ISOMORPHISM CONDITIONS FOR THE SIMPLE MODULAR LIE ALGEBRAS OF BLOCK TYPE

by

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DEDICATED TO

Wife: Shola

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CHAPTER I

INTRODUCTION

The purpose of the dissertation is to discover some necessary isomorphism conditions for the simple Lie algebras of Block. If L is any simple Lie algebra of Block, we will seek to imbed L in a smallest restricted Lie algebra R'(L) which has the property that any restricted Lie algebra in which L is imbedded also contains R'(L) (up to isomorphism). By examining the dimension of R'(L) we hope to find some necessary isomorphism conditions for the simple Lie algebras of Block type.

Since L is a simple Lie algebra, the center of L is zero. But the center of L is the kernel of the adjoint representation of L. Hence the kernel of the adjoint representation of L is zero. Thus we can identify L with ad L, the algebra of inner derivations of L; and then attempt to find a smallest restricted Lie algebra R(L) containing ad L.

If D(L) is the derivation algebra of L, then ad $L \subsetneq R(L) \subset D(L)$. With this we can compare the dimension of R(L) with that of D(L); and then determine when L itself is restricted (previously determined by Block in [2] using a different method). Furthermore, we seek to find a necessary condition for two algebras L and L' of Block type, which are of the same dimension, to be

isomorphic.

We also intend to discuss briefly how the work can be utilized to discover nonisomorphic simple modular Lie algebras among some of the different types of non-restricted simple Lie algebras of the same dimension.

CHAPTER II

PRELIMINARIES

It will be assumed throughout the chapter that Lie algebras are finite dimensional.

DEFINITION 1: A restricted Lie algebra L is a Lie algebra over a field Φ of prime characteristic p which has a "p-map" $a \to a^{[p]}$ from L into L such that

(i) $(\lambda a)^{[p]} = \lambda^p a^{[p]}$

(ii) $(\text{ad } a)^p = \text{ad}(a^{[p]})$ for all $a \in L$, $\lambda \in \Phi$;

(iii) $(a+b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a,b)$ for $a,b \in L$,

where $is_i(a,b)$ is the coefficient of X^{i-1} in the expansion of $a(\text{ad}(aX+b))^{p-1}$, and X is an indeterminate over Φ.

REMARKS 1: For Lie algebras with trivial center, and in particular for simple Lie algebras, condition (iii) can be dropped. A Lie algebra L of characteristic p can be given the structure of a restricted Lie algebra if and only if $(\text{ad } x)^p \in \text{ad } L$ for all $x \in L$. In fact it will be enough to have $(\text{ad } u_i)^p \in \text{ad } L$ for every u_i in a basis $\{u_i\}$ of L. The proof can be found in [6]; pg. 188-191.

DEFINITION 2: Let L be a Lie algebra over a field Φ of prime characteristic p. R'(L) is the smallest restricted Lie algebra containing L provided that any restricted Lie algebra in which L can be imbedded also contains an isomorphic copy of R'(L).

DEFINITION 3: If $R'(L) \tilde{=} L$, then L is said to be a restricted Lie algebra.

REMARK 2: If L is a simple Lie algebra, then L can be identified with ad L, the algebra of inner derivations of L; while R'(L) can be identified with R(L), the smallest restricted Lie algebra containing ad L. Since we will only be concerned with simple Lie algebras, our consideration will only be for the smallest restricted Lie algebra R(L) containing ad L. We will sometimes refer to R(L) as the smallest restricted Lie algebra of L.

LEMMA 1: Let L be a simple Lie algebra over a field Φ of prime characteristic p, R(L) the smallest restricted Lie algebra containing ad L, and D(L) the derivation algebra of L. Then $\text{ad } L \subsetneq R(L) \subseteq D(L)$; and consequently R(L) is a finite dimensional Lie algebra. Moreover,

$$R(L) = \sum_{j=0}^{k} (\text{ad } L)^{p^j},$$

where k is the least integer such that for all $i \geq 0$ and all $x \in L$,

$$(\text{ad } x)^{p^i} \in \sum_{j=0}^{k} (\text{ad } L)^{p^j}.$$

PROOF: Every $\delta \in D(L)$ satisfies the Leibniz' rule

$$([a,b])\delta^k = \sum_{j=0}^{k} \binom{k}{j} [a\delta^j, b\delta^{k-j}]$$

for $a,b \in L$. From char $\Phi = p>0$, we have $\delta^p \in D(L)$ also, since $([a,b])\delta^p = [a\delta^p,b] + [a,\delta b^p]$ by Leibniz' rule. Since, by the Jacobi identity in L, ad x is a derivation it follows by taking $\delta = \text{ad } x$ that $(\text{ad } x)^p$ is a derivation on L for all $x \in L$. Hence every vector in $\sum_{j=0}^{k} (\text{ad } L)^{p^j}$ is a derivation on L. We thus get $\text{ad } L \subseteq R(L) \subseteq D(L)$.

The finite dimensionality of R(L) is a consequence of that of D(L).

To show $R(L) = \sum_{j=0}^{k} (\text{ad } L)^{p^j}$, where k is the least positive integer such that $(\text{ad } x)^{p^i} \in \sum_{j=0}^{k} (\text{ad } L)^{p^j}$, for all $i \geq 0$ and all $x \in L$, we first show that $\sum_{j=0}^{k} (\text{ad } L)^{p^j}$ is a Lie algebra over Φ. It is obviously a vector space over Φ. Since $\text{ad } x \in \text{ad } L$ implies $(\text{ad } x)^p$ is a derivation on L it follows that $\sum_{j=0}^{k} (\text{ad } L)^{p^j}$ is a Φ-subspace of D(L). Hence to show that $\sum_{j=0}^{k} (\text{ad } L)^{p^j}$ is a Lie algebra we only need to show that it is closed under the bracket product.

To show $\sum_{j=0}^{k} (\text{ad } L)^{p^j}$ is closed under the bracket product, it

is sufficient to show that $[(\text{ad } x)^{p^r}, (\text{ad } y)^{p^s}]$ is in $\sum_{j=0}^{k} (\text{ad } L)^{p^j}$

for $1 \leq r, s \leq k$ and $x, y \in L$.

For any $\sigma, \tau \in \text{End}_{\phi}(L)$ and $n \geq 1$ we have, using a straightforward induction argument and the fact that $\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$, $\sigma\tau^n = \sum_{j=0}^{n} \binom{n}{j} \tau^{n-j} (\sigma(\text{ad } \tau)^j)$. Substituting $\sigma = (\text{ad } x)^{p^r}$, $\tau = \text{ad } y$ and $n = p^s$ we have $(\text{ad } x)^{p^r} (\text{ad } y)^{p^s} = (\text{ad } y)^{p^s} (\text{ad } x)^{p^r} + (\text{ad } x)^{p^r} \text{ad } (\text{ad } y)^{p^s}$.

Hence, $[(\text{ad } x)^{p^r}, (\text{ad } y)^{p^s}] = (\text{ad } x)^{p^r} \text{ad}(\text{ad } y)^{p^s}$. Let $\delta = \text{ad}(\text{ad } y)^{p^s}$.

Then δ is a derivation on ad L. It has been shown in [6]; pg. 186 that $(a^{p^j})\delta = (a\delta)^{p^j}$ for all $a \in$ ad L. Whence $(\text{ad } x)^{p^r} \delta = ((\text{ad } x)\delta)^{p^r}$.

We thus have $(\text{ad } x)^{p^r} \text{ad}(\text{ad } y)^{p^s} = \{(\text{ad } x)\text{ad}(\text{ad } y)^{p^s}\}^{p^r} = \{[\text{ad } x, (\text{ad } y)^{p^s}]\}^{p^r}$. For $z \in L; \delta \in D$ the following result is proved in [13]; pg. 22: $[\text{ad } z, \delta] = \text{ad}(z\delta)$. With the use of this and the fact that $(\text{ad } y)^{p^s}$ is a derivation on L we have

$$[(\text{ad } x)^{p^r}, (\text{ad } y)^{p^s}] = \{[\text{ad } x, (\text{ad } y)^{p^s}]\}^{p^r}$$

$$= \{\text{ad}(x(\text{ad } y)^{p^s})\}^{p^r} \in (\text{ad } L)^{p^r}.$$

Therefore $[(\text{ad } x)^{p^r}, (\text{ad } y)^{p^s}] \in (\text{ad } L)^{p^r} \subseteq \sum_{j=0}^{k} (\text{ad } L)^{p^j}$.

Consequently, $\sum_{j=0}^{k} (\text{ad } L)^{p^j}$ is closed under the bracket product; and therefore is a Lie algebra over Φ.

The fact that $\sum_{j=0}^{k} (\text{ad } L)^{p^j}$ is closed under the pth powers implies by remarks 1 that it is a restricted Lie algebra containing ad L. Therefore

$$R(L) \subseteq \sum_{j=1}^{k} (\text{ad } L)^{p^j}.$$

R(L) being a restricted Lie algebra with ad $L \subseteq R(L)$ implies by remarks 1 that $(\text{ad } L)^{p^i} \subseteq R(L)$ for all $i \geq 0$. Hence $\sum_{j=0}^{k} (\text{ad } L)^{p^j} \subseteq R(L)$; and the conclusion of the lemma follows.

LEMMA 2: If Φ is an algebraically closed field of characteristic $p>0$, and $a_1, a_2, \ldots, a_k$ are in Φ such that $a_1 \neq 0$, then the solutions of the equation

$$x^{p^k} + a_k x^{p^{k-1}} + a_{k-1} x^{p^{k-2}} + \ldots + a_2 x^p + a_1 x = 0$$

form a k-dimensional vector space over the prime field Z_p.

PROOF: The formal derivative of the polynomial $f(x) = x^{p^k} + a_k x^{p^{k-1}} + \ldots + a_2 x^p + a_1 x$ in $\Phi[x]$ is a_1. Hence the highest common factor of f(x) and its derivative is 1. But f(x) has a repeated root in Φ if and only if the degree of the highest common

factor of f(x) and its formal derivative is positive. We therefore conclude that f(x) has no repeated roots in Φ. Thus the equation

$$x^{p^k} + a_k x^{p^{k-1}} + a_{k-1} x^{p^{k-2}} + \ldots + a_2 x^p + a_1 x = 0$$

has p^k distinct roots in Φ.

Let γ_1 and γ_2 be two arbitrary roots of the equation in Φ, and let α_1 and α_2 be in the prime field Z_p. Then

$$(\alpha_1\gamma_1+\alpha_2\gamma_2)^{p^k} + a_k(\alpha_1\gamma_1+\alpha_2\gamma_2)^{p^{k-1}} + \ldots + a_1(\alpha_1\gamma_1+\alpha_2\gamma_2) =$$

$$\alpha_1(\gamma_1^{p^k} + a_k\gamma_1^{p^{k-1}} + \ldots + a_2\gamma_1^p + a_1\gamma_1) + \alpha_2(\gamma_2^{p^k} + a_k\gamma_2^{p^{k-1}} + \ldots$$

$$+ a_2\gamma_2^p + a_1\gamma_2)$$

$$= 0.$$

It then follows that $\alpha_1\gamma_1+\alpha_2\gamma_2$ is also a root of the equation.

Hence the set of all roots of the equation form a vector space over Z_p of dimension k.

COROLLARY 1: Let $a_1, a_2, \ldots, a_k$ be elements of Φ with a_1 non-zero, V be a vector space over Z_p spanned by the solutions in Φ of the equation $x^{p^k} + a_k x^{p^{k-1}} + \ldots + \ldots + a_2 x^p + a_1 x = 0$, and $\{v_i \mid i = 1,2,\ldots,k\}$ be a basis of V over Z_p. Then v_i; $i = 1,2,\ldots,k$ cannot all be roots of the equation of the form

$$x^{p^j} + b_j x^{p^{j-1}} + b_{j-1} x^{p^{j-2}} + \ldots + \ldots + b_2 x^p + b_1 x = 0$$

such that b_1 is nonzero, and $k>j$.

PROOF: Assume the result is not true. We know from lemma 2 that the roots of the equation form a j-dimensional vector space V_1 over Z_p. Therefore if v_i; $i=1,2\ldots,k$ are roots of this equation, then V will be a subspace of V_1. This says $j \geq k$; which contradicts $k>j$.

Thus, v_i; $i=1,2,\ldots,k$ cannot all be roots of the equation

$$x^{p^j} + b_j x^{p^{j-1}} + \ldots + \ldots + b_2 x^p + b_1 x = 0;$$

such that b_1 is nonzero and $k>j$.

LEMMA 3: Let Φ be an algebraically closed field of characteristic $p \geq 3$, G an elementary p-group of order p^n, where $n \geq 2$, $\{\beta_i | i = 1,2,\ldots n\}$ a basis of G, and f a skew-symmetric biadditive function on G with values in Φ. Then f is non-degenerate if and only if the matrix $(f(\beta_i,\beta_j))_{1 \leq i,j \leq n}$ has no non-trivial linear combination over Z_p of the rows equal to zero.

PROOF: It needs to be remarked that $f:GxG \to \Phi$ is a Z_p-bilinear map and not necessarily a bilinear form on G.

Since the conclusion of the lemma is a standard result of linear algebra, its proof will be omitted.

