STUDY ON LOMAX-KUMARASWAMY DISTRIBUTION: ITS PROPERTIES AND APPLICATIONS

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DECLARATION

I declare that the work in this dissertation titled "Study on Lomax-Kumaraswamy Distribution: Its Properties and Applications" has been carried out by me in the Department of Statistics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other institution.

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CERTIFICATION

This dissertation titled STUDY ON LOMAX-KUMARASWAMY DISTRIBUTION: IT'S PROPERTIES AND APPLICATIONS by MABUR Tajan Mashingil meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This work is dedicated to God Almighty, my strong pillar, source of inspiration, wisdom, knowledge and understanding.

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ABSTRACT

The Kumaraswamy probability distribuion has the same basic properties as the beta distribution. In this dissertation, we proposed a new extension of the Kumaraswamy distribution known as the Lomax-Kumaraswamy distribution, with two additional parameters using the Lomax generator proposed in earlier research. We derived the cumulative distribution function (cdf) and the probability density function (pdf). We also derived some basic statistical properties of the new distribution such as moments, moment generating function, quantile function, the characteristics function, reliability analysis, and the distribution of order statistics. Some plots of the distribution revealed that it is a positively skewed distribution. The parameters of the model have been estimated using maximum likelihood method. Two real data sets were used to test the performance of the new distribution, by comparing it with some generalizations of the Kumaraswamy distribution. The result shows that the proposed distribution provides better fits than the baseline distribution and some of the generalizations of Kumaraswamy distribution when the data sets are positively skewed.

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CHAPTER ONE: INTRODUCTION

1.1 Background of the Study

The Lomax or Pareto II (the shifted Pareto) distribution was pioneered to model business failure data by Lomax (1954). This distribution has found wide application in a variety of fields such as income and wealth inequality, size of cities, actuarial science, medical and biological sciences, engineering, lifetime and reliability modeling. It has been applied to model data obtained from income and wealth, firm size (Corbellini *et al.,* 2007), size of computer files on servers (Holland *et al.*, 1989), reliability and life testing (Hassan and Al-Ghamdi, 2009) and receiver operating characteristic (ROC) curve analysis, The Lomax distribution is known as a special form of Pearson type VI distribution and has also been considered as a mixture of exponential and gamma distributions. It belongs to the family of decreasing failure rate, in the lifetime context, (Chahkandi and Ganjali, 2009) and also arises as a limiting distribution of residual lifetimes at great age (Balkema and de Haan, 1974). This distribution has been suggested as heavy tailed alternative to the exponential, Weibull and gamma distributions. Furthermore, it is related to the Burr family of distributions (Tadikamalla, 1980) and as a special case, can be obtained from compound gamma distributions (Durbey, 1970).

Some details about the Lomax distribution and Pareto family are given in Arnold (1983). The distributional properties, estimation and inference of the Lomax distribution are described. In record value theory, some properties and moments for the Lomax distribution have been discussed in Ahsanullah (1991), Balakrishnan and Ahsanullah (1994) and Amin (2011). The comparison of Bayesian and non-Bayesian estimation from the Lomax distribution based on record values has been made in Abd-Ellah (2006). The moments and inference for the order statistics and generalized order statistics are given in Saran and Pushkarna (1999) and Moghadam *et al*. (2012), respectively. The estimation of parameters in case of progressive and hybrid censoring have been investigated in Asgharzadan and Valiollahi (2011) and Ashour *et al*. (2011). The problem of Bayesian prediction bounds for future observation based on uncensored and type-I censored sample from the Lomax model are dealt in Abd-Ellah (2003) and Al-Hussaini *et al*. (2001). Furthermore, the Bayesain and non-Bayesian estimators of the sample size in case of type-I censored samples for the Lomax distribution are obtained in Abd-Elfattah *et al*. (2007), and the estimation under step-stress accelerated life testing for the Lomax distribution was considered in Hassan and Al-Ghamdi (2009). The parameter estimation through generalized probability weighted moments (PWMs) is addressed in Abd-Elfattah and Alharby (2010). More recently, the second-order bias and bias-correction for the maximum likelihood estimators (MLEs) of the parameters of the Lomax distribution were determined in Giles *et al.* (2013).

The Kumaraswamy distribution describes a distribution in which outcomes are limited to a specific range, the probability density function within this range being characterized by two shape parameters. It is similar to the beta distribution but possibly easier to use because it has simpler analytical expressions for its probability density function and cumulative distribution function. The Kumaraswamy probability distribution was originally proposed by Kumaraswamy (1980) for double bounded random processes for hydrological applications. The Kumaraswamy double bounded distribution denoted by (a, b) distribution is a family of continuous probability distributions defined on the interval (0,1) with cumulative distribution function (cdf) given by

$$G(x) = 1 - \left(1 - x^{a}\right)^{b};\ 0 < x < 1, \quad a, b > 0 \tag{1.1}$$

and probability density function (pdf) given by

$$g(x) = ab\,x^{a-1}\left(1 - x^{a}\right)^{b-1};\ 0 < x < 1,\ \ a, b > 0 \tag{1.2}$$

where *a* and *b* are the shape parameters.

The Kumaraswamy probability density function has the same basic properties as the beta distribution. According to Jones (2009) and Cordeiro *et al*. (2010, 2012) it depends on the same way as the beta distribution on the values of its parameters. They investigated not only properties of the Kumaraswamy distribution but also some similarities and differences between the beta and Kumaraswamy distributions. According to them the Kumaraswamy distribution has many advantages over the beta distribution, such as, simple explicit formulae for the distribution, normalizing constant, quantile function, L-moments and moments of order statistics. On the other hand, the beta distribution has simpler formulae for moments and the moment generating function, a one-parameter sub-family of symmetric distributions, simpler moment estimation and more ways of generating the distribution through physical processes. The Kumaraswamy distribution is applicable to a number of hydrological problems and many natural phenomena whose process values are bounded on both sides.

The Kumaraswamy distribution has received considerable interest In hydrology and related areas., Fletcher and Ponnambalam (1996) presented for the first and second moments of the storage state distribution. Explicit expression for the maximum and minimum storage limits in the reservoir system was given based on traditional control theory. Estimates were obtained for the probability of spills and deficits and used for approximating the storage state distribution using a Kumaraswamy double bounded density function. Ganji *et al*. (2006) proposed a new method of reliability analysis for crop water production for evaluating the crop yield failure probability, using an advanced first-order second moment, which is use to determine the variance and the mean of actual evapo-transpiration. The violation probabilities of crop yield at different levels were obtained. Also using the optimization results and the double bounded density function estimation methodology, the weekly soil moisture density function is derived which can be used as a short term reliability index. Ponnambalam *et al*. (2001) presented a new method for finding optimal solutions of systems with design parameters which are random variables distributed with various general and possibly non- symmetrical distributions. A double bounded density function is used to approximate the distributions. Sundar and Subbiah (1989) presented a new application of the double bounded probability density function to describe the ocean wave statistics. The basic importance is to estimate the most probable maximum wave height for offshore structural designs. Seifi *et al.* (2000) presents a general method for maximizing manufacturing yield when the realizations of system components are independent random variables with arbitrary distributions. A double-bounded density function was used for the approximation.

According to Nadarajah (2008), many papers in the hydrological literature have used this distribution because it is deemed to be a “better alternative” to the beta distribution; see, for example, (Koutsoyiannis and Xanthopoulos 1989).

1.2 Statement of the Problem

The general problem in statistical modeling today is that a lot of skewed real life data sets do not fit most of the existing and well known standard probability distributions (models) and hence can not be used for valid statistical inference. Therefore this research work is to provide an alternative model that will improve the baseline Kumaraswamy distribution.

Since the introduction of the Lomax generator, there is no researcher that has extended the Kumaraswamy distribution using the Lomax generator, Therefore, we present a new extension of the Kumaraswamy distribution which have a heavy and flexible failure rate and may have bathtub shaped hazard function for some parameter values and performs better compared to the baseline and some other extensions discussed in the literature.

1.3 Aim and Objectives of the Study

The aim of this research work is to propose a Lomax Kumaraswamy distribution using the Lomax generator (Lomax-G).

Specific Objectives:

i. to obtain the statistical properties of the proposed distribution.
ii. to estimate the parameters of the new distribution using the method of maximum likelihood estimation (MLE).
iii. to evaluate the strength of the new model by comparing it with some existing generalizations of the Kumaraswamy distribution.

1.4 Significance of the Study

The study is another extension of the Kumaraswamy distribution which has not been discussed in the literature. The new Lomax-Kumaraswamy distribution has two additional parameters which will capture the main features of the data such as the skewness, which is an improvement on the baseline distribution.

CHAPTER TWO: LITERATURE REVIEW

2.1 Generalizations Based on Kumaraswamy Distribution

Cordeiro and De Castro (2009) proposed a new class of the Kumaraswamy generalized distributions (denoted by the Kw–G distribution) based on the Kumaraswamy distribution. They derived some mathematical properties of the Kw-G distributions, such as the ordinary moments, probability weighted moments and the order statistics. They also discussed the maximum likelihood estimation of the parameters. The Kw-G generalized distributions was used to obtain the following new distributions, the Kumaraswamy-normal (KwN), Kumaraswamy-Weibull (KwW), Kumaraswamy-gamma (KwGa), and the Kumaraswamy-Gumbel (KwGu) distributions.

El-sherpieny and Ahmed (2014) introduced a new distribution, referred to as Kumaraswamy-Kumaraswamy (Kw-Kw) distribution, from the class of Kumaraswamy Generalized (KW-G) distributions. The derived the cumulative distribution function (cdf) and the probability density function (pdf), as well as some properties such as moments, quantiles, mean deviation, median, mode, order statistics, the entropy, L-moments and parameters estimation using the method of maximum likelihood were obtained. They used numerical illustration by simulation in studying the properties of the parameters. It was found that the Kw-Kw distribution provides a flexible method of fitting a wide range of real data sets.

Javanshiri *et al*. (2015) proposed exp-kumaraswamy (exp-Kw) distribution using the exponentiated Generalized (exp-G) family of probability distributions. The new distribution is an alternative to beta and kumaraswamy distributions. Some basic properties of this proposed distribution, such as moments, moment generating function, quantile function, hazard rate function, skewness, kurtosis and maximum likelihood estimation were studied. Its applicability was also illustrated by means of two real data sets.

Khan *et al*. (2016) proposed a generalization of the Kumaraswamy distribution referred to as the Transmuted Kumaraswamy (TKw) distribution. The new distribution was obtained using the quadratic rank transmutation map studied by Shaw and Buckley (2007). A comprehensive account of the mathematical properties of the new distribution was provided. Explicit expressions for the moments, moment generating function, entropy, mean deviation, Bonferroni and Lorenz curves, and moments for order statistics were derived. The TKw distribution parameters were estimated by using the method of maximum likelihood. Monte Carlo simulation was performed in order to investigate the performance of the MLEs. A flood data and HIV/ AIDS data applications were used to illustrate the usefulness of the proposed model. TKw distribution produced monotonically increasing and decreasing hazard rates. In terms of the statistical significance of the model adequacy, the Transmuted Kumaraswamy distribution leads to a better fit than the baseline Kumaraswamy distribution.

