<0$ in the interval $0<\mu<\frac{1}{2}$.

Now, since the values of Δ when $\mu=0$ and $\mu=\frac{1}{2}$ have unlike signs and $\frac{d\Delta}{d\mu}<0$ in the interval $0<\mu<\frac{1}{2}$, there is a value of μ in the interval (0, 1/2) such that $\Delta=0$. This value denoted by μ_c is called the critical mass ratio.

Three cases can be examined for Δ in the interval $0<\mu<\frac{1}{2}$:

(a) If $0<\mu<\mu_c$, $\Delta>0$ and the roots (4.45)are distinct pure imaginary numbers, showing the triangular points are linearly stable.

(b) If $\mu_c<\mu<\frac{1}{2}$, $\Delta<0$ and the real parts of two of the roots (4.45)are positive. Therefore, the triangular points are unstable.

(c) When $\mu = \mu_c$, Δ=0 and the roots of Equation (4.45) are double, showing that the triangular points are unstable.

The critical mass ratio μ_c

The critical mass value μ_c is obtained from Equation (1) when Δ=0. That is, the solution of

$$\left(27+\varepsilon_1\right)\mu^2-\left(27+\varepsilon_2\right)\mu+\left(1+\varepsilon_1\right)=0$$

where,

$$\varepsilon_1 = 6p_1 + 6p_2 + \frac{801\sigma_1}{4} - \frac{333\sigma_2}{4} + \frac{801\sigma_1'}{4} - \frac{333\sigma_2'}{4} + \frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} + \frac{27M_b}{(r_c^2+T^2)^{5/2}} \ll 1,$$

$$\varepsilon_2 = 6p_1 + 6p_2 + \frac{891\sigma_1}{4} - \frac{447\sigma_2}{4} + \frac{711\sigma_1'}{4} - \frac{219\sigma_2'}{4} + \frac{66M_b(2r_c-1)}{(r_c^2+T^2)^{3/2}} + \frac{27M_b}{(r_c^2+T^2)^{5/2}} \ll 1,$$

(4.49)

$$\varepsilon_3 = \frac{57\sigma_1}{2} - \frac{63\sigma_2}{2} + 6\sigma_1' - 3\sigma_2' + \frac{2M_b(2r_c+3)}{(r_c^2+T^2)^{3/2}} - \frac{6M_b r_c^2}{(r_c^2+T^2)^{5/2}} \ll 1.$$

Thus, the mass ratio $\mu_c$is given by

$$\mu_c = \frac{27+\varepsilon_2}{2\left(27+\varepsilon_1\right)} \pm \frac{1}{2}\sqrt{\frac{\left(27+\varepsilon_2\right)^2 - 4\left(27+\varepsilon_1\right)\left(1+\varepsilon_3\right)}{\left(27+\varepsilon_1\right)^2}}.$$

$$\mu_c = \frac{27+\varepsilon_2}{2\left(27+\varepsilon_1\right)} \pm \frac{1}{2}\sqrt{\frac{\left(27+\varepsilon_2\right)^2 - 4\left(27+\varepsilon_1\right)\left(1+\varepsilon_3\right)}{\left(27+\varepsilon_1\right)^2}}.$$

$$= \frac{27+\varepsilon_2}{54\left(1+\frac{\varepsilon_1}{27}\right)} \pm \frac{1}{2}\sqrt{\frac{\left(27+\varepsilon_2\right)^2}{\left(27+\varepsilon_1\right)^2} - \frac{4\left(1+\varepsilon_3\right)}{27\left(1+\frac{\varepsilon_1}{27}\right)}} \quad .$$

Performing binomial expansion and retaining only linear terms in very small quantities, produces

$$\mu_c = \frac{27+\varepsilon_2-\varepsilon_1}{54} \pm \frac{1}{2}\sqrt{\frac{27^2+54\varepsilon_2}{27^2+54\varepsilon_1} - \frac{4\left(1+\varepsilon_3-\frac{\varepsilon_1}{27}\right)}{27}}.$$

$$= \frac{27+\varepsilon_2-\varepsilon_1}{54} \pm \frac{1}{2}\sqrt{\frac{27^2\left(1+\frac{2\varepsilon_2}{27}\right)}{27^2\left(1+\frac{2\varepsilon_1}{27}\right)} - \frac{4}{27} - \frac{4\left(\varepsilon_3-\frac{\varepsilon_1}{27}\right)}{27}}.$$

$$= \frac{27+\varepsilon_2-\varepsilon_1}{54} \pm \frac{1}{2}\sqrt{\frac{23}{27} + \frac{2\varepsilon_2}{27} - \frac{2\varepsilon_1}{27} - \frac{4}{27}\left(\varepsilon_3-\frac{\varepsilon_1}{27}\right)}.$$

$$= \frac{27+\varepsilon_2-\varepsilon_1}{54} \pm \frac{1}{2}\sqrt{\frac{23}{27} + \frac{2}{27}\left[\varepsilon_2-\varepsilon_1-2\left(\varepsilon_3-\frac{\varepsilon_1}{27}\right)\right]}.$$

$$= \frac{27+\varepsilon_2-\varepsilon_1}{54} \pm \frac{1}{2}\sqrt{\frac{23}{27}\left(1+\frac{2}{23}\left[\varepsilon_2-\frac{25\varepsilon_1}{27}-2\varepsilon_3\right]\right)}.$$

$$= \frac{27+\varepsilon_2-\varepsilon_1}{54} \pm \frac{1}{2}\sqrt{\frac{23}{27}}\left(1+\frac{2}{23}\left[\varepsilon_2-\frac{25\varepsilon_1}{27}-2\varepsilon_3\right]\right)^{1/2}$$

$$= \frac{27+\varepsilon_2-\varepsilon_1}{54} \pm \frac{1}{2}\sqrt{\frac{23}{27}}\left(1+\frac{1}{23}\left[\varepsilon_2-\frac{25\varepsilon_1}{27}-2\varepsilon_3\right]\right).$$

Since the mass ratio μ™*(0, ½)*, we consider only the minus. Therefore,

$$\begin{aligned}\mu_c &= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{23}{27}} + \frac{\varepsilon_2-\varepsilon_1}{54} - \frac{1}{2}\frac{1}{3\sqrt{3}}\frac{1}{\sqrt{23}}\left[\varepsilon_2-\frac{25\varepsilon_1}{27}-2\varepsilon_3\right]. \\ &= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{23}{27}} + \frac{\varepsilon_2-\varepsilon_1}{54} - \frac{1}{6\sqrt{69}}\left[\varepsilon_2-\frac{25\varepsilon_1}{27}-2\varepsilon_3\right].\end{aligned} \tag{4.50}$$

Making use of the values of $\varepsilon_1, \varepsilon_2$ *and* ε_3 of Equation (4.49) in Equation (4.50), the critical mass value μ_c becomes

$$\mu_c = \mu_0 + \mu_r + \mu_t + \mu_b \tag{4.51}$$

where,

$$\mu_0 = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right),$$

$$\mu_r = -\frac{2}{27\sqrt{69}}\left[(1-q_1) + (1-q_2)\right],$$

$$\mu_t = \left(\frac{5}{12} + \frac{59}{18\sqrt{69}}\right)\sigma_1 - \left(\frac{19}{36} + \frac{85}{18\sqrt{69}}\right)\sigma_2 - \left(\frac{5}{12} - \frac{59}{18\sqrt{69}}\right)\sigma_1' + \left(\frac{19}{36} - \frac{85}{18\sqrt{69}}\right)\sigma',$$

$$\mu_b = \frac{(76 - 8r_c)\left(r_c^2 + T^2\right)^{5/2} - 9(1 + 6r_c^2)\left(r_c^2 + T^2\right)^{3/2}}{27\sqrt{69}\left(r_c^2 + T^2\right)^4} M_b.$$

The value of the critical mass parameter to fourteen decimal places (T=0.01, r_c =0.99) is

$$\begin{aligned}\mu_c = {} & 0.03852089650455 - 0.00891747059895(1-q_1) - 0.00891747059895(1-q_2) \\ & + 0.81126474067002\sigma_1 - 1.09626652846058\sigma_2 - 0.02206859266331\sigma_1' \\ & - 0.04071097290502\sigma_2' + 0.02253016502653\,\mathrm{M_b}.\end{aligned}$$

4.4.3 Stability of collinear libration points

As in the case of the triangular libration points, to investigate the stability of any collinear libration point obtained in Table 4.1 we:

- evaluate the second partial derivatives Equations (4.38), (4.39) and (4.40) at the collinear point;

- substitute the values of the second partial derivatives in the characteristic equation (4.37);
- then determine the roots of the characteristic equation .

Now, any collinear libration point (x_0, 0) in Table 4.1, produces

$$
\begin{aligned}
\Omega^0_{xx} = n^2 &- \frac{(1-\mu)q_1}{|x_0+\mu|^3} + \frac{3(1-\mu)\left(x+\mu\right)^2 q_1}{|x_0+\mu|^5} - \frac{3q_1(1-\mu)(2\sigma_1-\sigma_2)}{2|x_0+\mu|^5} + \frac{15q_1(1-\mu)(x+\mu)^2(2\sigma_1-\sigma_2)}{2|x_0+\mu|^7} \\
&- \frac{\mu q_2}{|x_0+\mu-1|^3} + \frac{3\mu\left(x+\mu-1\right)^2 q_2}{|x_0+\mu-1|^5} - \frac{3\mu q_2(2\sigma_1'-\sigma_2')}{2|x_0+\mu-1|^5} + \frac{15\mu q_2(x+\mu-1)^2(2\sigma_1'-\sigma_2')}{2|x_0+\mu-1|^7} \\
&- \frac{M_b}{\left({x_0}^2+T^2\right)^{3/2}} + \frac{3M_b x_0^2}{\left({x_0}^2+T^2\right)^{5/2}},
\end{aligned}
$$

$$
\begin{aligned}
= n^2 &+ \frac{2(1-\mu)q_1}{|x_0+\mu|^3} + \frac{6q_1(1-\mu)(2\sigma_1-\sigma_2)}{|x_0+\mu|^5} + \frac{2\mu q_2}{|x_0+\mu-1|^3} + \frac{6\mu q_2(2\sigma_1'-\sigma_2')}{|x_0+\mu-1|^5} \\
&- \left(\frac{1}{\left({x_0}^2+T^2\right)^{3/2}} - \frac{3x_0^2}{\left({x_0}^2+T^2\right)^{5/2}}\right) M_b,
\end{aligned}
$$

(4.52)

$$
\Omega^0_{yy} = n^2 - \frac{(1-\mu)q_1}{|x_0+\mu|^3} - \frac{3(1-\mu)q_1(4\sigma_1-3\sigma_2)}{2|x_0+\mu|^5} - \frac{\mu q_2}{|x_0+\mu-1|^3} - \frac{3\mu q_2(4\sigma_1'-3\sigma_2')}{2|x_0+\mu-1|^5} - \frac{M_b}{({x_0}^2+T^2)^{3/2}} = 0,
$$

(4.53)

$$
\Omega^0_{xy} = \Omega^0_{yx} = 0. \tag{4.54}
$$

Substituting Equations (4.52), (4.53) and (4.54) in Equation (4.37), the characteristic equation becomes

$$
\lambda^4 + \left(4n^2 - \Omega^0_{xx} - \Omega^0_{yy}\right)\lambda^2 + \Omega^0_{xx}\Omega^0_{yy} = 0. \tag{4.55}
$$

The roots of the characteristic equation (4.55) for L_i (i = 1, 2, 3), L_{nj}(j=1, 2) of Table 4.1 are given in the Tables 4.2 to 4.6 respectively.

Table 4.2: Stability of the libration point L_1

Case	L_1	Ω_{xx}^0	Ω_{yy}^0	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	1.265858	5.0730	-1.0365	± 1.5203	$\pm 1.5083i$	Unstable
2	1.261034	5.2158	-1.0710	± 1.5449	$\pm 1.5298i$	Unstable
3	1.25723	5.3291	-1.0995	± 1.5654	$\pm 1.5463i$	Unstable
4	1.255981	5.3497	-1.1098	± 1.5721	$\pm 1.5499i$	Unstable
5	1.268700	5.5428	-1.1276	± 1.6203	$\pm 1.5429i$	Unstable
6	1.263284	5.5619	-1.1354	± 1.6261	$\pm 1.5454i$	Unstable
7	1.265086	5.6637	-1.1571	± 1.6417	$\pm 1.5592i$	Unstable
8	1.258540	5.7068	-1.1758	± 1.6548	$\pm 1.5652i$	Unstable

Table 4.3: Stability of the libration point L_2

Case	L_2	Ω_{xx}^0	Ω_{yy}^0	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	0.360743	16.0618	-6.5309	± 3.6571	$\pm 2.8006i$	Unstable

2	0.364839	16.6399	-6.7833	±3.7264	±2.8510i	Unstable
3	0.369905	17.4288	-7.0524	±3.8231	±2.9000i	Unstable
4	0.367537	17.2131	-6.9446	±3.7946	±2.8812i	Unstable
5	0.353408	18.1724	-7.2168	±3.9162	±2.9242i	Unstable
6	0.356370	18.0109	-7.1339	±3.8954	±2.9099i	Unstable
7	0.358440	19.0852	-7.5181	±4.0234	±2.9771i	Unstable
8	0.359093	18.6603	-7.3154	±3.9702	± 2.9428i	Unstable

Table 4.4: Stability of the libration point L_3

Case	L_3	Ω_{xx}^{0}	Ω_{yy}^{0}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	-1.103167	3.4940	-0.2470	±0.7911	±1.1742i	Unstable
2	-1.097847	3.5801	-0.2531	±0.8009	±1.1887i	Unstable
3	-1.099561	3.7050	-0.2682	±0.8339	±1.1955i	Unstable
4	-1.093774	3.7084	-0.2697	±0.8361	±1.1961i	Unstable
5	-1.092499	3.6471	-0.2595	±0.8106	±1.2000i	Unstable
6	-1.091890	3.6502	-0.2610	±0.8128	±1.2008i	Unstable
7	-1.094384	3.7738	-0.2749	±0.8440	±1.2066i	Unstable
8	-1.088040	3.7807	-0.2780	±0.8485	±1.2081i	Unstable

Table 4.5: Stability of the libration point L_{n1}

Case	L_{n1}	Ω_{xx}^{0}	Ω_{yy}^{0}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark

1	-	-	-	-	-	-
2	-0.001192	-9279.5000	-9838.9000	±96.0482i	±99.4823i	stable
3	-0.001583	-8771.7000	-9703.1000	±93.4567i	±98.7159i	stable
4	-0.001548	-8819.5000	-9717.1000	±93.6931i	±98.8059i	stable
5	-0.001191	-9280.4000	-9839.2000	±95.9990i	±99.5398i	stable
6	-0.001192	-9279.5000	-9838.8000	±96.0405i	±99.4898i	stable
7	-0.001582	-8772.9000	-9703.5000	±93.4437i	±98.7383i	stable
8	-0.001548	-8819.4000	-9717.1000	±93.6930i	±98.8059i	stable

Table 4.6: Stability of the libration point L_{n2}

Case	L_{n2}	Ω^0_{xx}	Ω^0_{yy}	$\lambda_{1,2}$	$\lambda_{3,4}$	Remark
1	-	-	-	-	-	-
2	-0.023389	1.064.5000	-671.5374	±32.5874	±25.9447i	Unstable
3	-0.019520	1512.3000	-1036.3000	±38.8556	±32.2176i	Unstable
4	-0.019774	1481.1000	-1006.1000	±38.4522	±31.7457i	Unstable
5	-0.023400	1063.7000	-670.8263	±32.5746	±25.9314i	Unstable
6	-0.023388	1064.6000	-671.5893	±32.5887	±25.9462i	Unstable
7	-0.019529	1511.4000	-1035.3000	± 38.8440	±32.2025	Unstable
8	-0.019773	1481.2000	-1006.2000	±38.4537	±31.7477i	Unstable

CHAPTER FIVE

5.0 RESULTS AND DISCUSSIONS

5.1 Introduction

Results, deducible from the problems studied in Chapters 3 and 4 will bepresented and discussed in this chapter. We begin with the examination of the one in chapter three.