LEMMA 4: Let Φ be any extension field of Z_p. If elements of $Z_p^{(n)}$

are linearly independent over Z_p, then they are linearly independent over Φ and conversely.

PROOF: Since $Z_p^{(n)}$ is a vector space over Z_p we can extend the space $Z_p^{(n)}$ into a vector space $(Z_p^{(n)})_\Phi = \Phi \otimes_{Z_p} Z_p^{(n)}$ over Φ relative to $\alpha(\Sigma \alpha_i \otimes x_i) = \Sigma \alpha\alpha_i \otimes x_i$ for all $\alpha, \alpha_i \in \Phi$ and $x_i \in Z_p^{(n)}$.

It will then follow that $\{v_1, v_2, \ldots, v_s\} \subset Z_p^{(n)}$ is linearly independent over Z_p if and only if

$$\{1 \otimes_{Z_p} v_1, 1 \otimes_{Z_p} v_2, \ldots, 1 \otimes_{Z_p} v_s\} \subset \Phi \otimes_{Z_p} Z_p$$

is linearly independent over Φ.

LEMMA 5: Let G be an elementary p-group of order p^n for $n \geq 2$, $\{\beta_i \mid i = 1,2,\ldots,n\}$ be a basis of G over Z_p, and f a nondegenerate and skew-symmetric biadditive function on G taking values in an algebraically closed field Φ of characteristic $p \geq 3$ such that the rank of f is t. Then at least n-t columns (respectively rows) of the matrix $(f(\beta_i, \beta_j))_{1 \leq i,j \leq n}$ contain elements not in Z_p.

PROOF: It is worthwhile noting that $f: G \times G \to \Phi$ is a Z_p-bilinear map, and not necessarily a Φ-bilinear map, on G. Consequently, the rank of the matrix $(f(\beta_i, \beta_j))_{1 \leq i,j \leq n}$ need not be n. But in the case where the image space of f is contained in Z_p, the rank will be n.

Let v_i; $i = 1,2,\ldots,n$, where

$$v_i = (f(\beta_1,\beta_i), f(\beta_2,\beta_i), \ldots, 0, -f(\beta_i,\beta_{i+1}), \ldots -f(\beta_i,\beta_n))$$

be the columns of the matrix $(f(\beta_i,\beta_j))_{1\leq i,j\leq n}$ in the basis $\{\beta_i | i=1,2,\ldots,n\}$, and $v_j; j=1,2,\ldots,t$ be the linearly independent columns over Φ.

Suppose, for some s, $v_{i_k}; k=1,2,\ldots,s$ have all their components from Z_p, where $i_1,i_2,\ldots,i_s \in \{1,2,\ldots,n\}$. Then $v_{i_k}; k=1,2,\ldots,s$ are in $Z_p^{(n)}$. Since f is nondegenerate over Z_p, it follows from the third lemma that $v_{i_k}; k=1,2,\ldots,s$ are linearly independent over Z_p. But, by lemma 4, if the elements of $Z_p^{(n)}$ are linearly independent over Z_p, then they are linearly independent over Φ, and conversely. Hence $v_{i_k}; k=1,2,\ldots s$ are linearly independent over Φ. Consequently $s\leq t$.

Therefore at least n-t columns of $(f(\beta_i,\beta_j)); 1\leq i,j\leq n$ contains entries outside of Z_p. By the skew-symmetry of f, the same conclusion also holds for the rows of $(f(\beta_i,\beta_j))$, $1\leq i,j\leq n$.

LEMMA 6: If Φ is an extension field of Z_p, A is a matrix with all its entries from Z_p, and L(A) is a system of homogeneous equation whose matrix of coefficients is A, then the solutions in $\Phi^{(n)}$ of the system L(A) are the linear combinations over Φ of solutions in $Z_p^{(n)}$ of the system L(A).

PROOF: Since the lemma is a standard result, its proof can be found in any standard textbook on linear algebra, or from the linear algebra references listed in the bibliography.

LEMMA 7: Let Φ be an extension field of Z_p, and let U_1, U_2 be subspaces of $\Phi^{(n)}$. If U_1, U_2 have bases in $Z_p^{(n)}$, then $U_1 \cap U_2$ will also have basis in $Z_p^{(n)}$.

PROOF: Let $V = \Phi^{(n)}$ and $V_0 = Z_p^{(n)}$. Then $V = V_0 \otimes \Phi$. The fact that U_1, U_2 are subspaces of V with bases in V_0 is equivalent to saying that U_1, U_2 are of the form $U_1 = U_{10} \otimes \Phi$, $U_2 = U_{20} \otimes \Phi$ for some subspaces U_{10}, U_{20} of V_0. Let $W_0 = U_{10} \cap U_{20}$. The conclusion of the lemma will hold if we can show that $U_1 \cap U_2 = W_0 \otimes \Phi$; for in that case $U_1 \cap U_2$ will have a basis in V_0.

$W_0 \otimes \Phi \subset U_1 \cap U_2$ is obvious. Hence we will have equality if $\dim_\Phi W_0 \otimes \Phi = \dim_\Phi U_1 \cap U_2$. But $\dim_\Phi U_1 \cap U_2 + \dim_\Phi (U_1 + U_2) = \dim_\Phi U_1 + \dim_\Phi U_2$. By hypothesis we have

$$\dim_\Phi U_1 + \dim_\Phi U_2 = \dim_{Z_p} U_{10} + \dim_{Z_p} U_{20} .$$

Therefore

$$\begin{aligned} \dim_\Phi U_1 \cap U_2 + \dim_\Phi (U_1 + U_2) &= \dim_{Z_p} U_{10} + \dim_{Z_p} U_{20} \\ &= \dim_{Z_p} U_{10} \cap U_{20} + \dim_{Z_p} (U_{10} + U_{20}) \\ &= \dim_{Z_p} W_0 + \dim_{Z_p} (U_{10} + U_{20}) . \end{aligned}$$

So it is sufficient to prove that $\dim_{Z_p}(U_{10}+U_{20}) = \dim_{\phi}(U_1+U_2)$

But $\dim_{Z_p}(U_{10}+U_{20}) = \dim_{\phi}(U_1+U_2)$ follows from the fact that U_1+U_2 is spanned by a set of vectors in V_0; and hence has a basis contained in V_0.

Therefore if U_1, U_2 have bases in $Z_p^{(n)}$, then $U_1 \cap U_2$ will also have basis in $Z_p^{(n)}$.

In [2] Block succeeded in showing that, with the stated restrictions on f, $L(G,\delta,f)$ is a central simple Lie algebra, and its dimension is p^n-2; except when $G=G_0$, in which case it is p^n-1.

For the entire discussion of $L(G,\delta,f)$, it will be assumed that $p>3$. This is due to the fact that characteristics 2 and 3 are exceptions to all the results. For those two cases there is no unified approach to their discussions; in some cases the algebras turn out to be isomorphic to known simple Lie algebras, while in other cases they are not isomorphic to any previously known simple Lie algebras. In the latter case our results will hold true. Some of the results on isomorphism between the algebras can be found in [2].

We shall denote, from now on, the smallest restricted Lie algebra containing ad $L(G,\delta,f)$ by $RL(G,\delta,f)$, and the derivation algebra of $L(G,\delta,f)$ by $DL(G,\delta,f)$.

Let $v(\alpha)$ be denoted by v_α. Block has defined the derivations $D(\gamma_k,-\delta_k)$, $D(\delta,0)$, and $D(\sigma_{ij},0)$ of $L(G,\delta,f)$ by $v_\alpha D(\gamma_k,-\delta_k) = f(\alpha_k,\gamma_k)\, v(\alpha-\delta_k)$, $v_\alpha D(\sigma_{ij},0) = s_{ij}(\alpha)v_\alpha$, and $v_\alpha D(\delta,0) = [-1+\sum_{i=1}^{m} s_i(\alpha)]v_\alpha$ if $G_0=0$; $D(\delta,0) = 0$ if $G_0 \neq 0$; where γ_k is any element in G_k for any k; $\sigma_{01},\sigma_{02},\ldots,\sigma_{0r_0}$ is the basis of G_0 over Z_p; $\sigma_{i1},\sigma_{i2},\ldots,\sigma_{ir_i},\delta_i$ a basis of G_i over Z_p such that $f(\sigma_{i1},\delta_i) \neq 0$; for $i = 1,2,\ldots;m$; $s_{ij}(\alpha)$ is the coefficient of σ_{ij} in α for $i = 0,1,2,\ldots,m$; and $s_i(\alpha)$ is the coefficient of δ_i in α. $DL(G,\delta,f)$ has the following representations:

CHAPTER III

MAIN RESULTS

Block constructed simple Lie algebras over a modular field by considering an additive group G of order $p^n>1$, which is a direct sum of a finite number of finite elementary p-groups $G_0,G_1,\ldots,G_m$. These groups are then finite dimensional vector spaces over Z_p. He assumed that G_0 can be allowed to be zero, and $p>2$, or $G \neq G_1$. Let $\delta = \delta_1 + \delta_2 + \ldots + \delta_m$ be a fixed element of G, where $0 \neq \delta_i \varepsilon G_i$ for $i = 1,2,\ldots,m$, and let L be a vector space over the modular field Φ whose basis is indexed by the elements of G other than 0 and $-\delta$. Denote by $v(\alpha)$ the basis element of L corresponding to α in G. Hence the basis of L is in one-one correspondence $v(\alpha) \leftrightarrow \alpha$ with elements of G such that $\alpha \neq 0,-\delta$. Furthermore, we assume given, for each i, a nondegenerate and skew-symmetric biadditive function f_i on $G_i x G_i$ to Φ, such that, for $1 \leq i \leq m$, there are additive functions g_i, h_i on G_i to Φ, with $g_i(\delta_i) = 0$ and $f_i(\alpha,\beta) = g_i(\alpha)h_i(\beta) - g_i(\beta)h_i(\alpha)$ for every α and β in G_i. Multiplication on L is defined by $v(\alpha)ov(\beta) = \sum_{i=0}^{m} f_i(\alpha_i,\beta_i)v(\alpha+\beta-\delta_i)$; where α_i and β_i are the components of α and β in G_i, respectively; and where δ_0 and $v(0)$ denote zero. With this multiplication L becomes a Lie algebra, which is represented by $L(G,\delta,f)$; where f is the biadditive function on G whose restriction to G_i is f_i.

In [2] Block succeeded in showing that, with the stated restrictions on f, $L(G,\delta,f)$ is a central simple Lie algebra, and its dimension is p^n-2; except when $G=G_0$, in which case it is p^n-1.

For the entire discussion of $L(G,\delta,f)$, it will be assumed that $p>3$. This is due to the fact that characteristics 2 and 3 are exceptions to all the results. For those two cases there is no unified approach to their discussions; in some cases the algebras turn out to be isomorphic to known simple Lie algebras, while in other cases they are not isomorphic to any previously known simple Lie algebras. In the latter case our results will hold true. Some of the results on isomorphism between the algebras can be found in [2].

We shall denote, from now on, the smallest restricted Lie algebra containing ad $L(G,\delta,f)$ by $RL(G,\delta,f)$, and the derivation algebra of $L(G,\delta,f)$ by $DL(G,\delta,f)$.

Let $v(\alpha)$ be denoted by v_α. Block has defined the derivations $D(\gamma_k,-\delta_k)$, $D(\delta,0)$, and $D(\sigma_{ij},0)$ of $L(G,\delta,f)$ by $v_\alpha D(\gamma_k,-\delta_k) = f(\alpha_k,\gamma_k)\, v(\alpha-\delta_k)$, $v_\alpha D(\sigma_{ij},0) = s_{ij}(\alpha)v_\alpha$, and $v_\alpha D(\delta,0) = [-1+\sum_{i=1}^{m} s_i(\alpha)]v_\alpha$ if $G_0=0$; $D(\delta,0) = 0$ if $G_0 \neq 0$; where γ_k is any element in G_k for any k; $\sigma_{01},\sigma_{02},\ldots,\sigma_{0r_0}$ is the basis of G_0 over Z_p; $\sigma_{i1},\sigma_{i2},\ldots,\sigma_{ir_i},\delta_i$ a basis of G_i over Z_p such that $f(\sigma_{i1},\delta_i) \neq 0$; for $i = 1,2,\ldots;m$; $s_{ij}(\alpha)$ is the coefficient of σ_{ij} in α for $i = 0,1,2,\ldots,m$; and $s_i(\alpha)$ is the coefficient of δ_i in α. $DL(G,\delta,f)$ has the following representations:

(a) $DL(G,\delta,f) = \left\langle \text{ad } v_\alpha, D(\sigma_{0i},0) \mid i=1,2,\ldots,n_0 \right\rangle$, if $G = G_0$

(b) $DL(G,\delta,f) = \left\langle \text{ad } v_\alpha, \text{ad } v_{-\delta}, D(\gamma_k,-\delta_k), D(\sigma_{0\ell},0), D(\sigma_{ij},0), D(\delta_k,-\delta_k) \mid \right.$

$\left. \ell = 1,2,\ldots,n_0; i,k = 1,\ldots m;\ j = 2,\ldots,n_i-1 \right\rangle$, if $G_0 \neq 0$, $G \neq G_0$

(c) $DL(G,\delta,f) = \left\langle \text{ad } v_\alpha, \text{ad } v_{-\delta}, D(\gamma_k,-\delta_k), D(\delta,0), D(\sigma_{ij},0), D(\delta_k,-\delta_k) \mid \right.$

$\left. i,k = 1,2,\ldots,m;\ j = 2,\ldots,n_i-1 \right\rangle$, if $G_0 = 0$.