Santana *et al*. (2012) introduced the Kumaraswamy log-logistic distribution based on the log-logistic and Kumaraswamy distributions. The new distribution was found to contain several important distributions discussed in the literature as sub-models such as the log-logistic, exponentiated log-logistic and Burr XII distributions, among several others. The interesting thing about the new distribution is its ability to model non-monotonic failure rate functions, which are quite common in reliability and lifetime data analysis. Some of its structural properties were studied. The proposed Kumaraswamy-logistic regression model which has, as sub-models, various widely-known regression models. They also used the method of maximum likelihood to estimate the model parameters and determined the observed information matrix. Two real data sets were used to illustrate the importance and flexibility of the proposed models.

Mandouh (2016) carried out a Bayesian analysis for the Kumaraswamy-Weibull (Kum-W) distribution. He used Gibbs sampling procedure to obtain the approximate Bayes estimates under the assumptions of non-informative priors. This procedure allowed for generating samples from posterior distributions. Also using Bayesian approach predictive density for a single future response, a bivariate future response, and several future responses were derived. The predictive means, standard deviations, highest predictive density intervals, and the shape characteristics for a single future response were also determined. Finally, applications to real data sets were utilized to illustrate the potentiality of the Bayesian analysis and the predictive results.

Adepoju *et al*. (2016) proposed the Kumaraswamy-F (KUMA-F) distribution, which is a generalization of the traditional Fisher Snedecor (F-distribution). It was obtained with the addition of two shape parameters to a continuous F-distribution, which is mostly used in the Analysis of Variance (ANOVA) to test the null hypothesis. Some statistical properties of the distribution such as the asymptotic behavior, moments, and moment generating function, among others were studied. Maximum likelihood estimation method was used to estimate the model parameters. The distribution was found to be more robust and flexible to regular assumptions of the traditional F-distribution. The new distribution was recommended for use in most applications where the assumption underlying the use of traditional F distribution for one-way analysis of variance are violated such as homogeneity of variance or normality assumption probably as a result of the presence of outlier(s). It is interesting to note that the new distribution preserves the data without any transformation.

Oguntunde *et al.* (2015) introduced the Kumaraswamy power distribution. The new distribution serves as a generalization of the two parameter power distribution using the Kumaraswamy generalized family of distributions. They investigated some of its statistical properties. The generalized power distribution, exponentiated power distribution and the Power distribution were found to be sub-models of the proposed distribution. The method of maximum likelihood estimation was proposed in estimating the parameters of the model.

Ramos *et al.* (2015) also proposed a new generalized family called the Kumaraswamy-G Poisson (Kw-GP) with three additional parameters. They introduced some special distributions in the new family such as the Kw-gamma Poisson Kw-Weibull Poisson, and Kw-beta Poisson distributions. They derived some mathematical properties of the new family including the ordinary moments, generating function and order statistics. The method of maximum likelihood was used to fit the distributions in the new family. They also illustrated its potentiality by means of an application to a real data set.

Handique and Chakraborty (2016) proposed a new generalization of the family of Kumaraswamy-G distribution that includes three recently proposed families namely the Garhy generated family, Beta-Dagum and Beta-Singh-Maddala distribution by constructing beta generated Kumaraswamy-G distribution. Useful expansions of the pdf and the cdf of the proposed family were derived and seen as infinite mixtures of the Kumaraswamy-G distribution. Order statistics, probability weighted moments, moment generating function, Rényi entropies, quantile power series, random sample generation, asymptotes and shapes were also investigated. Two methods of parameter estimation were presented. Suitability of the proposed model in comparisons to its sub models was carried out considering two real life data sets. Finally, some new classes of beta generated families were proposed for future investigations.

Bourguignon *et al*. (2012), proposed the Kumaraswany Pareto distribution, based on the family of Kumaraswamy generalized (Kw-G) distributions. They defined and expressed the pdf of the distribution as a linear combination of the Pareto densities, which allowed them to derive some of its mathematical properties like the moments, generating function and they provided the density function of the order statistics and obtain their moments. The method of maximum likelihood was used for estimating the model parameters and the observed information matrix was derived. An application to a real data set showed that the fit of the new model is superior to the fits of its main sub-model.

2.2 Generalizations Based on Lomax Distribution

Cordeiro *et al*. (2014) proposed a new generalized Lomax family of distributions called the generalized Lomax-G family, with two extra positive parameters to generalize any continuous baseline distribution. Some special models such as the Lomax-normal, Lomax–Weibull, Lomax-log-logistic and Lomax–Pareto distributions were discussed. Some mathematical properties of the new generator including ordinary and incomplete moments, quantile and generating functions mean and median deviations, distribution of the order statistics and some entropy measures were presented. They discussed the estimation of the model parameters using the method of maximum likelihood. They also proposed a magnification process based on the marginal Lomax-exponential distribution. They also defined a log-Lomax–Weibull regression model for censored data. The importance of the new generator was illustrated by three real data sets.

Gupta *et al*. (2015) introduced a new statistical distribution constructed by composition of the cumulative density functions (cdf) of Frechet probability distribution and Lomax probability distribution which was named as the Lomax-Frechet distribution. It may have wider applicability in engineering, order statistics and other fields as it has more number of parameters. They derived expressions for its moments, characteristic function, hazard rate function and survival function. They plot some graphs for its probability density function (pdf) using the software ‘Mathematica’ and also investigated the variation of the skewness and kurtosis and discuss estimation by the method of maximum-likelihood.

Gupta *et al*. (2016) developed a new technique for constructing a family of new statistical distributions by combining the cdf of two known statistical distributions. They defined and studied a new distribution by combining the cdf of Lomax and Gumbel distributions and named it as Lomax-Gumbel distribution. This distribution has four parameters, two from Lomax and two from Gumbel and was found to be more flexible compared to its constituent distributions. They derived explicit expressions for the moments, characteristic function, hazard rate function, and survivor function for the Lomax-Gumbel distribution. They also used the software ‘Mathematica’ to plot some graphs for its pdf which show the effect of variation of different parameters occurring in the definition and investigated the variation of the hazard rate function. The parameters of the distribution were estimated by the method of maximum-likelihood.

2.3 Generalizations of Other Distributions

There are several generalizations of some continuous probability distributions most of which have been found to perform better than the classical ones. These among others include the following:

Nadarajah and Eljabri (2013) studied the generalized Pareto (GP) distribution which is a popular model for extreme values by considering the recent work of Papastathopoulos and Tawn (2013), they proposed some generalizations of the GP distribution for improved modeling. They pointed out that Papastathopoulos and Tawn's generalizations are in fact not new and then proposed a tractable generalization of the GP distribution. For the latter generalization, they provided a comprehensive treatment of the mathematical properties, estimated parameters by the method of maximum likelihood and provided the observed information matrix. The proposed model was shown to give a better fit for the real data set used.

Mahmoudi (2011), introduced the beta generalized Pareto (BGP) distribution to extend the generalized Pareto (GP) distribution and they derived important properties of the new distribution and obtained closed-form expressions for its moments. Applications of the BGP distribution to three real data sets were given to show that the new distribution provides consistently better fits than the Gamma Pareto, Beta Pareto, Weibull and Pareto distributions.

Bourguignon *et al*. (2014) also developed a new family of distributions called the Weibull generalized (Weibull-G) family of distributions. They derived general mathematical properties of the new Weibull family of distributions. They provided some of this structural properties such as the ordinary and incomplete moments, Quantile function and order statistics. This study also proposed that for any Weibull-G family distributions, the estimation of model parameters should be done by method of maximum likelihood estimation. They also proposed that for any baseline distribution G, their methodology should be adopted.

Merovci and Elbatal (2015) used the methodology by Bourguignon *et al*., (2014) to obtain the Weibull-Rayleigh (WR) distribution. They derived expressions for the moments and the moments generating functions. Estimation of parameters of their model was done by the method of maximum likelihood estimation. Their model performance was compared to some baseline distributions such as Beta-Weibull, Exponentiated-Weibull and the standard Weibull distribution. The results of this comparison using the likelihood ratio statistics showed that the Weibull-

Rayleigh distribution can be used quite effectively to provide better fit than the Weibull distribution.

Oguntunde *et al*. (2015) also introduced the Weibull-Exponential distribution (WED) using the generator by Bourguignon *et al*., (2014). They also derived explicit expression for some of its basic mathematical properties like moments, moment generating function, reliability analysis, limiting behavior and order statistics. The method of maximum likelihood estimation was proposed for estimating its parameters and real life applications were provided to illustrate the flexibility of the new distribution over the standard exponential distribution.

Afify *et al*. (2016), introduced a four parameter distribution named the Weibull-Frechet distribution using the family by Bourguignon *et al*. (2014). They studied some of its mathematical and statistical properties. The applications of the new distribution with some baseline distributions showed that it is more fitted compared to kumaraswamy-Frechet (*KFr*), exponentiated-Frechet (*EFr*), Beta-Frechet (*BFr*), gamma extended Frechet (*GEFr*), transmuted Marshall-Olkin-Frechet (*TMOFr*) and Frechet (*Fr*) distributions.

Cordeiro *et al.* (2015), proposed a new generalized Weibull family of distributions with two extra positive parameters, the family can be express as a mixture of linear combination of exponentiated G densities function. This family was able to extend any continuous probability distribution and also provide explicit expressions for the ordinary and incomplete moments, generating function, mean deviations, probability weighted moments, entropies, reliability and distributions of order statistics. The work proposed that, the parameters of this distributions family can be estimated by the method of maximum likelihood estimation (MLEs).

CHAPTER THREE: RESAERCH METHODOLOGY

3.1 The Lomax-Kumaraswamy Distribution.

Here, we derived the cdf and pdf of the Lomax-Kumaraswamy distribution using the Lomax generator proposed by Cordeiro *et al.* (2014). According to them, the cumulative distribution function (cdf) and the probability density function (pdf) of the Lomax-G is defined for any continuous probability distribution as

$$F(x) = \int_{0}^{-\log[1-G(x)]} \alpha\beta^{\alpha} \frac{dt}{(\beta+t)^{\alpha+1}} \tag{3.1}$$

where G(x) is the cdf of any continuous distribution to be generalized, and $\alpha>0$ and $\beta>0$ are the two additional parameters responsible for the scale and shape of the distribution respectively,

The above integral yields

$$F(x) = 1 - \left\{ \frac{\beta}{\beta - \log[1-G(x)]} \right\}^{\alpha} \tag{3.2}$$

and

$$f(x) = \alpha\beta^{\alpha} \frac{g(x)}{[1-G(x)]\{\beta - \log[1-G(x)]\}^{\alpha+1}}, \tag{3.3}$$

respectively,

where $g(x) = G'(x)$ is the pdf of any continuous probability distribution to be generalized. Now the cdf and pdf of the Kumaraswamy distribution are given in equations (1.1) and (1.2) substituting the two equations in (3.2) and (3.3) and simplifying, we obtain the cdf and pdf of the Lomax-Kumaraswamy distribution as:

$$F(x) = 1 - \frac{\beta^{\alpha}}{\left[\beta - \log\left(1 - x^{a}\right)^{b}\right]^{\alpha}}; \quad 0 < x < 1 \tag{3.4}$$

and

$$f(x) = \alpha\beta^{\alpha} ab x^{a-1}\left(1 - x^{a}\right)^{-1}\left(\beta - \log\left(1 - x^{a}\right)^{b}\right)^{-(\alpha+1)}; \quad 0 < x < 1 \tag{3.5}$$

respectively,

for $a, b, \alpha, \beta > 0$, where $a, b, and \beta$ are the shape parameters, while α is the scale parameter.