5.2 Oblateness and Radiation of the Primaries with Potential from a Belt

5.2.1 Equations of motion

Equation (3.23) gives the equations that govern the motion of an infinitesimal body moving under the combined influences of radiation and oblateness up to the coefficients J_4 of the primaries; and gravitational potential from a belt in the restricted three-body problem. The presence of the radiation factors of the primaries (q_1, q_2), oblateness up to the coefficient J_4 of the primaries (A_1, A_2, B_1, B_2) and gravitational potential from the belt (M_b) in Equation (3.23), signify that the equations of motion of the infinitesimal body are affected by them. However, if the effects of the aforementioned perturbations are ignored (i.e. $q_1 = q_2 = 1, A_1 = A_2 = B_1 = B_2 = M_b = 0$), the equations of motion (3.23) boil down to those of the classical case of Szebehely (1967) and Valtonen and Karttunen (2005). In the presence of radiation of the more massive primary only (i.e. $q_2 = 1, A_1 = A_2 = B_1 = B_2 = M_b = 0$), the equations of (3.23) tally with those of Bhatnagar and Chawla (1979). In the absence of the oblateness up to J_4 of the primaries (i.e. $A_2 = B_2 = 0$), the Equations of (3.23) coincide with those obtained by Singh and Taura (2013). In the presence of the oblateness coefficients up to J_4 of the more massive primary only (i.e. $q_1 = q_2 = 1, B_1 = B_2 = M_b = 0$), equations of (3.23)agree with Abouelmagd (2012). Considering the radiation and oblateness up to J_2 of the primaries and ignoring the potential from the belt (i.e. $A_2 = B_2 = M_b = 0$), the equations are in agreement with those of Singh and Ishwar (1999), Abdulraheem and Singh (2006) in the absence of perturbations in the Coriolis and centrifugal forces and Singh and Umar (2013b) in the circular case.In the presence of the radiation of the more massive primary, oblateness up to J_2 of the less massive primary and the circular cluster of material points only $(i.e. q_2 = 1, A_1 = A_2 = B_2 = 0),$ the equations become analogous to those of Kushvah (2008). Disregarding the effects of all the perturbations except the potential from the circular cluster of material points (i.e. $q_1 = q_2 = 1, A_1 = A_2 = B_1 = B_2 = 0$), the equations of (3.23) diminish to those of Jiang and Yeh (2006) and Yeh and Jiang (2006).

5.2.2 Locations of libration points

5.2.2.2*Triangular libration points*

Equation (3.47) specifies the coordinates of the triangular libration points L_4 and L_5. It is evident that these coordinates, are influenced by radiation and oblateness up to J_4 of the primaries and the gravitational potential from the belt. However, the *x*-coordinates of these points are not affected by the potential spawned by the belt. The coordinates (3.47) are different from those obtained by Singh and Ishwar (1999) due to the occurrence of oblateness coefficients A_2, B_2 of the bigger and smaller primaries respectively and M_b, the potential from the circular cluster of material points. If the radiation of the primaries, oblateness of the less massive primary and the potential from the circular cluster of material points are neglected (i.e. $q_1 = q_2 = 1, B_1 = B_2 = M_b = 0$), these coordinates fully agree with those of Abouelmagd (2012). On ignoring the oblateness of the more massive primary, the radiation and the oblateness coefficient B_2 of the less massive primary (i.e. $q_2 = 1, A_1 = A_2 = B_2 = 0$), the coordinates (3.47) coincide with those of Kushvah (2008) when only linear terms in small quantities are considered. However, when $q_1 = q_2 = 1, A_1 = A_2 = B_1 = B_2 = M_b = 0$, the coordinates reduce to $x = \frac{1}{2}(1-2\mu),\ y = \pm\frac{\sqrt{3}}{2}$ which correspond to the classical case of Szebehely, (1967).

The effects of the various perturbations on the positions of the triangular libration points $L_{4(5)}$ are presented in Figures (5.1-5.6) when μ=0.25 and T = 0.01.

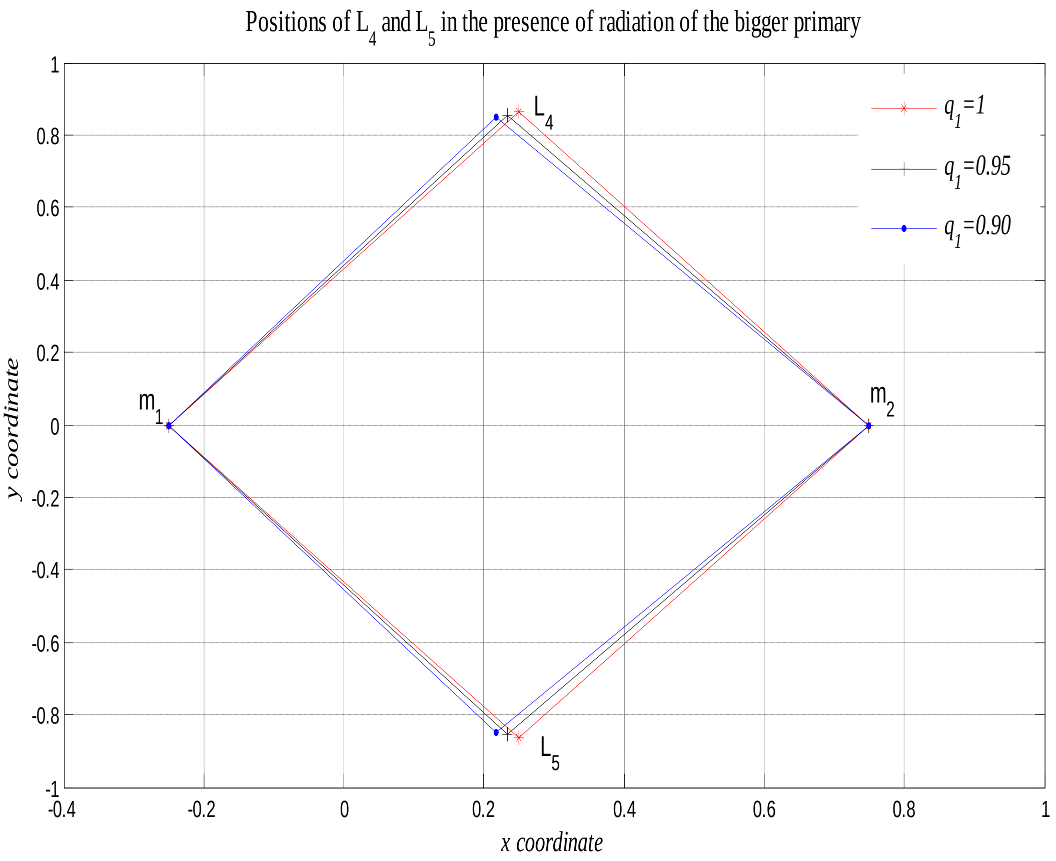


Figure 5.1: Effects of radiation of the bigger primaryon the location of L4,5 ($q_2 = 1, A_1 = A_2 = B_1 = B_2 = M_b = 0$).

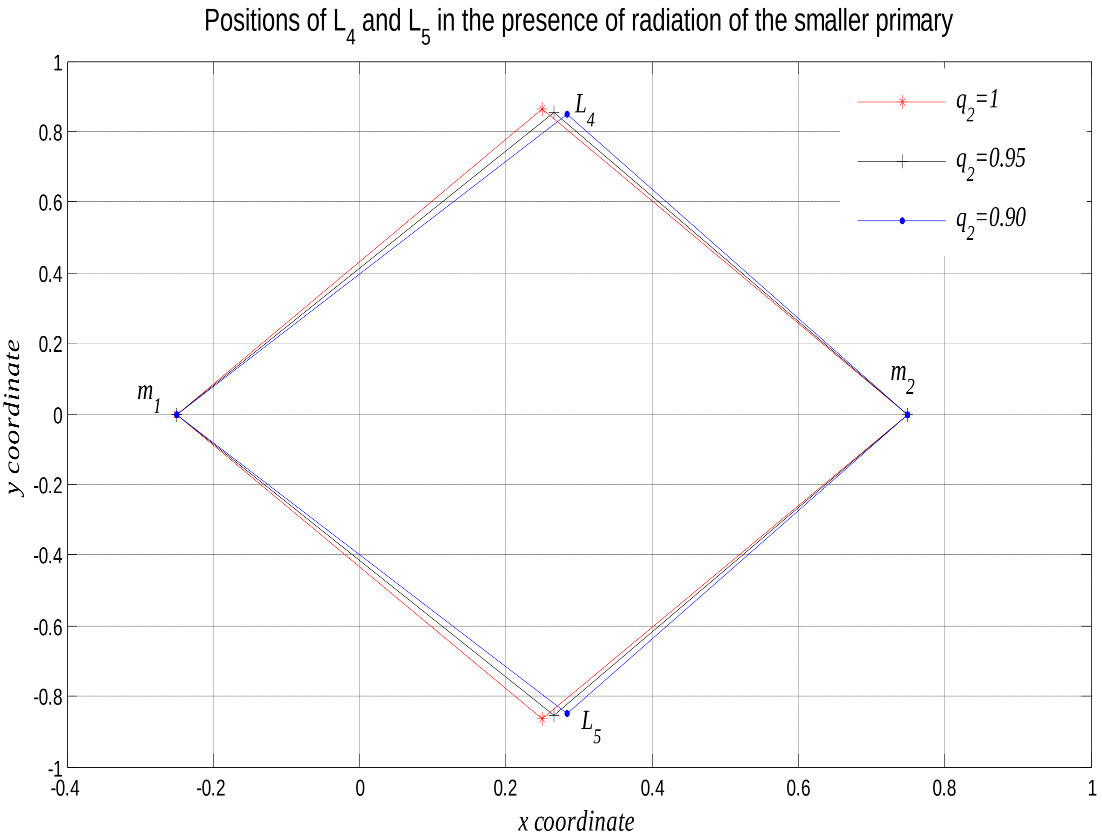


Figure 5.2: Effects of radiation of the smaller primaryon the location of $L_{4,5}$ ($q_1 = 1$, $A_1 = A_2 = B_1 = B_2 = M_b = 0$).

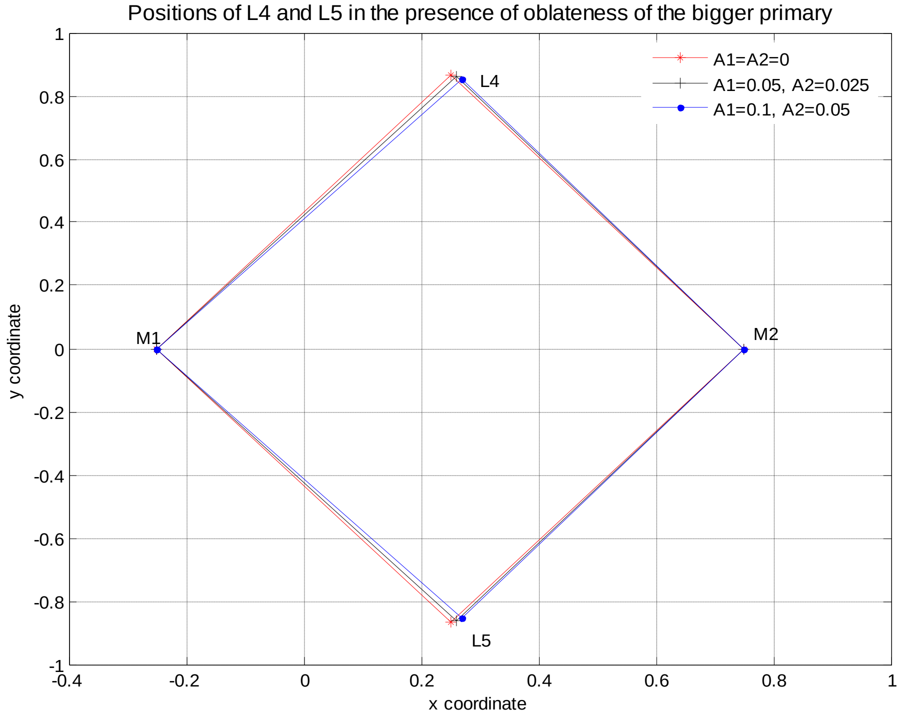


Figure 5.3: Effects of oblateness of the bigger primaryon the location of $L_{4,5}(q_1 = q_2 = 1, B_1 = B_2 = M_b = 0)$.

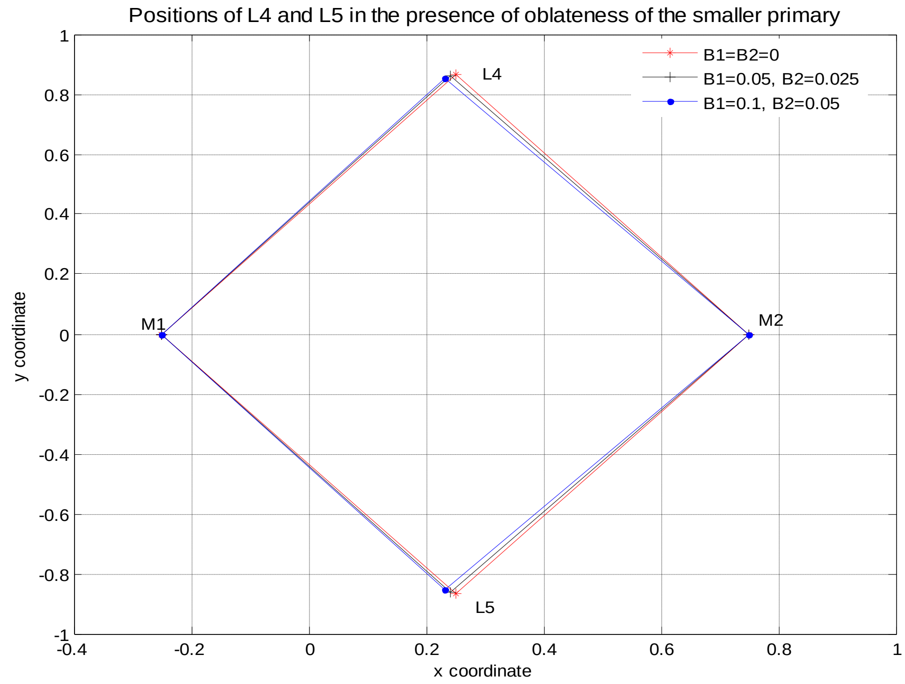


Figure 5.4: Effects of oblateness of the smaller primaryon the location of L$_{4,5}$ ($q_1 = q_2 = 1$, $A_1 = A_2 = M_b = 0$).