Let $|G| = p^n$. In the case where $G = G_0$ the outer derivation algebra $L^*(G,\delta,f)$ of $L(G,\delta,f)$ is an n-dimensional abelian algebra; thereby giving dim $DL(G,\delta,f) = p^n+n-1$. For $G_0 = 0$, $L^*(G,\delta,f)$ is (n+2)-dimensional, and dim $DL(G,\delta,f) = p^n+n$. In the situation where $G_0 \neq 0$ and $G \neq G_0$, dim $L^*(G,\delta,f) = n+1$, and dim $DL(G,\delta,f) = p^n+n-1$. $L^*(G,\delta,f)$ is always abelian except when $m = 1$.

We intend to use the multiplication table of Block to determine the dimension of $RL(G,\delta,f)$, and then compare it to the already known dimension of $DL(G,\delta,f)$.

If we denote $v_\alpha \text{ad } v_\beta$ for all $\alpha,\beta \neq -\delta$ in G by $v_\alpha R_\beta$, we obtain, using Block's multiplication, a general representation for $v_\alpha R_\beta^p$ as follows:

$$v_\alpha R_\beta^p = (v_\alpha \circ v_\beta) R_\beta^{p-1} = \left[\sum_{i_1=0}^{m} f(\alpha_{i_1},\beta_{i_1}) v(\alpha+\beta-\delta_{i_1}) \right] R_\beta^{p-1}$$

$$= \sum_{i_1=0}^{m} f(\alpha_{i_1},\beta_{i_1}) \left[v(\alpha+\beta-\delta_{i_1}) R_\beta^{p-1} \right] .$$

Continuing we get

$$v_\alpha R_\beta^p = \sum_{i_1,i_2,\ldots,i_p=0}^{m} f(\alpha_{i_1},\beta_{i_1})f(\alpha_{i_2}-\delta_{i_1},\beta_{i_2})\ldots$$

$$f(\alpha_{i_p}-\delta_{i_1}-\ldots-\delta_{i_{p-1}},\beta_{i_p})v(\alpha-\delta_{i_1}-\ldots-\delta_{i_p})$$

Letting the components of α and β in G_i to be α_i and β_i, respectively, we have the following:

$$v_\alpha R_{\beta_0}^p = f(\alpha_0,\beta_0)^p v_\alpha$$

$$v_\alpha R_{\beta_i}^p = [f(\alpha_i,\beta_i)^p - f(\beta_i,\delta_i)^{p-1} f(\alpha_i,\beta_i)]v_\alpha,$$

where $i = 1,2,\ldots,m$.

From now on we assume that the algebra $L(G,\delta,f)$ is defined with respect to a group G of order p^n which is a direct sum of elementary p-groups G_i of order p^{n_i}; $i = 0,1,2,\ldots,m$ and a nondegenerate and skew-symmetric biadditive function f defined on G and taking its values in the ground field Φ such that its restriction to G_i is f_i; $i = 0,1,2,\ldots,m$. In addition we require that the ground field Φ be algebraically closed of characteristic $p>3$, and $\delta = \delta_1+\delta_2+\ldots+\delta_m$; where $\delta_i \varepsilon G_i$. Unless otherwise stated, it is to be assumed that $n\geq 2$.

LEMMA 8: Let $n\geq 3$, $G \neq G_0$, and $\{\sigma_{0k},\sigma_{ij} | j = 1,2,\ldots,n_i;\ i = 1,2,\ldots,m;\ k = 1,2,\ldots,n_0\}$ be some basis of G over Z_p such that $\delta_r = \sigma_{rn_r}$ and $f(\sigma_{r1},\delta_r) \neq 0$; $r = 1,2,\ldots,m$. Then for each $i = 0,1,\ldots,m$ none of the columns of the matrix of the coefficients of $D(\sigma_{ij},0)$; $j = 1,2,\ldots,n_i-1$ in ad $v_{\sigma_{ij}}^p$; $j = 1,2,\ldots,n_i$ is zero.

PROOF: With the use of $v_\alpha \text{ad } v_\eta^p = \sum_{i=0}^{m} [f(\alpha_i,\eta_i)^p - f(\eta_i,\delta_i)^{p-1} f(\alpha_i,\eta_i)]^p v_\alpha$

we have

$$v_\alpha \text{ad } v_{\eta_0}^p = f_0(\alpha_0,\eta_0)^p v_\alpha$$

and

$$v_\alpha \text{ad } v_{\eta_i}^p = [f_i(\alpha_i,\eta_i)^p - f_i(\eta_i,\delta_i)^{p-1} f_i(\alpha_i,\eta_i)] v_\alpha,$$

where α_i,η_i are the components of α,η in G_i, respectively; and $i=1,2,\ldots,m$.

It should be remarked that the nondegeneracy of each f_i over $Z_p; i=0,1,2,\ldots,m$ has been established by Block in [2].

From the expressions of $v_\alpha \text{ad } v_{\eta_i}^p; i=0,1,\ldots,m$, it follows that to prove the lemma it is sufficient to consider each of the f_i on G_i in turn. In fact, for $i = 1,2,\ldots,m$, we only need to consider one of them.

Consider f_0 on G_0. Since $v_\alpha \text{ad } v_{\eta_0}^p = f_0(\alpha_0,\eta_0)^p v_\alpha$ for all $\alpha = (\alpha_0,\ldots,\alpha_m)$ in G, $\text{ad } v_{\sigma_{0j}}^p = \sum_{i=1}^{n_0} f_0(\sigma_{0i},\sigma_{0j})^p D(\sigma_{0i},0)$; $j = 1,2,\ldots,n_0$.

Hence whenever a column of the matrix of the coefficients of $D(\sigma_{0i},0); i=1,2,\ldots,n_0$ in $\text{ad } v_{\sigma_{0j}}^p$; $j = 1,2,\ldots,n_0$ is zero, a column of the matrix of f_0 in the basis $\sigma_{0j}; j = 1,2,\ldots,n_0$ will be zero. This will make f_0 degenerate over Z_p; thereby giving a contradiction. Therefore no column of the matrix of the coefficients of $D(\sigma_{0i},0); i=1,2,\ldots,n_0$

in ad $v_{\sigma_{0j}}^{p}$; $j = 1,2,\ldots,n_0$ will be zero.

We now consider f_1 on G_1. Using v_α ad $v_{\eta_1}^{p} = [f_1(\alpha_1,\eta_1)^p - f_1(\eta_1,\delta_1)^{p-1}f_1(\alpha_1\eta_1)]v_\alpha$, we get ad $v_{\sigma_{1j}}^{p} = \sum_{i=1}^{n_1-1} [f_1(\sigma_{1i},\sigma_{1j})^p - f_1(\sigma_{1j},\delta_1)^{p-1}f_1(\sigma_{1i},\sigma_{1j})]D(\sigma_{1i},0)$; $j = 1,2,\ldots,n_1$, where δ_1 is defined as σ_{1n_1}.

Suppose one of the columns of the matrix of the coefficients of $D(\sigma_{1i},0)$; $i = 1,2,\ldots,n_1-1$ in ad $v_{\sigma_{1j}}^{p}$; $j = 1,2,\ldots,n_1$ is zero. Since $f_1(\sigma_{11},\sigma_{1n_1}) \neq 0$, the first column of the matrix is not zero. Assuming the second column is zero, then $f_1(\sigma_{12},\sigma_{1n_1}) = 0$ and $f_1(\sigma_{12},\sigma_{1i})^p - f_1(\sigma_{1i},\sigma_{1n_1})^{p-1}f_1(\sigma_{12},\sigma_{1i}) = 0$; $i = 1,2,\ldots,n_1 - 1$. This says that whenever $f_1(\sigma_{1j},\sigma_{1n_1}) = 0$, then $f_1(\sigma_{12},\sigma_{1j}) = 0$.

It has been shown in [2] that the Jacobi identity will be satisfied in the algebra $L(G,\delta,f)$ if and only if $f(\alpha_i,\beta_i)f(\gamma_i,\delta_i) + f(\beta_i,\gamma_i)f(\alpha_i,\delta_i) + f(\gamma_i,\alpha_i)f(\beta_i,\delta_i) = 0$; for $i = 1,2,\ldots,m$ and all α, β and γ in G.

Suppose $f_1(\sigma_{12},\sigma_{1\ell}) = 0$, for some ℓ. By the nondegeneracy of f_1 over Z_p, there exists a σ_{1j} such that $f_1(\sigma_{12},\sigma_{1j}) \neq 0$. Then with the substitution of $\alpha_i = \sigma_{12}, \beta_i = \sigma_{1\ell}$, and $\gamma_i = \sigma_{1j}$ in the alternative Jacobi identity, we get the following:

$f_1(\sigma_{12},\sigma_{1\ell})f_1(\sigma_{1j},\sigma_{1n_1}) + f_1(\sigma_{1\ell},\sigma_{1j})f_1(\sigma_{12},\sigma_{1n_1}) +$

$f_1(\sigma_{1j},\sigma_{12})f_1(\sigma_{1\ell},\sigma_{1n_1}) = 0.$

Since $f_1(\sigma_{12},\sigma_{1\ell}) = f_1(\sigma_{12},\sigma_{1n_1}) = 0$ and $f_1(\sigma_{1j},\sigma_{12}) \neq 0$, the identity reduces to $f_1(\sigma_{1\ell},\sigma_{1n_1}) = 0$. We have thus proved that if $f_1(\sigma_{12},\sigma_{1\ell}) = 0$, then $f_1(\sigma_{1\ell},\sigma_{1n_1}) = 0$.

Combining this with the result that if $f_1(\sigma_{1\ell},\sigma_{1n_1}) = 0$, then $f_1(\sigma_{12},\sigma_{1\ell}) = 0$, we get $f_1(\sigma_{12},\sigma_{1\ell}) = 0$ if and only if $f_1(\sigma_{1\ell},\sigma_{1n_1}) = 0$.

If for some k, $f_1(\sigma_{12},\sigma_{1k}) \neq 0$, then $f_1(\sigma_{1k},\sigma_{1n_1}) \neq 0$, and $\{f_1(\sigma_{12},\sigma_{1k})[f_1(\sigma_{1k},\sigma_{1n_1})]^{-1}\}^{p-1} = 1$. This implies that $f_1(\sigma_{12},\sigma_{1k}) = a_k f_1(\sigma_{1k},\sigma_{1n_1})$ for some nonzero a_k in Z_p.

Suppose for some nonzero a_i, a_j in Z_p, $f_1(\sigma_{12},\sigma_{1i}) = a_i f_1(\sigma_{1i},\sigma_{1n_1})$ and $f_1(\sigma_{12},\sigma_{1j}) = a_j f_1(\sigma_{1j},\sigma_{1n_1})$. Then substituting $\alpha_1 = \sigma_{12}$, $\beta_1 = \sigma_{1i}$, and $\gamma_1 = \sigma_{1j}$ in the condition for the satisfaction of the Jacobi identity, we get $f_1(\sigma_{12},\sigma_{1i})f_1(\sigma_{1j},\sigma_{1n_1}) + f_1(\sigma_{1i},\sigma_{1j})f_1(\sigma_{12},\sigma_{1n_1})$ $+ f_1(\sigma_{1j},\sigma_{12})f_1(\sigma_{1i},\sigma_{1n_1}) = 0$. Whence $a_i f_1(\sigma_{1i},\sigma_{1n_1})f_1(\sigma_{1j},\sigma_{1n_1}) +$ $f_1(\sigma_{1i},\sigma_{1j})f_1(\sigma_{12},\sigma_{1n_1}) - a_j f_1(\sigma_{1j},\sigma_{1n_1})f_1(\sigma_{1i},\sigma_{1n_1}) = 0$. Since $f_1(\sigma_{12},\sigma_{1n_1}) = 0$, it follows that $(a_i - a_j)f_1(\sigma_{1i},\sigma_{1n_1})f_1(\sigma_{1j},\sigma_{1n_1}) = 0$.

By the assumptions that $f_1(\sigma_{1i},\sigma_{1n}) \neq 0$ and $f_1(\sigma_{1j},\sigma_{1n_1}) \neq 0$, we will have $a_i = a_j$.

Hence for each $i = 1,2,\ldots,m$, either $f_1(\sigma_{12},\sigma_{1i}) = 0$, or $f_1(\sigma_{12},\sigma_{1i}) = af_1(\sigma_{1i},\sigma_{1n_1})$ for some nonzero $a\varepsilon Z_p$. Since $f_1(\sigma_{12},\sigma_{1i}) = 0$ if and only if $f_1(\sigma_{1i},\sigma_{1n_1}) = 0$, we can write $f_1(\sigma_{12},\sigma_{1i}) = af_1(\sigma_{1i},\sigma_{1n_1})$ for all $i = 1,2,\ldots,m$. Inserting the deduction in the matrix of f_1 in the basis $\sigma_{1i}; i = 1,2,\ldots,n_1$, the second row (respectively column) of the matrix of f_1 will be a multiple over Z_p of the last row (respectively column). But lemma 3 says this can only hold if f_1 is degenerate.