Hence equation (3.4) and (3.5) are the cdf and pdf of the Lomax-Kumaraswamy distribution.

Model validity check

Recall that for any continuous probability distribution, a valid probability density function has the following properties:

1. $f(x) \geq 0$ (3.6)
2. $\int_{-\infty}^{\infty} f(x)dx = 1$ (3.7)
3. $P(a < X < b) = \int_{a}^{b} f(x)dx$ (3.8)

Proof

Considering the pdf of the Lomax-Kumaraswamy distribution from equation (3.5)

$$\int_0^1 f(x)dx = \int_0^1 \alpha\beta^{\alpha} ab\, x^{a-1}\left(1-x^a\right)^{-1}\left(\beta-\log\left(1-x^a\right)^b\right)^{-(\alpha+1)} dx \tag{3.9}$$

Using integration by substitution in equation (3.9)

Let

$$u = \beta - \log\left(1-x^a\right)^b \tag{3.10}$$

New Limit of u as $x \to 0$, $u = \beta - \log\left(1-0^a\right)^b$ (3.11)

$$= \beta - 0 = \beta \tag{3.12}$$

New Limit of u as $x \to 1$, $u = \beta - \log\left(1-1^a\right)^b$ (3.13)

$$= \beta + \infty = \infty \tag{3.14}$$

$$\frac{du}{dx} = \frac{abx^{a-1}}{\left(1-x^a\right)} \tag{3.15}$$

$$dx = \frac{\left(1-x^a\right)du}{abx^{a-1}} \tag{3.16}$$

Substituting the new limits of u and *dx* in (3.9) and simplifying the resulting expression, we obtained

$$= \int_{\beta}^{\infty} \alpha\beta^{\alpha} ab\, x^{a-1}\left(1-x^a\right)^{-1} u^{-(\alpha+1)} \frac{\left(1-x^a\right)du}{abx^{a-1}} \tag{3.17}$$

$$=\alpha\beta^{\alpha}\int_{\beta}^{\infty}u^{-(\alpha+1)}du \tag{3.18}$$

$$=-\beta^{\alpha}\left[u^{-\alpha}\right]_{\beta}^{\infty} \quad =\beta^{\alpha}\left[-\infty^{-\alpha}+\beta^{-\alpha}\right] \tag{3.19}$$

$$=\frac{-\beta^{\alpha}}{\infty^{\alpha}}+\frac{\beta^{\alpha}}{\beta^{\alpha}} \tag{3.20}$$

$$=0+1=1$$

Graphical analysis of the pdf and cdf.

Given some values for the parameters *a*, *b*, *α and β,* we provide some possible shapes for the pdf and the cdf of the LKUD as shown in figure 3.1 and 3.2 below:

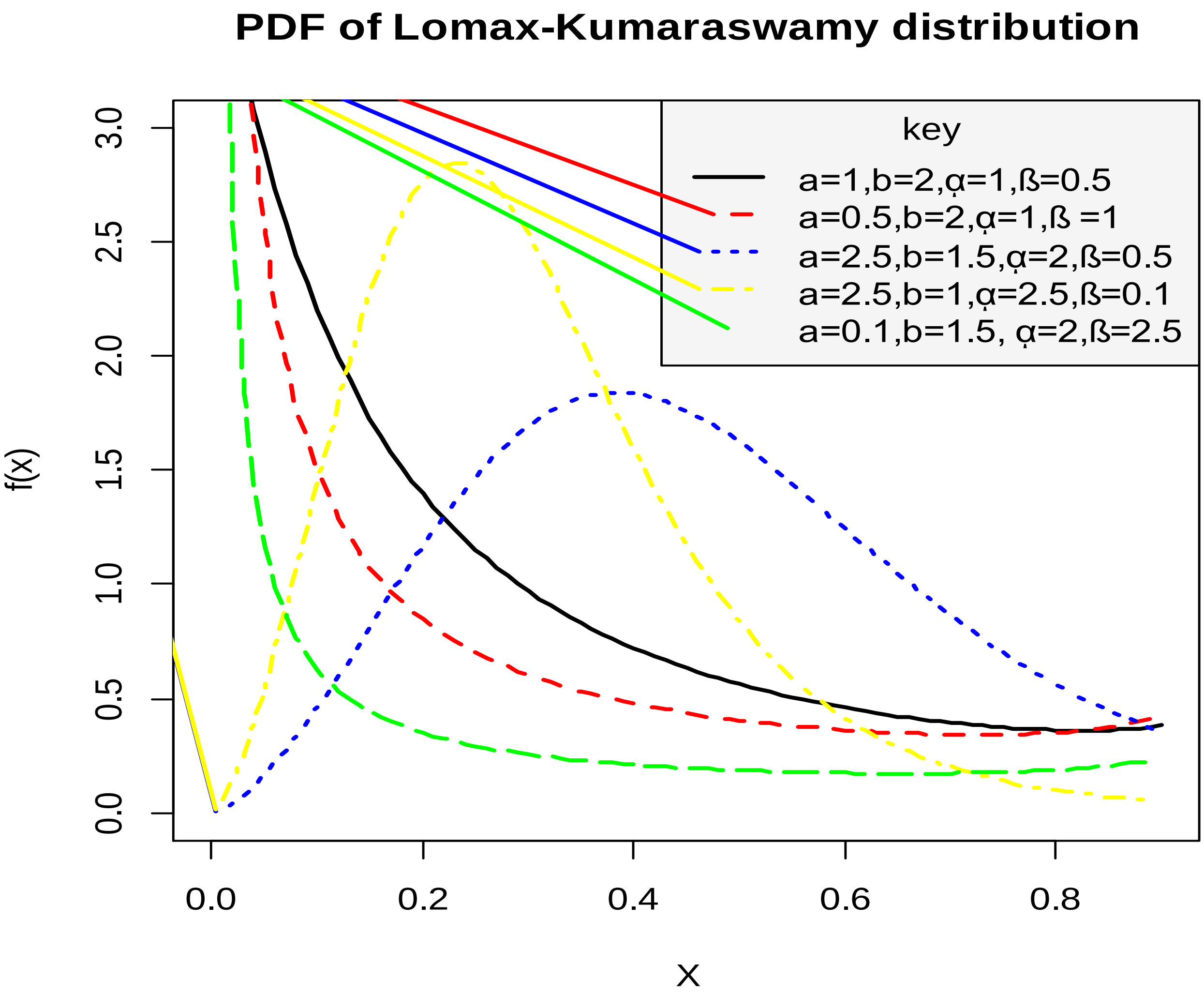

Fig. **3.1**: PDF plot of the LKuD for different values of the parameters.

Figure **3.1** indicates that the LKuD distribution has various shapes such as asymmetrical and right-skewed (positively skewed) shapes. This means that the distribution can be useful for modeling datasets with different shapes.

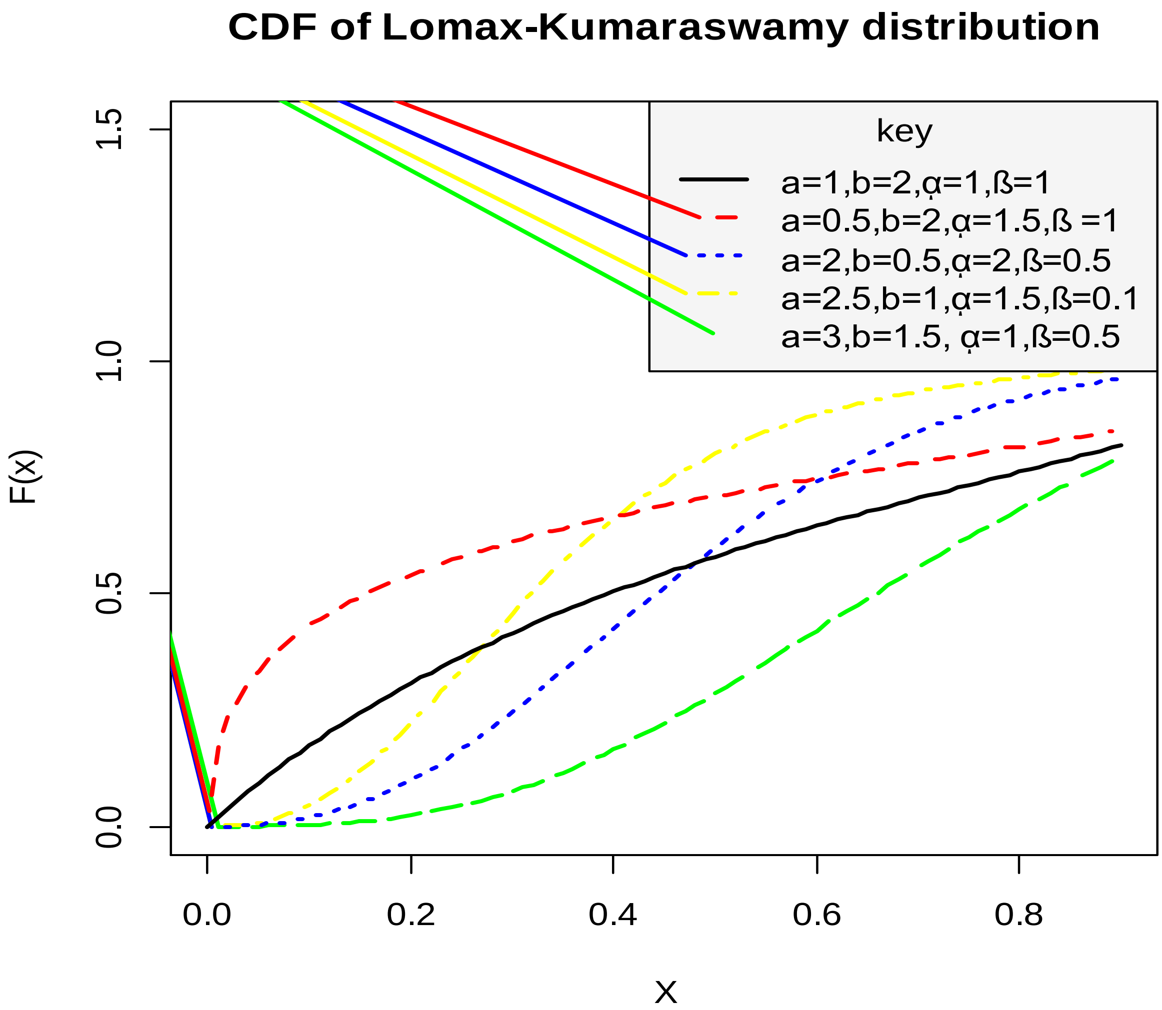

Fig. **3.2**: CDF plot of the LKuD for different values of the parameters.