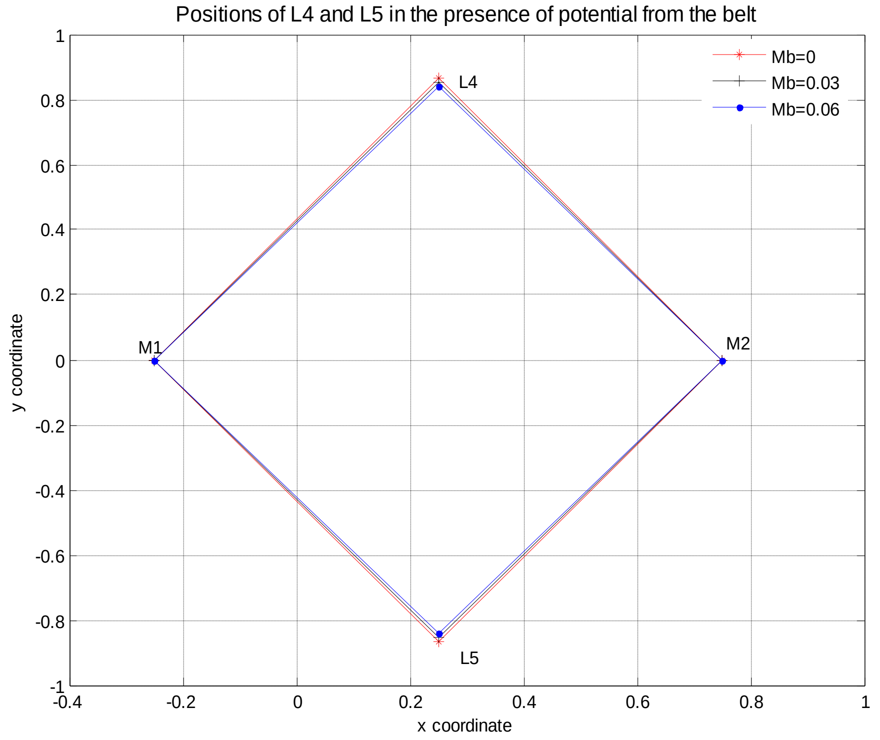


Figure 5.5: Effects of potential from the belt on the location of $L_{4,5}$($q_1 = q_2 = 1, A_1 = A_2 = B_1 = B_2 = 0$).

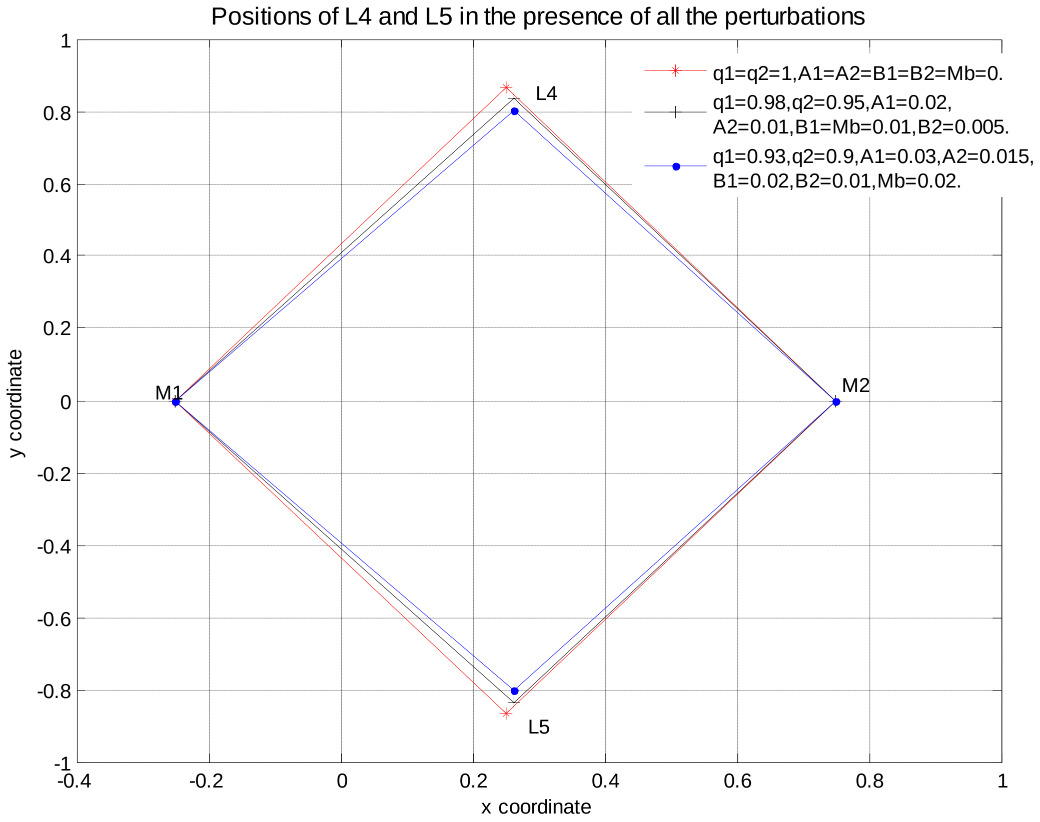


Figure 5.6: The combined effect of the perturbations on the location of $L_{4,5}$.

It is noted here that in this problem, the triangular points no longer form equilateral triangles with the primaries as they do in the classical case; rather they form scalene triangles (Figures 5.1, 5.2, 5.3, 5.4) or isosceles triangles (Figure 5.5) with the primaries. The triangular points move closer to the more massive primary due to radiation of the more massive primary (Fig. 5.1) and oblateness up to J_4 of the less massive primary (Figure5.4), whereas they shift away from it in the presence of radiation of the less massive primary (Figure 5.2) and oblateness up to J_4 of the more massive primary (Figure 5.3). The points draw closer to the *x*-axis due to the potential from the belt (Figure 5.5); and in the presence of all the perturbations the triangular points tend towards the less massive primary (Figure 5.6).

5.2.2.2*Collinear libration points*

The coordinates of the collinear libration points are obtained numerically in Table 3.1, which shows that the points are also affected by the perturbations. However, in the absence of the perturbations (i.e. $A_1 = A_2 = B_1 = B_2 = M_b = 0, q_1 = q_2 = 1$) Table 3.1 Case 1, it is seen that there are three collinear libration points (L_i, i = 1, 2, 3) which correspond to the classical case of Szebehely (1967). On considering the effect of the potential from the belt only (i.e. $A_1 = A_2 = B_1 = B_2 = 0, q_1 = q_2 = 1, M_b = 0.01$) Case 2, there exist five collinear libration points (L_{n1}, L_{n2} *and* L_i, i = 1, 2, 3). This finding is in concordance with Yeh and Jiang (2006) and Singh and Taura (2013).In the presence of the oblateness of the bigger primary up to J_2 and the potential from the belt only (i.e. $A_1 = M_b = 0.01, A_2 = B_1 = B_2 = 0, q_1 = q_2 = 1$) Case 3, the new collinear point L_{n1} shifted toward the bigger primary while L_{n2} stepped away from it. In Case 4, due oblateness of the bigger primary up to J_4 with potential from the belt (i.e. $A_1 = M_b = 0.01, A_2 = 0.005, B_1 = B_2 = 0, q_1 = q_2 = 1$), L_{n1} moved away from the bigger primary while L_{n2} stepped towards it. Similarly, owing to the oblateness of the smaller primary up to J_2 with potential from the belt only (i.e. $A_1 = A_2 = B_2 = 0, q_1 = q_2 = 1, B_1 = M_b = 0.01$) Case 5, L_{n1} moved away from the bigger primary while L_{n2} stepped towards it. On considering the oblateness of the smaller primary up to J_4 with potential from the belt only (i.e. $A_1 = A_2 = 0, q_1 = q_2 = 1, B_1 = M_b = 0.01, B_2 = 0.005$) Case 6, L_{n1} shifted toward the bigger primary while L_{n2} stepped away from it and additional two new collinear points $L_{n3}$and L_{n4} appeared.

In Case 7 $(A_1 = B_1 = A_2 = B_2 = 0, q_2 = 1, q_1 = 0.98, M_b = 0.01)$ and Case 8 $(A_1 = B_1 = A_2 = B_2 = 0, q_1 = 1, q_2 = 0.95, M_b = 0.01)$, L_{n1} still sifted towards the bigger primary and L_{n2} stepped towards the smaller primary. However, due to the radiation and oblateness of the primaries up to J_4 with potential from the belt (i.e. $A_1 = B_1 = M_b = 0.01,\ A_2 = B_2 = 0.005, q_1 = 0.98, q_2 = 0.95$) Case 9, $L_{n3}$and L_{n4} exist with L_{n1} stepping away from the bigger primary while L_{n2} towards it.

Figure 3.4, shows the positions of the three collinear points of the classical case (i.e. $A_1 = A_2 = B_1 = B_2 = M_b = 0,\ q_1 = q_2 = 1)$ for $\mu \in (0,1/2)$. While Figure 3.5, shows the positions of the collinear points under the combined effect of radiation and oblateness up to J$_4$ of the primaries and the potential from the belt (i.e. $A_1 = B_1 = 0.01, A_2 = B_2 = 0.005,$ $q_1 = 0.98,\ q_2 = 0.95,\ M_b = 0.01)$ for $\mu \in (0,1/2)$. It reveals that the number of the collinear points increases with increase in the value of the mass parameter. There are three collinear points for lower mass ratio, five for about average mass ratio and seven for higher mass ratio. It further shows that, the collinear points L_1 and L_2 do not exist for lower mass ratio. Figure 3.6 exposes that the effect of the oblateness of the smaller primary up to J$_4$ (i.e. $A_1 = A_2 = M_b = 0, q_1 = q_2 = 1,\ B_1 = 0.01, B_2 = 0.005$), is responsible for the non existence of the collinear points L_1 and L_2 for the lower mass ratio. It portrays that in the range $0 < \mu \le 0.17291$ there exists only one libration point L_3, $0.17292 \le \mu \le 0.31296$ there are three libration points (L_1, L_3, L_{n3}) and five libration points $(L_1, L_2, L_3, L_{n3}, L_{n4})$ in the interval $0.31297 \le \mu \le 0.5.$

Now, since due oblateness of the smaller primary, two new collinear points $L_{n3}$and L_{n4} exist (Table 3.1 Case 6), we further examine the effect due to oblateness of the smaller primary in Table 5.1; and the combined effect of the perturbations on the collinear libration points in Table 5.2.

Table 5.1: Effects of oblateness of the smaller on the collinear points μ=0.4, A_1 = A_2 = M_b=0, q_1=q_2=1.

B_1	B_2	L_1	L_2	L_3	L_{n1}	L_{n2}	L_{n3}	L_{n4}
0	0	1.23081	0.14162	-1.16205				
0.0001	0.00005	1.23074	0.14178	-1.16201			0.69814	0.50184
0.001	0.0005	1.23016	0.14324	-1.16188			0.77395	0.42566
0.01	0.005	1.22400	0.16373	-1.16049			0.90953	0.27895
0.015	0.0075	1.22021	0.19489	-1.15972			0.94372	0.21865
0.016	0.008	1.21940		-1.15956			0.94959	
0.018	0.009	1.21777		-1.15926			0. 96065	
0.02	0.01	1.21606		-1.15892			0.97098	

Table 5.2: Combined effect of the perturbations on the collinear points when μ =0.4, T=0.01.

	L_1	L_2	L_3	L_{n1}	L_{n2}	L_{n3}	L_{n4}
$q_1 = q_2 = 1$, $A_1=B_1=M_b = 0$, $A_2 = B_2 = 0$.	1.23081	0.14162	-1.16205	-	-	-	-
$q_1 = 0.99, q_2 = 0.97$, $A_1=B_1=M_b=0.001$, $A_2 = B_2 = 0.0005$.	1.22209	0.14762	-1.15736	-0.00294	-0.57347	0.77400	0.42563
$q_1 = 0.98, q_2 = 0.96$, $A_1=B_1=M_b=0.005$, $A_2 = B_2 = 0.00075$.	1.21697	0.15668	-1.15125	-0.00052	-0.58498	0.78565	0.41386
$q_1 = 0.97, q_2 = 0.95$, $A_1=B_1= M_b= 0.01$, $A_2 = B_2 = 0.005$.	1.20306	0.18826	-1.14242	-0.00016	-0.70500	0.91080	0.27392

Studying Table 5.1 reveals that with increase in the oblateness factors B_1 and $B_{2,}$ the collinear libration points L_2 move towards m_2, while L_{n4} move away from it. L_1 and L_3 draw closer to the primaries, while L_{n3} moves away from the primaries. With further increase in the oblateness factors B_1 and B_2, it is observed that the libration points L_2 and L_{n4} coincide and disappear leaving no libration point between the primaries.

Examining Table 5.2 exposes that owing to the combined effect of the perturbations with $\mu = 0.4,$ there are seven collinear points with only three $L_{n1,} L_2, L_{n4}$ between the primaries. It further reveals that with increase in the perturbations, the collinear libration points L_1, L_3 draw closer to the primaries, while L_{n2}, L_{n3} move away from the primaries; and $L_{n1,} L_2$ tend towards m_2, while L_{n4} moves away from m_2.

5.2.3 Stability of libration points

5.2.3.1*Stability of triangular libration*

The characteristic equation which is used for the determination of stability of the libration points is given by Equation (3.59). The characteristic equation for the triangular libration points is specified in Equation (3.61). It differs from that of Singh and Taura (2013) only due to the presence of the zonal harmonics coefficients J_4 of the primaries. If the effect of the belt and the oblateness coefficients A_2, B_2 are neglected, the characteristic equation (3.61) corresponds to those of Singh and Ishwar (1999) and AbdulRaheem and Singh (2006) in the absence of perturbations in the Coriolis and centrifugal forces.