Therefore the matrix of the coefficients of $D(\sigma_{1j},0)$; $j = 1,2,\ldots,n_1-1$ in ad $v^p_{\sigma_{1i}}$; $i = 1,2,\ldots,n_1$ cannot have a zero column.

Using an identical argument, we can deduce that for each $i = 1,2,\ldots,m$, the matrix of the coefficients of $D(\sigma_{ij},0)$; $j = 1,2,\ldots,n_i-1$ in ad $v^p_{\sigma_{ij}}$; $\gamma = 1,2,\ldots,n_i$ cannot have a zero column.

LEMMA 9: Let $G \neq G_0$, and $n \geq 3$. If $\{\sigma_{0k},\sigma_{ij} | j = 1,2,\ldots,n_i; i = 1,2,\ldots,m; k = 1,2,\ldots,n_0\}$ is a basis of G over Z_p, then for each $i = 1,2,\ldots,m$ the columns of matrix of the coefficients of $D(\sigma_{ij},0)$; $j = 1,2,\ldots,n_i-1$ in ad $v^p_{\sigma_{ij}}$; $j = 1,2,\ldots,n_i$ are linearly independent over Z_p; and the columns of the matrix of the coefficients of $D(\sigma_{0j},0)$; $j = 1,2,\ldots,n_0$ in ad $v^p_{\sigma_{0j}}$; $j = 1,2,\ldots,n_0$ are also linearly independent over Z_p.

PROOF: Since f_i is the restriction of f to G_i; where i = 0,1,...,m it then follows that

$$\text{ad } v_{\sigma_{0j}}^{p} = \sum_{i=1}^{n_0} f_0(\sigma_{0i},\sigma_{0j})^{p} D(\sigma_{0i},0);$$

where $j = 1,2,\ldots,n_0$ and

$$\text{ad } v_{\sigma_{ij}}^{p} = \sum_{i=1}^{n_i-1} [f_i(\sigma_{ik},\sigma_{ij})^{p} - f_i(\sigma_{ij},\delta_i)^{p-1} f_i(\sigma_{ik},\sigma_{ij})] D(\sigma_{ik},0)$$

for $j = 1,2,\ldots,n_i$; $i = 1,2,\ldots,m$.

To prove the lemma we need to consider the matrices of the coefficients of $D(\sigma_{0j},0)$; $j = 1,2,\ldots,n_0$ in ad $v_{\sigma_{0j}}^{p}$; $j = 1,2,\ldots,n_0$, and $D(\sigma_{ik},0)$; $k = 1,2,\ldots,n_i-1$ in ad $v_{\sigma_{ij}}^{p}$; $j = 1,2,\ldots,n_i$ for $i = 1,2,\ldots,m$.

Suppose the columns of the matrix of the coefficients of $D(\sigma_{0i},0)$; $i = 1,2,\ldots,n_0$ in ad $v_{\sigma_{0j}}^{p}$; $j = 1,2,\ldots,n_0$ are linearly dependent over Z_p. Then by the skew-symmetry of the matrix the rows will also be linearly dependent over Z_p. Consequently the rows of the matrix of f_0 will be linearly dependent over Z_p; and thus a contradiction to lemma 3. Hence the columns of the matrix of the coefficients of $D(\sigma_{0i},0)$; $i = 1,2,\ldots,n_0$ in ad $v_{\sigma_{0j}}^{p}$; $j = 1,2,\ldots,n_0$ are linearly independent over Z_p.

We now consider the matrix of the coefficients of $D(\sigma_{1k},0)$;

$k = 1,2,\ldots,n_1-1$ in ad $v^p_{\sigma_{ij}}$; $j = 1,2,\ldots,n_1$. Assuming the rows of the matrix are linearly dependent over Z_p; then we can find η_i; $i = 1,2,\ldots,n_1-1$ in Z_p not all of which are zero such that they give a dependence relation over Z_p between the columns of the matrix of the coefficients of $D(\beta_{1j},0)$; $j = 1,2,\ldots,n_1-1$ in ad $v^p_{\beta_{1i}}$; $i = 1,2,\ldots,n_1$. If $\gamma = \sum_{i=1}^{n_1-1} \eta_i \beta_{1i}$, then $f_1(\beta_{1n_1},\gamma) = 0$, and

$$f_1(\beta_{1i},\gamma)^p = f_1(\beta_{1i},\beta_{1n_1})^{p-1} f_1(\beta_{1i},\gamma);\ i = 1,2,\ldots,n_1-1.$$

We note that γ is not a multiple of β_{11} over Z_p as otherwise $f(\beta_{1n_1},\beta_{11})$ will be zero; which is not possible. Assuming that the coefficient of β_{12} in γ is not zero, then $\beta_{11},\gamma,\beta_{13},\ldots,\beta_{1n_1}$ is again a basis of G_1 over Z_p.

With this new basis for G_1, (ad $L(G_1,\delta_1,f_1)$)^p will be spanned by ad v^p_γ, ad $v^p_{\beta_{1j}}$; $j=1,\ldots,n_1-1$; $i \neq 2$. The following matrix representation for the expansions of ad v^p_γ, ad $v^p_{\beta_{1i}}$; $i = 1,2,\ldots,n_1-1$; $i \neq 2$ in $D(\beta_{1j},0), D(\gamma,0)$; $j = 1,2,\ldots,n_1-1$; $j \neq 2$ will then arise.

$$
\begin{bmatrix} \operatorname{ad} v^p_{\beta_{11}} \\ \operatorname{ad} v^p_{\gamma} \\ \operatorname{ad} v^p_{13} \\ \vdots \\ \operatorname{ad} v^p_{\beta_{1n_1-1}} \\ \operatorname{ad} v^p_{\beta_{1n_1}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & X^p_{13}-X^{p-1}_{n_11}X_{13} & \cdots & X^p_{1n_1-1}-X^{p-1}_{n_11}X_{1n_1-1} \\ -f(\gamma,\beta_{11})^p & 0 & f(\beta_{13},\gamma)^p & \cdots & f(\beta_{1n_1-1},\gamma)^p \\ -(X^p_{13}-X^{p-1}_{n_{13}}X_{13}) & 0 & 0 & \cdots & X^p_{3n_1-1}-X^{p-1}_{n_13}X_{3n_1-1} \\ \vdots & \vdots & \vdots & & \vdots \\ -(X^p_{1n_1-1},X^{p-1}_{n_1n_1-1}X_{1n_1-1}) & 0 & -(X^p_{3n_1-1}-X^{p-1}_{n_{1n_1-1}}X_{3n_1-1}) & \cdots & 0 \\ -X^p_{1n_1} & 0 & -X^p_{3n_1} & \cdots & -X^p_{n_1-1n_1} \end{bmatrix} \begin{bmatrix} D(\beta_{11},0) \\ D(\gamma,0) \\ D(\beta_{13},0) \\ \vdots \\ D(\beta_{1n_1-1},0) \\ \end{bmatrix}
$$

Therefore we have found a basis $\{\beta_{11},\gamma,\beta_{13},\ldots,\beta_{1n_1}\}$ of G_1 over Z_p such that the matrix of the coefficients of $D(\gamma,0),D(\beta_{1j},0)$; $j = 1,2,\ldots,n_1-1$; $j \neq 2$ in ad v_γ^p,ad $v_{\beta_{1i}}^p$; $i = 1,2,\ldots,n_1-1$; $i \neq 2$ has a zero column. But according to lemma 8 this is possible only if f_1 is degenerate over Z_p. Since f_1 is nondegenerate over Z_p, we will end up with a contradiction.

Repeating the same proof for the matrix of the coefficient of $D(\sigma_{ik},0)$; $k = 1,2,\ldots,n_i-1$ in ad $v_{\sigma_{ij}}^p$; $j = 1,2,\ldots,n_i$, where $i = 2,3,\ldots,m$, we conclude that the columns of the matrix of the coefficients of $D(\sigma_{ij},0)$; $j = 1,2,\ldots,n_i-1$ in ad $v_{\sigma_{ij}}^p$; $j = 1,2,\ldots,n_i$, for $i = 1,2,\ldots,m$ are linearly independent over Z_p.

From now on we denote the vector space $(\text{ad } L(G,\delta,f))^p + (\text{ad } L(G,\delta,f))^{p^2} + \ldots + (\text{ad } L(G,\delta,f))^{p^i}$ by D_i for all $i \geq 1$.

LEMMA 10: Let $G = G_0$, and let the rank of f be t. Then dim D_i = dim $D_{i-1} + x_i$ such that $1 \leq x_i \leq t$, provided the dimension of the space D_{i-1} does not attain the dimension of the outer derivation algebra which is n.

Consequently, the dimension of $(\text{ad } L(G,\delta,L))^p + (\text{ad } L(G,\delta,f))^{p^2} + \ldots$ is n.

PROOF: Let $G = G_0 = Z_p \times Z_p \times \ldots \times Z_p$ (n times), and $\{\beta_i | i=1,2,\ldots,n\}$ be a basis of G over Z_p consisting of the elements β_i with 1 in the ith position and zeros elsewhere. Using the formula that for any $\alpha \in G$ and any $\eta \in G, v_\alpha \text{ad } v_\eta^p = f(\alpha,\eta)^p v_\alpha$, we get the following:

$$v_\alpha \text{ad } v_{\beta_i}^p = \sum_{j=1}^{n} i_j f(\beta_j,\beta_i)^p v_\alpha = \sum_{j=1}^{n} f(\beta_j,\beta_i)^p v_\alpha D(\beta_j,0);\ i = 1,2,\ldots,n,$$

where $\alpha = (i_1,i_2,\ldots,i_n)$. Consequently, ad $v_{\beta_i}^p =$

$\sum_{j=1}^{n} f(\beta_j,\beta_i)^p D(\beta_j,0)$, i = 1,2,...,n will be the expressions for ad $v_{\beta_i}^p$; i = 1,2,...,n in $D(\beta_j,0)$; j = 1,2,...,n. Denoting $f(\beta_j,\beta_i)$ by X_{ij} will reduce the identity to ad $v_{\beta_i}^p = \sum_{j=1}^{n} X_{ij}^p D(\beta_j,0)$;

such that: i = 1,2,3,...,n, where the matrix of the coefficients of $D(\beta_j,0)$; j = 1,2,...,n in ad $v_{\beta_i}^p$; i = 1,2,...,n has rank t.

Let V_1 be a subspace of $\Phi^{(n)}$ spanned by $r_i^{(1)}$; i = 1,2,...,n with $r_i^{(j)} = (-X_{1i}^{p^j}, -X_{2i}^{p^j}, \ldots, -X_{i-1i}^{p^j}, 0, X_{ii+1}^{p^j}, \ldots, X_{in}^{p^j})$ for $j \geq 1$.

Then the dimension of V_1 over Φ is t. We can without any loss of generality assume that $r_i^{(1)}$; i = 1,2,...,t are linearly independent vectors over Φ. If $r_i^{(2)}$ is to lie in V_1 we must be able to find θ_{ij}; j = 1,2,...,n in Φ not all zero such that $r_i^{(2)} =$

$\sum_{j=1}^{n} \theta_{ij} r_j^{(1)}$. This gives rise to the following system of n equations in θ_{ij}; $j = 1,2,\ldots,n$: $\sum_{j=1}^{n} X_{hj}^{p} \theta_{ij} = X_{hj}^{p^2}$; $j = 1,\ldots,n$. Thus we get the matrix identity below:

$$\begin{pmatrix} 0 & -X_{12}^{p} & -X_{13}^{p} & \cdots\cdots & -X_{1n}^{p} \\ X_{12}^{p} & 0 & -X_{23}^{p} & \cdots\cdots & -X_{2n}^{p} \\ \vdots & \vdots & \vdots & & \vdots \\ X_{1n}^{p} & X_{2n}^{p} & X_{3n}^{p} & \cdots\cdots & 0 \end{pmatrix} \begin{pmatrix} \theta_{i1} \\ \theta_{i2} \\ \vdots \\ \theta_{in} \end{pmatrix} = \begin{pmatrix} -X_{1i}^{p^2} \\ -X_{2i}^{p^2} \\ \vdots \\ -X_{i-1i}^{p^2} \\ 0 \\ \vdots \\ X_{in}^{p^2} \end{pmatrix}$$

We observe that our matrix of the coefficients is independent of i, and it is the transpose of the matrix of the coefficients of $D(\beta_j,0)$; $j = 1,2,\ldots,n$ in ad $v_{\beta_i}^{p}$; $i = 1,2,\ldots,n$. Hence it has rank t.

Due to the fact that for any $a \in Z_p$, $a^{p^s} = a$ for $s \geq 0$; it will follow that whenever $r_j^{(1)}$ has all its components from Z_p, then $r_j^{(k)} = r_j^{(1)}$ for all $k \geq 1$; and hence ad $v_{\beta_j}^{p^k}$; $k \geq 1$ will all be elements of D_1. The biadditive function f will then be a nondegenerate

bilinear form; and therefore will have rank n. Hence dim D_1 = n = dim $L^*(G,\delta,f)$. Consequently, whenever t < n, we can use lemma 5 to conclude that at least n - t rows (respectively columns) of the matrix of f will contain a component not from Z_p. Using the fact that if $\alpha \varepsilon Z_p$ such that $x^p = \alpha$ for some x, then $x = \alpha$, it follows that at least n-t rows (respectively columns) of the matrix of the coefficients of $D(\beta_k,0)$; k = 1,2,...,n in ad $v^p_{\beta_j}$; j = 1,2,...,n have entries not from Z_p. Therefore some of ad $v^{p^2}_{\beta_i}$; i = 1,2,...,n are not in D_1. Hence dim D_2 = dim D_1 + X_2 for $1 \leq x_2 \leq t$.

