MULTISET THEORY:
A New Paradigm of Science

An Inaugural
LECTURE

By

Professor D. Singh
Professor of Mathematics,
Department of Mathematics
Ahmadu Bello University, Zaria - Nigeria

20th September, 2006

Multiset Theory: *A New Paradigm of Science*

AN
INAUGURAL LECTURE

BY

PROFESSOR D. SINGH
Professor of Mathematics.
Department of Mathematics,
Ahmadu Bello University,
Zaria, Nigeria

Wednesday, 20th September, 2006.



Professor D. Singh

Professor of Mathematics, Department of Mathematics,

Ahmadu Bello University, Zaria - Nigeria

Copyright © University Organized Lectures Committee, 2006

All Rights Reserved.

No part of the publication may be photocopied, recorded or otherwise, produced, stored in a retrieval system, or transmitted, in any form or by any means electronic, mechanical means without the prior permission of the copyright Author.

First Published 2006

A Publication of
The University Organized Lectures Committee
Vice-Chancellor's Office,
Ahmadu Bello University,
Zaria, Nigeria.

All enquires should be directed to:
The Chairman,
University Organized Lectures Committee
Vice-Chancellor's Office,
Ahmadu Bello University, Zaria, Nigeria.

Printed by
Ahmadu Bello University Press Limited,
P. M. B. 1094, Zaria, Kaduna State, Nigeria.
Telephone: 069-550054
E-Mail: abupresslimited@yahoo.com

TABLE OF CONTENTS

Introduction 1
Some early history of multisets..... 3
Theory of Multisets 7
Introduction 7
Representations of mset 9
Operations under Mset 12
Definition 1. Multiset 12
Definition 2. Dressed epsilon symbol, $\in_+$ 13
Definition 3. Equal msets 13
Definition 4. Multisubsets (or msubsets, for short)..... 14
Definition 5. Similar msets 14
Definition 6. Ordered pair of two mset terms 14
Definition 7. Power multiset. 15
Definition 8. Union ($\cup$), Intersection ($\cap$) and
addition or sum or merge (+) or $\uplus$ 15
Definition 9. Difference and complementation 19
Definition 10. Functions between msets 20
Definition 11. Multiset Ordering 22
Applications of multisets 23
Applications in Mathematics 24
Applications in Computer Science 27
Concluding remarks and some future directions 33
Some future directions 33
Acknowledgement 35
References 36

Multiset Theory: A New Paradigm of Science

Introduction

Set theory was discovered (or invented) by George Ferdinand Ludwig Philip Cantor (1845 – 1918), a German Mathematician. In post-Sputnik era, besides numerous publications appearing in the core area of set theory, one can hardly come across a book in other areas of mathematics which does not begin with some discussion of set theory. In fact, set theory has eventually become the language of science.

Cantor's concept of a set:

> *"By a 'set', we are to understand any collection M of definite and distinct objects m of our intuition or thought (which will be called the 'elements' of M) into a whole."*

Accordingly, one of the underlying assumptions of Cantorian set theory dictates that no element shall be allowed to appear more than once. The collection {a, b, b, c} consequently becomes a set only after deleting the repeated element viz, {a, b, c}. Indeed, this aspect of Cantorian set theory did not go hand in hand with the requirements of various other sciences in seeking mathematical formulation of some of the challenging problems. For example, repeated roots of polynomial equations, repeated observations in statistical samples, repeated hydrogen atoms in a water molecule, H_2O, etc. need to be counted for attaining adequacy and exactness. The concept of multiset ensues once the repeated elements are admitted in a set.

A multiset (mset, for short) is an unordered collection of objects (called the elements) in which, unlike a standard (Cantorian) set,

elements are allowed to repeat. In other words an mset is a set in which elements may belong more than once and hence it is a non-Cantorian set. Knuth ([knu 81], P.636) notes that "despite frequent occurrences of multisetlike structures in mathematics, there is currently no structured way to deal with multisets" and similar observations have been voiced by many others.

It is gratifying to note that during the recent years a sizeable number of papers, specially dealing with the development of the theory of multisets, have appeared and, not very surprisingly, some of the outcomes turned out to be exceedingly vexing. In part, it conforms to Cantor's resolute insistence on not admitting repeated elements in a set. One is tended to believe that time is getting ripe to have a clearer perspective of Cantor's compassion for delimiting himself to the development of set theory in its simplest form, called the *standard* or *crisp* set theory. In our opinion, a convergent view point of multitude of researches that have gone into developing a comprehensive theory of multisets dictates that it should be Cantor's fascinating theory of infinity that could not have straightforwardly got through in the generalized cases, such as multisets. Over the years, besides sporadic evidences of applications of mulisets in philosophy, Logic, Linguistics and Physics, a good number of them witnessed in mathematics and computer science which have led to the formulation of a comprehensive theory of multisets.

In this paper, we endeavour to present some of the history of multisets, a systematization of various approaches toward formalizing the concept of muliset and operations under multisets, an overview of recent applications of multisets and finally, propose some further research problems.

Some early history of multisets

The concept of multiple-membership collection is as old as the concept of number itself. For example, evidences of representing a number by a collection of tally marks or units are found in the work of the Babylonians (200 B.C.), Egyptians (3500 – 1700 B.C) and Greeks (see [Hal 94], p.132 and [Ifr 85] for details). Knuth ([Knu 81], p. 23) notes that enumeration of permutations of a set was known in ancient times and historically the first known document is the Hebrew Book of Creation (C. 100 A. D.), followed by the Indian Classic *Anuyogadvarā-sutra* (C.500 A. D.); and the corresponding results for multiset seems to have appeared first in another Indian Classic *Lilávati of Bháscara Achárya* (C. 1150). He further notes that Kircher (C. 1650, pp. 5 – 7) correctly gave the number of permutations of multiset {m.C, n.D} for several values of m and n, though without revealing his method of calculation except when n = 1. A generalization of the rule for enumerating the permutations of multiset appeared in *Prestet's Elémens de mathématiques* (Paris 1675, 351 – 352), and later in John Walli's Treatise of Algebra 2 (Oxford 1685, pp. 117– 118). It is heartening to note that, by exploiting Dominique Floata's work done in 1965, Knuth ([Knu 81], pp. 24 – 31) presents a good number of significant results on multiset permutations.

An early reference to multiset is also found in the work of Marius Nizolius (1498 – 1576), cf. [Ang 65] and [Sin 94]. "It is not perhaps clear whether Nizolius' multitude comes closer to 'class' or 'heap'.... But Nizolius' multitudines, might still be heaps in the sense of Quine or Goodman" ([Ang 65], p.319 – 320).

In ([Bri 87], p.1), it is noted with 'emphasis' (rightly, I think), which is missing in ([Bli 91], p.330): "...attempting to 'make sense' of Boole's algebra of Logic that multisets do have a history, and ...that honour should go to George Boole's [1854] Laws of Thought".

DOCUMENT UNIT
RABIM LIBRARY
A.B.U. Zaria

[Hai 76] has justified (by introducing signed heaps) that Boole's Laws of Thought may be interpreted as a treatise dealing with multisets. However, on the question whether Boole himself had this interpretation in mind, ([Bri 87], P. 2) notes that "Hailperin is silent on this point, but strong evidence comes from another quarter: at least one eminent 19th century Logician read Boole in precisely right way, and that was Charles Sanders Pierce". ([Bri 78], p. 294) notes that "the axiom system for Boole's original system is an axiom system for signed heaps".

Cantor himself, in his first (1895) definition of a set, defines the cardinality M^* of a set M as a collection of 'units' (one unit for each element of M), admitting repetitions (for details, see [Hal 84], pp. 128-142 and [Can 55], p. 11)

Dedekind, in his 1888 masterpiece "*was sind und was sollen die zahlen*?" ("The nature and meaning of numbers", English translation, cf. [Bli 91], p. 80) introduces the notion of msets. He considers a non-injective function ψ from the set Σ (with n elements) onto the set $\psi(\Sigma)$ (with m elements) to conclude that n is also the number of elements in $\psi(\Sigma)$ counted in this sense, while the number of its actually different elements is m. Dedekind remarks: "In this way, we reach the notion, very useful in many cases, of systems [sets] in which every element is endowed with a certain frequency number which indicates how often it is to be reckoned as element of the system". In fact, Dedekind's approach has been found to be the most fundamental.