Section 3.5.3 establishes that the triangular points are only stable when the mass ratio $\mu \in (0, \mu_c)$. Equation (3.69) provides the value of the critical mass parameter μ_c, which is used for the determination of the bound of stability for the triangular libration points. It mirrors the influence of radiation of the primaries, oblateness up to J_4 of the primaries and the gravitational potential from the belt. Nevertheless, on ignoring the effect of these perturbations (i.e. $q_1 = q_2 = 1, A_1 = A_2 = B_1 = B_2 = M_b = 0$), μ_c μ_c becomes

$$\mu_c = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right),$$

which tallies with the classical problem (Szebehely, 1967, Valtonen and Karttunen, 2005). If the radiation of the primaries, oblatenesss of the smaller primary and potential from the belt are neglected (i.e. $q_1 = q_2 = 1, B_1 = B_2 = M_b = 0$), μ_c confirms the result of Abouelmagd (2012). Taking in to account the oblateness up to J_2 only of the primaries and neglecting the potential from the belt (i.e. $A_2 = B_2 = M_b = 0$), μ_c agrees with those of Singh and Ishwar (1999), Abdulraheem and Singh (2006) when the perturbations in the Coriolis and the centrifugal forces are neglected, Singh (2009) in the case of linear stability, Singh and Umar (2012) in the circular case and Singh and Haruna (2014a) when the infinitesimal mass is a point mass and in absence of the perturbations in the Coriolis and the centrifugal forces. If the primaries are spherical and the potential from the belt is ignored (i.e. $A_1 = A_2 = B_1 = B_2 = M_b = 0$), μ_c verifies the result of Haque and Kumar (2015) when the perturbations in the Coriolis and the centrifugal forces are discarded. On considering the oblateness of the primaries only (i.e. $q_1 = q_2 = 1, M_b = 0$), μ_c agrees with that of Abouelmagd, *et al.* (2015).

The effects of the radiation of the primaries, oblateness up to J_4 of the primaries and potential from the belt on the critical mass value, are portrayed in Table 5.3.

Table5.3: The effects of the perturbations on the critical mass value, T=0.01, r_c=0.99.

Case	q_1	q_2	A_1	A_2	B_1	B_2	M_b	μ_c
1	1	1	0	0	0	0	0	0.038520896504551
2	0.98	1	0	0	0	0	0	0.038342547092572
	0.93	1	0	0	0	0	0	0.037896673562625
	0.88	1	0	0	0	0	0	0.037450800032678
3	1	0.95	0	0	0	0	0	0.038075022974604
	1	0.90	0	0	0	0	0	0.037629149444657
	1	0.85	0	0	0	0	0	0.037183275914709
4	1	1	0.01	0	0	0	0	0.035670878626646
	1	1	0.02	0	0	0	0	0.032820860748740
	1	1	0.03	0	0	0	0	0.029970842870835
	1	1	0.01	0.005	0	0	0	0.039149799687163
	1	1	0.02	0.01	0	0	0	0.039778702869774
	1	1	0.03	0.015	0	0	0	0.040407606052385
5	1	1	0	0	0.005	0	0	0.038206998676710
	1	1	0	0	0.01	0	0	0.037893100848868
	1	1	0	0	0.015	0	0	0.037579203021026
	1	1	0	0	0.005	0.0025	0	0.038557570318079
	1	1	0	0	0.01	0.005	0	0.038594244131607
	1	1	0	0	0.015	0.0075	0	0.038630917945135
6	1	1	0	0	0	0	0.01	0.038746198154817
	1	1	0	0	0	0	0.02	0.038971499805082
	1	1	0	0	0	0	0.03	0.039196801455347
7	0.99	0.98	0.01	0.005	0.005	0.0025	0.01	0.039144251032987
	0.98	0.97	0.02	0.01	0.01	0.005	0.02	0.039856780267413
	0.97	0.96	0.03	0.015	0.015	0.0075	0.03	0.040569309501839

Studying Table 5.3indicates that considering any of the oblateness up to J_2 of the primaries (Cases 4 and 5) or any of the radiation of the primaries (Cases 2 and 3), reduces the value of the mass parameter μ_c; while oblateness up to J_4 of the primaries (Cases 4 and 5), the potential from the circular cluster of material points (Case 6) and the combined effect of the perturbations (Case 7), increase the value μ_c. Now, since the triangular libration points are stable for $0 < \mu < \mu_c$, it follows that the radiation and oblateness up to J_2 of the primaries reduce the range of stability; while oblateness up to J_4 of the primaries and the potential from the circular cluster of material points increase the size of stability.

5.2.3.2*Stability of the collinear points*

From the Tables 3.2, 3.3and 3.4, it is clear that all the collinear libration points $\mathrm{L_i}\,(i = 1, 2, 3)$ are unstable. In Table 3.5, L_{n1} is stable due to the oblateness up to J_4 of the smaller primary (Case 6), radiation of the primaries (Cases 7and 8) and the combine effects of the perturbations (Case 9). The collinear point L_{n2} is stable only when considering the oblateness up to J_4 of the bigger primary (Case 4) and the radiation and oblateness up to J_4 of the primaries (Table 3.6: Case 9). InTable 3.7 and Table 3.8 the additional new collinear points L_{n3} and L_{n4} are stable.

5.2.4 Periodic orbits around the triangular libration points

As stated in Equation (3.76), the motion around the stable triangular points of theCR3BP under the combined effect of radiation and oblateness up to the coefficients J_4 of the primaries together with gravitational potential from a belt is bounded and composed of two harmonic motions; the short and the long periodic orbits. Equations (3.90) and (3.91) specify the frequencies for the short and long periodic orbits, correspondingly. Section 5.3 has shown that these orbits are ellipses centered at the triangular points. Equation (3.97) specifies the directions of the principal axes of the ellipses. The eccentricity, lengths of semi–major and semi–minor axes of the short periodic orbits is stated in Equations (3.122), (3.130) and (3.131) respectively. Equations (3.127), (3.132) and (3.133) give the eccentricity, lengths of semi–major and semi–minor axes of the long periodic orbits respectively.

The elements of periodic motions (i.e. frequencies, eccentricity,and the directions of the principal axes of the ellipses) reduce to those of classical problem (Szebehely, 1967) when the oblateness and radiation of the primaries and the gravitational potential from the belt are ignored. In the absence of the potential from the belt and considering oblateness up J_2 of the primaries only (i.e. $A_2 = B_2 = M_b = 0$), the elements of the periodic motion agree with those of Singh and Haruna (2014b) when the infinitesimal body is spherical and the perturbations in the Coriolis and centrifugal forces are ignored.

The numerical analyses (Tables 3.9-3.13) emphasize the effects of each perturbing force on the periodic elements: frequency, angle of rotation of the principal axis, eccentricity and lengths of the semi-axes of the elliptic orbits of the infinitesimal mass.Studying Table 3.9 reveals that the frequency of the long periodic orbit s_1 decreases in the presence of oblateness up to the coefficient J_4 of the primaries (Cases 4 and 5) and due to the combined effect of radiation and oblatenes up J_4 of the primaries with the potential from belt (Case 7). While the frequency of the short periodic orbit s_2 decreases only due to the radiation of the bigger primary (Case 2) or radiation of the smaller primary (Case 3) or due to the oblatness up to J_2 of the bigger primary.

Examining Table 3.10 shows that radiation of the bigger primary (Case 2), oblateness up to J_2 and up to J_4 of the smaller primary (Case 5) reduces the angle of the rotation of the principal axes of the ellipses, while radiation of the smaller primary (Case 3), oblateness up to J_4 of the bigger primary, potential from the belt (Case 6) and the combined effect of all the perturbations (Case 7), increases the angle of rotation of the principal axes.

Table 3.11 depicts that the value of the eccentricity $e_1$of the long periodic orbit reduces in the presence of radiation of the bigger primary (Case 2) or radiation of the smaller primary (case 3) or oblateness of the smaller primary up to J_2 (Case5), while it increases due to oblateness up to J_4 of the primaries (cases4 and 5) or potential from the belt (Case 6) or due to combined effect of the perturbations (Case 7). It further, reveals that the eccentricity e_2 of the long periodic orbit increases owing to the radiation of the primaries (Cases 2 or 3) or oblateness up to J_2 of the primaries (Cases 4 or 5), whereas its value decreases due to oblateness up to J_4 of the primaries (cases4 or 5) or due to the potential from the belt (Case 6) or due to combined effect of the perturbations (Case 7).

Table 3.12 shows that the length of the semi-major axis a_1 of the long periodic orbit increases in the presence of any or all of the perturbations except due to the radiation of the bigger primary (Case 2) or radiation of the smaller primary (Case 3), whereas the length of the semi-minor axis $b_1$of the long periodic orbit decreases in the presence of any or all of the perturbations.

Table 3.13 indicates that the length of the semi-major axis a_2 of the short periodic orbit, increases only due to oblateness up to J_2 of the bigger primary (Case4) or due to oblateness up to J_2 of the smaller primary (Case5) or due to the potential from belt (Case 6) or due to the combined effect of all the perturbations (Case7). Furthermore, the table shows that the length of the semi-minor axis b_2 of the short periodic orbit decreases in the presence of any one or all of the perturbations.

5.3 Triaxiality and Radiation of the Primaries with Potential from a Belt

5.3.1 Equations of motion

The position of the infinitesimal body moving subject to the influences of radiation and triaxiality of the primaries; and gravitational potential created by the belt, within the frame work of the CR3BP is given by Equation (4.14). The presence of the aforementioned perturbations in Equation (4.14) connote that the motion of the infinitesimal body is affected by them.

Now, if the primaries are oblate instead of triaxial $(i.e.\ \sigma_1 = \sigma_2 = A_1\ and\ \sigma_1' = \sigma_2' = B_1\ say)$, then Equation (4.14) verifies Equation (3.23) when considering oblateness up $J_2$of the primaries and that of Singh and Taura (2013). In the absence of the potential created by the belt $(i.e. M_b = 0,\ \sigma_1 \neq 0, \sigma_2 \neq 0,\ \sigma_1' \neq 0, \sigma_2' \neq 0,\ q_1 \neq 1, q_2 \neq 1),$ Equation (4.14) confirms those of Sharma *etal.* (2001*b*) and Singh (2013) in the absence of perturbations in the Coriolis and centrifugal forces. If however, the influences of the aforesaid perturbations are ignored $(i.e.\ \sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = M_b = 0,\ q_1 = q_2 = 1),$ Equation (4.14) reduces to those of the classical case of Szebehely (1967) and Valtonen and Karttunen (2005).

5.3.2 Locations of libration points

5.3.2.1*Triangular libration points*

The triangular libration points L_4 and L_5 are given in Equation (4.32), and they are functions of $q_1, q_2, \sigma_1, \sigma_2,\ \sigma_1', \sigma_2'\ and\ M_b$. It is remarkable that the y -coordinates of the triangular points depend on the mass parameter μ, such that as $\mu \to 0$ the coordinates in Equation (4.32) do not exist, which is contrary to those of Equation (3.69) and Szebehely (1967), but it is in agreement with Sharma *et al.* (2001b, 2001c) and Singh (2013). If the primaries are oblate instead of triaxial, then the coordinates of Equation (4.32) coincide with those of Equation (3.47) when considering oblateness up $J_2$of the primaries and Singh and Taura (2013). On ignoring the effect of the potential from the belt, Equation (4.32) tallies with those of Sharma *et al.* (2001*b*) and Singh (2013) in the absence of perturbations in the Coriolis and centrifugal forces. Nevertheless, Equation (4.32) diminishes to those of the classical case of Szebehely (1967) and Valtonen and Karttunen (2005) if the primaries are non-luminous spheroids and the potential from the belt is ignored ($i.e.$ $\sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = M_b = 0,\ q_1 = q_2 = 1$).

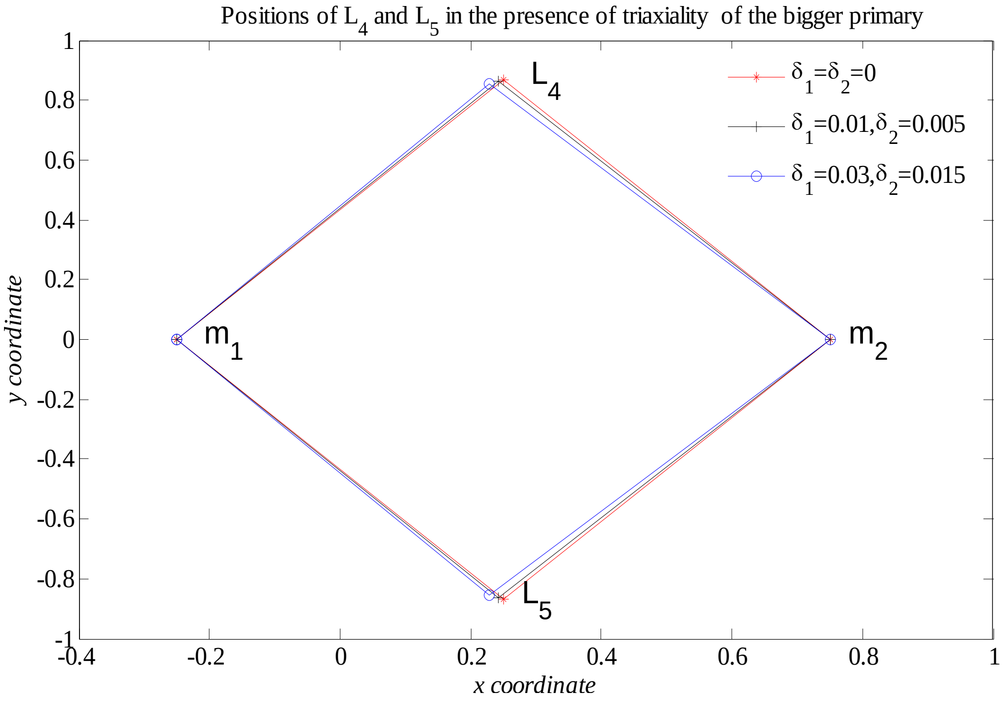


Figure 5.7: Effects of triaxiality of the bigger primaryon the location of $L_{4,5}$ (i.e. $\sigma_1' = \sigma_2' = M_b = 0,\ q_1 = q_2 = 1$).