Supposing some two rows of the matrix of coefficients of $D(\beta_k,0)$; k = 1,2,...,n in ad $v^p_{\beta_j}$; j = 1,...,n are linearly dependent over Z_p. Assume the first two rows are linearly dependent over Z_p. Since $x^p = y$ if and only if x = y for all x and y in a field of characteristic p > 0, it follows that the two corresponding rows of the matrix of f will also be linearly dependent over Z_p. But this will contradict the nondegeneracy of f over Z_p as proved in lemma 3. Hence no two of the rows of the matrix of the coefficients of $D(\beta_k,0)$ in ad $v^p_{\beta_j}$; k,j=1,2,...,n can be linearly dependent over Z_p.

Next we consider the case when the rows of the matrix of the coefficients are linearly dependent over Φ. Let λ_i; i = 1,2,...,n be some elements of Φ such that $\sum_{i=1}^{n} \lambda_i r_i^{(1)} = 0$. We note that, for any i, the matrix of the coefficients of θ_{ik}; k = 1,2,...,n is unchanged;

and due to the nondegeneracy of f over Z_p, one of $X^{p^2}_{ij}$; $j = 1,2,\ldots,n$

is nonzero. Hence the linear system of equations is nonhomogeneous.

Therefore we can use the well-known fact that a linear system

of n nonhomogeneous equations $\sum_{\nu=1}^{m} a_{\mu\nu}\psi_\nu = b_\mu$ $(\mu = 1,2,\ldots,n)$ in m

unknowns $\psi_\nu (\nu = 1,2,\ldots,m)$ has a solution if and only if every

solution $\eta_\mu (\mu = 1,2,\ldots,n)$ of the transposed homogeneous system

$\sum_{\mu=1}^{n} a_{\mu\nu}\psi_\mu = 0$ $(\nu = 1,2,\ldots,m)$ satisfies the orthogonality-relation

given by $\sum_{\mu=1}^{n} b_\mu \eta_\mu = 0$ to deduce that the system $\sum_{j=1}^{n} X^p_{kj}\theta_{ij} = X^{p^2}_{ki}$;

$k = 1,2,\ldots,n$ has a solution if and only if any solution ψ_{ij};

$j = 1,2,\ldots,n$ of the transposed homogeneous system $\sum_{k=1}^{n} X^p_{kj}\theta_{ik} = 0$;

$j = 1,2,\ldots,n$ satisfies $\sum_{k=1}^{n} X^{p^2}_{ki}\psi_{ik} = 0$ for each $i = 1, 2,\ldots,n$.

The skew-symmetry of the matrix of the coefficients of θ_{ij}; $j = 1,2,\ldots,n$ enables the solutions of the transposed homogeneous system to coincide with the solutions of the homogeneous system, without transposition.

It then follows that λ_j; $j = 1,2,\ldots,n$ must satisfy the system

of equation given by $\sum_{k=1}^{n} X^{p^2}_{ki}\psi_{ik} = 0$; $i = 1,2,\ldots,n$.

If the vectors $r_i^{(2)}$ lie in V_1, then ad $v_{\beta_i}^{p^{h+1}}$ will all be

vectors in D_1 for $h \geq 1$. That is $D_{h+1} \subseteq D_1$ for all $h \geq 1$. Consequently,

$r_i^{(h+1)}$ will be in V_1 for all $h \geq 1$. Since the matrix of the coefficients of θ_{ij}; $j = 1,2,\ldots,n$ is independent of i we have

$$\sum_{k=1}^{n} X_{ki}^{p^{h+1}} \lambda_k = 0; \quad i = 1,2,\ldots,n, \quad h \geq 1.$$

Therefore, if $i = 1,2,\ldots,n$, every solution λ_j; $j = 1,2,\ldots,n$ of the homogeneous part of the nonhomogeneous linear system will be a solution to each of the following n homogeneous systems:

$$\sum_{k=1}^{n} X_{ki}^{p^{h+1}} \psi_k = 0; \quad h \geq 1.$$

By the nondegeneracy of f over Z_p, none of the n systems will be a trivial system; and, by all the rows of their matrices of the coefficients being in V_1, the rank of each of the n systems is at most t.

Let W be the space of solutions of $\sum_{k=1}^{n} X_{kj}^{p} \psi_k = 0$; $j = 1,2,\ldots,n$; and U_i be the space of solutions of $\sum_{k=1}^{n} X_{ki}^{p^{h+1}} \psi_k = 0$; $h \geq 1$, where $i = 1,2,\ldots,n$. Then W is $(n-t)$-dimensional, and $W \subseteq U_i$ for each $i = 1,2,\ldots,n$.

We note that none of the n systems with the solution-spaces as U_i; $i = 1,2,\ldots,n$ can have rank t. The reason for this is that if the rank of the matrix of the coefficients of $\sum_{k=1}^{n} X_{ks}^{p^{h+1}} \psi_k = 0$; $h \geq 1$ is t for some s, then dim U_s = dim W. But $W \subseteq U_s$. Hence $W = U_s$. Since the vector $(0,\ldots,0,1,0,\ldots,0)$ in $\Phi^{(n)}$ with 1 in the sth place

and zeros elsewhere is in U_s, it will also be in W. This will force the s-column of the matrix of the coefficients of $\sum_{k=1}^{n} X_{kj}^{p}\theta_{ik} = 0$; $j = 1,2,\ldots,n$ to be zero, and thus contradicting the nondegeneracy of f over Z_p.

Assume all the matrices of the coefficients of the n systems $\sum_{k=1}^{n} X_{k_i}^{p^{h+1}}\psi_h = 0$; $h \geq 1$ have rank less than t, and consider the system $\sum_{k=1}^{n} X_{k1}^{p^{h+1}}\psi_h = 0$; $h \geq 1$. Let the matrix of its coefficients have rank r_1. It can, as will be explained later, always be assumed that the first r_1 rows are linearly independent over Φ. With this assumption there will exist γ_j; $j = 1,2,\ldots,r_1$ in Φ such that X_{1i}; $i = 2,\ldots,n$ satisfy the equation $x^{p^{r_1}} + \gamma_{r_1} x^{p^{r_1-1}} + \ldots + \gamma_2 x^{p} + \gamma_1 x = 0$.

By lemma 2 and its corollary, the roots of such an equation will form at most an r_1-dimensional vector space over Z_p. Therefore there will exist at most r_1 linearly independent elements over Z_p among X_{1i}; $i = 2,3,\ldots,n$. From the fact that a relation between the X_{1i}; $i = 2,3,\ldots,n$ with coefficients in Z_p holds if and only if the same relation holds among $X_{1i}^{p^h}$; $i = 2,\ldots,n$ with the same coefficients, where $h \geq 1$, it follows that the number of linearly independent column vectors over Z_p is at most r_1, and hence must be exactly r_1.

We now explain why the first r_1 rows must always be linearly independent over Φ. Suppose they are linearly dependent over Φ.

Then X_{1i}; $i = 2,\ldots,n$ will be roots of an equation over Φ whose degree is lower than p^{r_1}, and thus making the number of linearly independent column vectors over Z_p less than r_1. This will give linear dependence relation over Z_p between any r_1 columns, which will be in contradiction to the existence of r_1 linearly independent columns over Φ.

Denote the columns of the matrix of the coefficients of $\sum_{k=1}^{n} X_{k1}^{p^{h+1}} \psi_k = 0$; $h \geq 1$ by C_i; $i = 1,2,\ldots,n$. Without any loss of generality assume that the last r_1 columns are linearly independent over Φ. We observe that W_1 has dimension $n-r_1$.

It then follows that for each $i = 1,2,\ldots,n-r_1$, we have a relation $\alpha_{1i}C_{n-r_1+1} + \alpha_{2i}C_{n-r_1+2} + \ldots + \alpha_{r_1 i}C_n + C_i = 0$, where α_{ji} are in Z_p. The meaning of this is that the system admits the following $n-r_1$ linearly independent solutions over Φ:
$\alpha_i = (0,\ldots,0,1,0,\ldots,0,\alpha_{1i},\alpha_{2i},\ldots,\alpha_{r_1 i})$; $i = 1,2,\ldots,n-r_1$; where 1 is the ith component of α_i. Hence α_i; $i = 1,2,\ldots,n-r_1$ form a basis of U_1 over Φ. Thus U_1 will have a basis in $Z_p^{(n)}$.

Let r_i be the rank of the matrix of the coefficients of $\sum_{k=1}^{n} X_{ki}^{p^{h+1}} \psi_k = 0$; $h \geq 1$, where $i = 2,3,\ldots,n$. Then U_i is $(n-r_i)$-dimensional over Φ. With the same arguments as for the system $\sum_{k=1}^{n} X_{k1}^{p^{h+1}} \psi_k = 0$; $h \geq 1$ it can be shown that every U_i has a basis in $Z_p^{(n)}$.

Due to the assumption that the rows of the matrix of coefficients of $\sum_{j=1}^{n} X_{kj}^{p}\theta_{ij} = X_{ki}^{p^2}$; $k = 1,2,\ldots,n$ are linearly dependent over Φ, we have $n > t$. Therefore W is never trivial; and has in fact infinite elements. Consequently $\bigcap_{i=1}^{n} U_i$ is not zero.

Consider the space $U_1 \cap U_2$. Since U_1 and U_2 have bases in $Z_p^{(n)}$, lemma 7 can be invoked to get a basis of $U_1 \cap U_2$ in $Z_p^{(n)}$.

The consideration of $U_1 \cap U_2$ can be extended, inductively, to $\bigcap_{i=1}^{n} U_i$; thereby getting a basis of $\bigcap_{i=1}^{n} U_i$ in $Z_p^{(n)}$. Let $\gamma = (\gamma_1,\gamma_2,\ldots,\gamma_n)$ be in $Z_p^{(n)}$ such that γ is in $\bigcap_{i=1}^{n} U_i$. Then, for any $h \geq 1$, γ is a solution of the homogeneous linear system $\sum_{k=1}^{n} X_{kj}^{p^{h+1}}\psi_k = 0$; $j = 1,2,\ldots,n$. From $0 = \sum_{k=1}^{n} X_{kj}^{p^{h+1}}\gamma_k = (\sum_{k=1}^{n} X_{kj}\gamma_k)^{p^{h+1}}$, we get $\sum_{k=1}^{n} X_{kj}\gamma_k = 0$; $j = 1,2,\ldots,n$. Since the matrix of the coefficients of $\sum_{k=1}^{n} X_{kj}\gamma_k = 0$; $j = 1,2,\ldots,n$ is the same as the matrix of f, then γ_i; $i = 1,2,\ldots,n$ will give a non-trivial linear relation over Z_p among the rows of the matrix of f. But the nondegeneracy of f, as shown in lemma 3, prohibits any linear dependence over Z_p among the rows (respectively columns) of the matrix of f.

Therefore there exists a k such that $r_k^{(2)}$ is not in V_1, and

hence ad $v^{p^2}_{\beta_k}$ is a vector in D_2 which is not in D_1. This may turn out to be the only vector in D_2 which is not in D_1, and there may be as many as t of them. Thus, as a result of adding $(\text{ad } L(G,\delta,f))^{p^2}$ to $(\text{ad } L(G,\delta,f))^p$, the following holds: $\dim D_2 = \dim D_1 + x_2$ for $1 \leq x_2 \leq t$.

The case where the n rows are linearly independent over Φ will automatically give $(\text{ad } L(G,\delta,f))^p = \text{ad } v^p_{\beta_i} \mid i = 1,2,\ldots,n$.

We now consider the vector space D_3. Let V_2 be the space spanned over Φ by $r_i^{(1)}$, $r_i^{(2)}$; $i = 1,2,\ldots,n$. We want to show that in the situation where the dimension of D_2 does not give us the dimension of the outer derivation algebra, we can still find some ad $v^{p^3}_{\beta_k}$ such that ad $v^{p^3}_{\beta_k}$ is in D_3 but not in D_2.

Suppose $r_k^{(3)}$ were to lie in V_2, there should exist $\theta^{(2)}_{kj}, \theta^{(1)}_{kj}$; $j = 1,2,\ldots,n$ in Φ some of which are not zero such that $r_k^{(3)} = \sum_{i=1}^{n} [\theta^{(2)}_{ik} r_i^{(2)} + \theta^{(1)}_{ik} r_i^{(1)}]$. From this we obtain the following n linear nonhomogeneous equations in $\theta^{(2)}_{kj}, \theta^{(1)}_{kj}$; $j = 1,2,\ldots,n$:

$$\sum_{j=1}^{n} [X^{p^2}_{ij} \theta^{(2)}_{kj} + X^p_{ij} \theta^{(1)}_{kj}] = X^{p^3}_{ik}; \quad i = 1,2,\ldots,n.$$

Let $U^{(i)}$; $i = 1,2,\ldots,n$ denote the rows of the matrix of its coefficients, and let $C_i^{(j)}$; $i = 1,2,\ldots,n$; $j = 1,2$ denote its columns.