Weierstrass defines real numbers as certain msets of rational numbers in which finitely many repetitions are allowed. For example, the quantity $\pi = 3.141\ldots$ can be identified with a multiset containing the number 1 with multiplicity 3, the element $\frac{1}{10}$ with

multiplicity 1, the element $\frac{1}{100}$ with multiplicity 4, etc. (see [Hal 84], P.134; [Bli 91], PP.325-326; and Jourdian's observations in ([Can 55], pp.16 – 18) for details).

[Whi 33] is indeed the first place where msets receive a substantive mathematical treatment, particularly in terms of algebra of characteristic functions of sets and subsequently, that of 'generalized' sets (sets whose characteristic functions are integer valued). He cites "chains in analysis situs" in which "each element is counted any number of times" ([Whi 33], p. 412).

The appendix B, entitled *Ars combinatoria*, of [Wey 49] is another rich place where msets receive utmost vindication, in theoretical as well as applied directions. Weyl defines mset as a set with an equivalence relation defined on it. Equivalent elements are said to be "in the same state" ("in the same sort" in [Mon 87]). ([Wey 49], pp. 238 – 239) notes: "... electrons may be in this or that position; atoms in a molecule may be N, He, Li, ...atoms. ... no artificial differences between elements are introduced by their levels ... and merely the intrinsic differences of states are made use of ...". Weyl applies these concepts to a variety of problems in physics, chemistry and genetics: for example, in physics, "Two individuals in the same 'complete state' (no further refinement is possible) are indiscernible by any intrinsic characters although they may not be the same thing" ([Wey 49], p.245). ([Wey 49], p. 238) notes: "...one has to distinguish between equal (= of the same kind) and identicals". Thus, an aggregate is identified by its distinct states which is precisely the concept of multiset.

Rado [Rad 75] defines an mset to be any cardinal-valued function whose nontrivial domain (the collection of elements not mapped to zero) is a set. The class of msets is called the cardinal module, like "a module over the semi-group of all cardinals" (p.135), An mset f

represents a family of sets $\alpha = (x_i)$, $i \in I$ just in case $f(x_i)$ equals the number of times x_i occurs in α (that is, $f(x) = |\{i \in I : x_i = x\}|$ for all x). Rado also makes use of signed msets (functions which may have negative cardinal values).

Parker-Rhodes ([Par 81], p. xiii) observes that "... there exists no branch of mathematics in which a third parity-relation, besides equality and inequality, is admitted". Parker-Rhodes [Par 81] develops a theory of "sorts" (collections of indistinguishables), an elaborate mathematical system, and applies it to explicate some fundamental problems of physics. "Copies of elements behave as identicals when they appear as elements of different classes, but as a plurality (each of them contributes to cardinality) when they are elements of the same class" (p.7). Elements of multisets conform to the Parker-Rhodes principle of indistinguishables. However, the system of Parker-Rhodes is a radical departure from classical mathematics because of its triparitous nature (objects may be identical, distinct or twins). The underlying idea lies in treating equality and identity as different relations.

In course of formalizing Category theory, Corcoran ([Cor 80], pp. 199 – 201) introduces msets, for example, $S_a\ [a, b]_{2,1} = [a, b]_{3,1}$, where S_a = the successor function of a.

Meyer and MacRobie [MMc 82] make use of multisets in the study of relevant implication by way of reckoning how often a premise is repeated in the course of derivation.

The principle of indistinguishability of [Par 81] has been re-emphasized in [Wil 03]. Wildberger argues that two physical objects are either different or they are the same (or equal) but separate or they are coinciding (and identical). He takes the example of a water molecule with two hydrogen atoms, say H^1 and H^2 and one oxygen atom say O and concludes that H^1 and O are

obviously different (and, for that matter, H^2 and O are different), however H^1 and H^2 are the same but separate, while H^1 and H^1 are coinciding and identical.

Wildberger ([Wil 03], p.3), by taking some examples from the real world, such as "....to a shopkeeper, any two dollar coins are equal even if their years of minting are different, where as a coin collector would regard them essentially different, etc"., proposes some very innovative conclusions like "the notion of equality is often a relative one", "traditional mathematics works with the implicit assumption that all objects exist uniquely in some ideal sense" and "relies heavily on equivalence relation", etc., which, for the time being, appear to us only a contrived idea.

In order to gauze the pervasiveness of the use of msets, we endeavour to present in page 26, an overview of results pertaining to application of msets, especially in mathematics (combinatorial and computational) and computer science, that have appeared in the recent years.

Theory of Multisets

Introduction

As to the development of a formal (axiomatic) theory of multisets, despite having voiced by a number of authors ([Sin 85], ([FBA 73], for example), their disagreement with Cantor's insistence on disallowing repeated elements in a set, nothing substantial came through for a considerable period of time. [Lak 76], inspired by Rado's remark that "there is no axiomatization for msets" during his 1974 London Mathematical Society lecture entitled " Multisets and multicardinals" , is known to be the first axiomatization of multiset theory. In [Lak 76], function (like von Neumann's axiomatization) is taken as primitive. Lake admits: '.... It might be

though desirable to have an axiomatization which does not go via functions, such an axiomatization ...could be conveniently written out using $x\in_z y$ (intended to stand for "x belongs to y precisely z times', p. 325). This is the approach taken in [Bli 89], the most recent and sustained work on this line. Unlike many other formalizations of non-Cantorian set theory, Blizard [Bli 89] formulates a formal theory of msets as a conservative extension of standard set theory". Bunder [Bun 87], using BCK-linear logic (weakened first-order logic), develops an elementary theory of msets (infact, finite msets). [Bli 91] provides an exhaustive survey of the literature dealing with the development of the theory of msets.

The term multiset as knuth ([knu 81], p. 36} notes, was first suggested by N.G.de Bruijn in a private communication to him. Owing to its aptness, it has replaced a variety of terms viz. list, heap, bunch , bag, sample , weighted set , occurrence set and fireset (finitely repeated element set) used in different contexts but conveying synonimity with mset.

As mentioned earlier, elements are allowed to repeat in an mset, finitely in most of the known application areas, albeit in a theoretical development infinite multiplicities of elements are also dealt with (see [Bli 93], [Mon 83], [Hic 80], [Lak 76], [Rad 75] and [Eil 74], in particular).

The number of copies (([Bri 87], p. 5) prefers to call it 'multiples') of an element appearing in an mset is called its multiplicity. Moreover, multiple occurrences of an element in an mset are treated without preference (perhaps to retain the force of classical concept of identity). We mention [Par 81] for an earliest extensive treatment of indistinguishability of repeated elements without any preference, and [Mon 83] and [Wil 03] for an alternative treatment.

The number of distinct elements in an mset M (which need not be finite) and their multiplicities jointly determine its cardinality, denoted by C(M). In other words the cardinality of an mset is the sum of multiplicities of all its elements. An mset M is called finite if the number of distinct elements in M and their multiplicities are both finite, it is infinite otherwise. Thus, an mset M is infinite if either the number of elements in M is infinite or the multiplicity of one or more of its elements is infinite i.e., $C(M) \geq \aleph_0$. The root or support or carrier of an mset M, denoted by M^* is the set containing the distinct elements of M. The elements of the root set of an mset are called the generators of that mset.

A considerable amount of efforts have also gone into the study of msets with negative multiplicities (see [Bli 90], [Hai 86], [Rei 85], [Rad 75], [Eil 74], [Whi 33], [Wil 03], in particular).

Representations of mset

1. **Multiplicative form**
   Following Meyer and McRobbie [1982], the use of square brackets to represent an mset has become almost standard. Thus, an mset containing one occurrence of a, two occurrences of b, and three occurrences of c is notationally written as [[a,b,b,c,c,c,]] or [a,b,b,c,c,c,] or $[a,b,c,]_{1,2,3}$ or $[a^1,b^2,c^3]$ or [a1,b2,c3] , depending on one's taste and convenience.