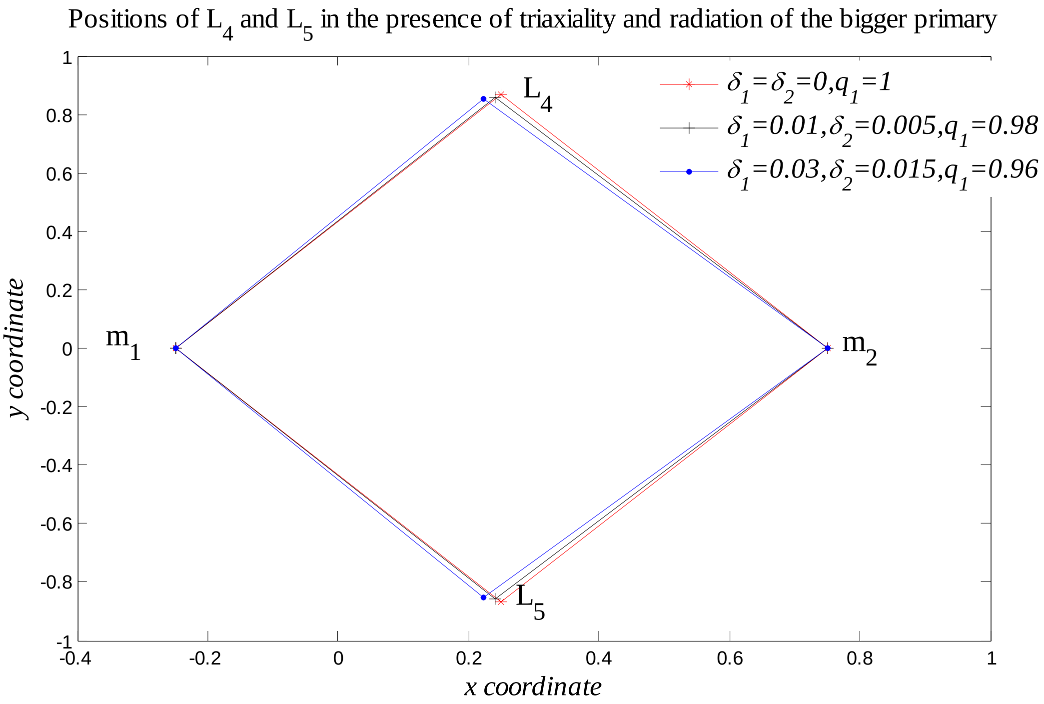


Figure 5.8: Effects of triaxiality and radiation of the bigger primaryon the location of $L_{4,5}$ (i.e. $\sigma_1' = \sigma_2' = M_b = 0,\ q_2 = 1$).

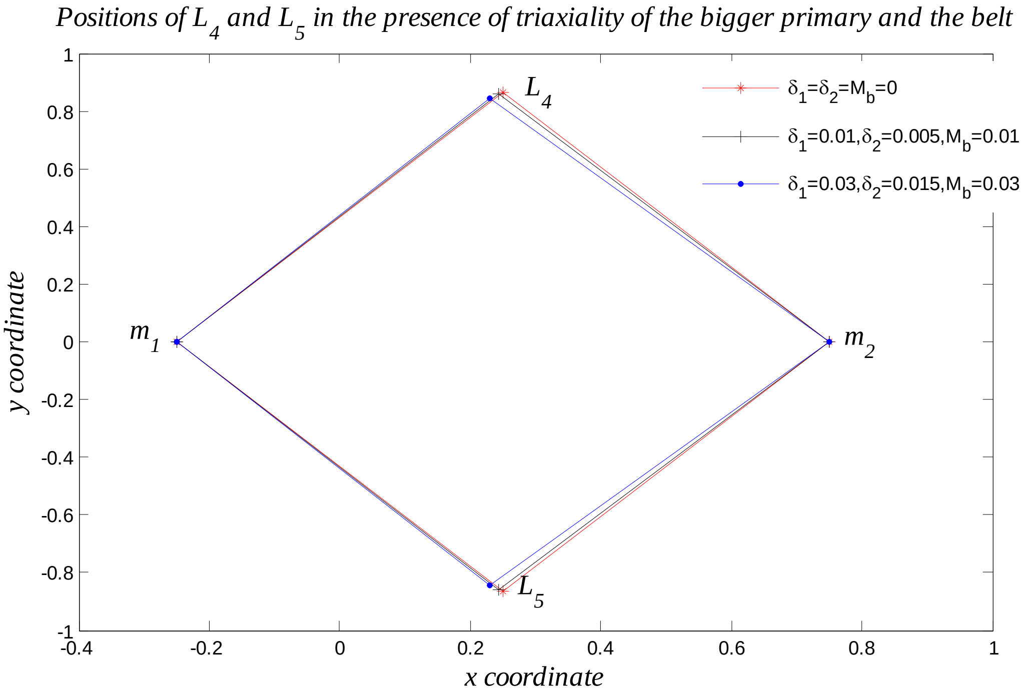


Figure 5.9: Effects of triaxiality of the bigger primary and the belton the location of $L_{4,5}$ ((i.e. $\sigma_1' = \sigma_2' = 0,\ q_1 = q_2 = 1$).

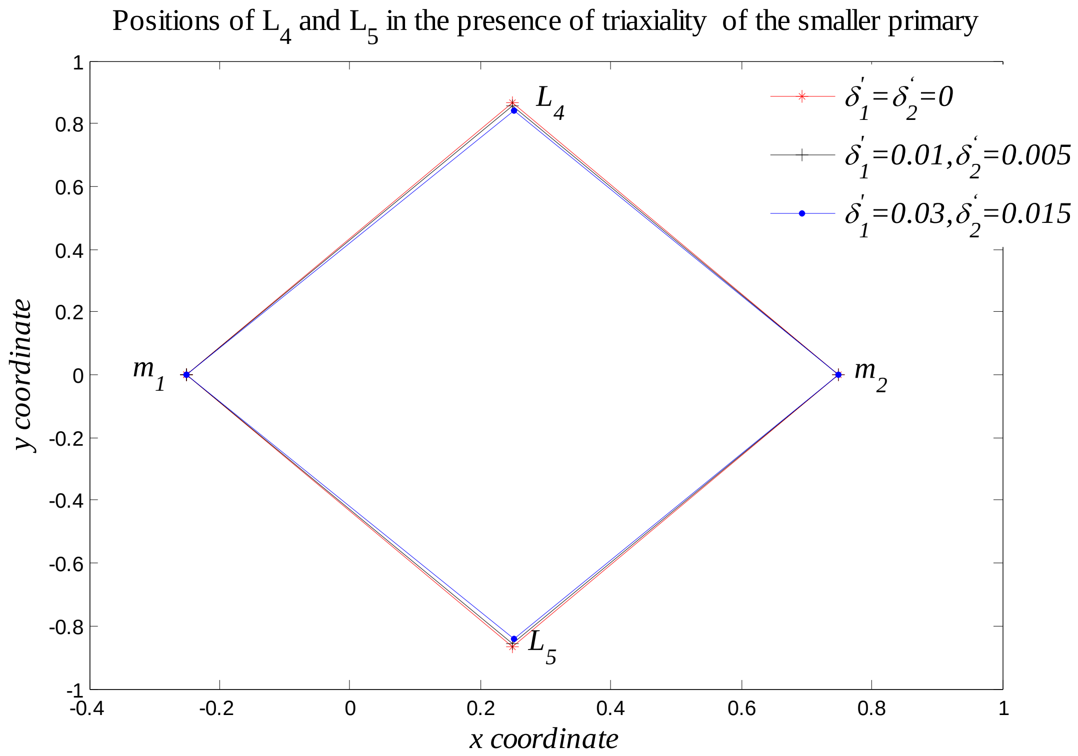


Figure 5.10: Effects of triaxiality of the smaller primaryon the location of L$_{4,5}$ (i.e $\sigma_1 = \sigma_2 = M_b = 0,\ q_1 = q_2 = 1$) .

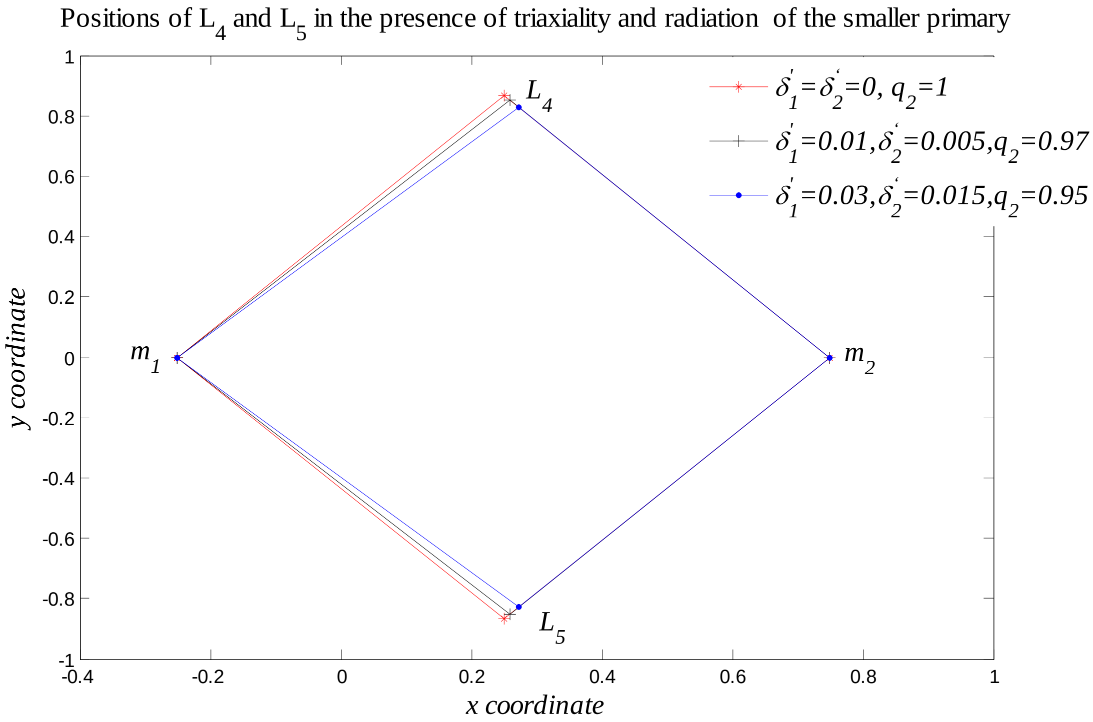


Figure 5.11: Effects of triaxiality and radiation of the smaller primaryon the location of L$_{4,5}$ (i.e $\sigma_1 = \sigma_2 = M_b = 0,\ q_1 = 1$) .

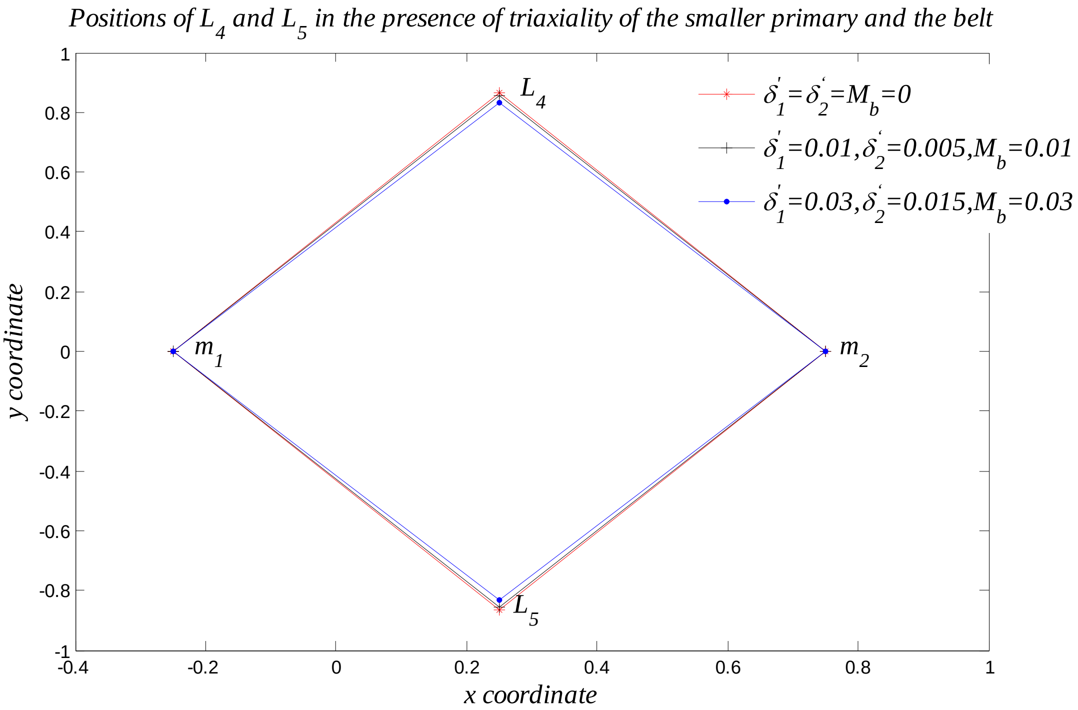


Figure 5.12: Effects of triaxiality of the smaller primary and the belton the location of $L_{4,5}$ (i.e. $\sigma_1 = \sigma_2 = 0$, $q_1 = q_2 = 1$).

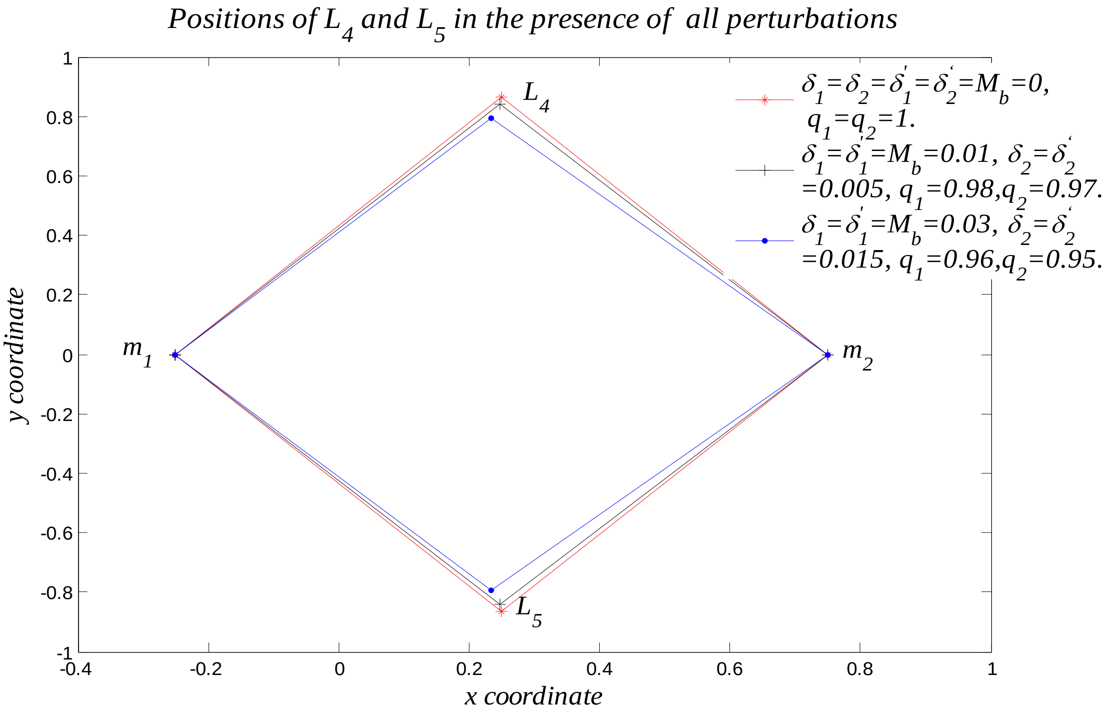


Figure 5.13: The combined effect of the perturbations on the location of $L_{4,5}$.