Suppose $U^{(i)}$; $i = 1,2,\ldots,n$ are linearly independent over Φ, then the row rank of the matrix of the coefficients is n. Therefore its column rank is also n; and some n of the 2n columns will be linearly independent over Φ. But the columns of the matrix of the coefficients are all vectors in V_2. Therefore some n of ad $v_{\beta_i}^{p^j}$; $j = 1,2$; $i = 1,2,\ldots,n$ will form a basis of D_2, and thereby contradicting the assumption that ad $v_{\beta_k}^{p^3}$ for some k, is in D_3 but not in D_2.

If $U^{(i)}$; $i = 1,2,\ldots,n$ were linearly dependent over Φ, then for some α_j; $i = 1,2,\ldots,n$ in Φ, we will have $\sum_{i=1}^{n} \alpha_i U^{(i)} = 0$. It would then follow from the consideration of V_1 that the linear nonhomogeneous system in $\theta_{kj}^{(2)}, \theta_{kj}^{(1)}$; $j = 1,2,\ldots,n$ will have a solution if and only if the solutions α_i; $i = 1,2,\ldots,n$ of the transposed homogeneous system $\sum_{k=1}^{n} X_{ki}^{p^j} \psi_k = 0$; $i = 1,2,\ldots,n$; $j = 1,2$, of the homogeneous system $\sum_{j=1}^{n} [X_{ij}^{p^2} \theta_{kj}^{(2)} + X_{ij}^{p} \theta_{kj}^{(1)}] = 0$; $i = 1,2,\ldots,n$ satisfy $\sum_{k=1}^{n} X_{ki}^{p^{h+2}} \alpha_k = 0$; $i = 1,\ldots,n$; $h \geq 1$.

Let the rank of the matrix of the coefficients of $\sum_{k=1}^{n} X_{ki}^{p^j} \alpha_k = 0$; $i = 1,2,\ldots,n$; $j = 1,2$, be t_1. If for some s, the system $\sum_{k=1}^{n} X_{ks}^{p^{h+2}} \phi_k = 0$; $h \geq 1$ has rank t_1, then its space of solutions will

coincide with the space of solutions of $\sum_{k=1}^{n} X_{ki}^{p^j}\psi_k = 0$; $i = 1,\ldots,n$; $j = 1,2$. This will make the matrix of f to have a zero column, and thereby contradicting the nondegeneracy of f over Z_p. On the other hand, if all the n systems $\sum_{k=1}^{n} X_{ki}^{p^{h+2}}\phi_k = 0$; $h \geq 1$ have rank less than t_1; the arguments of V_1 can be applied almost directly to deduce that $r_k^{(3)}$ can only be in V_1, for all $k = 1,2,\ldots,n$, if there is linear dependence relation over Z_p between the rows of the system $\sum_{k=1}^{n} X_{ki}^{p^j}\psi_k = 0$; $i = 1,2,\ldots,n$; $j = 1,2$. But this will imply a dependence relation between the rows of the matrix of f, thus giving a contradiction to the nondegeneracy of f over Z_p.

Then there is some i such that $r_i^{(3)}$ is not in V_2. Hence ad $v_{\beta_i}^{p^3}$ will not be in D_2. As a result of this, we get the following:

$\dim D_3 = \dim D_2 + x_3$; $1 \leq x_3 \leq t$.

In general we consider s such that $\dim D_{s-1} < n$. Let V_{s-1} be spanned by $r_i^{(j)}$; $j = 1,2,\ldots,s-1$; $i = 1,\ldots,n$ over Φ. For $r_k^{(s)}$ to be in V_{s-1} we need $\theta_{ki}^{(j)}$; $j = 1,2,\ldots,s-1$; $i = 1,2,\ldots,n$ in Φ such that $r_k^{(s)} = \sum_{i=1}^{n} [\theta_{ki}^{(s-1)} r_i^{(s-1)} + \theta_{ki}^{(s-2)} r_i^{(s-2)} + \ldots + \theta_{ki}^{(1)} r_i^{(1)}]$. From which we get the following n systems in $\theta_{ki}^{(j)}$:

$$\sum_{j=1}^{n} [X_{ij}^{p^{s-1}} \theta_{kj}^{(s-1)} + X_{ij}^{p^{s-2}} \theta_{kj}^{(s-2)} + \ldots + X_{ij}^{p} \theta_{kj}^{(1)}] = X_{ik}^{p^{s}}; \; i = 1,2,\ldots,n.$$

Denote the rows of the matrix of the coefficients by $U^{(i)}$; $i = 1,2,\ldots,n$, and the columns by $C_i^{(j)}$; $i = 1,2,\ldots,n$; $j = 1,2,\ldots,s-1$. If the n rows of the matrix of the coefficients are linearly independent over Φ, then there will also be n linearly independent columns over Φ. But the columns of the matrix of the coefficients are also vectors in the space V_{s-2} spanned by $r_i^{(j)}$; $j = 1,2,\ldots,s-2$; $i = 1,\ldots,n$ over Φ. Hence V_{s-2} will have n vectors which are linearly independent over Φ. This is equivalent to saying that there are some n of ad $v_{\beta_i}^{p^j}$;

$j = 1,2,\ldots,s-2$; $i = 1,2,\ldots,n$, which are linearly independent over Φ. If this happens, then D_{s-1} will have its dimension equal to the dimension of the outer derivation algebra; contrary to our initial assumption. Hence the rows of the matrix of the coefficients must be linearly dependent over Φ.

Supposing $U^{(i)}$; $i = 1,2,\ldots,n$ were linearly dependent over Φ, then there would exist α_i; $i = 1,2,\ldots,n$ in Φ such that

$\sum_{i=1}^{n} \alpha_i U^{(i)} = 0$; and α_i; $i = 1,2,\ldots,n$ constitute a solution to the transposed homogeneous system, consisting of n(s-1) equations, of the homogeneous part of the above nonhomogeneous system.

The nonhomogeneous system in $\theta_{ki}^{(j)}$ will have a solution if and only if any solution α_i; $i = 1,2,\ldots,n$ of the transposed homogeneous system $\sum_{k=1}^{n} X_{ik}^{p^j} \psi_k = 0$; $i = 1,2,\ldots,n$; $j = 1,2,\ldots,s-1$ satisfy

$\sum_{k=1}^{n} X_{ik}^{p^{h+s-1}} \phi_k = 0$; $h \geq 1$. Let the rank of the matrix of the coefficients of $\sum_{k=1}^{n} X_{ik}^{p^j} \psi_k = 0$; $i = 1,2,\ldots,n$; $j = 1,2,\ldots,s-1$ be t_{s-2}. Then similar arguments to that of V_1 will show that for all the rows of the matrices of coefficients of $\sum_{k=1}^{n} X_{ik}^{p^{h+s-1}} \phi_k = 0$; $h \geq 1$ to be in V_{s-1}, the rows of the matrix of f must be linearly dependent over Z_p. This will be in contradiction to the nondegeneracy of f over Z_p as was proved in the third lemma.

Therefore if D_{s-1} does not have n linearly independent elements over Φ, we can, as a result of adding $(\text{ad } L(G,\delta,f))^{p^s}$ to D_{s-1}, find a vector in D_s which is not in D_{s-1}. Hence $\dim D_s = \dim D_{s-1} + x_s$; $1 \leq x_s \leq t$.

As a consequence of the result we have the dimension of $(\text{ad } L(G,\delta,f))^p + (\text{ad } L(G,\delta,f))^{p^2} + \ldots$ equal to n; and hence, for $G = G_0$, the dimension of $(\text{ad } L(G,\delta,f))^p + (\text{ad } L(G,\delta,f))^{p^2} + \ldots$ for any simple Lie algebra $L(G,\delta,f)$ of Block type is independent of the nondegenerate and skew-symmetric biadditive function f with respect to which $L(G,\delta,f)$ is defined.

LEMMA 11: Let rank $f = t$, rank $f_i = t_i$, and $G_0 \neq 0$, $G \neq G_0$. If $L(G_i,\delta_i,f_i)$ is the simple Lie algebra which arises as a result of identifying G with G_i for $i = 0,1,2,\ldots,m$, then for each $i = 0,1,2,\ldots,m$, $\dim[D_{j-1} + (\text{ad } L(G_i,\delta_i,f_i))^{p^j}] = \dim D_{j-1} + x_{ij}$; $1 \leq x_{ij} \leq t_i$ provided $\dim[(\text{ad } L(G_i,\delta_i,f_i))^p + \ldots + (\text{ad } L(G_i,\delta_i,f_i))^{p^{j-1}}] < n_i - 1$ for $i = 1,2,\ldots,m$, and $\dim[(\text{ad } L(G_0,\delta_0,f_0))^p + \ldots + (\text{ad } L(G_0,\delta_0,f_0))^{p^{j-1}}]$

$< n_0$.

Consequently, the dimension of $(\text{ad } L(G,\delta,f))^p + (\text{ad } L(G,\delta,f))^{p^2}$ + ... is n-m.

PROOF: If the order of G_i is p^{n_i} for each i = 0,1,...,m, then $n = \sum_{i=0}^{m} n_i$, and $t = \sum_{i=0}^{m} t_i$. Let $\{\beta_{ij} | j = 1,2,...,n_i;\ i = 0,1,...,m\}$ be a basis of G consisting of β_{ij} in G_i with 1 in the jth position and zeros elsewhere. Then $\{\beta_{ij} | j = 1,2,...,n_i\}$ will be a basis of G_i. Every α_i in G_i is of the form $(i_1,i_2,...,i_{n_i})$. Let α and η be some two elements of G. From $v_\alpha \text{ad } v_\eta^p = \sum_{i=0}^{m} [f(\alpha_i,\eta_i)^p - f(\eta_i,\delta_i)^{p-1} f(\alpha_i,\eta_i)]v_\alpha$, we have $v_\alpha \text{ad } v_{\eta_0}^p = f_0(\alpha_0,\eta_0)^p v_\alpha$, and $v_\alpha \text{ad } v_{\eta_i}^p = [f_i(\alpha_i,\eta_i)^p - f_i(\eta_i,\delta_i)^{p-1} f_i(\alpha_i,\eta_i)]v_\alpha$, where α_i,η_i are the components of α,η in G_i, respectively; and i = 1,2,...,m.

Since f_i has been shown in [2] to be nondegenerate over Z_p, $L(G_i,\delta_i,f_i)$ is a simple Lie algebra.

It follows from the expressions of $\text{ad } v_{\eta_i}^p$; i = 0,1,...,m that to find the dimension of $(\text{ad } L(G,\delta,f))^p + (\text{ad } L(G,\delta,f))^{p^2} + ...$ we need to find the dimension of $(\text{ad } L(G_i,\delta_i,f_i))^p + (\text{ad } L(G_i,\delta_i,f_i))^{p^2}$ + ... for each i = 0,1,...,m. The sum of all the dimensions will then be the dimension of $(\text{ad } L(G,\delta,f))^p + (\text{ad } L(G,\delta,f))^{p^2} + ...$

Let $F_{ij} = (\text{ad } L(G_i,\delta_i,f_i))^p + ... + (\text{ad } L(G_i,\delta_i,f_i))^{p^j}$. To

show, for a fixed $i\varepsilon\{0,1,\ldots,m\}$, that $\dim[D_{j-1} + (\text{ad } L(G_i,\delta_i,f_i))^{p^j}] =$ $\dim D_{j-1} + x_{ij}$; $1 \leq x_{ij} \leq t_i$, whenever $\dim F_{0,j-1} < n_0$ if $i = 0$, or $\dim F_{i,j-1} < n_i - 1$ if $i = 1,2,\ldots,m$, it is sufficient to show that $\dim F_{ij} = \dim F_{i,j-1} + x_{ij}$; $1 \leq x_{ij} \leq t_i$, whenever $\dim F_{0,j-1} < n_0$, where $i = 0$, or $\dim F_{i,j-1} < n_i - 1$ if $i = 1,2,\ldots,m$.

For $L(G_0,\delta_0,f_0)$, we have $\text{ad } v_{\beta_{0i}}^p = \sum_{j=1}^{n_0} f_0(\beta_{0j},\beta_{0i})^p D(\beta_{0j},0)$; $i = 1,2,\ldots,n_0$. Hence f_0 will have a similar setting to that of the function considered in lemma 10. We can then conclude that $\dim[F_{0,j-1} + (\text{ad } L(G_0,\delta_0,f_0))^{p^j}] = \dim F_{0,j-1} + x_{0j}$; $1 \leq x_{0j} \leq t_0$, whenever $\dim F_{0,j-1} < n_0$. Hence $\dim[D_{j-1} + (\text{ad } L(G_0,\delta_0,f_0))^{p^j}] =$ $\dim D_{j-1} + x_{0j}$; $1 \leq x_{0j} \leq t_0$, if $\dim F_{0,j-1} < n_0$.

In $L(G_i,\delta_i,f_i)$ for $i = 1,2,\ldots,m$, we have $\text{ad } v_{\beta_{ik}}^p =$

$$\sum_{j=1}^{n_i} [f_i(\beta_{ij},\beta_{ik})^p - f_i(\beta_{ik},\delta_i)^{p-1} f_i(\beta_{ij},\beta_{ik})] D(\beta_{ij},0) \text{ for } k = 1,2,\ldots,n_i.$$

From the similarity in the expressions of $\text{ad } v_{\beta_{ik}}^p$; $k = 1,2,\ldots,n_i$ in $D(\beta_{ij},0)$; $j = 1,2,\ldots,n_i$ for each $i = 1,2,\ldots,m$, it suffices to consider the effect of a particular f_i on G_i, and then to extend the result to all the others. In other words, we only need to work in the algebra $L(G_i,\delta_i,f_i)$ for some $i\varepsilon\{1,2,\ldots,m\}$.