2. **Linear form**
   Wildberger [Wil 03] puts forward a linear notation for multisets which seems quite innovative, especially when negative multiplicities (integral as well as rational) are to be dealt with, for example,
   the mset $M = [a, b, c]_{1, 2, 3}$ can be written as

M = [a] + 2[b] + 3[c].
Similarly, a rational mset can be represented. For example,

$$N = \frac{2}{3}[5] - \frac{1}{2}[1_8].$$

In order to accommodate negative multiplicities round brackets are used: (a) in an mset stands for negative of a; for example,
[2, 4, (5), (5), 4] = [2] + 2[4] - 2[5].

In the same place, the distinction between the terms 'element' and 'object' occurring in an mset is made explicit as fallows:

Each individual occurrence of an object x in an mset A is called an element of A. Thus in the linear notation of M above; b, for example, is an object appearing twice, and every occurrence of b is an element of M. It follows that the distinct elements of an mset are the objects. An object is an element if its multiplicity is unity.

Further, the following notations used in ([Wil 03], PP. 5–6) to represent data structures of set,ordered set, multiset and list, are quite instructive:

A collection containing x, y, z,... is denoted by {xyz...} or {x_y_z...} *if it is a set*; {x, y, z,...} *if it is an ordered* set; [xyz...] = [x_y_z_...] if it is a multiset; and [x, y, z,....] if it is a list.

Note that a list is an ordered sequence of elements with repetitions allowed, whereas an mset is a sequence with its ordering stripped off.

3. **Multiset as a Sequence**

A multiset can also be represented as a sequence in which the multiplicity of an element equals the number of times the element occurs in the sequence, which is exactly Dedekind's 'frequency-number'. The idea is to construct an mset as a sequence (a function with domain $\aleph$, the set of natural numbers) and ignore the ordering of its elements, which can be done by taking all permutations of the domain of the sequence.

4. **Multiset as a Family of Sets**

A multiset can also be represented as a family of sets, which is altogether a generalization of the idea of a sequence described above. Thus, the family of sets
$F = \{Fi\}, i \in I$, where $Fi = Fj$, if $i = j$, which identifies a repeated element, represents an mset. Clearly, such a family F is a function: $I \rightarrow \{Fi \mid i \in I\}$, which in turn, is a sequence if $I = \aleph$.

5. **Multiset as a Numeric-valued function**

Representation of an mset as a numeric-valued or cardinal-valued function abounds, especially in the application areas. Formally, an mset is just a mapping from some ground or generic or universal set into some set of numbers. For example, an mset $\alpha = [x, y, z]_{1,2,3}$ is a mapping from a ground set S to $\aleph$, the set of natural numbers with zero, defined by

$$\alpha(t) = \begin{cases} 1, & \text{if } t = x \\ 2, & \text{if } t = y \\ 3, & \text{if } t = z \\ 0, & \text{for all the remaining } t \in S. \end{cases}$$

DOCUMENT UNIT

In general terms, for a given ground set S and a numeric set T, we call a mapping $\alpha: S \rightarrow T$,

$$\begin{cases} \text{a set, if } T = \{0, 1\}; \\ \text{a multiset, if } T = \aleph, \text{ the set of natural numbers;} \\ \text{a signed multiset (or, hybrid/shadow set) if } T = Z, \\ \qquad \text{the set of integers;} \\ \text{a fuzzy (or hazy) set if } T = [0,1] \subseteq \Re, \\ \qquad \text{a two – valued Boolean algebra.} \end{cases}$$

6. **Multiset as a generalized characteristic function**
Similar to the representation of a set by its characteristic function (function whose range is {0,1}), a multiset or hybrid set is determined by its generalized characteristic function (whose range is the set of integers, positive, negative or zero), see [Whi 33] for details.

Operations under Mset
[knu 81] can be considered as the earliest reference describing intuitively properties of msets in a sufficient detail. During the recent years, a good number of papers ([Lak 76], [Hai 76], [Hic 80], [Bli 89], [Wil 03], and others) have appeared. We endeavour to present an overview of various approaches in this regard. We will adhere to function- approach and use Dom(f), Ran(f) to denote the domain and range respectively of a given function f.

Definition 1. Multiset
Let $D = \{x_1, x_2, \ldots, x_j, \ldots\}$ be a set. An mset A over D is a cardinal-valued function i.e., $A: D \rightarrow \aleph = \{0, 1, 2, \ldots\}$ such that for $x \in$ Dom(A) implies A(x) is a cardinal and $A(x) = m_A(x) > 0$, where $m_A(x)$ denotes the number of times an object x occurs in A

i.e., a counting function of A. The set D is called the ground or generic set of the class of all msets containing objects from D.

An mset A can also be represented by the set of pairs as follows:

$$A = \{\langle m_A(x_1), x_1\rangle, \ldots, \langle m_A(xj), xj\rangle, \ldots\}$$

or, $A = \{m_A(x_1)\ .\ x_1, \ldots, m_A(xj).\ xj, \ldots\}$

Relatedly, an mset is called 'regular' or 'constant' if all its elements occur with the same multiplicity. Also an mset is called *'simple' if all its elements are the same, for example, $[x]_3$ is simple* mset containing x as its only object.

Definition 2. Dressed epsilon symbol, $\in_+$

The symbol $\in_+$ was first introduced by Singh and Singh [SS, to appear]. For any object x occurring as an element of an mset A i.e., $m_A(x) > 0$, we write $x \in_+ A$, where $\in_+$ (dressed epsilon is a binary predicate intended to be 'belongs to at least once', as $\in$ is 'belongs to only once' in the case of sets. Thus, $m_A(x) = 0$ implies $x \notin A$, and $x \in_+^k A$ implies 'x belongs to A at least k times', however $x \in^k A$ means 'x belongs k times to A'. The mset for any ground set D is called empty, denoted by $\varnothing$ or [ ], if $m_\varnothing(x) = 0$ for all $x \in D$.

Further, in order to make our presentation concise, we shall follow some terminologies introduced in ([Hic 80], pp. 212 – 213): A(x) denotes the number of copies of x, including x itself, belonging to Dom (A), which is exactly the Dedekind's frequency number.

Definition 3. Equal msets

Two msets A and B are equal or the same, written as A = B, iff for any object $x \in D$, $m_A(x) = m_B(x)$ or A(x) = B(x). Equivalently, A = B if every element of A is in B and conversely. Clearly, $A = B \Rightarrow A^* = B^*$, however the converse need not hold.

Definition 4. Multisubsets (or msubsets, for short)
Let A and B be two msets, A is an msubset or a submultiset of B, written as $A \subseteq B$ or $B \supseteq A$, if $m_A(x) \leq m_B(x)$ for all $x \in D$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submset of B. An mset is called the parent in relation to its msubsets.

It is easy to see that $\subseteq$ is antisymmetirc i.e.,
$A \subseteq B$ and $B \subseteq A \Rightarrow A = B$, and it is a partial ordering on the class of msets defined on a given generic domain. Clearly, $\varnothing$ is a submset of every mset.

Note that the terms 'element' and 'object' are being distinguished throughout, and coincide if a generic set is in consideration. We wish to emphasize that introduction of $\in_+$ greatly enhance the language of msets. For example, $A \subseteq B$ stands for
$\forall z \forall_k (z \in^k A \rightarrow z \in_+^k B)$.

Relatedly, a 'whole' msubset of a given mset contains all multiplicities of common elements; while a 'full' msubset contains all objects of the parent mset and accordingly, every mset contains a unique full msubset, called its root set. Clearly, for any two msets A and B, if $A \subseteq B$ and Dom (A) = Dom (B), then A is a full msubset of B.

Definition 5. Similar msets
Two msets A and B are said to be 'cognate' or similar if $\forall x(x \in A \Leftrightarrow x \in B)$, where x is an object. Thus, similar msets have equal root sets but need not be equal themselves.