From the above cdf plot, the cdf increases when *X* increases, and approaches 1 when *X* becomes large, as expected.

3.2 Some Properties of the Proposed Distribution

3.2.1 Moments

Moments are the constants of a population which are used to study the various characteristics of a random variable such as mean, variance, skewness and kurtosis of a distribution.

Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu_n' = E\left[X^n\right] = \int_{-\infty}^{\infty} x^n f(x)dx \tag{3.21}$$

Where *f(x)* is the pdf of the Lomax-Kumaraswamy distribution is given in equation (3.5).

$$\mu_n' = E\left[X^n\right] = \int_{0}^{1} x^n f(x)dx \tag{3.22}$$

$$= \sum_{l=0}^{\infty} W_{i,j,k} ab \int_0^1 x^{n+a-1}\left(1-x^a\right)^{bl-1} dx \tag{3.23}$$

Recall,

$$f(x) = \alpha\beta^{\alpha} ab x^{a-1}\left(1-x^a\right)^{-1}\left(\beta - \log\left(1-x^a\right)^b\right)^{-(\alpha+1)}$$

$$= \alpha\beta^{\alpha} ab x^{a-1}\left(1-x^a\right)^{-1}\left(\beta\left(1-\beta^{-1}\log\left(1-x^a\right)^b\right)\right)^{-(\alpha+1)} \tag{3.24}$$

$$= \alpha\beta^{\alpha} ab x^{a-1}\left(1-x^a\right)^{-1}\beta^{-(\alpha+1)}\left(1-\beta^{-1}\log\left(1-x^a\right)^b\right)^{-(\alpha+1)} \tag{3.25}$$

Let

$$A = \left(1-\beta^{-1}\log\left(1-x^a\right)^b\right)^{-(\alpha+1)} \tag{3.26}$$

Using the generalized binomial theorem on *A* above, we have:

$$\left(1-\beta^{-1}\log\left(1-x^a\right)^b\right)^{-(\alpha+1)} = \sum_{i=0}^{\infty}\frac{\Gamma(\alpha+i+1)}{j!\Gamma(\alpha+1)}\beta^{-i}\left(\log\left(1-x^a\right)^b\right)^i \tag{3.27}$$

Now, consider the following formula which holds for *i≥1*, using power series expansion which was used by El-Sherpieny and Ahmed (2014)

we can write the last term in (3.27) above as

$$\left(\log\left(1-x^{a}\right)^{b}\right)^{i} = \sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\left[\left(1-x^{a}\right)^{b}\right]^{l} \tag{3.28}$$

Where for (for j≥0) $P_{j,0}=1$ and (for k=1,2,…..)

$$P_{j,k} = k^{-1}\sum_{m=1}^{k}(-1)^{m}\frac{\left[m(j+1)-k\right]}{(m+1)}P_{j,k-m} \tag{3.29}$$

Combining equation (3.27) and (3.28) and inserting the above power series in equation (3.25), we have:

$$f(x)=\alpha\beta^{\alpha}ab\,x^{a-1}\left(1-x^{a}\right)^{-1}\beta^{-(\alpha+1)}\sum_{i=0}^{\infty}\frac{\Gamma(\alpha+i+1)}{j!\,\Gamma(\alpha+1)}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\left(1-x^{a}\right)^{bl} \tag{3.30}$$

$$=ab\alpha\beta^{-1}\sum_{i=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{\Gamma(\alpha+i+1)\beta^{-i}}{j!\,\Gamma(\alpha+1)}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\,x^{a-1}\left(1-x^{a}\right)^{bl-1} \tag{3.31}$$

$$=\alpha\beta^{-1}\sum_{i=0}^{\infty}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{\Gamma(\alpha+i+1)\beta^{-i}}{j!\,\Gamma(\alpha+1)}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}ab\,x^{a-1}\left(1-x^{a}\right)^{bl-1} \tag{3.32}$$

$$=\sum_{l=0}^{\infty}w_{i,j,k}ab\,x^{a-1}\left(1-x^{a}\right)^{bl-1} \tag{3.33}$$

Where

$$w_{i,j,k} = \alpha\beta^{-1}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{\Gamma(\alpha+i+1)\beta^{-i}}{j!\,\Gamma(\alpha+1)}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k} \tag{3.34}$$

Hence,

$$\mu_{n}^{'} = E\left[X^{n}\right]=\int_{0}^{1}x^{n}f(x)dx \tag{3.35}$$

$$=\sum_{l=0}^{\infty} w_{i,j,k}ab\int_0^1 x^{n+a-1}\left(1-x^a\right)^{bl-1}dx \qquad (3.36)$$

Recall for the Kumaraswamy distribution;

$$E\left[X^r\right]=\int_0^1 x^r f(x)dx \qquad (3.37)$$

Hence, this implies that

$$\mu_n^{'}=E\left[X^n\right]=\int_0^1 x^n f(x)dx$$

$$=ab\int_0^1 x^{r+a-1}\left(1-x^a\right)^{b-1}dx=b\beta\left(\tfrac{r}{a}+1,b\right) \qquad (3.38)$$

$$=\sum_{l=0}^{\infty} w_{i,j,k}ab\int_0^1 x^{n+a-1}\left(1-x^a\right)^{bl-1}dx \qquad (3.39)$$

$$=\sum_{l=0}^{\infty} w_{i,j,k}b\beta\left(\tfrac{n}{a}+1,bl\right) \qquad (3.40)$$

The Mean

The mean of the LKuD can be obtained from the *n*th moment of the distribution when *n=1* as follows:

$$\mu_n^{'}=E\left[X^n\right]=\sum_{l=0}^{\infty} w_{i,j,k}b\beta\left(\tfrac{n}{a}+1,bl\right)$$

$$\mu_1^{'}=E\left[X^1\right]=\sum_{l=0}^{\infty} w_{i,j,k}b\beta\left(\tfrac{1}{a}+1,bl\right) \qquad (3.41)$$

Also the second moment of the LKuD is obtained from the *n*th moment of the distribution when *n=2* as

$$E\left[X^{2}\right]=\sum_{l=0}^{\infty} w_{i,j,k} b\beta\left(\tfrac{2}{a}+1, bl\right) \tag{3.42}$$

The Variance

The n^{th} central moment or moment about the mean of *X*, say μ_n, can be obtained as

$$\mu_n = E\left[X-\mu_1^{'}\right]^{n}=\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}\mu_1^{'i}\mu_{n-i}^{'} \tag{3.43}$$

The variance of *X* for *LKuD* is obtained from the central moment when *n=2,* that is,

$$Var(X)=E[X^{2}]-\{E[X]\}^{2} \tag{3.44}$$

$$=\sum_{l=0}^{\infty} w_{i,j,k} b\beta\left(\tfrac{2}{a}+1, bl\right)-\left\{\sum_{l=0}^{\infty} w_{i,j,k} b\beta\left(\tfrac{1}{a}+1, bl\right)\right\}^{2} \tag{3.45}$$

The variation, skewness and kurtosis measures can also be calculated from the non-central moments using some well-known relationships.

3.2.2 Moment Generating Function (mgf)

This is a way in which the distribution of a random variable is often characterized in terms of its moments, a real function whose derivatives at zero are equal to the moments of the random variable, that is for any real number say k, the k^{th} derivative of $M_X(t)$ evaluated at $t = 0$ is the k^{th} moment $\mu_k^{'}$ of *X*.

The mgf of a random variable *X* can be obtained by

$$M_x(t)=E\left[e^{tx}\right]=\int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{3.46}$$

Recall that by power series expansion,

$$e^{tx}=\sum_{n=0}^{\infty}\frac{(tx)^{n}}{n!}=\sum_{n=0}^{\infty}\frac{t^{n}}{n!}x^{n} \tag{3.47}$$

Using the result in equation (3.47) and simplifying the integral in (3.46), therefore we have;

$$M_{x}(t)=E\left[e^{tx}\right]=\sum_{n=0}^{\infty}\frac{(tx)^{n}}{n!}=\sum_{n=0}^{\infty}\frac{t^{n}}{n!}\int_{-\infty}^{\infty}x^{n}f(x)dx \tag{3.48}$$

$$=\sum_{n=0}^{\infty}\frac{(tx)^{n}}{n!}=\sum_{n=0}^{\infty}\frac{t^{n}}{n!}\mu_{n}^{'} \tag{3.49}$$

$$M_{x}(t)=\sum_{n=0}^{\infty}\frac{t^{n}}{n!}\mu_{n}^{'}=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{t^{n}}{n!}w_{i,j,k}bB\left(\tfrac{n}{a}+1,bl\right) \tag{3.50}$$

Where *n* and *t* are constants, *t* is a real number and μ_{n}' denotes the n^{th} ordinary moment of *X*.

3.2.3 Characteristics Function

The characteristics function of any real-valued random variable completely defines its probably distribution. It is particularly useful for generating moments, characterization of distributions and in analysis of linear combination of independent random variables.

The characteristics function of a random variable *X* is given by;

$$\varphi_{x}(t)=E\left[e^{itx}\right]=E\left[\cos(tx)+i\sin(tx)\right]=E\left[\cos(tx)\right]+E\left[i\sin(tx)\right] \tag{3.51}$$

Recall from power series expansion that

$$\cos(tx)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}x^{2n} \tag{3.52}$$

$$E\left[\cos(tx)\right]=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}\mu'_{2n} \tag{3.53}$$

And also

$$\sin(tx)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}x^{2n+1} \tag{3.54}$$

$$E\left[\sin(tx)\right]=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\mu'_{2n+1} \tag{3.55}$$

Simple algebra and power series expansion proves that

$$\phi_x(t)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}\mu'_{2n}+i\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\mu'_{2n+1} \tag{3.56}$$

$$\varphi_x(t)=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{t^{2n}}{(2n)!}w_{i,j,k}bB\left(\tfrac{2n}{a}+1,bl\right)+\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}w_{i,j,k}bB\left(\tfrac{2n+1}{a}+1,bl\right) \tag{3.57}$$

Where μ'_{2n} and $\mu'_{2n+1}$are the moments of X for $n=2n$ and $n=2n+1$ respectively.

3.2.4 Quantile Function

The Quantile function n is used to partition probability distributions. It is also used to obtain the median of a distribution and for simulation of random numbers. Quantile Function is obtained by taking the inverse of the cumulative distribution function.

Let $F(x)=u$ such that the quantile function $Q(u)$ is given as

$$F(Q(u)) = u \Rightarrow Q(u) = F^{-1}(u) \tag{3.58}$$

for $0 < u < 1$.