As in the previous problem, the triangular points no longer form equilateral triangles with the primaries as they do in the classical problem (Figures 5.7-5.13). They form scalene triangles and move towards the bigger primary in the presence of triaxiality of the bigger primary (Figure 5.7) or in the presence of radiation and triaxiality of the bigger primary (Figure 5.8) orin the presence of triaxiality of the bigger primary with potential from the belt (Figure 5.9).Due totriaxiality of the smaller primary (Figure 5.10) or due to radiation and triaxiality of the smaller primary (Figure 5.11), the triangular points form scalene triangles and move towards the smaller primary. However, in the presence of triaxiality of the smaller primary with the potential from the belt (Figure 5.12), the points tend towards the *x*-axis and form isosceles triangles with the primaries. The amalgamated effect of all theperturbations (Figure 5.13) causes the triangular points to draw closer to the bigger primary and form scalene triangles with the primaries.

5.3.2.2*Collinear libration points*

Table 4.1 shows the positions of the collinear libration points subject to the influence of radiation and triaxiality of the primaries, together with gravitational potential created by the belt. It indicates that the positions of the collinear libration points are also affected by the aforementioned perturbations.Nonetheless, in the absence of these perturbations $(i.e.\ \sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = M_b = 0,\ q_1 = q_2 = 1)$ Table 4.1 Case 1, there are only three collinear libration points (L_i, i = 1, 2, 3) which correspond to the classical case of Szebehely, (1967). As in the previous problem, under the effect of the gravitational potential created by the belt only $(i.e. M_b = 0.01,\ \sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = 0,\ q_1 = q_2 = 1)$ Case 2, there exist five collinear libration points (L_{n1}, L_{n2} *and* L_i, i = 1, 2, 3). On considering the effect due to triaxiality of the bigger primary with the potential created by the belt only $(i.e.\ \sigma_1 = M_b = 0.01,\ \sigma_2 = 0.008,\ \sigma_1' = \sigma_2' = 0,\ q_1 = q_2 = 1)$ Case 3 or radiation and triaxiality of the bigger primary with the potential created by the belt only $(i.e. q_1 = 0.98,\ \sigma_1 = M_b = 0.01,\ \sigma_2 = 0.008,\ \sigma_1' = \sigma_2' = 0,\ q_2 = 1)$ Case 4 or radiation and triaxiality of the smaller primary with the potential created by the belt only $(i.e. q_2 = 0.97,\ \sigma_1' = M_b = 0.01,\ \sigma_2' = 0.008,\ \sigma_1 = \sigma_2 = 0,\ q_1 = 1)$ Case 6 or triaxiality of the primaries with the potential created by the belt only $(i.e.\ \sigma_1 = \sigma_1' = M_b = 0.01,\ \sigma_2 = \sigma_2' = 0.008,\ q_1 = q_2 = 1)$ Case 7 or radiation and triaxiality of the primaries with the potential created by the belt

$(i.e.\ \sigma_1 = \sigma_1' = M_b = 0.01,\ \sigma_2 = \sigma_2' = 0.008,\ q_1 = 0.98,\ q_2 = 0.97)$ Case 8, the new collinear point L_{n1} moved toward the bigger primary whileL_{n2} moved away from it. In Case 5, owing to triaxiality of the smaller primary with the potential created by the belt only $(i.e. \sigma_1' = M_b = 0.01,\ \sigma_2' = 0.008,\ \sigma_1 = \sigma_2 = 0,\ q_2 = q_1 = 1)$, L_{n1} sifted away from the bigger primary whereas L_{n2} stepped towards it. The combined effect of the perturbations on the positions of the collinear libration points arein Table 5.4.

Table 5.4. : Influences of the perturbations on the collinear points when μ =0.25, T=0.01.

	$q_1 = q_2 = 1$, σ_1=σ_1'=M$_b$=0, σ_2 = σ_2' = 0.	$q_1 = 0.98, q_2 = 0.97$, σ_1=σ_1'=M$_b$= 0.01, σ_2 = σ_2' = 0.008.	$q_1 = 0.96, q_2 = 0.95$, σ_1=σ_1'= M$_b$= 0.02, σ_2 = σ_2' = 0.018.	$q_1 = 0.94, q_2 = 0.93$, σ_1=σ_1'=M$_b$ = 0.03, σ_2 = σ_2' = 0.028.
L_1	1.2658581025	1.2585397955	1.2529854356	1.2478791666
L_2	0.3607434284	0.3590930537	0.3580343024	0.3573063535
L_3	-1.1031668488	-1.0880401250	-1.0749059246	-1.0627421281
L_{n1}	-	-0.0015477633	-0.0008762383	-0.0006591986
L_{n2}	-	-0.0197733105	-0.0263703668	-0.0299135903

With increase in the perturbations factors Table 5.4,the collinear libration points

L_1, L_3 tend closer to the primaries, and L_{n2}, L_2 draw towards m_1, while L_{n1} moves away from m_1.

On examining the positions of the collinear points under the combined effect of radiation and triaxiality of the primaries and the potential from the belt $(i.e.\ \sigma_1 = \sigma_1' = M_b = 0.01,\ \sigma_2 = \sigma_2' = 0.008,\ q_1 = 0.98,\ q_2 = 0.97)$ for $\mu \in (0,1/2)$ Figure 4.3, the new collinear points L_{n1} and L_{n2} do not exist for lower mass ratio. However, Figures 5.14 and 5.15 which provide the locations for the collinear points L_{n1}, $L_{n2}$for $\mu \in (0,1/2)$, Mb = 0.01, 0.02, 0.03, show that with increase in the mass of the belt, L_{n1} and L_{n2} exist for the lower mass ratio. The figures further indicate that L_{n2} comes nearer to the more massive primary while L_{n1} moves away from it, with the increase in the mass of the belt or the mass ratio.

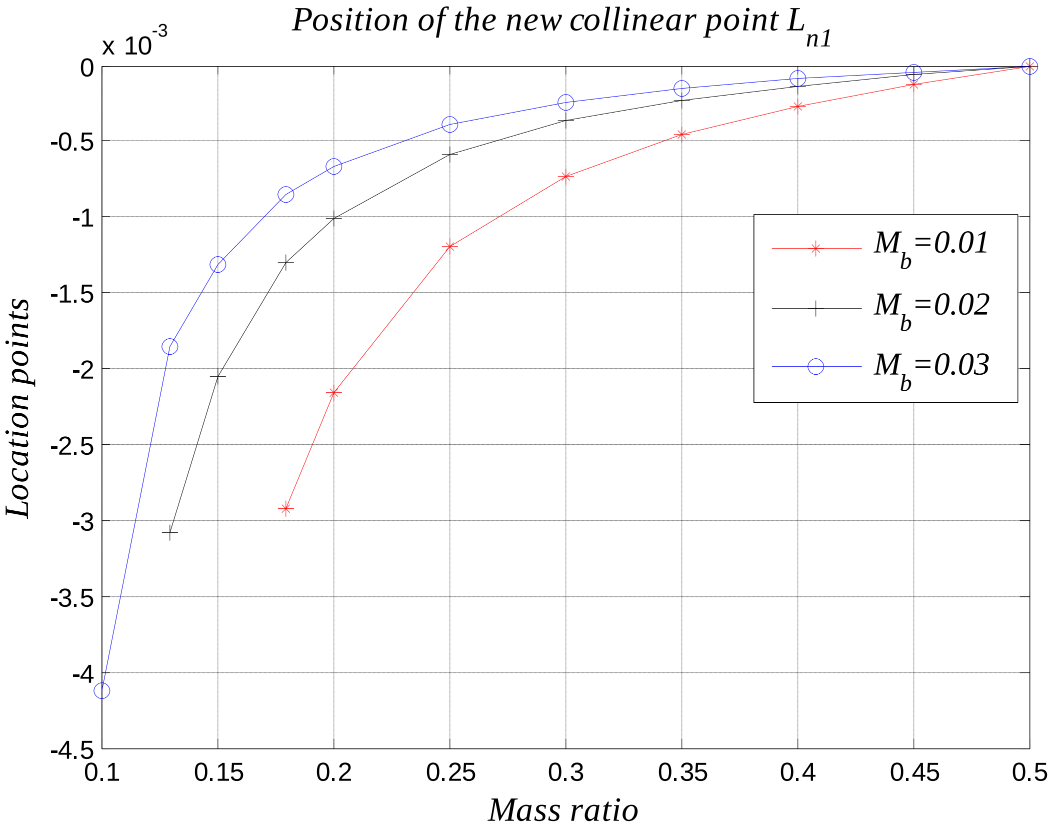


Figure 5.14: Effects of the potential from the belt on L_{n1}
$(i.e.\sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = 0,\ q_1 = q_2 = 1)$.

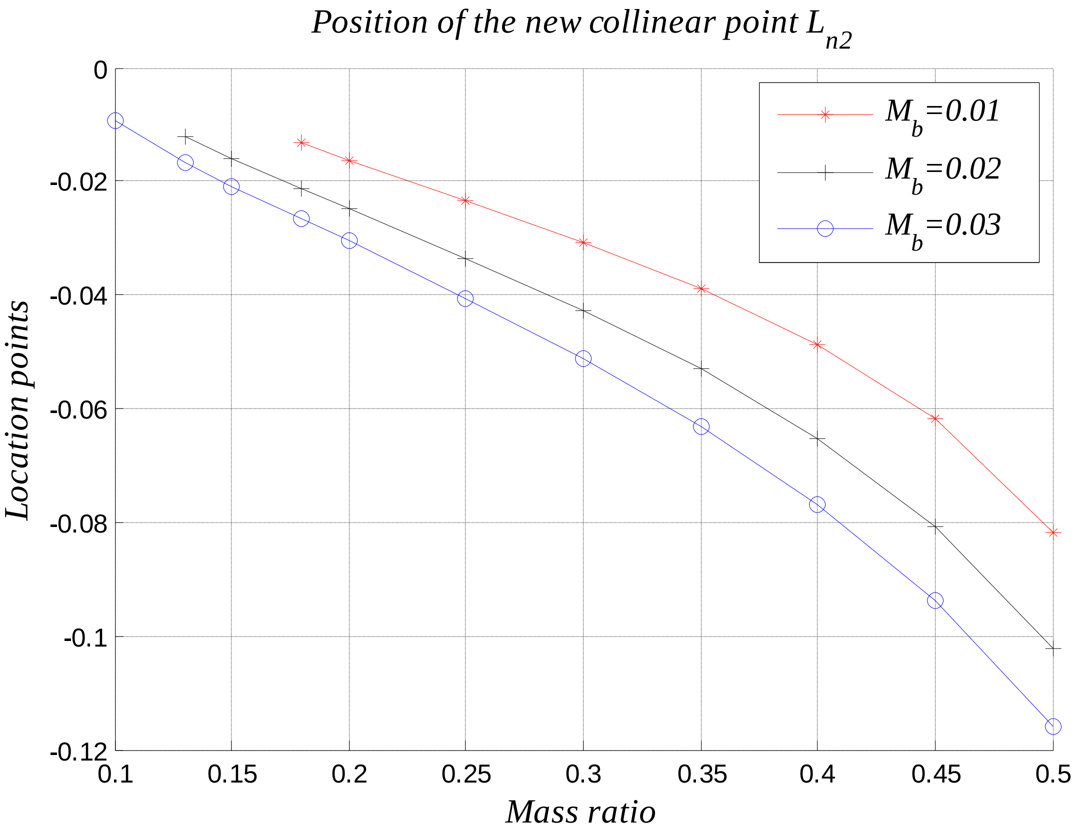


Figure 5.15: Effects of the potential from the belt on L_{n2}
$(i.e.\ \sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = 0,\ q_1 = q_2 = 1)$.

5.3.3 Stability of libration points

5.3.3.1*Stability of triangular libration*

The roots of the characteristic equation (4.37) are used to discuss the stability of the libration points. Section 4.4.2 proves that the triangular libration points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \le \mu \le \frac{1}{2}$, where μ_c is the critical mass ratio influenced by the radiation and triaxiality of the primaries; and the potential from the belt. Equation (4.51) offers the value of the critical mass parameter μ_c. It reflects the effects of the radiation and triaxiality of the primaries, and the gravitational potential from the belt. If the primaries are oblate instead of triaxial ($i.e.\ \sigma_1 = \sigma_2 \neq 0\ and\ \sigma_1' = \sigma_2' \neq 0$), μ_c verifies Equation (3.69) when considering oblateness coefficients up $J_2$of the primaries and that of Singh and Taura (2013). On ignoring the potential created by the belt $(i.e. M_b = 0,\ \sigma_1 \neq 0, \sigma_2 \neq 0,\ \sigma_1' \neq 0, \sigma_2' \neq 0,\ q_1 \neq 1, q_2 \neq 1),\ \mu_c$ confirms that obtained by Singh (2013) in the absence of perturbations in the Coriolis and centrifugal forces.Neglecting the effects of radiation primaries, triaxiality of the primaries and the gravitational potential from the belt $(i.e.\ \sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = M_b = 0,\ q_1 = q_2 = 1)$,the value of μ_c becomes $\mu_c = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{23}{27}}$ which confirms that of the classical problem (Szebehely, 1967 and Valtonen and Karttunen, 2005).

The effect of triaxiality of the bigger primary, the effect of triaxiality of the smaller primary and the combined effect of radiation and triaxiality of the primaries coupled with the gravitational potential from the belt on the position of the critical mass value are depicted in Figures 5.16, 5.17 and 5.18 respectively.

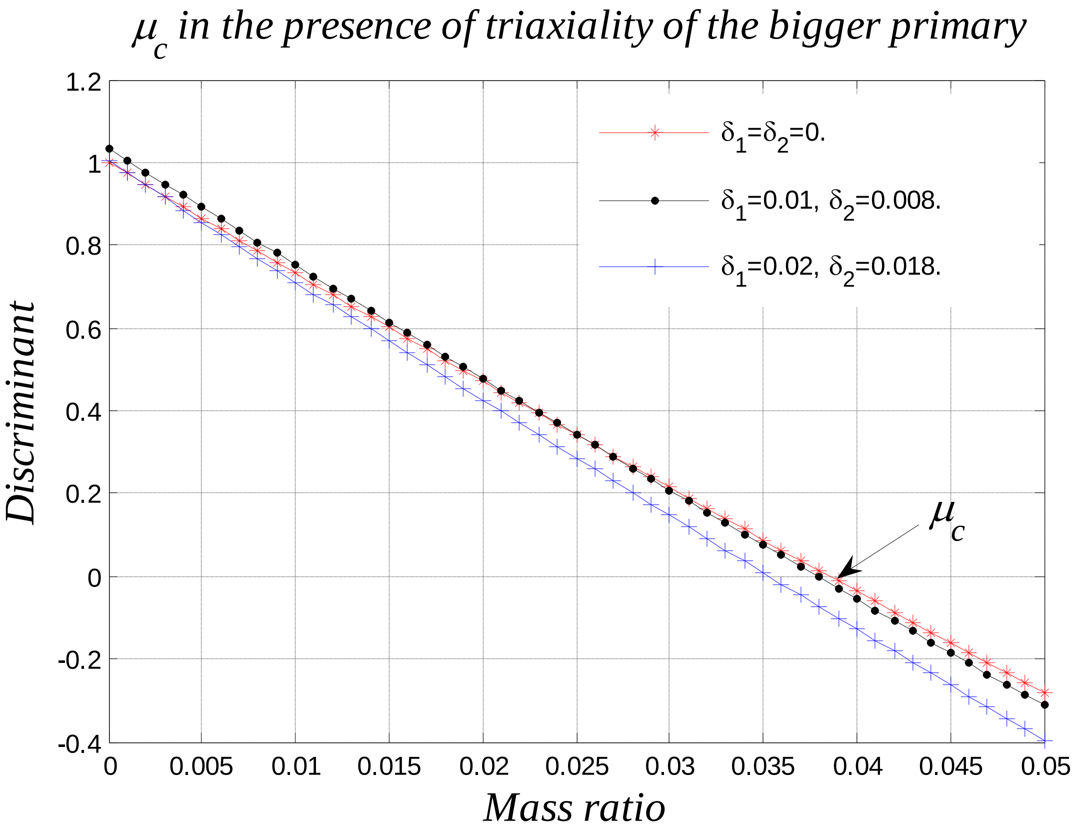


Figure 5.16: The effects of the triaxiality of the bigger primary on the critical mass parameter $(\sigma_1' = \sigma_2' = M_b = 0,\ q_1 = q_2 = 1)$.