Definition 6. Ordered pair of two mset terms
Ordered pair of two mset terms u and v, denoted by [u, v], can be defined as follows:

$$[u, v] = \{u, v\} \text{ if } u \neq v, \text{ and } [u, v] = \{[u]_2\} \text{ if } u = v.$$

Here, [u] is written as {u}, and ⟨u, v⟩ is actually the ordered pair set, where set (u) stands for $u = \phi \vee \forall x\ \forall n\ (x \in^n u \Rightarrow n = 1)$, though x itself may be an mset term.(see [Bli 89] pp. 42 – 44, for details).

Definition 7. Power multiset.
In Cantorian spirit, the power multiset of a given mset A, denoted by $\widetilde{\wp}(A)$ to distinguish it from the symbol $\wp(A)$ used for power set of A, is the multiset of all submultisets of A. For example, let $A = [x, y]_{2,1} = [x, x, y]$. Then,

$$\widetilde{\wp}(A) = [\phi, \{x\}, \{x\}, [x]_2, \{y\}, \{x, y\}, \{x, y\}, [x, y]_{2,1}].$$

In this sense, $C(\widetilde{\wp}(A)) = 2^{C(A)}$, for any mset A.

However, as has been voiced by many researchers in the area of msets and their applications (see [Hic 80], p. 213 and [Bli 89], p. 45, in particular), there is no 'good' reason for admitting repeated elements into a power multiset. Hence, a power multiset needs to be called a power set only and denoted by $\wp(A)$. Accordingly, for $A = [x, y]_{2,1}$,
$\wp(A) = \{\phi, \{x\}, [x]_2, \{y\}, \{x, y\}, [x, y]_{2,1}\}$ and hence $C(\wp(A)) < 2^{C(A)}$ which implies that Cantor's power set theorem: $C(A) < C(\wp(A))$ fails. However, for finite msets, Cantor's theorem holds for power mset (see [Bli 89], p.45, for related inherent difficulties if the mset in consideration is infinite).

Definition 8. Union (∪), Intersection (∩) and addition or sum or merge (+) or ⊎
Let A and B be two msets over a given domain set D.

1. $A \cup B$ is the mset defined by
   $m_{A \cup B}(x) = m_A(x) \cup m_B(x) =$ maximum $(m_A(x), m_B(x))$, being the union of two numbers. That is, an object z

occurring a times in A and b times in B, occurs maximum (a, b) times in A∪B, if such a maximum exists; otherwise the minimum of (a, b) is taken which always exists.

It follows that for any given mset x there exists an mset y which contains elements of elements of x, where the multiplicity of an element z in y is the maximum multiplicity of z as an element of elements of x along with the above stipulation on the existence of such a maximum. We denote this fact by $y = \cup x$

Clearly, $Dom(\cup x) = \cup\{Dom(A), A \in x\}$ and that the multipliticity of z in y is the maximum of its multiplicities as an element of elements of x, if it exists, otherwise, the minimum is taken.

For example, if A = [2 3 4 4], B = [1 4 3 3] then A∪ B = [1 2 3 3 4 4].

Also, it follows that for a finite mset x, the maximum multiplicity of elements of elements of x always exists. However, for certain infinite sets like
$x = \{\{y\}, [y]_2, [y]_3 \ldots\}$, the maximum multiplicity of elements of elements of x does not exist, and hence $\cup x = \{y\}$. It is obvious by definition of the union that multiplicity of any $y \in x \neq \emptyset$ is irrelevant to $\cup x$, and hence $\cup x = \cup x^*$ (see [Bli 89], pp. 48 – 49, for details).

1. $A \cap B$ is the mset defined by
$m_{A\cap B}(x) = m_A(x) \cap m_B(x)$ = minimum $(m_A(x), m_B(x))$, being the intersection of two numbers.

That is, an object x occurring a times in A and b times in B, occurs minimum (a, b) times in $A \cap B$, which always exists.

In general, for a given mset x, $Dom(\cap x) = \cap \{Dom(A): A \in x\}$ and $z \in \cap x$ implies that the multiplicity of z is the minimum of its multiplicities as element of elements of x.

For example, if A = [3 3 3 4 4], B = [1 4 3 3],
then $A \cap B$ = [3 3 4].
Note that for any mset x, we have $\cap x \subseteq \cup x$.

3. A + B or $A \uplus B$ is the mset defined by
$m_{A+B}(x) = m_A(x) + m_B(x)$, for any $x \in D$, direct sum of two numbers.

That is, an object x occurring a times in A and b times in B, occurs a+b times in $A \uplus B$.

For example if A = [1 1 2 2 4 4 4], B = [1 2 3 3] then

$A \uplus B$ = [1 1 1 2 2 2 3 3 4 4 4].

Clearly, $C(A \uplus B) = C(A \cup B) + C(A \cap B)$.
Note that if x be an infinite mset, then the multiplicity of some mset $z \in \uplus x$ may not be finite. In that case, the multiplicity of z in $\cup x$ is used, For example,

if x = $\{\{z\}, [z]_2, [z]_3 \ldots\}$ then $\uplus x = \cup x = \{z\}$
(see ([Bli 89], p. 51, for details).

4. **Some Properties holding for mset operations**
(see [knu 81] p. 636; [Bli 89], p. 53, in particular).

(i) Commutativity:

$$A + B = B + A,\ A \cup B = B \cup A,$$
$$A \cap B = B \cap A.$$

(ii) Associativity:

$$A \uplus (B \uplus C) = (A \uplus B) \uplus C,$$
$$A \cup (B \cup C) = (A \cup B) \cup C,$$
$$A \cap (B \cap C) = (A \cap B) \cap C.$$

(iii) Idempotency:

$$A \cup A = A,\ A \cap A = A, \text{ but } A \uplus A \neq A.$$

In fact, as it has been suggested recently (see [Wil 03], p. 9, in particular), in order to obtain a linear combination of msets, kA may be interpreted to denote the sum of k number of A's, where k is a natural number.

(iv) Identity laws:

$$A \cup \phi = A,\ A \cap \phi = \phi,\ A + \phi = A.$$

(v) Distributivity:

$$A \uplus (B \cup C) = (A \uplus B) \cup (A \uplus C),$$
$$A \uplus (B \cap C) = (A \uplus B) \cap (A \uplus C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

The proof of all these identities follows from the interpretation of $\cup$, $\cap$ and $\uplus$ of two natural numbers as maximum, minimum and (direct) sum respectively.

It is easy to see that $\uplus$ is stronger than both $\cup$ and $\cap$ in the sense that neither $\cup$ nor $\cap$ distributes over $\uplus$, whereas $\uplus$ distributes over both $\cup$ and $\cap$. Also,

$$\cap x \subseteq \cup x \subseteq \uplus_x$$

It is promising to observe that multiset operations form a "realm" ([Wil 03], P.9).

Definition 9. Difference and complementation
Let A and B be two msets over D, and $B \subseteq A$, then
$m_{A-B}(x) = m_A(x) - m_{A \cap B}(x)$, for all $x \in D$.

It is sometimes called the arithmetic difference of B from A. Note that even if $B \not\subset A$, this definition holds good.

It can be seen quickly that some of the consequences of the aforesaid definition are disturbing ([Hic 80], P. 214).

For example, if $A = [a, b]_{4,5}$; $B = [a , b]_{2,3}$
then $A - B = [a\ b]_{2,\ 2} \subseteq B$, contradicting the classical law:
$(A - B) \cap B = \phi$.

In order to define the complement, we follow Petrovsky ([Pet 97], pp. 3 – 4):

Let $\Im = \{A_1, A_2, \ldots\}$ be a family of multisets composed of the elements of the generic set D. Then, the maximum multiset z is defined by $m_Z(x) = \max_{A \in \Im} m_A(x)$ for all $x \in D$ and all $A \in \Im$.

Now, the complement of an mset A, denoted by $\overline{A}$, is defined as fallows:

$$\overline{A} = Z - A = \left\{ m_{\overline{A}}(x) \,.\, x / m_{\overline{A}}(x) = m_Z(x) - m_A(x),\ for\ all\ x\ \varepsilon\ D \right\}$$

It is understood that some new operations like arithmetic multiplication, raising to the arithmetic power, direct product, raising to the direct power, defined by Petrovsky, can be gainfully exploited for further research.