Taking *F(x)* to be the cdf of the Lomax-Kumaraswamy distribution and inverting it as above give the quantile function as follows.

$$F(x) = 1 - \frac{\beta^{\alpha}}{\left[\beta - \log\left(1 - x^{a}\right)^{b}\right]^{\alpha}}$$

Inverting, *F(x) = u*

$$F(x) = 1 - \frac{\beta^{\alpha}}{\left[\beta - \log\left(1 - x^{a}\right)^{b}\right]^{\alpha}} = u \tag{3.59}$$

Simplifying equation (3.58) above, we obtained:

$$1 - u = \frac{\beta^{\alpha}}{\left[\beta - \log\left(1 - x^{a}\right)^{b}\right]^{\alpha}} \tag{3.60}$$

$$\left[\beta - \log\left(1 - x^{a}\right)^{b}\right]^{\alpha} = \frac{\beta^{\alpha}}{1-u} \tag{3.61}$$

$$\beta - \log\left(1 - x^{a}\right)^{b} = \left(\frac{\beta^{\alpha}}{1-u}\right)^{\frac{1}{\alpha}} \tag{3.62}$$

$$\beta - \log\left(1 - x^{a}\right)^{b} = \frac{\beta}{\left(1-u\right)^{\frac{1}{\alpha}}} \tag{3.63}$$

$$\beta - \frac{\beta}{\left(1-u\right)^{\frac{1}{\alpha}}} = \log\left(1 - x^{a}\right)^{b} \tag{3.64}$$

Take the exponential of both sides of equation (3.64)

$$\exp\left\{\beta-\frac{\beta}{(1-u)^{\frac{1}{\alpha}}}\right\}=\left(1-x^{a}\right)^{b} \tag{3.65}$$

$$1-\left(\exp\left\{\beta-\frac{\beta}{(1-u)^{\frac{1}{\alpha}}}\right\}\right)^{\frac{1}{b}}=x^{a} \tag{3.66}$$

$$Q(u)=X_{q}=\sqrt[a]{1-\left(\exp\left\{\beta-\frac{\beta}{(1-u)^{\frac{1}{\alpha}}}\right\}\right)^{\frac{1}{b}}} \tag{3.67}$$

Hence, the median of *X* from the *LKUD* is simply $X_{\frac{1}{2}}=Q(1/2)$ is obtained by setting *u*=0.5 in equation (3.67).

3.2.5 Skewness and Kurtosis

Skewness and kurtosis are two commonly listed values used in descriptive statistics function, which gives insights into the shape of the distribution. Skewness is a measure of symmetry of a distribution. A symmetrical data set will have a skewness of 0. So a normal distribution will have a skewness of 0. It essentially measures the relative size of the two tails.

kurtosis was originally thought to measure the peakedness of a distribution, is a measure of combined sizes of the two tails which measures the amount of probability in the tails, the value is always compared to kurtosis of a normal distribution which is 3.

In this dissertation, the quantile based measures of skewness and kurtosis will be employed due to non-existence of the classical measures in some cases. The Bowley's measure of skewness (Kennedy and Keeping, 1962) based on quartiles is given by;

$$SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{3.68}$$

And the Moors (1998) kurtosis is on octiles and is given by;

$$KT = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + \left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{1}{4}\right)} \tag{3.69}$$

3.3 Reliability Analysis

3.3.1 Survival Function

Survival function is the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1-F(x) \tag{3.70}$$

Where $F(x)$ is the *cdf* of the proposed Lomax-Kumaraswamy distribution.

$$F(x) = 1 - \frac{\beta^{\alpha}}{\left[\beta - \log\left(1-x^{a}\right)^{b}\right]^{\alpha}}$$

$$S(x) = 1 - \left\{1 - \frac{\beta^{\alpha}}{\left[\beta - \log\left(1-x^{a}\right)^{b}\right]^{\alpha}}\right\} \tag{3.71}$$

$$=\frac{\beta^{\alpha}}{\left[\beta-\log\left(1-x^{a}\right)^{b}\right]^{\alpha}} \tag{3.72}$$

Below is a plot of the survival function at chosen parameter values in figure **3.3**

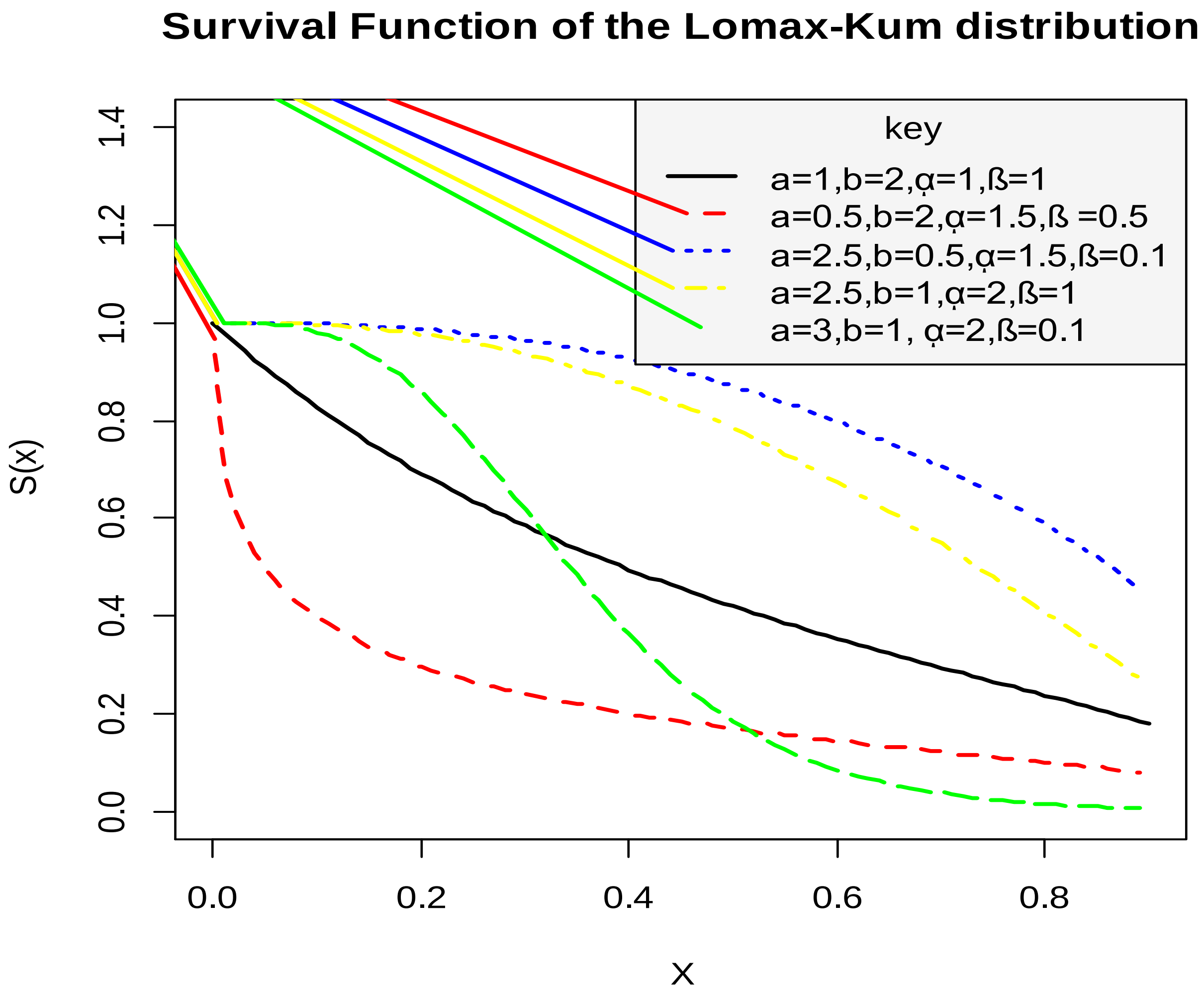

Figure 3.3: The survival function of the LKuD for different values of the parameters.

Figure 3.3 revealed that the probability of survival for any random variable following a Lomax-Kumaraswamy distribution decreases as the values of the random variable increases, that is, as time goes on, probability of life decreases. This implies that the Lomax-Kumaraswamy distribution can be used to model random variables whose survival rate decreases as their age grows.

3.3.2 Hazard Function

Hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x) = \frac{f(x)}{1-F(x)} \tag{3.73}$$

Where *f(x)* and *F(x)* the pdf and cdf of the proposed Lomax-Kumaraswamy distribution are given as

$$f(x) = \alpha\beta^{\alpha} ab x^{a-1}\left(1-x^{a}\right)^{-1}\left(\beta-\log\left(1-x^{a}\right)^{b}\right)^{-(\alpha+1)}$$

and

$$F(x) = 1-\frac{\beta^{\alpha}}{\left[\beta-\log\left(1-x^{a}\right)^{b}\right]^{\alpha}}$$

respectively.

Substituting for *f(x)* and *F(x)* and simplifying gives

$$h(x) = \frac{\alpha\beta^{\alpha} ab x^{a-1}\left(1-x^{a}\right)^{b-1}\left(\beta-\log\left(1-x^{a}\right)^{b}\right)^{-(\alpha+1)}\left[\beta-\log\left(1-x^{a}\right)^{b}\right]^{\alpha}}{\left(1-x^{a}\right)^{b}\beta^{\alpha}} \tag{3.74}$$

$$= \frac{ab\alpha x^{a-1}\left(\beta-\log\left(1-x^{a}\right)^{b}\right)^{-1}}{\left(1-x^{a}\right)} \tag{3.75}$$

The following is a plot of the hazard function at chosen parameter values in figure **3.4**