Let us consider $L(G_1,\delta_1,f_1)$. Then

$$\text{ad } v^p_{\beta_{1i}} = \sum_{j=1}^{n_1} [f_1(\beta_{1j},\beta_{1i})^p - f_1(\beta_{1i},\delta_1)^{p-1} f(\beta_{1j},\beta_{1i})] D(\beta_{1j},0),$$

where $i = 1,2,\ldots,n_1$; and, from the definition of $D(\beta_{1j},0)$; $j = 1,2,\ldots,n_1$, $f(\beta_{11},\delta_1)$ is assumed to be nonzero.

Denoting $f(\beta_{1j},\beta_{1i})$ by X_{ij}, and defining δ_1 as β_{1n}, we get the following matrix identity:

$$
\begin{pmatrix} \operatorname{ad} v^p_{\beta_{11}} \\ \operatorname{ad} v^p_{\beta_{12}} \\ \vdots \\ \operatorname{ad} v^p_{\beta_{1n_1-1}} \\ \operatorname{ad} v^p_{\beta_{1n_1}} \end{pmatrix} = \begin{pmatrix} 0 & X^p_{12}-X^{p-1}_{n_1 1}X_{12} & \cdots & X^p_{1n_1-1}-X^{p-1}_{n_1 1}X_{1n_1-1} \\ -(X^p_{12}-X^{p-1}_{n_1 2}X_{12}) & 0 & \cdots & X^p_{2n_1-1}-X^{p-1}_{n_1 2}X_{2n_1-1} \\ \vdots & \vdots & & \vdots \\ -(X^p_{1n_1-1}-X^{p-1}_{n_1 n_1-1}X_{1n_1-1}) & -(X^p_{2n_1-1}-X^{p-1}_{n_1 n_1-1}X_{2n_1-1}) & \cdots & 0 \\ -X^p_{1n_1} & -X^p_{2n_1} & \cdots & -X^p_{n_1-1,n_1} \end{pmatrix} \begin{pmatrix} D(\beta_{11},0) \\ D(\beta_{12},0) \\ \vdots \\ D(\beta_{1n_1-1},0) \end{pmatrix}
$$

Since X_{1n_1} is not zero, ad $v^p_{\beta_{1n_1}}$ cannot be zero. The rank of the matrix of the coefficients of $D(\beta_{1j},0)$; $j = 1,2,\ldots,n_1-1$ in ad $v^p_{\beta_{1i}}$; $i = 1,2,\ldots,n_1$ is then at least one and at most n_1-1.

If n_1-1 of the rows of the matrix of the coefficients of $D(\beta_{1j},0)$; $j = 1,2,\ldots,n_1-1$ in ad $v^p_{\beta_{1i}}$; $i = 1,2,\ldots,n_1$ are linearly independent over Φ, then some n_1-1 of ad $v^p_{\beta_{1i}}$; $i = 1,2,\ldots,n_1$ will be a basis of $(\text{ad } L(G_1,\delta_i,f_1))^p$, and in this case we will have dim $F_{11} = n_1-1$.

Suppose the rank of the matrix of coefficients of $D(\beta_{1j},0)$; $j = 1,2,\ldots,n_1-1$ in ad $v^p_{\beta_{1i}}$; $i = 1,2,\ldots,n_1$ is less than n_1-1, then the dimension of F_{11} will be less than n_1-1. In this case we can investigate the vector space F_{12} to see if we can find a vector in it which does not lie in F_{11}.

Identifying the matrix of the coefficients of $D(\beta_{1j},0)$; $j = 1,2,\ldots,n_1-1$ in ad $v^p_{\beta_{1i}}$; $i = 1,2,\ldots,n_1$ by (a^p_{ij}); $1 \leq i \leq n_1$; $1 \leq j \leq n_1-1$, we will have ad $v^{p^j}_{\beta_{1i}} = \sum_{j=1}^{n-1} a^{p^j}_{ij} D(\beta_{1i},0)$; $i = 1,2,\ldots,n_1$; $j \geq 1$. Let U_1 be a subspace of $\Phi^{(n-1)}$ spanned by $r^{(1)}_{1i}$; $i = 1,2,\ldots,n_1$ over Φ, where $r^{(j)}_{1i} = (a^{p^j}_{i1}, a^{p^j}_{i2}, \ldots, a^{p^j}_{i,n_1-1})$, $j \geq 1$. It then follows that $\dim_\Phi U_L < n_1-1$.

If $r^{(2)}_{1i}$ is to be in U_1 there will be θ_{ij}; $i = 1,2,\ldots,n_1$ in

Φ not all zero such that $r_{1i}^{(2)} = \sum_{j=1}^{n_1} \theta_{ij} r_{1j}^{(1)}$. This will give rise to the following system of equations in θ_{ij}; $j = 1,2,\ldots,n_1$:

$$\sum_{j=1}^{n_1} a^p_{jk} \theta_{ij} = a^{p^2}_{ik}; \quad k = 1,2,\ldots,n_1-1.$$

Let the rows of the matrix of the coefficients of θ_{ij}; $j = 1,2,\ldots,n_1$ be linearly independent over Φ. Then some n_1-1 of its n_1 columns are linearly independent over Φ. But the column vectors coincide with the vectors $r_{1i}^{(1)}$; $i = 1,\ldots,n_1$ in U_1. This will say that U_1 is (n_1-1)-dimensional, which contradicts an earlier assumption.

Then the rows of the matrix of the coefficients of θ_{ij}; $j = 1,2,\ldots,n_1$ are linearly dependent over Φ. Therefore any solution λ_i; $i = 1,2,\ldots,n_1-1$ of the system $\sum_{j=1}^{n_1-1} a^p_{ij}\psi_j = 0$; $i = 1,2,\ldots,n_1$, which is the transposed homogeneous system of the system in θ_{ij}; $i = 1,2,\ldots,n_1$, will satisfy $\sum_{j=1}^{n_1-1} a^{p^2}_{ij}\lambda_j = 0$.

If $r_{1i}^{(2)}$; $i = 1,2,\ldots,n_1$ all belong to the vector space U_1, then ad $v^p_{\beta_{1i}}{}^{h+1}$; $i = 1,2,\ldots,n_1$ will all be in F_{1h} for all $h \geq 1$. Hence $r_{1i}^{(h)}$; $i = 1,2,\ldots,n_1$; $h \geq 1$ will all be elements of U_1. Since the matrix of the coefficients of θ_{ij}; $j = 1,2,\ldots,n_1$ is not dependent on i, it will follow that $\sum_{j=1}^{n_1-1} a^{p^{h+1}}_{ij}\psi_j = 0$; $i = 1,2,\ldots,n_1, h \geq 1$. Thus

if $i = 1,2,\ldots,n_1$, every solution of $\sum_{j=1}^{n_1-1} a_{ij}^{p}\psi_j = 0$; $i = 1,2,\ldots,n_1$ will be a solution of each of the homogeneous systems: $\sum_{j=1}^{n_1-1} a_{ij}^{p^{h+1}}\lambda_j = 0$, $h \geq 1$.

It will then follow from the discussions in lemma 10 that the columns of the matrix of the coefficients of $\sum_{j=1}^{n_1-1} a_{ij}^{p}\psi_j = 0$; $i = 1,2,\ldots,n_1$ will be linearly dependent over Z_p.

Since the matrix of the coefficients of $\sum_{j=1}^{n_1-1} a_{ij}^{p}\psi_j = 0$; $j = 1,2,\ldots,n_1$ is the same as the matrix of the coefficients of $D(\beta_{1j},0)$; $j = 1,2,\ldots,n_1-1$ in $\operatorname{ad} v_{\beta_{1i}}^{p}$; $i = 1,2,\ldots,n_1$, then the columns of the matrix of the coefficients of $D(\beta_{1j},0)$; $j = 1,2,\ldots,n_1-1$ in $\operatorname{ad} v_{\beta_{1i}}^{p}$; $i = 1,2,\ldots,n_1$ will be linearly dependent over Z_p. But according to lemma 9 this cannot be possible.

Hence we can find a k such that $r_{1k}^{(2)}$ is not in U_1, and consequently $\operatorname{ad} v_{\beta_{ik}}^{p}$ will be a vector in the vector space F_{12} which is not in the space F_{11}. Therefore, as a result of adding $(\operatorname{ad} L(G_1,\delta_1,f_1))^{p^2}$ to $(\operatorname{ad} L(G_1,\delta_1,f_1))$, the following will hold: $\dim F_{12} = \dim F_{11} + x_{12}$ such that $1 \leq x_{12} \leq t_1$. Hence for D_1 we have

$\dim[D_1 + (\operatorname{ad} L(G_1,\delta_1,f_1))^{p^2}] = \dim D_1 + x_{12}$; $1 \leq x_{12} \leq t_1$.

We are now in a position to consider F_{13}. We want to show that in the event of the dimension of F_{12} being less than n_1-1, we

will have the following identity:

$$\dim[F_{12} + (\text{ad } L(G_1,\delta_1,f_1))^{p^3}] = \dim F_{12} + x_{13};\ 1 \leq x_{13} \leq t_1.$$

Let U_2 be the space spanned over Φ by $r_{1i}^{(1)}, r_{1i}^{(2)}$; $i = 1,2,\ldots,n_1$. Suppose $r_{1k}^{(3)}$; $k = 1,2,\ldots,n_1$ were all in U_2, then for each k we can find $\theta_{kj}^{(2)}, \theta_{kj}^{(1)}$; $j = 1,2,\ldots,n_1$ in Φ some of which are not zero such that $r_{1k}^{(3)} = \sum_{i=1}^{n_1} [\theta_{ik}^{(2)} r_{1i}^{(2)} + \theta_{ik}^{(1)} r_{1i}^{(1)}]$. This will give rise to the following equations in $2n_1$ variables:

$$\sum_{j=1}^{n_1} [a_{ji}^{p^2} \theta_{kj}^{(2)} + a_{ji}^{p} \theta_{kj}^{(1)}] = a_{ki}^{p^3};\ i = 1,2,\ldots,n_1-1.$$

Suppose the rows of the matrix of the coefficients of $\theta_{kj}^{(2)}, \theta_{kj}^{(1)}$; $j = 1,2,\ldots,n_1$ are linearly independent over Φ. Then n_1-1 of its columns will also be linearly independent over Φ. But all the columns of the matrix are vectors in U_2. Hence U_2 will contain n_1-1 linearly independent vectors over Φ, which is contrary to the assumption that U_2 has dimension less than n_1-1. Therefore the matrix of the coefficients cannot have n_1-1 linearly independent rows over Φ.

Since the rank of the matrix of coefficients of $\theta_{kj}^{(2)}, \theta_{kj}^{(1)}$; $j = 1,2,\ldots,n_1$ is less than n_1-1, it follows that any η_i; $i = 1,2,\ldots,n_1-1$, in Φ satisfying $\sum_{k=1}^{n_1-1} a_{ik}^{p^j} \eta_k = 0$; $i = 1,2,\ldots,n_1-1$ $j = 1,2$ will also satisfy $\sum_{k=1}^{n_1-1} a_{ik}^{p^{h+2}} \eta_k = 0$; $i = 1,2,\ldots,n_1$; $h \geq 1$.

In exactly the same manner as in lemma 10 we will find linear

dependence over Z_p among the columns of the matrix of the coefficients of $D(\beta_{1i},0)$; $i = 1,2,\ldots,n_1-1$ in ad $v^p_{\beta_{1j}}$; $j = 1,2,\ldots,n_1$.

But linear dependence over Z_p among the columns will give rise to a contradiction of lemma 10. Therefore not all of $r^{(3)}_{1i}$; $i = 1,2,\ldots,n_1$ can be in U_2.

Consequently there exists a k such that ad $v^{p^3}_{\beta_{1k}}$ is not in F_{12}. It then follows that $\dim[F_{12} + (\text{ad } L(G_1,\delta_1,f_1))^{p^3}] = \dim F_{12} + x_{13}$; $1 \leq x_{13} \leq t_1$.

In the general case we let $\dim F_{1,s-1} < n_1-1$, and U_{s-1} be the space spanned over Φ by $r^{(j)}_{1i}$; $j = 1,2,\ldots,s-1$; $i = 1,2,\ldots,n_1$. With similar arguments as in the consideration of U_1 and U_2, we will get a system of n_1-1 equations in $s(n_1-1)$ variables.

If the rows of the matrix of the coefficients of the system are linearly independent over Φ, then some n_1-1 of its $s(n_1-1)$ columns will also be linearly independent over Φ. Since all the columns are vectors in U_{s-1}, this will mean U_{s-1} is (n_1-1)-dimensional; thereby contradicting the fact that $\dim F_{1,s-1} < n_1-1$.