Definition 10. Functions between msets
The underlying assumption in defining a function between msets has been invariably not to allow mapping of identical elements to non-identical elements and hence, it amounts to defining the function between their root sets, which is just the classical definition of a function.

The function f: A → B is an injection iff
(i) f: A^* → B^* is an injection, and
(ii) $\forall z\,(z \in A^* \Rightarrow m_A(z) \leq m_B(f(z)))$

The function f: A → B is a surjection iff
(i) f: A^* → B^* is a surjection, and
(ii) $\forall z\,(z \in A^* \Rightarrow m_A(z) \geq m_B(f(z)))$.

The function f: A → B is a bijection iff
(i) f: A^* → B^* is a bijection and
(ii) $\forall z\,(z \in A^* \Rightarrow m_A(z) = m_B(f(z)))$.

For example, f: $[x]_3 \rightarrow [x]_{10}$ is an injection, f: $[x]_5 \rightarrow [y]_4$ is a surjection, and f: $[x,y]_{6,2} \rightarrow [x, y]_{3,4}$ is neither an injection nor a surjection.

For various other details, see [Bli 89] and [Hic 80].

Note that some of the consequences of the aforesaid definitions are conflicting with some fundamental theorems of the classical set theory.

1. Having defined functions between msets as above, it can be proved that Cantor's theorem does not hold, there is no injection from $A \rightarrow \wp(A)$, see ([Hic 80]) p. 215).

2. Msets of equal cardinality need not have a bijection between them. For example, $[a, b]_{1, 2}$ and [a, b, c] both contain three elements, but there can be no bijection between them because the objects and their multiplicities are different in the domain and the range of any such function. In the other words, there is no bijection between their root sets viz; f: {a, b}→ {a, b, c} can not be a bijection.

3. Schroder–Bernstein theorem fails (see [Hic 80]), p. 215] and ([Bli 89], p. 47).
   Let $A = [x_1, x_2, \ldots]_{2,4,6,\ldots}$ and
   $B = [y_0, y_1, y_2, \ldots]_{1,3,5,\ldots}$

   The function f: $A^* \rightarrow B^*$ defined as $f(x_n) = y_n$ makes f: $A \rightarrow B$ an injection so that $A \leq B$. The function g: $B^* \rightarrow A^*$ defined by $g(y_n) = x_{n+1}$ makes g: $B \rightarrow A$ an injection so that $B \leq A$. But there cannot be a bijection h: $A \rightarrow B$ since all multiplicities in A are even and that in B are odd. Note that $\leq$ is the standard dominance relation.

Definition 11. Multiset Ordering

It seems really surprising that the seminal work of Knuth ([Knu 73], pp. 213 – 214, 241 – 242), related to multiset orderings and their applications, has escaped the attention of most of us until quite recently. According to Knuth:

> *"Multiset μ_1 dominates μ_2 if both μ_1 and μ_2 contain the same number of elements and the K^{th} largest element of μ_1 is greater than or equal to the K^{th} largest element of μ_2 for all K"* (p.214).
>
> *"If a and b are multisets of m numbers each, we say that $a << b$ iff $a \hat{\wedge} b = a$ (equivalently, $a \check{\vee} b = b$, the largest element of a is less than or equal to the smallest of b). Thus $a \hat{\wedge} b << a \check{\vee} b$"(p. 241).*
>
> *"An nth level 'cascade distribution' is a multiset..." (p. 299).*

In fact, on a serious note, it is out and out a pioneering work on multiset ordering and its applications. However, [DM 79] is the earliest reference known to introducing multiset ordering and using it for proving termination of programs and term rewriting systems. In fact, it has served as a basis for host of orderings introduced in this context. We endeavour to outline the Dershowitz-Manna multiset ordering as follows:

Let S be a set equipped with a partial ordering < (irreflexive and transitive relation or, equivalently, a transitive but not an equivalence relation). Let $\mathcal{M}(S)$ be the set of all the finite msets M on S, and let << be the associated (induced by <) mset ordering on $\mathcal{M}(S)$. It is easy to see that each M is an mset with a finite carrier viz: $\{x \in S: M(x) \neq 0\}$.

The Dershowtiz-Manna Ordering

M<<N if there exist two msets X and Y in $\mathcal{M}$(S) satisfying:

(i) $\{\} \neq X \subseteq N$,

(ii) $M = (N - X) + Y$,

(iii) $(\forall y \in Y)(\exists x \in X)[y < x]$

In other words, M<<N if M is obtained from N by removing none or at least one element (those in X) from N, and replacing each such element x by zero or any finite number of elements (those in Y), each of which is strictly less than (in the ordering <) one of the elements x that have been removed. Informally, we say that M is smaller than N in this case. Similarly, >> on $\mathcal{M}$(S) with (S, >) can be defined.

For example, let S = ({0, 1, 2,...}= $\aleph$), then under the corresponding multiset ordering >> over $\aleph$, the mset [3 3 4 0] is greater than each of the following msets: [3 4], [3 2 2 1 1 1 4 0] and [3 3 3 3 2 2]. The empty set { } is smaller than any multiset.

It is also easy to observe that:

$[\forall y\ y \in N \Rightarrow \exists x\ x \in M \wedge x > y] \Rightarrow M >> N$.

For various ramifications of the Dershowitz-Manna Ordering, see [JL 82] and [Mar 89], in particular.

Applications of multisets

In this section, a modest attempt is made to present a comprehensive survey of various applications of multisets which is arranged under two major headings: Mathematics (especially, combinatorial and computational aspects) and computer science; along with some overlapping results placed appropriately.

Applications in Mathematics

An early elaborate reference is ([Knu 81], pp. 441 – 466; 636-667). Some of the Knuth's findings are as follows:

The prime factorization of an integer n > 0 is a multiset N whose elements are primes, where $\pi_{p \in N} = N$. Accordingly, as every positive integer can be uniquely factored into primes, one can obtain a bijiction between the positive integers and the finite msets of prime numbers. For example, if $n = 2^2 . 3^3 . 17$, the corresponding mset is N = [2, 2, 3, 3, 3, 17]. A simple algebra of msets is also developed.

The natural correspondence between a monic polynomial over the complex numbers and the unique mset containing its roots exists. That is, every monic polynomial f (z) over the complex numbers corresponds in a natural way to the mset F of its roots. Hence, if f(z) and g (z) are the polynomials corresponding to the finite msets F and G of complex numbers, then

$$F \uplus G = f(z)\, g(z), \quad F \cup G = \text{lcm}\,(f(z), g(z)),$$
$$\text{and} \quad F \cap G = \text{gcd}\,(f(z), g(z)).$$

Zeros and poles of meromorphic functions, invariants of matrices in canonical form and invariants of finite abelian groups do correspond to multiset representations.

Generating functions and nonnegative integer coefficients correspond one-to-one with msets of nonnegative integers. Thus, if A and B are msets of nonnegative integers, and if $G(z) = \sum_{n \in A} z^n$ and $H(z) = \sum_{n \in B} z^n$ be generating functions

corresponding to A and B, then G(z) + H(z) corresponds to $A \uplus B$, etc. Also, for msets A and B, the product of Dirichlet generating

functions $g(z) = \sum_{n \in A} \frac{1}{n^z}$ and $H(z) = \sum_{n \in B} \frac{1}{n^z}$ corresponds to the mset product AB.

Not much is known about the history of permutations of an mset. Knuth ([Knu 73], pp. 22 – 34) in line with some early references cited by himself, expounds the area of permutations of msets.

Goguen ([Gog 74], pp. 513 – 561), in course of developing a category – theoretic foundation of fuzzy set theory, investigates properties of semiring sets, as in [Eil 74], along with various applications to msets. He also discusses usefulness of msets in the study of combinatorics and formal languages.