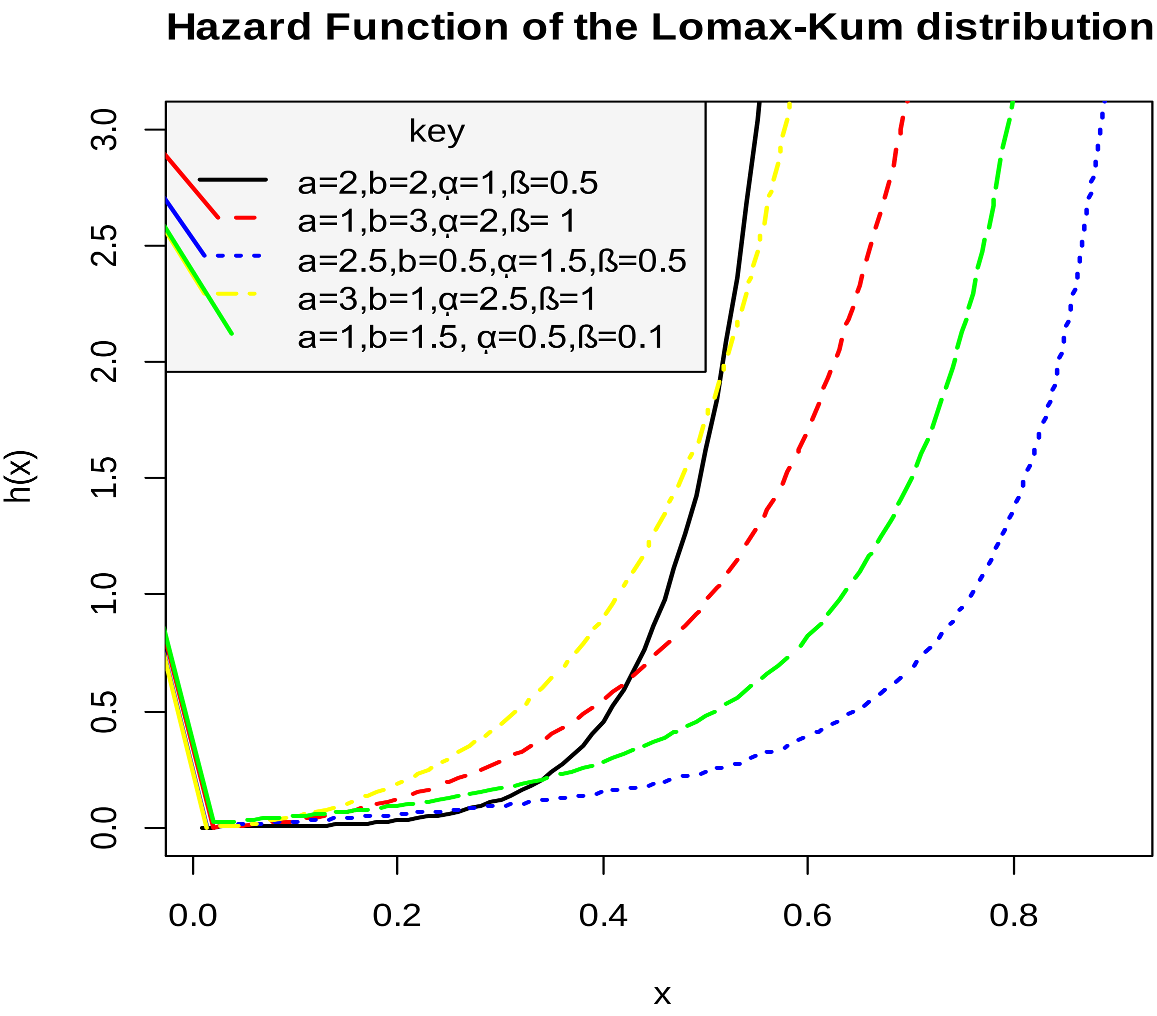

Figure 3.4: The hazard function of the LKuD for different values of the parameters.

Figure 3.4 above revealed that the probability of failure for any random variable following a Lomax-Kumaraswamy distribution increases as the values of the random variable increases, that is, as time goes on the probability of a component to fail or an individual to die increases. This implies that the Lomax-Kumaraswamy distribution can be used to model random variables whose failure rate increases as their age grows.

3.4 Order Statistics

Sample values such as the smallest, largest, or middle observation from a random sample provide important information. For example, the highest rainfall, flood or minimum temperature recorded during past years might be useful when planning for future emergencies. Let $X(1)$ denote the smallest of $\{X1, X2, \dots , Xn\}$, $X(2)$ denote the second smallest of $\{X1, X2, \dots , Xn\}$, and similarly $X(\mathrm{i})$ denote the ith smallest of $\{X1, X2, \dots , Xn\}$.Then the random variables $X(1), X(2), \dots , X(n)$, called the order statistics of the sample $X1, X2, \dots , Xn$, has probability density function of the ith order statistic, $X(\mathrm{i})$, as:

$$f(i{:}n) = \frac{n!}{(i-1)!\,(n-i)!} f(x) F(x)^{i-1} [1 - F(x)]^{n-i} \tag{3.76}$$

Where *f(x)* and *F(x)* are the pdf and cdf of the proposed distribution respectively.

Using equations (3.4) and (3.5), the pdf of the a^{th} order statistics$X_{a:n}$, can be expressed from (3.76) as;

$$f_{a:n}(x) = \frac{n!}{(a-1)!(n-a)!} \sum_{k=0}^{n-a} (-1)^k \binom{n-a}{k} \left[\sum_{l=0}^{\infty} W_{i,j,k}\, ab\, x^{a-1} \left(1-x^a\right)^{b-1} \left[\left(1-x^a\right)^b \right]^{l-1} \right] * \left[\frac{\left[\beta - \log\left(1-x^a\right)^b \right]^{\alpha} - \beta^{\alpha}}{\left[\beta - \log\left(1-x^a\right)^b \right]^{\alpha}} \right]^{a+k-1} \tag{3.77}$$

Hence, the pdf of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the LKuD are given by;

$$f_{1:n}(x) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left[\sum_{l=0}^{\infty} W_{i,j,k}\, ab\, x^{a-1} \left(1-x^a\right)^{b-1} \left[\left(1-x^a\right)^b \right]^{l-1} \right] * \left[\frac{\left[\beta - \log\left(1-x^a\right)^b \right]^{\alpha} - \beta^{\alpha}}{\left[\beta - \log\left(1-x^a\right)^b \right]^{\alpha}} \right]^{k} \tag{3.78}$$

and

$$f_{n:n}(x)=n\left[\sum_{l=0}^{\infty}w_{i,j,k}abx^{a-1}\left(1-x^{a}\right)^{b-1}\left[\left(1-x^{a}\right)^{b}\right]^{l-1}\right]\left[\frac{\left[\beta-\log\left(1-x^{a}\right)^{b}\right]^{\alpha}-\beta^{\alpha}}{\left[\beta-\log\left(1-x^{a}\right)^{b}\right]^{\alpha}}\right]^{n-1} \tag{3.79}$$

respectively.

3.5 Estimation of Parameters

Let $X_{1,}$ - - -,X_n be a sample of size ‘n’ independently and identically distributed random variables from the LKuD with unknown parameters *α, β, a,* and *b* defined previously. The pdf of the LKuD is given as

$$f(x;\alpha,\beta,a,b)=\alpha\beta^{\alpha}abx^{a-1}\left(1-x^{a}\right)^{-1}\left(\beta-\log\left(1-x^{a}\right)^{b}\right)^{-(\alpha+1)} \tag{3.80}$$

The likelihood function is given by; $L\left(x_1,x_2,.....,x_n/\alpha,\beta,a,b\right)=\prod_{i=1}^{n}f(x_i;\alpha,\beta,a,b)$ (3.81)

$$L\left(x_i/\alpha,\beta,a,b\right)=\left(\alpha\beta^{\alpha}ab\right)^{n}\prod_{i=1}^{n}\left[x_i^{a-1}\left(1-x_i^{a}\right)^{-1}\left(\beta-\log\left(1-x_i^{a}\right)^{b}\right)^{-(\alpha+1)}\right] \tag{3.82}$$

$$L\left(x_i/\alpha,\beta,a,b\right)=\left(\alpha\beta^{\alpha}ab\right)^{n}\prod_{i=1}^{n}x_i^{a-1}\prod_{i=1}^{n}\left(1-x_i^{a}\right)^{-1}\prod_{i=1}^{n}\left(\beta-\log\left(1-x_i^{a}\right)^{b}\right)^{-(\alpha+1)} \tag{3.83}$$

Let the log-likelihood function be $l=\log L\left(x_i/\alpha,\beta,a,b\right)$ (3.84)

Therefore taking the natural logarithm of the likelihood function, we obtain

$$l=n\log\alpha+n\alpha\log\beta+n\log a+n\log b+(a-1)\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}\log\left(1-x_i^{a}\right)-(\alpha+1)\sum_{i=1}^{n}\log\left(\beta-\log\left(1-x_i^{a}\right)^{b}\right) \tag{3.85}$$

Differentiating *l* partially with respect to *α, β, a* and *b* respectively gives;

$$\frac{\partial l}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\left\{\frac{x_i^{a}\ln x_i}{\left(1-x_i^{a}\right)}\right\}+b(\alpha+1)\sum_{i=0}^{n}\left\{\frac{x_i^{a}\ln x_i}{\left(\beta-\log\left(1-x_i^{a}\right)^{b}\right)\left(1-x_i^{a}\right)}\right\} \tag{3.86}$$

$$\frac{\partial l}{\partial b}=\frac{n}{b}+(\alpha+1)\sum_{i=0}^{n}\left\{\frac{\log\left(1-x_i^{a}\right)}{\left(\beta-\log\left(1-x_i^{a}\right)^{b}\right)}\right\} \tag{3.87}$$

$$\frac{\partial l}{\partial \alpha}=\frac{n}{\alpha}+n\log\beta-\sum_{i=1}^{n}\log\left(\beta-\log\left(1-x_i^{a}\right)^{b}\right) \tag{3.88}$$

$$\frac{\partial l}{\partial \beta}=\frac{n\alpha}{\beta}-(\alpha+1)\sum_{i=0}^{n}\left\{\frac{1}{\left(\beta-\log\left(1-x_i^{a}\right)^{b}\right)}\right\} \tag{3.89}$$

The solution of the non-linear system of equations of $\frac{\partial l}{\partial \alpha}=0, \frac{\partial l}{\partial \beta}=0, \frac{\partial l}{\partial a}=0$ and $\frac{\partial l}{\partial b}=0$ will give the maximum likelihood estimates of parameters α,β,a and b . However, the solution cannot be obtained analytically except numerically with the aid of suitable statistical software. In this dissertation we used R package.

3.6 Model Selection Criteria and Data Sets

3.6.1. Model selection criteria

For us to assess these models, we will use some model selection criteria, the bench mark is the model with the lowest value for these statistics would be chosen as the best model to fit the data.

Akaike Information Criterion (AIC): This is the most popular information criterion. It compensates for the number of parameters in the model to encourage parsimony and estimates the relative kullback-Leibler (KL) distance of the likelihood function specified by a fitted candidate model from the unknown true likelihood function that generated the data.

$$AIC = -2ll + 2k \tag{3.90}$$

Where *ll* denotes the log-likelihood function evaluated at the maximum likelihood estimates, while k is the number of model parameters. As parameters richness is increased, log-likelihood is expected to decrease while the penalty term increases so the model with the lowest AIC will be the balance between parameter richness and the informativeness of additional parameters.

Bayesian Information Criterion (BIC*)*: Bayesian Information Criterion, the Schwartz (1978), approximates marginal log-likelihood for the candidate model a prior probability is set for each model Mi, and prior distributions are also set for the non zero coefficients in each model. We can use Bayes theorem, assuming that one and only one model with its associated priors is true,.

$$BIC = -2ll + k\log(n), \tag{3.91}$$

Where ll denotes the log-likelihood function evaluated at the *MLEs*, k is the number of model parameters and n is the sample size.

Consistent Akaike Information Criterion (CAIC*)*: Bozdogan (1987), extended the *AIC* known as the consistent *AIC*, because the *AIC* has often been criticized for lack of consistency.

$$CAIC = -2ll + \tfrac{2kn}{(n-k-1)} \tag{3.92}$$

Hannan-Quinn information criterion (HQIC): For a fixed number of samples the BIC tends to underestimate the dimension *n*, while the AIC tends to overestimate the number of sample of the necessary parameters. Hannan and Quinn (1979), developed the $HQIC$ which provides values between the AIC and BIC.

$$HQIC = -2ll + 2k\log\left[\log(n)\right] \tag{3.93}$$

3.6.2 Data sets

Data set I**:** This data is flood data with 20 observations measured in cubic meters per second (cm/s) was obtained from Dumonceaux and Antle (1973) and was used by Khan *et al.* (2016), as follows: 0.265 0.269 0.297 0.315 0.324 0.338 0.379 0.379 0.392 0.402 0.412 0.416 0.418 0.423 0.449 0.484 0.494 0.613 0.654 0.740.