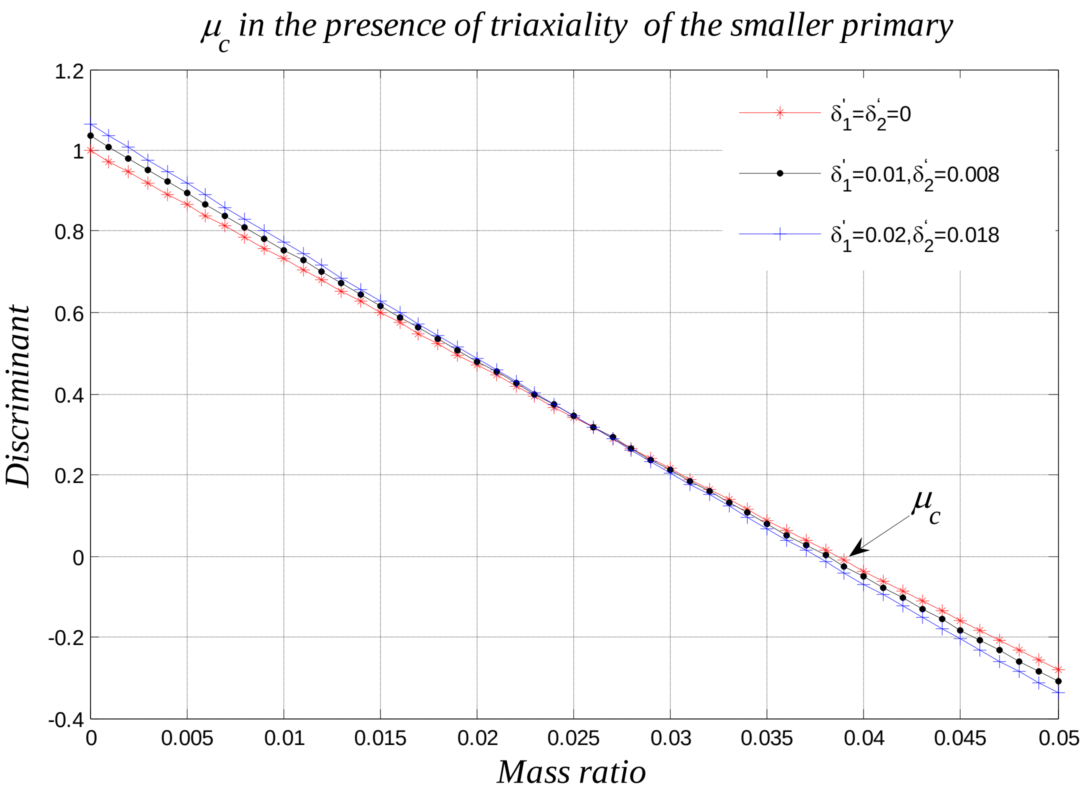


Figure 5.17: The effects of the triaxiality of the smaller primary on the critical mass parameter ($\sigma_1 = \sigma_2 = M_b = 0,\ q_1 = q_2 = 1$).

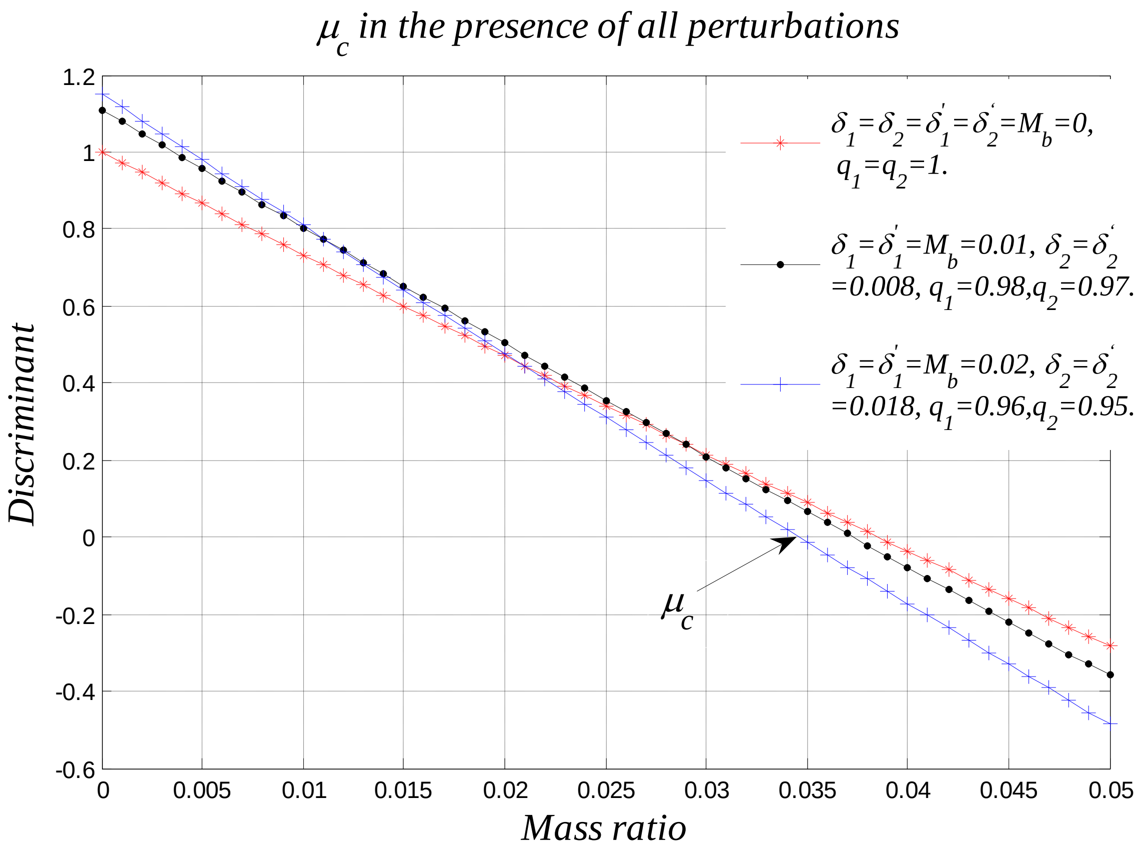


Figure 5.18: The combined effect of the perturbations on the critical mass parameter.

the value of the critical mass ratio decreases due to triaxiality of the bigger primary (Figure 5.16) or of triaxiality of the smaller primary (Figure 5.17) or the combined effect of radiation and triaxiality of the primaries with the gravitational potential from the belt (Figure 5.18). Thus, since the triangular libration points are stable in the range $0 < \mu < \mu_c$, it shows that triaxiality of the bigger primary, triaxiality of the smaller primary and the combined effect of radiation and triaxiality of the primaries with the gravitational potential from the belt each reduces the range of stability.

5.3.3.2*Stability of the collinear points*

As regards the stability of collinear points, Tables 4.2-4.6 show that all the collinear libration points are unstable stable except the new collinear point L_{n1}.

CHAPTER SIX

6.0 SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

The summary, conclusion and recommendations as regards to this study are given in this chapter.

6.1 Summary

We have investigated the libration points in a generalized restricted three-body problem. The restricted three-body problem is generalized to include the effect of radiation and asphericity of the primaries and gravitational potential from a belt. We have derived the equations that govern the motion of the infinitesimal body under the influence of: (i) oblateness up to J_4 of the primaries, radiation of the primaries and the gravitational potential from the belt; (ii) triaxiality and radiation of the primaries and the gravitational potential from the belt, within the structure of the circular restricted three-body problem. The positions of the libration points (triangular, collinear) and their linear stability have been determined. Also, the periodic orbits in the proximity of the stable triangular points under the combined effect of oblateness and radiation of the primaries and potential from the belt are studied.

We have seen that the positions of the triangular libration points (L_4, L_5) are affected by the perturbations such that they no longer form equilateral triangles with the primaries as they do in the classical case; rather they form scalene triangles or isosceles triangles with the primaries. They come closer to the smaller primary due to the resultant effect of oblateness up to J_4 of the primaries, radiation of the primaries and the gravitational potential from the belt whereas triaxiality and radiation of the primaries and the gravitational potential from the belt, cause the triangular points to come nearer to the bigger primary.

The collinear libration points are also influenced by the perturbations, such that there exist four new collinear points in addition to the three libration points $(L_{1,} L_2, L_3)$ in the classical problem. Two of the new libration points L_{n1} and $L_{n2}$are due to the potential from the belt, while the other two L_{n3} and L_{n4}appear due to the zonal harmonics up to J_4 of the smaller primary. Owing to the combined effect of the perturbations with $\mu = 0.4$, the collinear points $L_{n1,} L_2, L_{n4}$ are disposed between the primaries, L_{n2}, L_3 and L_{n3}, L_1 are beyond m_1 *and* m_2 respectively. With increase in the perturbations, L_1, L_3 come closer to the primaries while L_{n2}, L_{n3} shift away from the primaries; and $L_{n1,} L_2$ tend towards m_2 whereas L_{n4} moves away from it. The number of the collinear points decreases from seven $(L_{1,} L_2, L_3, L_{n1,} L_{n2}, L_{n3}, L_{n4})$ to five $(L_{1,} L_3, L_{n1,} L_{n2}, L_{n3})$ then to three $(L_3, L_{n1,} L_{n2})$, with the decrease in the value of the mass parameter. The oblateness up to J_4 of the smaller primary is accountable for this reduction in the number of the collinear points.

Under the combined effect of radiation and triaxiality of the primaries and the potential from the belt there are five collinear points $(L_{1,} L_2, L_3, L_{n1,} L_{n2})$. With increase in the perturbations factors, L_1 and L_3 come closer to the primaries, L_{n2}, L_2 shift towards m_1, while L_{n1} moves away from m_1. The libration points L_{n1} and L_{n2} exist for the lower mass ratio only with the increase in the mass of the belt.

The triangular points (L_4, L_5) are unstable for $\mu_c \le \mu \le \frac{1}{2}$ and stable for $0 < \mu < \mu_c$, where μ_c is the critical mass parameter affected by the aforesaid perturbations. The potential from the belt and the respective oblateness coefficients A_2 and B_2 of the more massive and less massive primaries and potential from the belt have stabilizing propensities; while the radiation of the primaries, the oblateness up to J_2 of the primaries and triaxiality of the primaries have destabilizing propensities. The resultant effect of the potential from the belt, radiation and oblateness up to J_4 of the primaries on μ_c has stabilizing tendency; while the combined effect of radiation and triaxiality of the primaries with the gravitational potential from the belt has destabilizing tendency.

Subject to the combined effect of the potential from the belt, radiation and oblateness up to J_4 of the primaries, the collinear libration points L_i $(i = 1, 2, 3)$ remain unstable, while two of the additional new collinear points L_{n3} and L_{n4} are stable. L_{n1} is stable due to the oblateness up to J_4 of the smaller primary with potential from the belt or due to radiation of any of the primaries with potential from the belt or due to the combined effect of all the perturbations. The collinear point L_{n2} is stable when considering the zonal harmonics up to J_4 of the bigger primary with potential from the belt and due to the joint influence of all the perturbations.All the collinear libration points under the combined effect of triaxiality and radiation of the primaries and the gravitational potential from the belt are unstable, except the new collinear point L_{n1}.

The periodic orbits in the vicinity of the triangular points for $0 < \mu < \mu_c$, where $\mu_c \in (0,1/2)$ is the critical mass ratio influenced by the oblateness and radiation of the primaries and potential from the belt, are ellipses. The values of the angular frequency of the long elliptic orbit s_1, the eccentricity of the short elliptic orbit e_2, the semi-minor axes of the elliptic orbits b_1, $b_2$decrease, whereasthe frequency of the short periodic orbits s_2, the eccentricity of the long periodic orbit e_1, the semi-major axes of the orbits a_1, a_2 and the angle of rotation of the principal axes β increase owing to the combined effect of the radiation of the primaries, oblateness up to the coefficients J_4 of the primaries and the gravitational potential from the belt.

6.2 Conclusions

The problem studied is a modification of the RTBP by considering the gravitational potential from a belt, the radiation and the asphericity of the primaries. We have derived the equations that regulate the motion of the infinitesimal body moving in the gravitational field of the radiating and oblate/triaxial primaries, together with the influence of gravitational potential generated by the belt within the framework of the restricted three-body problem. We have also investigated the effects of the aforementioned perturbations on the locations of the libration points and their linear stability and established that:

i. The equations that govern the motion of the infinitesimal body are affected by the aforesaid perturbations.
ii. The triangular libration points form scalene triangles or isosceles triangles with the primaries. They come closer to the less massive primary due to the resultant effect of the perturbations when the primaries are oblate; and move towards the more massive primary when the primaries are triaxial.

iii. There exist four additional collinear points in addition to the three libration points $(L_{1,} L_2, L_3)$ in the classical problem when the primary bodies are oblate. However, the number of the collinear points reduces with decrease in the value of the mass ratio, such that $L_1\ and\ L_2$ do not exist for lower mass ratios. When the primaries are triaxial, there exist only two additional collinear points.

iv. The triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \le \mu \le \frac{1}{2}$, where μ_c is the critical mass parameter influenced by the radiation and triaxial/oblateness up to J_4 of the primaries and potential from the belt. The coalesced effect of these perturbations has stabilizing tendency when the primaries are oblate and destabilizing when they are triaxial.

v. As in the classical problem, the collinear libration points $(L_{1,} L_2, L_3)$ remain unstable in the presence of these perturbations. Nonetheless, due to the amalgamated effect of the perturbations, there are new collinear points that are stable.

vi. The periodic orbits around the triangular points in the range $0 < \mu < \mu_c$, where μ_c is the critical mass parameter influenced by the radiation and oblateness up to J_4 of the primaries and potential from the belt are ellipses.