Chapin ([Cha 74] pp. 619 – 634; [Cha 75], pp. 255 – 267), while formalizing the theory of fuzzy sets and Boolean-lattice – valued sets, seems to have mset-model in mind. The interpretation of the atomic formula $\in$ (x, y, z) as "x is an element of y with degree of membership at least z" reflects seminality, (see page 16, definition 3, of this paper, cf. [SS ...], to appear). Mathematics of multisets proves a potential tool to study computing of Gröbner Bases and straightening laws in polynomial rings (see [Buc 76] and [Bac 81], in particular).

Brink ([Bri 87], pp. 1 – 13) outlines an algebra of msets and shows that it could model relevant structures to a great extent, yet falls short of DeMorgan monoid.

Anderson [And 87] can be considered as the first sustained development of conbinatorics of msets, in particular. The area of concentration includes the study of sizes, numbers and properties of 'chains' (collections of pair wise $\subseteq$ comparable subsets) and 'antichains' (collections of pairaise $\subseteq$ incomparable subsets) in the

subset and msubset lattices (see [Wey 49], [RND 77], [Ben 84], [Cle 88, 88a], [Bli 90], in particular).

Martin ([Mar 89], pp. 37 – 54) observes that many well-founded partial orderings of the set of finite msets on a given set have appeared that played significant roles in the study of invariant theory (see [Mui 03] and [Ger 61] for early references), ring theory (see [Bac 81], and [Pop 00] for some recent developments) and the theory of partitions (see [Ben 74], [Mac 79], [Bra 82], [Gar 83], [BDR 84] and [Bli 93] for various recent developments). It is also mentioned (P. 37) that msets can be gainfully exploited in the study of measure theory which, however, would require tools of functional analysis, especially when the msets considered are infinite.

Singh and Singh [SS 03] point out a fundamental problem. It is observed that the best known efficient formula ${}^{n}H_{r} = {}^{r+n-1}C_{r}$, designed to compute the number of combinations of n distinct objects taken r at a time when objects may occur with repetitions, includes terms of the type $[a_i]_n$, for each i, that are also msubsets of an mset in which each distinct element occurs unlimited number of times (hence, help determining the cardinality of the power set of an mset), turns out to be unworkable and computationally inefficient even if adjusted to be applicable, if the mset in consideration is finite or infinite, but with finite multiplicities. For example, for a 6 – element mset $\alpha = [x, y, z]_{1,2,3}$ the number of all combinations of size – 6 (repetitions allowed and consequently $[x]_6$ – like combinations included which is not an msubset of α) is ${}^{3}H_{6} = {}^{8}C_{6} = 28$ whereas the number of 6 – msubsets of α is only one and that is α itself. Also, besides discussing computational inefficiency of Inclusion – Exclusion principle, the paper puts forward a reasonably efficient and workable formula.

Petrovsky ([Pet 94], pp. 129 – 140) outlines some very innovative mathematical applications of msets. An axiomatic foundation for metrization of multiset spaces is developed.

Poplis [Pop 00] develops a multiset algebra (chapter 8, pp. 58 – 129) in his thesis work for the award of Ph. D degree to him. The thesis, titled "The semiring of multisets", explicates that msets provide a connection between eigen values/eigen vectors equations for the Max-Plus and the nonnegative real number systems. It is shown that Max-Plus, Max-Times and the nonnegative real numbers can be viewed as a special case of msets. The guiding factors for undertaking research in the area of mathematics abound. For example, the (Max, +) algebra has been extensively used in Discrete event systems, transportation networks, parallel computations, project management, machine scheduling, to name a few (see [Pop 00], pp. 1 – 2).

The interest in multisets and subsets of commutative monoids has increased in the recent years (see [CPRS 01] for various deliberations).

Hegarty and Larson ([HL 06], pp. 1 – 25) study permutations π of the natural numbers for which the numbers $\pi(n) - n$ belongs to a given (multi) subset M of Z (the set of integers), for all $n \in S$ (a given subset of the set of natural numbers)

Applications in Computer Science

[Knu 73] and [Knu 81] are the early known references to applications of msets in computer science. Knuth notes:

> *The terminal string of a non-circular context free grammar form an mset that is a set if and only if the grammar is unambiguous* ([Knu 73], p. 636).

He introduces multisets into algorithms that compute values of x^n where x is a real quantity and n is a large positive integer ([Knu 81], pp.441-466). Multisets and permutations of multisets are applied in a variety of *search* and *sort* procedures ([Knu 73], p.717 for various page numbers).

Eilenberg ([Eil 74], chapter vi) describes a general theory of msets and applies it to automata. Various semiring structures are exploited and also, extensions to the cases of msets and field structures are studied.

Shoesmith and Smiley [SS 78], pp. 66 – 69, 113 – 114, 164, 211, 224), while studying multiple – conclusion logic, find application of msets quite useful to account for "arrays" of formulae.

Dersowitz and Manna [DM 79] introduce mset ordering and ingeniously exploit it for proving termination of certain programs, which has served a basis for many alternative and equivalent orderings subsequently proposed to date. The major intent of this seminal work can be seen in the following: "... the mset ordering ... permits the use of relatively simple and intuitive termination functions in otherwise difficult termination proofs" ([DM 79], p. 188; see also [BDH 86], pp. 346 – 357).

Huet and Oppen [HO 80], and Jouannuad and Lescanne [JL 82] introduced significant refinements of Dersowitz-Manna mset ordering. In [JL 82], it is shown that the standard mset ordering is a maximal extension function. ([JL 82], pp. 61 – 62) also provides an efficient implementation of Dersowitz – Manna mset ordering.

Peterson ([Pet 81], pp. 237 – 240) shows that the very foundation of Petri net theory, introduced by C.A. Petri in 1962, needs the use of msets (see also, [Gra 81] and [Pra 87]).

Mayer and McRobbie [MMc 82] find use of msets quite appropriate to account for how often a premise is repeated in characterizing relevance aspect of an argument. Thistlewaite, McRobbie and Meyer [TMcM 88] make use of algebra of msets developed in [MMc 82] in explicating automated theorem proving for relevance logics, especially in implementation while using the program KRIPKE. They also recognise that msets have been taken as *datatypes* in a number of programming languages ([TMcM 88], p. 26).

Bundy ([Bun 83], Chapter 13, pp. 225 – 240) exploits mset of numbers to illustrate the definition-and-conjecture-formation program of Lenat.

Manna and Waldinger [MW 85] can be considered as a sustained exposition to substantiate the fact that mathematical logic plays a fundamental role in the realm of theoretical computer science. Chapter II, pp. 505 – 527 of [MW 85] is solely devoted to explicating fundamentals of msets. A theory of BAG is developed

by way of introducing a novel binary primitive operation, $\odot$:
if an atom u has multiplicity $n \geq 0$ in a bag x, then u has

multiplicity n +1 in the bag $u \odot x$.

Reisig [Rei 85] uses msets to define "relation nets". Interestingly, exploiting mathematical intent of the definition of msets as generalized (integer-valued) characteristic functions given in [Whi 33], the novel idea of "negative multiplicity" in an mset is introduced ([Rei 85], p.126; see [Bli 90] for a detailed exposition). Reisig also outlines the concept of "multirelations" – an mset whose domain is the Cartesian product of a set of sorts ([Rei 85]. pp. 126 – 131).

Yager ([Yag 86], pp. 23 – 27; [Yag 87] pp. 441 – 446), after developing an algebra of msets, introduces the notion of "fuzzy bag" as follows:" ...to each element x in a fuzzy bag A is associated a multiset containing elements α (real number in the interval [0, 1]) with multiplicities n (non-negative integers). The number n indicates the number of times the element x appears with membership grade α in the fuzzy bag A" ([Yag 86], p. 33). Yager notes that Zadeh fuzzy sets are special cases of fuzzy bags ([Yag 86], p. 35).

Grzymala – Busse ([Grz 87], pp. 325 – 332) extends the notion of "rough" set (introduced by Pawlak) to rough mset (see also, [Miy 96, 99]).