Data set II: The second data set is on shape measurements of 48 rock samples in kilograms (kg) from a petroleum reservoir. Extracted from BP research image analysis was obtained from Katz, R. (1995) and was used by Javanshiri *et al*. (2015) as follows: 0.0903296 0.189651 0.228595 0.200071 0.280887 0.311646 0.176969 0.464125 0.148622 0.164127 0.231623 0.144810 0.179455 0.276016 0.438712 0.420477 0.183312 0.203654 0.172567 0.113852 0.191802 0.19753 0.163586 0.200744 0.117063 0.162394 0.153481 0.291029 0.133083 0.326635 0.253832 0.262651 0.122417 0.150944 0.204314 0.240077 0.225214 0.154192 0.328641 0.182453 0.167045 0.148141 0.262727 0.161865 0.341273 0.276016 0.230081 0.200447.

CHAPTER FOUR: ANALYSIS AND DISCUSSION

4.1 Introduction to Some Generalizations of Kumaraswamy Distribution

We have used three generalizations of Kumaraswamy distribution for the analysis to compare the performance of the Lomax-Kumaraswamy distribution to those of the three models as follows;

4.1.1 The Kumaraswamy-Kumaraswamy Distribution (KKuD)

The pdf of the KKuD according to El-Sherpieny and Ahmed (2014) is given as;

$$f(x;a,b,\alpha,\beta)=ab\alpha\beta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta-1}\left[1-\left(1-x^{\alpha}\right)^{\beta}\right]^{a-1}\left\{1-\left[1-\left(1-x^{\alpha}\right)^{\beta}\right]^{a}\right\}^{b-1};\ 0<x<1 \quad (4.1)$$

For $a,b,\alpha,\beta>0$.

4.1.2 The Transmuted Kumaraswamy Distribution (TKuD)

Khan *et al*. (2016) defined the pdf of the TKuD as follows;

$$f(x;\alpha,\theta,\lambda)=\alpha\theta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\theta-1}\left\{1+\lambda-2\lambda\left(1-\left(1-x^{\alpha}\right)^{\theta}\right)\right\};\ 0<x<1 \quad (4.2)$$

For $\alpha,\theta>0,$ and $-1\leq\lambda\leq1$.

4.1.3 The Kumaraswamy Distribution (KuD)

According to Kumaraswamy (1980), the pdf of the Kumaraswamy distribution is given by

$$f(x;a,b)=ab x^{a-1}\left(1-x^{a}\right)^{b-1};\quad 0<x<1 \quad (4.3)$$

For $a,b,>0$

4.2 Analysis and Discussion

The summary of the two data sets in chapter 3 is provided as follows

Table 4.1: Summary of the two data sets

Parameters	Data set I	Data set II
n	20	48
Minimum	0.265	0.0903
Q_1	0.3345	0.1623
Median	0.4070	0.1988
Q_3	0.4578	0.2627
Mean	0.4232	0.2181
Maximum	0.7400	0.4641
Variance	0.0157	0.0069
Skewness	1.0677	1.1694
Kurtosis	0.5999	1.1099

We also provide some histograms and densities for the two data sets as shown in Figures 4.1 and 4.2 respectively.

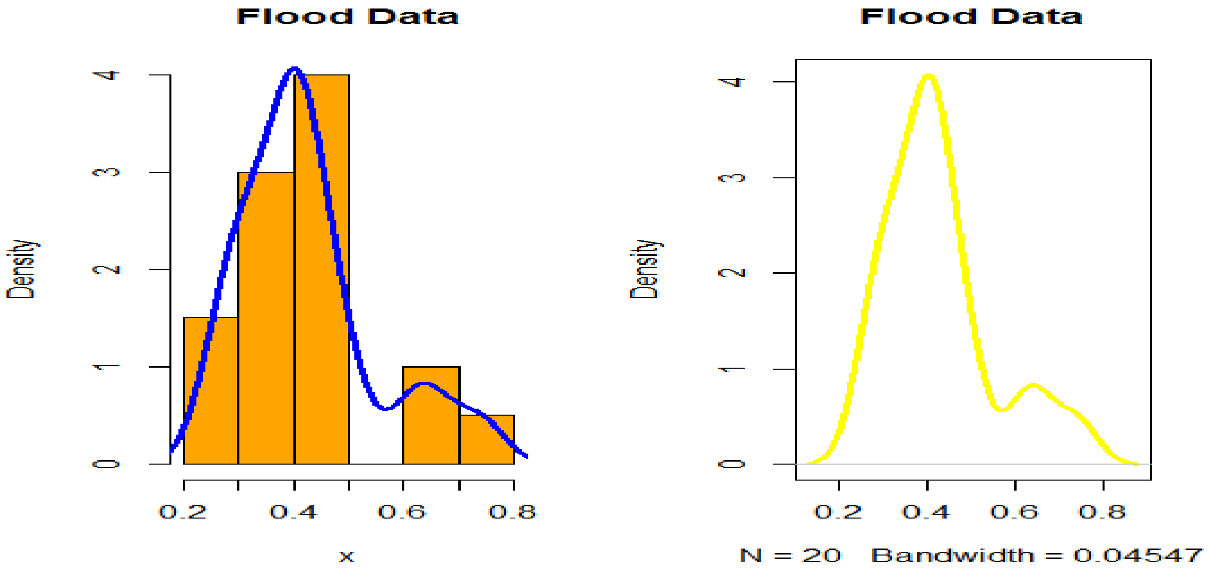

Figure 4.1: A histogram and density plot for the flood data (Data set I)

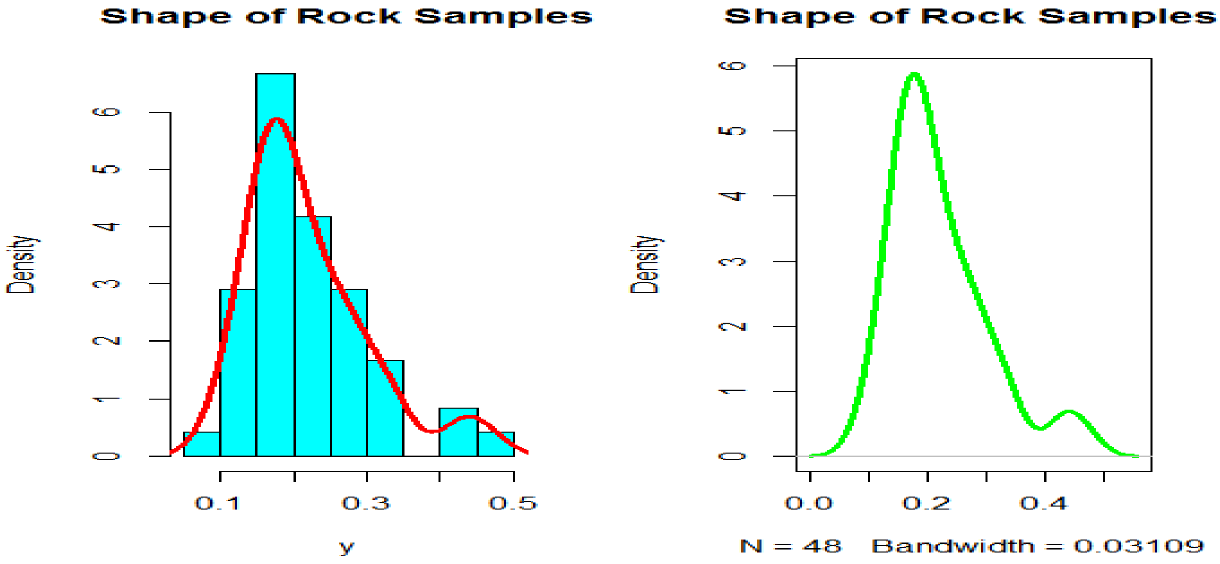

Figure 4.2: A Histogram and density plot for the rock sample data (Data set II)

From the descriptive statistics, histograms and densities shown above for the two data sets, we observed that both the first and second data sets are positively skewed and therefore suitable for distributions that are skewed to the right.

The estimates and performance measures of the above distributions are listed in tables 4.1 and 4.2 for data sets **I** and **II** respectively.

Table 4.2: Performance of the distribution based on data set I.

Distri-butions	Parameter estimates	-ll=(log-likelihood value)	*AIC*	*BIC*	*CAIC*	*HQIC*	Ranks of models performance
LKuD	$\hat{a}$=7.3061 $\hat{b}$=6.2384 $\hat{\alpha}$=0.7269 $\hat{\beta}$=0.0779	-15.9278	-23.8556	-26.6515	-21.1889	-30.9413	**2**
TKuD	$\hat{a}$=2.1658 $\hat{b}$=8.9877 $\hat{\lambda}$=0.4875	-14.1321	-22.2642	-24.3611	-21.142	-27.57	**3**
KuD	$\hat{a}$=2.7925 $\hat{b}$=10.2153	-12.8416	-21.6832	-23.0811	-20.9773	-25.2261	**4**
KKuD	$\hat{a}$=2.0168 $\hat{b}$=6.1956 $\hat{\alpha}$=3.3247 $\hat{\beta}$=1.8551	-16.3247	-24.6494	-27.4453	-21.9827	-31.7351	**1**

Table 4.2, Shows the parameter estimates for each of the four fitted distributions for the first data set (data set I), the table also provide the values of *-ll*, *AIC*, *BIC*, *CAIC* and *HQIC* of the fitted models.

The values in table 4.2 shows that the best performing model is the KKuD, while LKuD has better performance compared to the TKuD and KuD. It agrees with the fact that generalizing any continuous distribution provides a distribution with a better fit than the classical distribution in the sense that the LKuD distribution outperformed the classical KuD itself.

Table 4.3: Performance of the distributions based on data set II.

Distri-butions	Parameter estimates	-ll=(log-likelihood value)	*AIC*	*BIC*	*CAIC*	*HQIC*	Ranks based on models performance
LKuD	$\hat{a}$=4.6170 $\hat{b}$=9.1743 $\hat{\alpha}$=1.0974 $\hat{\beta}$=0.0081	-57.3044	-106.6088	-107.8838	-105.6786	-112.8038	**1**
TKuD	$\hat{a}$=1.8535 $\hat{b}$=2.3715 $\hat{\lambda}$=0.0799	-50.2138	-94.4276	-95.3839-	-93.882	-99.0739	**3**
KuD	$\hat{a}$=1.8978 $\hat{b}$=12.5398	-47.3652	-90.7304	-91.3679	-90.4637	-93.8279	**4**
KKuD	$\hat{a}$=9.2497 $\hat{b}$=1.9360 $\hat{\alpha}$=0.2917 $\hat{\beta}$=15.9876	-52.4279	-96.8558	-98.1308	-95.9256	-103.0508	**2**

Table 4.3 shows the parameter estimates for each of the four fitted distributions for the second data set (data set II), the table also provide the values of *-ll*, *AIC*, *BIC*, *CAIC* and *HQIC* of the fitted models.

The values in Table 4.3 indicate that the Lomax-Kumaraswamy distribution has better performance with the lowest values of *AIC*, *CAIC*, *BIC* and *HQIC*. The LKuD performed better than the KKuD, TKuD and KuD.

The following figures displayed the histogram and estimated densities of the fitted models for the two real life data sets used in this study.