6.3 Recommendations

This research has extensively explored the CR3BP when the primaries are radiating oblate/triaxial bodies enclosed by a belt of homogeneous cluster of material points. Further extension of the work can include:

i. Perturbations in the Coriolis and centrifugal forces.
ii. Variable mass problem.
iii. The Robe’s CR3BP.
iv. P-R (Poynting-Robertson) drag forces
v. Elliptic R3BP.

REFERENCES

AbdulRaheem, A. and Singh, J. (2006). Combined effects of perturbations, radiation, and oblateness on the stability of equilibrium points in the restricted three-body problem. *The Astronomical Journal,* 131:1880–1885.

AbdulRaheem, A. and Singh, J. (2008). Combined effects of perturbations, radiation and oblateness on the periodic orbits in the restricted three-body problem.*Astrophysics and Space Science*, 317: 9-14.

Abouelmagd, E. I. (2012). Existence and stability of triangular points in the restricted three-body problem with numerical applications. *Astrophysics and Space Science,* 342(1): 45-53.

Abouelmagd, E. I. (2013). The effect of photogravitational force and oblateness in the perturbed restricted three-body problem. *Astrophysics and Space Science***,** 346(1)**:** 51-69**.**

Abouelmagd, E. I. and El–Shaboury, S.M. (2012). Periodic orbits under combined effects of oblateness and radiation in the restricted problem of three bodies. *Astrophysics and Space Science*,341:331–341.

Abouelmagd, E.I. and Sharaf, M.A. (2013). The motion around the libration points in the restricted three-body problem with the effect of radiation and oblateness. *Astrophysics and Space Science*,344:321–332.

Abouelmagd, E. I., Alhothuali , M.S., Guirao , J.L.G. and Malaikah, H.M.(2015). The effect of zonal harmonic coefficients in the framework of the restricted three-body problem. *Advances in Space Research,* 55: 1660–1672.

Abouelmagd, E. I., Asiri, H. M. and Sharaf, M. A. (2013).The effect of oblateness in the perturbed restricted three-body problem. *Meccanica,* 48(10): 2479-2490.

Aumann, H.H., Beichman, C. A., Gillett, F. C., De-Jong, T., Houck,J.R., Low,F.J., Neugebauer, G., Walker, R.G. and Wesselius, P.R. (1984). Discovery of a shell around Alpha Lyrae.*Astrophysical Journal*, 278:L23-L27.

Baltagiannis, A. N. and Papadakis, K. E. (2011). Families of periodic orbits in the restricted four-body problem. *Astrophysics and Space Science*, 336(2):357- 367.

Bhatnagar, K. B. (1972). Periodic orbits of collision in the restricted problem of three bodies in a three dimensional co-ordinate system. *Indian Journal of Pure and Applied Mathematics,* 3(1):101-117.

Bhatnagar, K.B.and Chawla, J.M. (1979). A study of the Lagrangian points in the photogravitational restricted three-body problem. *Indian Journal of Pure and Applied Mathematics*, 10(11):1443–1451.

Bruno, A. D. and Varin, V. P. (2008). On families of periodic solutions of the restricted three-body problem. *Solar System Research*, 42(2): 154-176.

Das, M.K. Narang, P. Mahajan, S. and Yuasa, M. (2008). Effect of radiation on the stability of equilibrium points in the binary stellar systems: RW-Monocerotis, Krüger 60. *Astrophysics and Space Sci*ence, 314:261-274.

Elipe, A. and Lara, M. (1997). Periodic orbits in the restricted three body problem with radiation pressure. *Celestial Mechanics and Dynamical Astronomy*, 68**:**1–11.

Elshaboury, S. M. and Mostafa, A. (2011). Stability conditions of L4 in the RTBP when the smaller primary and the infinitesimal body are oblate spheroids. I*nternational Journal of Contemporary Mathematical Sciences*,6(22): 1091 – 1098.

Gomez, G. and Noguera, M.(1985). Some manifolds of periodic orbits in the restricted three-body problem. *Celestial Mechanics*,35: 235-255.

Greaves, J. S., Holland, W. S., Moriarty-Schieven, G. , Jenness,T., Dent,W. R. F., Zuckerman,B. , Mccarthy, C., Webb, R. A., Butner, H. M., Gear, W. K. and Walker, H. J. (1998). A Dust Ring around ϵEridani: Analog to the young Solar System.*The Astrophysical Journal*, 506:L133–L137.

Guillaume, P. (1969). Families of symmetric periodic orbits of the restricted problem, when the purturbating mass is small. *Astronomy and Astrophysics*, 3: 57-76.

Guillaume, P. (1973). Periodic symmetric solutions of restricted problem. *Celestial Mechanics*, 8:199-206.

Hadjidemetriou, J. D. (1984). Periodic orbits. *Celestial Mechanics*, 34(1-4): 379-393.

Haque, M. N. and Kumar , V. (2015). Stability of Triangular Libration Points in the Perturbed Photogravitational Restricted Problem of Three Bodies.*International Journal of Science and Research*, 4(2): 1197-1199.

Ishwar, B. **(**1997**)**. Non-linear stability in the generalized restricted three body problem. *Celestial Mechanics*,65(3): 253-289.

Ishwar, B. **(**1998**)**. First order normalization in the generalized restricted three body problem. *Review Bulletin of the Calcutta Mathematical*6(2): 73-78.

Jain, S., Kumar,A. and Bhatnagar, K. B. (2006). *Bulletin of the Astronomical Society of India*, 34: 211-223.

Jiang, I.G. and Yeh, L. C. (2003). Bifurcation for dynamical systems of planet-belt interaction. *International Journal Bifurcation and Chaos,13*(3): 617-630.

Jiang, I.G. and Yeh, L.C. (2004a). The modified restricted three body problems. *RevMexAA (Serie de Conferencias),* 21: 152–155.

Jiang, I.G. and Yeh, L.C. (2004b). On the chaotic orbits of disk-star-planet systems. *Astronomical Journal,*128: 23–932.

Jiang, I. G. and, Yeh, L.C. (2006). On the Chermnykh-like problems: I. The mass parameter $\mu = 0.5$. *Astrophysics and Space Science,* 305:341–348.

Khanna, M. and Bhatnagar, K.B. (1999). Existence and stability of libration points in the restricted three body problem when the smaller primary is a triaxial rigid body and the bigger one an oblate sheriods, *Indian Journal of Pure and Applied Mathematics.* 30(7):721-733.

Kushvah, B.S. (2008). Linear stability of equilibrium points in the generalized photogravitational Chermnykh's problem.*Astrophysics and Space Science,* 318: 41–50.

Kushvah, B.S. (2011**).**Trajectory and stability of Lagrangian point L_2 in the Sun-Earth system.*Astrophysics and Space Science,* 332(1): 99-106.

Lanzano, P. (1965). Periodic motion about a lagrangian triangular point. *Icarus,* 4: 223-241.

Markellos, V.V., Black, W. and Moran, P.E. (1974). A grid search for families of periodic orbits in the restricted problem of three bodies. *Celestial Mechanics,* 9:507-512.

McCuskey, S.W. (1963). *Introduction to Celestial Mechanics*. Addison-Wesley New York, USA.

Meyer, K. R. and Schmidt, D. S. (1971). Periodic orbits near L_4 for mass ratio near the critical mass ratio of Routh. *Celestial Mechanics,* 4: 99-109.

Mittal, A., Ahmad, I. and Bhatnagar, K.B. (2009). Periodic orbits generated by Lagrangian solution of the restricted three body problem when one of the primaries is an oblate body. *Astrophysics and Space Science,* 319**:**63–73.

Miyamoto, M. and Nagai, R. (1975). Three-dimensional models for the distribution of mass in galaxies. *Publications of the Astronomical Society of Japan,* **27**:533–543.

Moulton, F. R. (1914). *An Introduction to Celestial Mechanics 2nd ed*. Dover, New York USA.

Papadakis, K.E. and Rodi, M. I. (2010).Asymmetric periodic solutions in the restricted problem of three bodies. *Earth, Moon and Planets,*106, 37–53.

Perdios, E. A. (2003). Critical symmetric periodic orbits in the photogravitational restricted three-body problem. *Astrophysics and Space Science*, 286: 501-513.

Perdios, E. A., Kanavos, S. S. and Markellos,V. V. (1998). Bifurcations of plane to 3d periodic orbits in the restricted three-body problem with oblateness.*Astrophysics and Space Science*, 262(1): 75-87.

Perdios, E. and Zagouras, C. G. (1991). Vertical stability of periodic solution around the triangular equilibrium points. *Celestial Mechanics and Dynamical Astronomy,* 51: 75-81.

Perdios, E., Zagouras, C. G. and Ragos, O. (1991).Three-dimensional bifurcations of periodic solution around the triangular equilibrium points of the restricted three-body problem. *Celestial Mechanics and Dynamical Astronomy*, 51:349-362.

Peter, I. D. and Lissauer, J.J. (2007). *Planetary Sciences.* Cambridge University Press, New York, USA, pp. 30-33.

Poddar, A. K., Taqvi, Z.A., Jain, S. and Bhatnagar, K. B. (2002). Existence of periodic orbits in the restricted problem of three bodies when the more massive body is a source of radiation and the smaller primary is a triaxial rigid body. *IndianJournal of Pure and Applied Mathematics,* 33(10): 1575-1594.

Poincare, H. (1958). *The Value of Science*. Dover Publications Inc., New York, USA, pp.8.

Poynting, J.H. (1903). Radiation in the solar system: its effect on temperature and its pressure on small bodies. *PhilosophicalTransactions of the Royal Society of London***,** 202: 525–552.

Rabe, E. and Schanzle, A. (1962). Periodic librations about the triangular solutions of the restricted earth-moon problem and their orbital stability. *AstronomicalJournal,* 67: 732-739.

Radzievskii, V. V. (1950). The restricted problem of three-body taking account of light pressure. *Astronomical Journal Zh,* 27(5): 250-256.

Sharma, R. K. (1976). Periodic orbits of first kind in the restricted three-body problem when the more massive primary is an oblate spheroid. *Astronomy and Astrophysics*, 50: 257-258.

Sharma, R.K. (1987). The linear stability of libration points of the photogravitational restricted three- body problem when the smaller primary is an oblate spheroid. *Astrophysics and space Science* 135: 271-281.

Sharma, R.K., Taqvi, Z.A. and Bhatnagar, K.B. (2001a). Existence and stability of libration points in the restricted three body problem when the bigger primary is a triaxial rigid body and source of radiation. *Indian Journal of Pure and Applied Mathematics.*32(2): 255-266.

Sharma, R.K., Taqvi, Z.A. and Bhatnagar, K.B. (2001b). Existence and stability of libration points in the restricted three body problem when the primaries are triaxial rigid bodies and source of radiations. *Indian Journal of Pure and Applied Mathematics.* 32(7): 981-994.

Sharma, R.K., Taqvi, Z.A. and Bhatnagar, K.B. (2001c). Existence and stability of libration points in the restricted three body problem when the primaries are triaxial rigid bodies. *Celestial Mechanics and Dynamical Astronomy,* 79(2):119-133.

Simmons, J. F. L., McDonald, J. C. and Brown, J. C. (1985). The three-body problem with radiation pressure. *Celestial Mechanics,*35: 145-187.

Singh, J. (2009). Combined effects of oblateness and radiation on the nonlinear stability of L_4 in the restricted three-body problem. *The Astronomical Journal*, 137:3286–3292.

Singh, J. (2013). The equilibrium points in the perturbed R3BP with triaxial and luminous primaries. .*Astrophysics Space Science*, 346:41–50.

Singh, J. and Begha, J.M. (2011a). Stability of equilibrium points in the generalized perturbed restricted three-body problem.*Astrophysics Space Science,* 331:511-519.

Singh, J. and Begha, J.M. (2011b). Periodic orbits in the generalized perturbed restricted three-body problem. *Astrophysics and Space Science*, 332:319-324.

Singh, J. and Haruna, S. (2014a). Equilibrium points and stability under effect of radiation and perturbing forces in the restricted problem of three oblate bodies. *Astrophysics Space Science*, 349(1): 107-116.

Singh, J. and Haruna, S. (2014b). Periodic orbits around triangular points in the restricted problem of three oblate bodies. *American Journal of Astronomy and Astrophysics*, 2(2): 22-26.

Singh, J. and Ishwar, B. (1999). Stability of triangular points in the generalized photogravitational restricted three body problem. *Bulletin of the Astronomical Society of India,* 27: 415-424.

Singh, J. and Leke, O. (2014). Analytic and numerical treatment of motion of dust grain particle around triangular equilibrium points with post-AGB binary star and disc. *Advances in Space Research,* 54:1659–1677.

Singh, J. and Taura, J. J. (2013). Motion in the generalized restricted three-body problem.*Astrophysics Space Science*, 343(1): 95–106.

Singh, J. and Umar, A. (2012). On the stability of triangular points in the elliptic R3BP under radiating and oblate primaries. *Astrophysics and Space Science,*341:349–358.

Singh, J. and Umar, A. (2013a). Application of binary pulsars to axisymmetric bodies in the Elliptic R3BP. *Astrophysics and Space Science,*348: 393-402.

Singh, J. and Umar, A. (2013b). Collinear equilibrium points in the elliptic R3BP with oblateness and radiation. *Advances in Space Research,* 52: 1489–1496.

Szebehely, V. (1967). *Theory of Orbits: The Restricted Problem of Three Bodies*. Academic Press, New York, USA, pp.7-318.

Taqvi, Z. A., Poddar, A. K., Bhatnagar, K. B. and Jain, S. (2002a). Existence of periodic orbits in the restricted problem when the smaller primary is a triaxialrigid body. *Indian Journal of Pure and Applied Mathematics* ,33(7): 977-991.

Taqvi, Z. A., Poddar, A. K., Bhatnagar, K. B. and Jain, S. (2002b). Existence of periodic orbits in the restricted problem when the more massive body is a source of radiation and smaller primary is a triaxial rigid body. *IndianJournal of Pure and Applied Mathematics*, 33(10): 1575-1594.

Trilling, D. E., Stansberry J. A., Stapelfeldt K. R., Rieke G. H., Su K. Y. L., Gray R. O., Corbally C. J., Bryden G., Chen C.H., Boden A. and Beichman, C. A. (2007). Debris disks in main-sequence binary systems. *The Astrophysical Journal*, 658: 1289-1311.

Umar, A. and Singh, J. (2014). Periods, eccentricities and axes around $L_{4,5}$ in the ER3BP under radiating and oblate primaries.*International Journal of Astronomy and Astrophysics***,** 4:668-682.

Valtonen, M. and Karttunen, H. (2005). *The three-body problem*. Cambridge University Press, New York, USA, pp. 115-130.

Yeh, L. C. and Jiang, I.G. (2006). On the Chermnykh-like problems: II. The equilibrium points. *Astrophysics and Space Science,*306: 189–200.

Zagouras, C. G. (1991). Periodic motion around the triangular equilibrium points of the photogravitational restricted problem of three bodies. *Celestial Mechanics and Dynamical Astronomy*, 51: 331-348.