Martin [Mar 87] constructs a number of extension functions for mset orderings. Also, martin [Mar 89] introduces various wellfounded partial orderings of the set of all finite msets whose elements are taken from a given set. Martin exploits the notion of *cone* in R^n (n-dimensional real space) and provides a systematization of the construction and classification of various well founded partial orderings underlying in proofs of program termination (see [DM 79], [HO 80], [JL 87], [MS 93], etc.), in term rewriting systems (see [Der 85], [KB 70], [MO 87], [Oka 87], [Mar 87], [DJ 90], [BPM 96], [CFG 96], [CDLMS 00], etc.), and computer algebra (see [Eil 74], [Buc 76], [DPR 98], [Pop 00], etc.).

Pratt ([Pra 87], pp. 33 – 71]), besides giving several arguments for the use of *pomsets*, specially showing how pomsets can be used to represent *parallel processes*. He also describes how Petri nets can be modelled as pomsets.

Gischer ([Gis 88] pp. 199 – 224) exploits the notion of a partial string and a partial language introduced in [Gra 81] to show how pomsets can be used as a model of concurrency.

Applications of msets in Logic Programming languages is found to overcome "computational inefficiency" inherent in otherwise situation, especially in solving a sweep of real-life problems where multiple occurrences of an identical element are persistent. In fact, both the usual options viz, "attaching different levels" or "assigning numbers" to account for multiplicity turn out to be computationally inefficient. Representation of msets in a logic Programming language can be effected, for example, by introducing a binary associative and commutative function symbol ○, which also admits a unit element. In a constraint logic programming language, constraints are built up from mset operations and relations (see [GHSST 92], pp.177 – 191, and also an extended abstract on "Logical Programming with multisets" pp. 1 – 3). Kiziltan ([Kiz 04], Chapter 7, pp. 165 – 202) is a recent work on multiset and strict multiset ordering constraints. Rule based multiset programming paradigm is extensively exploited to study synthetic biology (see [Ban 93], [CPRS 01] and [Kri 05] for details). Basically, multisets are interpreted to represent biological systems, such as molecules in a biochemical system. A host of simulators for biological systems has been developed and found useful to several fields of biology such as biochemistry, microbiology and evolutionary biology. For example, a system could be a multiset [A, A, B, B, C], where A, B and C are elements that evolves by means of application of a set of rewriting rules (say, of the biological systems) viz;

$\{\{A, B\} \rightarrow \{C\}, \{C\} \rightarrow \{A, B\}\}$. If we choose the first rule to apply to the multiset in the example, the multiset {A, B, C, C} is obtained (see [BMMT 05], for details). The interaction among elements in a multiset object space, which includes the

environment, are like chemical reactions and the evolution of multiset can mimic the biological evolution leading to plausibility of DNA computing and programmable living machines (see [Kri 05], for details).

The specification of an abstract data type (ADT) for msets has been studied in ([BC 93] pp. 1 – 29). Dovier, Policriti and Rossi ([DPR 98], pp. 208 – 234), after developing mathematics of msets, substantiate that msets are the fundamental data structures for P systems. In fact, in computational sense, an mset is just a data structure, which differs from a set in permitting repetition of some of its elements and that from a *list* in being unordered, and hence it turns out to be a suitable modelling tool for a large class of real-life phenomena. Ross and Stoyanovich [RS 04], study cardinality bounded msets in Database systems to overcome consistency and performance problems that conventional representations in relational database suffered from.

Petrovsky ([Pet 97], pp. 174 – 184) explores a new dimension. He, after providing an extensive treatment of mset operations, discusses "Cluster" analysis in mset spaces, and demonstrates their application in augmenting Decision Support System (DSS). It is shown that "a multi-aspect analysis of problem and structuring alternatives allow us to gain an insight into the problem nature and find better decisions. ...suggested the tools for structuring a collection of objects represented by many qualitative attributes when a lot of copies of objects or values of attributes describing them exist. ...approach is based on a theory of multiset metric spaces and indexes of difference/similarity between mutisets".

A "rough" set approach to multi–attribute decision analysis of Pawlak and Slowinski ([PS 94], pp. 443 – 459) is a pioneering work in this direction.

5. Concluding remarks and some future directions

Multiset theory, owing to its multitudinal applications, has occupied a central place among a number of non-classical (nonstandard) set theories developed during the last few decades:

> *Boolean– and Heyting-valued set theory, Zadeh's Fuzzy set theory, Vopenka's Alternative (or Semi-) set theory, Intuitionistic and constructive set theory, Quantum set theory,. daCosta's paraconsistent set theory, Pawlak's Rough set theory, Tarski's set theory without variables, Aczel's Non-well-founded (or Hyper) set theory, Extremal set theory, etc.*

None of these theories endeavours to replace classical set theory rather intend to supplement or generalize it. Zermelo – Fraenkel (ZF) set theory is still the best candidate for a secure foundation for mathematics. Category theory is emerging as a strong alternative: Epsilons are replaced by "Arrows", but that time is not yet ripe. Currently, all efforts seem to point to discovering some primitive theory of structures (a pre-set theoretic) to which all set theories can be alluded, (see [Bli 91], pp. 321-322, for details).

Some future directions

1. Pursuing the techniques followed in [TMcM 88], an extension of the program KRIPKE can be achieved by interpreting a multiset of formulae viz., [A, B^m, C^n, ...] as an 'Ordinal' tree viz, [$A.B^m.C^n$, ...], where A, B, C, ... stand for 'events' and A^k for 'A occurs K times'. ([Sin 95], p. 14).

2. A reasonable expectation is that a set theory based on ω – valued logic ("$x \in y$" can have a truth value $n \in \omega$) and a multiset theory based on classical 2-valued logic ('$x \in^n y$ can

DOCUMENT UNIT
LIBRARY
Paris

have only two truth values) are equivalent theories ([Bli 91], pp. 337 – 338). [See Skolem ([Sko 62], chapter 18): "... it seems to be possible to obtain a consistent set theory with an unrestricted axiom of comprehension if all rational numbers ≥ 0 and ≤ 1 are allowed as truth values" (p.69).

3. "Multiset as a model for Multi-Attribute objects" is expected to revolutionize the area of mathematical modelling, discovery of intelligent systems, control of non-linear mechanical systems, just to mention a few.

4. Discovering competing simulators for biological systems seems to dominate the researches in biotechnology for a foreseeable future

Acknowledgement

Despite my own observations presented in my Ph. D. Thesis (1970) and later in [Sin 85], it was [Bli 89] and [Bli 91] that motivated my research in this area. Our list of references is largely an extension of [Bli 91] to date. However, as all round numbers are false (Samuel John, C. 1750), no such list can be claimed to be complete at any point of time and hence, our profound apologies are extended to those who might have been missed out.

I sincerely wish to thank my two current Ph. D scholars Mr. Tela Yohanna and Mal. Ibrahim Adeku Musa, who contributed a lot in preparation of this paper by way of procuring relevant materials and providing stimulating comments throughout. I must thank the Ahmadu Bello University administration for inviting me to a Senate Inaugural Lecture to present this paper. Finally I wish to dedicate this work to my beloved wife.