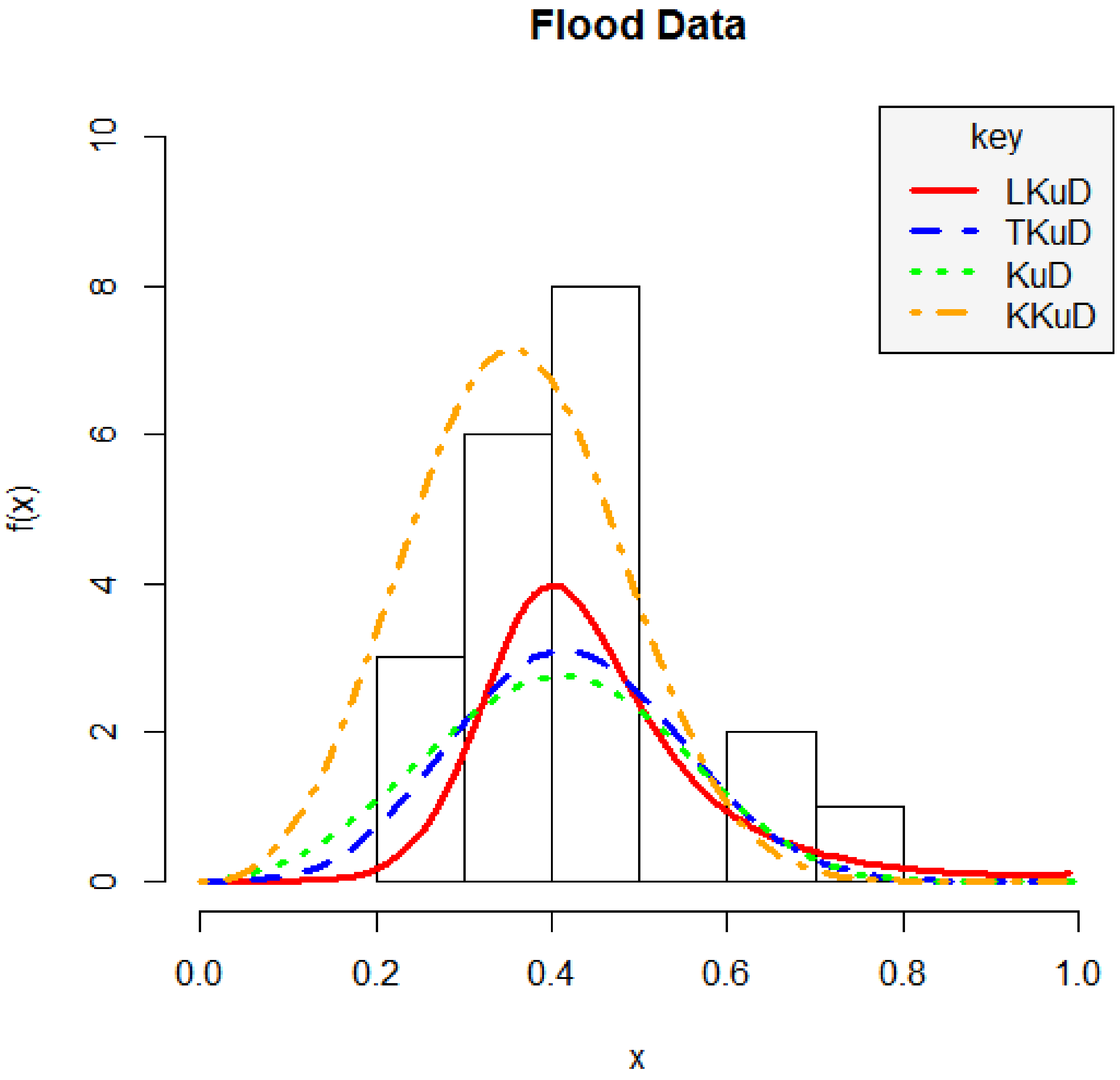

Figure 4.3: Histogram and estimated pdf plots of the Lomax-Kumaraswamy distributions (LKuD), transmuted Kumaraswamy distribution (TKuD), Kumaraswamy-Kumaraswamy distribution (KKuD) and the baseline Kumaraswamy distribution (KuD) for (data set I)

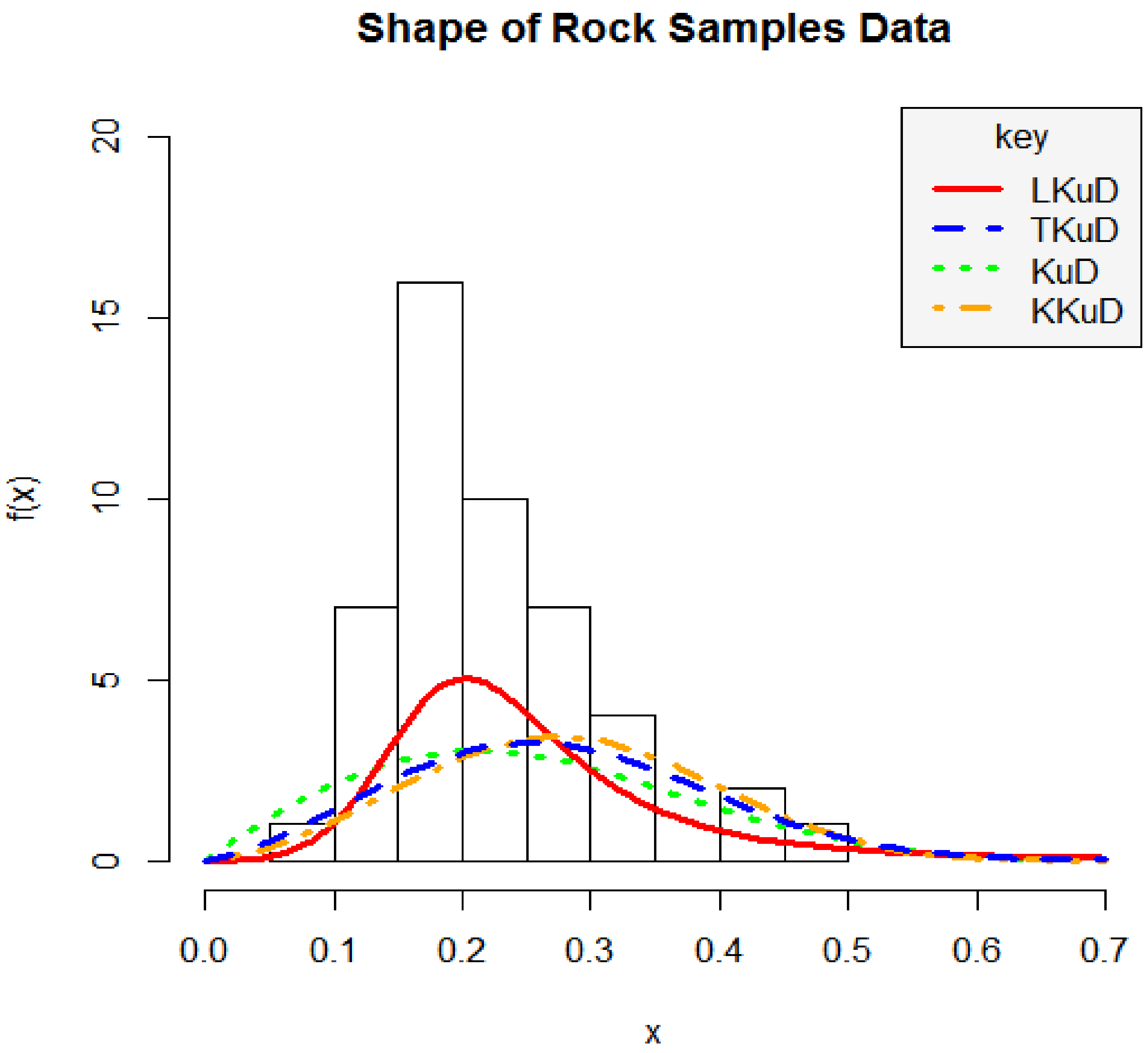

Figure 4.4: Histogram and estimated pdf plots of the Lomax-Kumaraswamy distribution (LKuD), transmuted Kumaraswamy distribution (TKuD), Kumaraswamy-Kumaraswamy distribution (KKuD) and the baseline Kumaraswamy distribution (KuD) for (data set II).

From figures 4.3 and 4.4, we can clearly see that the performance of the Lomax-Kumaraswamy distribution (LKuD), and the Kumaraswamy-Kumaraswamy (KKuD), remained consistent as the competing distributions that best fit the two data sets, followed by the Transmuted Kumaraswamy (TKuD), distribution while the baseline Kumaraswamy distribution (KuD) came last.

Having demonstrated earlier in table 4.2 and 4.3 with the values of their AIC, CAIC, BIC and HQIC, and we have a similar conclusion based on figure 4.3 and 4.4. The Lomax-Kumaraswamy distribution performs averagely better for the two data sets considered in this study

CHAPTER FIVE: SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 Summary

This dissertation proposed a distribution called Lomax-Kumaraswamy distribution. An obvious reason for generalizing a classical distribution is the fact that the generalization provides more flexibility to analyze real life data better than the classical distribution. We proposed some of its statistical properties like its ordinary moments, moment generating function, characteristics function, reliability analysis, quantile function and order statistics. The estimation of parameters has been done using the method of maximum likelihood estimation. The performance of the new distribution is illustrated by some applications to two real data sets. The results based on model selection criteria used and that of the histogram and the estimated densities of the distributions, showed that the new distribution (Lomax-Kumaraswamy distribution) performs better than the baseline Kumaraswamy distribution (KuD) and the transmuted-Kumaraswamy (TKuD), while it competed favuorably with the Kumaraswamy-Kumaraswamy distribution (KKuD) when the data sets are positively skewed.

5.2 Conclusion

In this dissertation, a new distribution has been proposed. Some statistical properties of the proposed distribution have been studied appropriately. The derivation of some expressions for its moments, moment generating function, characteristics function, survival function, hazard function, quantile function and ordered statistics have been provided. Some plots of the distribution revealed that it is a positively skewed distribution. The model parameters have been estimated using the method of maximum likelihood estimation. The implications of the plots for the survival function indicated that the Lomax-Kumaraswamy distribution could be used to model time or age-dependent events, where survival rate decreases with time or age.

5.3 Recommendations

The findings of this research recommend that the proposed distribution should be used to model positively skewed data sets with higher peak. It was also found that the distribution can be used to model age dependent events, systems and components.

5.4 Contribution to Knowledge

In this study, we have proposed a new distribution for modeling positively skewed data sets with high kurtosis. We also provided expressions for some of its properties which are useful for studying the shape characteristics of the distribution as well as the reliability functions which are useful in the field of engineering. The model parameters are also estimated clearly by using the method of Maximum Likelihood Estimation.

Finally, the adequacy of the new model has been tested in comparison with some existing distributions and we found it to be a better model when dealing with positively skewed data sets.

5.5 Suggestions for Further Research

i. We recommend that, further studies should be on the estimation of confidence intervals for the parameters of the proposed distribution.

ii. Also, researchers should estimate the parameters of the new distribution using either a different classical approach such as methods of moments, percentile based, method of least squares, maximum product of spacing method e.t.c., or a non-classical method for such as Bayesian approach under some priors and loss functions.

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