When the rows are linearly dependent over Φ, we can apply the same arguments as in the case of U_1 and U_2 to show that not all of the vectors $r^{(s)}_{1i}$; $i = 1,2,\ldots,n_1$ can be in U_s. Then there will exist a k such that ad $v^{p^s}_{\beta_{1k}}$ is not in $F_{1,s-1}$. Hence $\dim F_{1s} = \dim F_{1,s-1} + x_{1s}$; $1 \leq x_{1s} \leq t_1$. This shows that $\dim[D_{j-1} + (\text{ad } L(G_1,\delta_1,f_1))^{p^j}] = \dim D_{j-1} + x_{1j}$; $1 \leq x_{1j} \leq t_1$ until the dimension of $F_{1,j-1}$ reaches

the maximum possible dimension, which in this case is n_1-1.

Extending the proof to $i = 2,3,\ldots,m$, it follows that for each $i = 0,1,2,\ldots,m$, $\dim[D_{j-1} + (\text{ad } L(G_i,\delta_i,f_i))^{p^j}] = \dim D_{j-1} + x_{ij}$; $1 \leq x_{ij} \leq t_i$, whenever $\dim F_{i,j-1} < n_i-1$ for $i = 1,2,\ldots,m$, and $\dim F_{0,j-1} < n_0$.

Consequently, for $G_0 \neq 0$, $G \neq G_0$, the dimension of $(\text{ad } L(G,\delta,f))^p + (\text{ad } L(G,\delta,f))^{p^2} + \ldots$ is $n_0 + \sum_{i=0}^{m} (n_i-1) = n-m$; and hence the dimension is independent of the nondegenerate and skew-symmetric biadditive function f with respect to which $L(G,\delta,f)$ is defined.

LEMMA 12: Let the rank of f be t, the rank of f_i be t_i, and G be a group such that G_0 is assumed to be zero. If $L(G_i,\delta_i,f_i)$ is the simple Lie algebra which arises from the identification of G with G_i; $i = 1,2,\ldots,m$, then for each $i = 1,2,\ldots,m$

$$\dim[D_{j-1} + (\text{ad } L(G_i,\delta_i,f_i))^{p^j}] = \dim D_{j-1} + x_{ij},$$

where $1 \leq x_{ij} \leq t_i$; provided that

$$\dim[(\text{ad } L(G_i,\delta_i,f_i))^p + \ldots + (\text{ad } L(G_i,\delta_i,f_i))^{p^{j-1}}] < n_i-1$$

for $i = 1,2,\ldots,m$.

Therefore the dimension of $(\text{ad } L(G,\delta,f))^p + (\text{ad } L(G,\delta,f))^{p^2} + \ldots$ is $\sum_{i=1}^{m} (n_i-1) = n-m$.

PROOF: The proof is the same as in Lemma 11 except in this case n_0 is zero. Hence whenever n_0 appears in the proof of Lemma 11, it is replaced by zero.

REMARK 3: In view of the results achieved in lemmas 8 through 12, to prove the theorem which follows we only need to consider an arbitrary nondegenerate and skew-symmetric biadditive function defined on G.

THEOREM: The smallest restricted Lie algebra $RL(G,\delta,f)$ of $L(G,\delta,f)$ has dimension equal to $\dim[L(G,\delta,f)] + (n-2m)$.

PROOF: The proof of the theorem will consist of the following two cases: Case I, $G = G_0$; and Case II, $G_0 \neq 0$, $G \neq G_0$; or $G_0 = 0$.

Case I: Assume $G = G_0$. Since $L(G,\delta,f)$ is a simple Lie algebra we can identify $L(G,\delta,f)$ with ad $L(G,\delta,f)$, the algebra of inner derivations of $L(G,\delta,f)$. Then $\dim(\text{ad } L(G,\delta,f)) = \dim L(G,\delta,f) = p^n-1$. By lemma 1 we have

$$RL(G,\delta,f) = \sum_{i=0}^{k} (\text{ad } L(G,\delta,f))^{p^i} \text{ for some } k,$$

where $(\text{ad } L(G,\delta,f))^{p^j} \subseteq RL(G,\delta,f)$ for all $j \geq 0$. Hence

$$RL(G,\delta,f) = \sum_{i=0}^{\infty} (\text{ad } L(G,\delta,f))^{p^i}.$$

Since for any $\alpha,\beta \in G$, $v_\alpha o v_\beta = v_\alpha \text{ad } v_\beta = f(\alpha,\beta)v_{\alpha+\beta}$, and $v_\alpha(\text{ad } v_\beta)^{p^j} = f(\alpha,\beta)^{p^j}v_\alpha$ for $j \geq 1$, it follows that the only vector belonging to $[\text{ad } L(G,\delta,f)] \cap \sum_{j=1}^{\infty} (\text{ad } L(G,\delta,f))^{p^j}$ is the zero vector. Hence

$$\dim[\text{ad } L(G,\delta,f)] \cap \sum_{j=1}^{\infty} (\text{ad } L(G,\delta,f))^{p^j} = 0.$$

Consequently,

$$\dim RL(G,\delta,f) = \dim L(G,\delta,f) + \dim \sum_{j=1}^{\infty} (\text{ad } L(G,\delta,f))^{p^j}.$$

According to lemma 10 the dimension of $\sum_{j=1}^{\infty} (\text{ad } L(G,\delta,f))^{p^j}$ is n. Hence dim $RL(G,\delta,f)$ = dim $L(G,\delta,f)$ + n; and this proves the result in the case of $G = G_0$.

Case II: Let $G_0 \neq 0$, $G \neq G_0$; or $G_0 = 0$. $L(G,\delta,f)$ being a simple Lie algebra implies that $L(G,\delta,f)$ can be identified with ad $L(G,\delta,f)$. Hence dim(ad $L(G,\delta,f)$) = dim($L(G,\delta,f)$) = p^n-2. Using the first lemma we have $RL(G,\delta,f) = \sum_{j=0}^{k} (\text{ad } L(G,\delta,f))^{p^j}$ for some k, where $(\text{ad } L(G,\delta,f))^{p^i} \subseteq RL(G,\delta,f)$ for all $j \geq 0$. Hence $RL(G,\delta,f) = \sum_{j=0}^{\infty} (\text{ad } L(G,\delta,f))^{p^j}$. From lemmas 11 and 12 we have

$$\dim \sum_{j=1}^{\infty} (\text{ad } L(G,\delta,f))^{p^j} = n-m.$$

Let $\alpha,\beta \in G$ such that $\alpha = (\alpha_0,\alpha_1,\ldots,\alpha_m)$ and $\beta = (\beta_0,\beta_1,\ldots,\beta_m)$, where $\alpha_i,\beta_i \in G_i$; i = 0,1,...,m. From the multiplication table $v_\alpha o v_\beta = v_\alpha \text{ad } v_\beta = \sum_{i=0}^{m} f_i(\alpha_i,\beta_i) v_{\alpha+\beta-\delta_i}$ in $L(G,\delta,f)$ we will have $v_\alpha(\text{ad } v_{\beta_0})^{p^j} = f(\alpha_0,\beta_0)^{p^j} v_\alpha$, and $v_\alpha(\text{ad } v_{\beta_r})^{p^j} = [f(\alpha_r,\beta_r)^p - f(\beta_r,\delta_r)f(\alpha_r,\beta_r)]^{p^{j-1}} v_\alpha$; $j \geq 1$, and r = 1,2,...,m. It then follows

that $v_\alpha(\text{ad } v_\beta)^{p^j} = v_\alpha(\sum_{i=0}^{m} (\text{ad } v_{\beta_i})^{p^j})$ for $j \geq 1$. Consequently,

$$[\text{ad } L(G,\delta,f)] \cap [\sum_{j=1}^{\infty} (\text{ad } L(G,\delta,f))^{p^j}] = \{0, \text{ad } v_{\delta_i}^{p^h} \mid i = 1,2,\ldots,m;\ h \geq 0\}.$$

Since, for any pair of nonnegative integers r,s and any $i \in \{1,2,\ldots,m\}$, $\text{ad } v_{\delta_i}^{p^s}$ and $\text{ad } v_{\delta_i}^{p^r}$ are linearly dependent over Φ and, from the construction of the algebra $L(G,\delta,f)$, $\text{ad } v_{\delta_k}$ for $k = 1,2,\ldots,m$ are linearly independent over Φ, it follows that $\text{ad } v_{\delta_i}$; $i = 1,2,\ldots,m$ will constitute a basis of $[\text{ad } L(G,\delta,f)] \cap \sum_{j=1}^{\infty} (\text{ad } L(G,\delta,f))^{p^j}$. Hence

$$\dim[\text{ad } L(G,\delta,f)] \cap \sum_{j=1}^{\infty} (\text{ad } L(G,\delta,f))^{p^j} = m.$$

Since

$$\dim RL(G,\delta,f) = \dim[\text{ad } L(G,\delta,f) + \sum_{j=1}^{\infty} (\text{ad } L(G,\delta,f))^{p^j}] =$$

$$\dim \text{ad } L(G,\delta,f) + \dim \sum_{j=1}^{\infty} (\text{ad } L(G,\delta,f))^{p^j} -$$

$$\dim[\text{ad } L(G,\delta,f)] \cap \sum_{j=1}^{\infty} (\text{ad } L(G,\delta,f))^{p^j},$$

we have $\dim RL(G,\delta,f) = \dim \text{ad } L(G,\delta,f) + (n-m) - m$. Hence $\dim RL(G,\delta,f) = \dim L(G,\delta,f) + (n-2m)$ as required.

We deduce as immediate consequences of the theorem the following corollaries.

COROLLARY 1: The simple Lie algebra $L(G,\delta,f)$ is restricted if and only if $G_0 = 0$ and G_i; $i = 1,2,\ldots,m$ have order p^2.

The conclusion of Corollary 1 has been reached previously by Block in [2] using a different method.

COROLLARY 2: If $L(G,\delta,f)$ is simple, then $RL(G,\delta,f) = DL(G,\delta,f)$ if and only if $G = G_0$.

The truth of Corollary 2 follows from the fact that for $G = G_0$, $\dim DL(G,\delta,f) = \dim L(G,\delta,f) + n$; for $G_0 \neq 0$, $G \neq G_0$, $\dim DL(G,\delta,f) = \dim L(G,\delta,f) + n+1$; and for $G_0 = 0$, $\dim DL(G,\delta,f) = \dim L(G,\delta,f) + n+2$.

COROLLARY 3: Let G and G' be finite abelian groups which are direct sums of $m+1$ and $m'+1$ elementary p-groups, respectively. Then a necessary condition for any two simple algebras $L(G,\delta,f)$ and $L(G',\delta',f')$ of the same dimension to be isomorphic is either $G_0 = 0$, $G_0' = 0$ and $m = m'$; or $G_0 \neq 0$, $G_0' \neq 0$ and $m = m'$.

REMARK 4: The knowledge of the dimension of $RL(G,\delta,f)$ can hopefully be used to examine nonisomorphic simple non-restricted modular Lie algebras. The question as to when such known simple non-restricted algebras of the same dimension are nonisomorphic is a very deep one, which is yet to be settled.

In general, the Cartan subalgebra approach used in characteristic zero has been inadequate for modular Lie algebras, especially when two Cartan subalgebras of a modular Lie algebra need not have the

same dimension. In fact for any positive integer s, Block has exhibited a simple Lie algebra $L(G,\delta,f)$ with Cartan subalgebras of s distinct dimensions. Block algebras are also found to have Cartan subalgebras which are non-abelian; although all have a Cartan subalgebra which is abelian.

There is some dimensional coincidence between some algebras of type $L(G,\delta,f)$ and the non-restricted simple algebras of Jennings and Ree. By determining the dimensions of the smallest restricted Lie algebras of the algebras of Jennings and Ree one may hope to discuss isomorphism conditions for the two different classes of algebras. The same type of work can be extended to other classes of simple non-restricted modular Lie algebras, where conditions for isomorphism have not been derived.

BIBLIOGRAPHY

[1] Block, R. E., On torsion-free abelian groups and Lie algebras, Proc. A.M.S. vol 9 (1958), pp. 613-620.

[2] _____, New simple Lie algebras of prime characteristic, Trans. A.M.S. vol 89 (1958), pp. 421-449.

[3] Godement, R., Algebra, Hermann Publishers, Paris (1968).

[4] Greub, W. H., Linear Algebra, Springer-Verlag, New York, Inc. (1967).

[5] Jacobson, N., Classes of restricted Lie algebras of char. p,
I. Amer. J. Math vol 53 (1941), pp. 481-515
II. Duke Math J. vol 10 (1943), pp. 107-121.

[6] _____, Lie Algebras, Interscience Publishers, New York (1962).

[7] Kaplansky, I., Seminar on Simple Lie algebras, Bull. A.M.S. vol 60 (1954), pp. 470-471.

[8] Raikov, D. A., Vector Spaces, Groningen, The Netherlands (1965).

[9] Ree, R., On generalized Witt algebras, Trans. A.M.S. vol 83 (1956), pp. 510-546.

[10] Ree, R., and Jennings, S. A., On a family of Lie algebra of char. p, Trans. A.M.S. vol 84 (1957), pp. 192-207.

[11] ___, Note on Generalized Witt algebras, Canadian J. Math., vol 11 (1959), pp. 345-352.

[12] Seligman, G. B., Modular Lie algebras, Springer-Verlag, New York, Inc. (1967).

[13] Winter, D. J., Abstract Lie algebras, The MIT Press, Cambridge, Massachusetts (1972).

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