34. [Gis 88] Gischer L. Jay,The Equational Theory of POMSETS. *Theoretical Computer Science Vol. 61, pp. 199 – 224. North-Holland, 1988.*
35. [Gog 74] Goguen J. A., Concepts Representation in natural and artificial Languages: Axioms, Extensions and Applications for Fuzzy Sets, *International Journal of Man-machine Studies, Vol. 6, pp. 513 – 561, 1974.*
36. [Grz 87] Gryzymala-Busse J. W., Learning from Examples based on rough multisets, in Z.W. Ras and M. Zeamankova (editors), *Methodologies for Intelligent Systems, North Holland, pp. 325 – 332, 1987.*
37. [Hai 86] Hailperin T., Boole's logic and probability, *Second edition, North Holland, 1986.*
38. [Hal 84] Hallet M., Cantorian Set theory and Limitation of Size, *Clarendon Press, Oxford, 1984.*
39. [Hic 80] Hickman J.L., A Note on the concept of Multiset, *A bulletin of the Australian Mathematical society, Vol. 22, pp. 211 – 217, 1980.*
40. [Hig 52] Higman G., Ordering by Divisibility in an Abstract Algebra, *Proceedings, London Mathematical Society (3), Vol.2, pp. 326 – 336, 1952.*
41. [Ifr 85] Ifrah G., From one to zero: A universal History of numbers, *Viking Penguin, New York, 1985.*
42. [JL 82] J-P Jouannaud and P. Lescanne. On multiset orderings, *Information Processing letters Vol. 15 ,pp. 57 – 63, 1982.*
43. [KB 70] Knuth D. E. and Bendix P.B., Simple Word problems in Universal Algebras, *in: J. Leech, ed, Computationals problems in Abstract Algebra (Pergamon, Oxford), pp. 263 – 297, 1976.*
44. [Knu 73] Knuth D.E, The Art of computer programming, *Vol. 3, (Sorting and Searching), Addison-Wesley, Reading, Mass., 1973.*
45. [Knu 81] Knuth D.E., The Art of computer programming, *Vol. 2 (Seminumerical Algorithms), Second Edition, Addison-Wesley, Reading, Mass., 1981.*
46. [Kri 05] Krishinamwithy E. V., Rule-based Multiset Programming Paradigm: Applications to Synthetic Biology, *Computer Sciences Laboratory, Austrialian National University, Carberra, ACT 0200, Australia, 2005.*
47. [Lak 76] Lake J., Sets, Fuzzy sets, Multisets and functions, *Journal of the London Math. Society (2), Vol. 12, pp. 323 – 326, 1976.*

48. [Mac 79] Macdonald I.G, Symmetric Functions and Hall Polynomials, *Oxford University Press, 1979.*
49. [Mar 87] Martin U., How to choose the weights in the Knuth-Bendix ordering, *in: Proc. Second International Conference on Rewriting Techniques and Applications, Lecture Notes in Computer Science, Vol. 256, (Springer, Berlin), pp. 42 – 53, 1987.*
50. [Mar 87a] Martin U., Extension Functions for Multiset orderings, *Information Processing Letters, Vol. 26, pp.181 – 186, 1987.*
51. [Mar 89] Martin U., A geometrical approach to Multiset orderings, *Theoretical Computer Science, Vol.67, pp. 37 - 54, 1989.*
52. [MK 95] Murthy V. K. and Krishnamurthy E. V., Probabilistic Parellel Programming based on multiset transformation, future generation computer systems, 11, pp. 283 295, 1995.
53. [MM 82] Meyer R. K and McRobbie M. A., Multisets and Relevant Implication I and II, *Australasian Journal of Philosophy, Vol. 60, pp. 107 139 and pp. 265 281, 1982.*
54. [Mon 87] Monro G.P., The Concept of Multiset, *Zeitschrift fur Mathematische Logik und Grundlagen der Mathematik, Vol 3, pp. 171 – 178, 1987.*
55. [MW 85] Manna Z. and Waldinger R., The Logical Basis for Computer Programming: Deductive Reasoning,*Vol.2, Addison-Wesley, Reading, Mass., 1985.*
56. [Par 81] Parker-Rhodes A.F., The theory of indistinguishables, *D. Reidel, Dordrecht, 1981.*
57. [Pet 81] Peterson J. L., Petri Net Theory and the Modelling of Systems, *Prentice- Hall, Englewood Cliffs, N. J., 1981.*
58. [Pet 94] Petrovsky A. B., An axiomatic approach to Metrization of Multiset spaces. *Springer, pp. 265 – 270, 1994.*
59. [Pet 97] Petrovsky A. B., Structuring techniques in multiset spaces. *Springer, pp. 174 184, 1997.*
60. [Pop 00] Poplin, P. L., The Semiring of Multisets, Ph. D. Theses, *North Carolina State University, 2000.*
61. [Pra 87] Pratt V. R., Modelling Concurrency with Partial Orders, *International Journal of Parallel Programming, Vol. 15, pp. 33 - 71, 1987.*
62. [PS 94] Pawlak Z. and Slowinski Y. R., Rough Set approach to Multi-Attribute Decision Analysis. *E. J. Op. Res. Vol. 72, pp. 443 – 459, 1994.*

63. [RS 04] Ross K. A. and Stoyanovich A., Symmetric Relations and the Cardinality-Bounded Mutlisets in Database Systems, *Proceedings, 30th VLDB conference, Toronto, Canada, 2004.*

64. [Rad 75] Rado R., The Cardinal Module and some Theorems on Families of Sets, *Annali di Matematica pura ed applicata (4) Vol.102, pp 135-154, 1975.*

65. [RND 77] Reingold E.N., Nievergelt J. and Deo N., Combinatorial Algorithms: Theory and Practice, *Prentice-Hall, Englewood Cliffs, N. J., 1977.*

66. [Rei 85] Reisig W., Petri Nets: An Introduction, *Springer-Verlag, Berlin, 1986.*

67. [Ren 96] Rensink A., Algebra and theory of order-deterministic partially ordered multiset, *Notre Dame Journal of formal logic, Vol.37, NO.2, 1996.*

68. [SS 78] Shoesmith D. J. and Smiley T. J., Multiple Conclusion Logic, *Cambridge University Press, Cambridge, 1978.*

69. [Sin 70] Singh D., A Systematization of set theory, *Ph.D. Thesis, Patna Univ, India, 1970.*

70. [Sin 85] Singh D., On Cantor's Concept of Set, *International Logic Review, Vol. 32, pp. 72 – 78, 1985.*

71. [Sin 94] Singh D., A Note on the Development of Multiset Theory, *Modern Logic, Vol. 4, pp. 405 – 406, 1994.*

72. [Sin 95] Singh D., An extension of the PROGRAM KRIPKE (Abstract), *Journal of Symbolic Logic 1, Vol. 1, pp. 114, 1995.*

73. [SS 03] Singh D. and Singh J. N., Some Combinatorics of Multisets, *International Journal of Mathematical Education in Science and Technology, Vol. 34, No.4, pp. 489-499, 2003.*

74. [SS ...] Singh D. and Singh J. N., A note on the definition of multisubset (to appear).

75. [SSC 00] Singh D., Singh J. N. and Chinyo D.T., An outline of multiset theory, *MC 19, 2000.*

76. [Sko 62] Skolem T. A., Abstract Set Theory, *Notre Dame Mathematical Lectures, Number 8, University of Notre Dame, Notre Dame, Indiana, 1962*

77. [TMcM 88] Thistlewaite P. B., McRobbie M. A. and Meyer R. K., Automated Theorem Proving in Non-Classical Logics, *Research Notes in Theoretical Computer Science, Pitman, London, 1988.*

78. [Und 95] Underwood J., Typing Abstract Data Types, *Workshop on Algebraic specifications of Abstract Data Types (10), 1995.*

79. [Wei 81] Weidner A. J., Fuzzy Sets and Boolean-Valued Universe, *Fuzzy Sets and Systems, Vol. 6, pp. 61 – 72, 1981.*
80. [Wes 88] West D.B., Review of Combinatorics of Finite Sets, *Bulletin (New Series) of the American Mathematical Society, Vol. 19, pp 515-520, 1988.*
81. [Wey 49] Weyl H., Philosophy of Mathematics and Natural Science, *Princeton University Press, Princeton, N. J., 1949.*
82. [Whi 33] Whitney H., Characteristic Functions and the Algebra of Logic, *Annals of Mathematics, Vol. 34, pp. 405 – 414, 1933.*
83. [Wil 03] Wildberger N.J., A new look at multiset. *School of Mathematics, UNSW Sydney 2052, Australia, 2003.*
84. [Yag 86] Yager R.R., On the Theory of Bags, *International Journal of General Systems, Vol. 13, pp. 23 – 37, 1986.*
85. [Yag 87] Yager R. R., Cardinality of Fuzzy Sets Via Bags, *Mathematical Modelling, Vol. 9, pp. 441 – 446, 1987.*



Printed by Ahmadu Bello University Press Limited,
P.M.B. 1094, Samaru, Zaria, Kaduna State.
Telephone; 069-550054 E-mail: abupresslimited@yahoo.com